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EFFECT OF VARIATIONS IN THE ASSUMED FIGURE OF THE EARTH ON THE MAPPING OF A LARGE AREA

BY

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PREFACE.

This article was written to give some sort of answer to the questions: (1) How much might the best obtainable spheroid for the earth as a whole differ from the Clarke Spheroid of 1866, now officially adopted for the whole of North America? (2) How much difference would the adoption of such a spheroid make in the geographic coordinates of points now referred to the Clarke Spheroid? (3) Is it probable that, in the foreseeable future, it will be necessary or desirable to give up the Clarke Spheroid for geographic purposes and to adopt some other? (4) Are present methods of adjusting triangulation for obtaining the figure of the earth and the deflections of the vertical adequate when extended to a territory so large as the whole of North America?

The questions involve matters of opinion and the answers to them must necessarily involve opinion to a certain extent and must therefore be rather general in character. If the questions were made more specific, the answer would involve a great deal of numerical calculation, tedious to do and equally tedious to read about. It was not intended to overload this article with matter of that sort. Such calculations as are explained in detail have been placed separately at the end.

Dr. O. S. Adams and F. W. Darling, mathematicians, gave valuable assistance in making and checking these calculations.

The last chapter contains the formulas for transferring geographic coordinates over a long geodesic line of given length and direction and for the inverse problem, that of finding the length and azimuth of a geodesic line when the geographic coordinates of its end points are given. Numerical examples of the use of these formulas are also given. The calculations contained in other chapters of this publication were mostly made by these formulas. This matter was put in as an afterthought, since it appeared not to be readily accessible except in languages other than English.

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EFFECT OF VARIATIONS IN THE ASSUMED FIGURE OF THE EARTH ON THE MAPPING OF A LARGE AREA.

By Walter D. Lambert, *Mathematician, U. S. Coast and Geodetic Survey.*

VARIOUS DETERMINATIONS OF THE EARTH'S MEAN FIGURE.

The following table gives the principal determinations of the earth's mean figure, beginning with the work of Méchain and Delambre. Their work was undertaken for the purpose of obtaining the length of the meter, which was at that time defined as one ten-millionth part of the earth's meridian quadrant.[1] Their work forms a convenient transition between the older and the newer epochs in geodesy. The modern era may be considered as fully established by the time of Bessel.

Elements of the earth's mean figure.

Author.	Date.	Semi-major axis (*a*).	Reciprocal of the flattening (1/*f*).	Author.	Date.	Semi-major axis (*a*).	Reciprocal of the flattening (1/*f*).
		Kilometers.				*Kilometers.*	
Commission générale des poids et mesures (for the metric system).	1799	6,375.739	334.29	Clarke	1858	6,378.294	294.26
Delambre	1810	6,376.428	311.5	Do	1863	6,378.288	294.36
Do	1810	6,376.523	308.65	Pratt	1863	6,378.245	295.26
Walbeck	1819	6,376.895	302.78	Clarke	1866	6,378.207	294.98
Schmidt	1828	6,376.959	297.65	Fischer	1868	6,378.338	288.50
Everest	1830	6,377.253	300.80	Clarke	1880	6,378.249	293.47
Bessel	1841	6,377.397	299.15	Harkness	1891	6,377.972	300.20
Everest	1847	6,376.634	311.04	Helmert	1907	6,378.200	298.3
Airy	1849	6,377.491	299.32	Hayford	1907	6,378.283	297.8
James and Clarke	1856	6,377.936	297.72	Hayford(1)	[1] 1909	6,378.388	297.0
				Hayford(2)	[2] 1909	6,378.062	298.2

[1] Using deflections of the vertical corrected for topography and isostatic compensation.
[2] Using deflections of the vertical left uncorrected and the spheroid determined to fit as nearly as possible to the uncorrected deflections.

The data in the above table are taken chiefly from La Figure de la Terre (Revue de Géographie Annuelle—Tome II, 1908), by Capt. (now Col.) G. Perrier. Another similar list is given in Professional Paper No. 16 of the Survey of India, The Earth's Axes and Triangulation, by J. de Graaff Hunter, Dehra Dun, 1918. Other values of the semimajor axis differing slightly from the foregoing may sometimes be found in other sources of information. The discrepancies will usually be due to the use of relations between the foot, toise, and meter differing from those here assumed. The above table is based mainly on the relations between these quantities determined by Clarke.

[1] The length of the meter is now defined by the International standard meter kept at Sèvres, near Paris.

The semiminor axis (b) corresponding to the Hayford (1) Spheroid 1909 is 6,356.909 km. (See, however, the footnote on p. 10.) The relation between the semiaxes a and b and the flattening f is

$$b = a - af$$

If the flattening be held fixed, and a be varied, then b varies in the same direction as a by an almost equal amount (less than the change in a by only about one three-hundredth part). If a be held fixed, and the flattening f varied, then a change of one unit in the *reciprocal* of the flattening, i.e., in $1/f$, will change b by about 72 m.; the *reciprocal* of the flattening and the semiminor axis b increase and decrease together, a being held constant.

The determination in the preceding table that is in all probability nearest to the truth is the Hayford (1) result of 1909 ($a = 6{,}378.388$ km., $1/f = 297.0$). These values were adopted by the Paris (Astronomic) Conference of 1911, and are used in the American Ephemeris and Nautical Almanac. The probable errors[2] of these determinations of a and $1/f$ are, respectively, 0.035 km. and 0.8. The reciprocal of the flattening is further confirmed by (1) the results of pendulum observations and (2) deductions from the precession of the equinoxes. The value 298.3 for the reciprocal of the flattening attributed to Helmert in the table is from pendulum observations and a later determination from a greater number of observations taken all over the world (without correction for topography and isostatic compensation, but using only stations chosen where this correction is certainly small) is 296.7, and Bowie's value[3] from observations throughout the world, fewer in number but corrected for topography and compensation is 297.4; the mean of these two results from pendulum observations is almost exactly 297.0. The flattening deduced from the precession of the equinoxes depends on the law assumed for the variation of density within the earth, but only to a surpisingly slight extent Véronnet has shown[4] that with any law of density consistent with hydrostatic equilibrium in the interior of the earth we have from the precession

$$1/f = 297.12 \pm 0.38$$

Here the quantity following the double sign indicates not the usual probable error, but the range due to possible variations in the law of density. The precessional constant adopted by Véronnet is itself subject to error, which fact widens the limits of uncertainty to perhaps ± 1.0. The best available evidence, therefore, points to a flattening the reciprocal of which is about 297.

It will be noted that the value of the semimajor axis a deduced by Hayford is greater than any other value in the table. This quantity a can not be deduced from pendulum observations. This larger value of a is due to the correction for topography and isostatic compensation. The neglect of this correction in previous discussions has left outstanding in general the error due to the deflection of the plumb line toward the interior of the continental land masses, which

[2] Helmert: Sitzungsberichte der königlich preussischen Akademie der Wissenschaften, 1911, p. 19.
[3] Investigations of gravity and isostasy. U. S. Coast and Geodetic Survey Special Publication No. 40, pp. 131–134.
[4] Journal des mathématiques pures et appliquées, vol. 77 (1912), p. 416.

fact results in greater apparent curvature over continental areas than over the earth as a whole and diminishes the radii of curvature deduced from the uncorrected deflections. This fact appears when the figures of Hayford (1) and Hayford (2) are compared. It is to be expected, therefore, that future determinations in which the correction for isostasy will be made, will make the dimensions of the earth's figure greater than previous ones other than Hayford's and give results closer to Hayford's. The probable error of ± 0.035 km. (35 m.) comes from the observations themselves, and it is well known that probable errors so deduced give too low an estimate of the real uncertainty; this is due to the presence of systematic errors. It is probably safe to put the practical limits of uncertainty of a in Hayford (1) as less than ± 150 m.

SPHEROIDS ACTUALLY IN USE FOR GEOGRAPHIC PURPOSES.

Many of the spheroids given in the preceding table represent attempts to determine the earth's figure for scientific purposes only, and were never actually used for geographic purposes. The following table gives the elements of some of the spheroids actually in use. It makes no pretensions to completeness, but is intended rather to illustrate the variety of spheroids used. It contains special spheroids not found in the preceding table; these were presumably determined to fit local conditions rather than the general figure of the earth.

Spheroids used for geographic purposes.

Author and date.	Semimajor axis (a).	Reciprocal of flattening ($1/f$).	Countries using spheroid (with reference).
	Kilometers.		
Clarke, 1866	6,378.206	295.0	United States (1), Canada (2), Mexico (2).
Clarke, 1880	6,378.249	293.47	France (3), South Africa (4).
Everest, 1830	*6,377.253	300.8	India (5).
Plessis	6,376.523	308.64	Map of France (6).
Bessel, 1841	6,377.397	299.15	Germany (7), Austria (8), Dutch East Indies (9).
Kraijenhoff	6,376.950	309.65	Holland (9).
Danish Survey	6,377.019	300	Denmark (10).

* 20,922,931.80 feet is the official value.

(1) U. S. C. and G. S. Special Publication No. 8 (6th ed.), Formulae and Tables for the Computation of Geodetic Positions, p. 5.

(2) U. S. C. and G. S. Special Publication No. 19, Primary Triangulation on the 104th Meridian, etc., p. 80.

(3) Nouvelles tables de logarithmes à cinq et à quatre décimales. Service Géographique de l'Armée, 1889, near end (no paging) under "Nombres Usuels et leurs Logarithmes."

(4) Report International Geodetic Association, 16th Conference (London and Cambridge, 1909), vol. II, p 278.

(5) Auxiliary Tables to Facilitate Calculations of the Survey of India, 4th ed., 1906, p. 8.

(6) U. S. C. and G. S. Special Publication No. 49, Lambert Projection Tables with Conversion Tables, p. 5.

(7) Albrecht: Formeln and Hilfstafeln für geographischen Ortsbestimmungen.

(8) Die Ergebnisse der Triangulierungen des k. u. k. Geographischen Institutes, vol. 1, Vienna, 1901, p. IX.

(9) Tafelen benoodigd bij het hydrographisch opnemen. Department van Marine. The Hague, 1913, p. 81.

(10) Report International Geodetic Association, 16th Conference (London and Cambridge, 1909), vol. II, p. 272.

A spheroid with a flattening equal to that of the Bessel Spheroid, but with its major axis greater than the major axis of the Bessel Spheroid by 1 part in 10,000 has been extensively used by the Central Bureau of the International Geodetic Association for geodetic calculations relating to Europe.[5] This choice was made because many

[5] Encyclopedia Brittanica, 11th ed., under Earth, figure of, p. 813.

tables to facilitate computation have been based on the Bessel Spheroid, and these can be adapted to the increased major axis with comparative ease, while a change in the flattening would require a recomputation of the whole set of tables.

EFFECT OF A CHANGE IN THE DIMENSIONS OF THE SPHEROID ON THE COMPUTATION OF A TRIANGULATION.

In computing a triangulation the following quantities are used which involve the dimensions of the spheroid adopted: (1) The correction to the azimuths for the elevation of the object sighted, (2) correction to the azimuths to reduce from the azimuth of a vertical section to that of a geodetic line, (3) the spherical (or spheroidal) excess of the triangles, and (4) the reduction of horizontal angles from the geoid to the spheroid of reference.

(1) The correction (x_1) to an azimuth for the elevation of the object sighted is always small and is given by the formula,

$$x_1 = \frac{fh}{a \sin 1''} \cos^2 \phi \sin 2\alpha.$$

In this formula h is the elevation of the object sighted above sea level, a is the semimajor axis of the spheroid used (in the same units as h), f is the flattening, ϕ is the latitude of the point of observation, and α is the azimuth of the line. The largest values of x_1 that occur in the United States Transcontinental Triangulation are where it goes through the Rocky Mountains, and they are less than 0''.30. To change a correction of 0''.30 by 0''.01 or 1/30 of itself would require a change of about 10 units in $1/f$, which is very unlikely, or a change of over 200 km. in a, which is absolutely out of the question. A change of 0''.01, even in the larger values of x_1, is therefore to be considered as improbable, and the effect of a change in the dimensions of the spheroid upon this correction may be considered negligible for practical purposes.

(2) The correction (x_2) of an azimuth to reduce it from the azimuth of a vertical section to that of a geodetic line is given by the formula

$$x_2 = \frac{-fs^2}{6\, a^2 \sin 1''} \cos^2 \phi \sin 2\alpha.$$

The notation is the same as before, and s denotes the distance from observer to object sighted expressed in the same units as a. For a line 294 km. long, which is the longest side of a triangle found in the United States Transcontinental Triangulation, the maximum value of this correction is about 0''.25 at the Equator and less in higher latitudes. This correction is always so small that it is neglected entirely in computations of the Coast and Geodetic Survey, and, of course, changes in the correction due to changes in the spheroid are negligible a fortiori. It may be noted that the corrections x_1 and x_2 both reach their maximum numerical amounts in a direction inclined 45° to the meridian, but they are of opposite sign, so that in practice there is considerable compensation even of the small changes due to a change of spheroid.

(3) The spheroidal[6] excess of a triangle is affected only slightly by a change in the spheroid. Clarke[7] computes the effect on the excess of a large triangle in Ireland of varying each of the semiaxes of the ellipsoid 1,000 feet. Although the signs of the variations of the semiaxes were so taken as to act in the same way on the excess, the total effect was about 1/6800 part, or since the excess was 49″, the effect was about 0″.007. Spheroidal excesses of this magnitude are rare even in precise triangulation in mountain country, although there is one case in the transcontinental triangulation where the excess is 120″. For this triangulation a change of 1,000 feet in the semiaxes would make a change of 1/10000 of the excess, or 0″.012. Furthermore, 1,000 feet is an excessive estimate of the probable correction to the Clarke Spheroid used in the United States, so that as a practical conclusion we may say that the spheroidal excesses would be practically unaffected by a change in the spheroid.

(4) In measuring horizontal angles the graduated circle of the theodolite is leveled with reference to the actual plumb line of the point of observation. The triangles to be computed, however, lie on the spheroid of reference and to measure their angles directly the graduated circle should be leveled with reference to the normal to the spheroid of reference. The directions of the normal to the spheroid and of the plumb line do not, in general, coincide; the difference in direction is the deflection of the vertical. Since it is not practicable to level the theodolite with reference to the normal to the spheroid, a correction to the horizontal angles actually measured should in strictness be applied in order to reduce them to the spheroid. This correction is zero for observations on points the zenith distance of which is 90°.

The correction (x_4) to a direction for the noncoincidence of the normal to the spheroid of reference with the actual plumb line is given by the formula

$$x_4 = \cot \zeta \ (\xi \sin \alpha + \eta \cos \alpha) + \eta \tan \phi.$$

In this equation ζ is the zenith distance of the object sighted, α is its azimuth reckoned from the south through the west, ϕ is the latitude of the point of observation, and ξ and η are the deflections of the vertical in the meridian and the prime vertical, respectively, or more specifically,

ξ = Astronomic latitude *minus* geodetic latitude,
η = (Astronomic longitude *minus* geodetic longitude) cos ϕ,

in which west longitude is positive.

The sign of the correction x_4 as given is such that when applied to the astronomic azimuth it gives the geodetic azimuth. When the correction to an angle is computed as the difference of the corrections to the two directions of its sides, the term $\eta \tan \phi$ disappears.

To make this correction it would be necessary to have observations of latitude and longitude (or azimuth) at every triangulation station. As this ideal is not practicable on account of the expense,

[6] In practice the spheroidal excess is computed as if it were a spherical excess; this process is amply accurate except for precise work with large triangles. In any case for the purpose in hand, which is to estimate the change in excess due to a change in spheroid, it is sufficient to calculate the excess as spherical, the curvature of the sphere used being the average of the mean curvatures at the vertices of the triangle.

[7] Geodesy, p. 292.

this correction has never been applied on an extensive scale. It is easy to see that for observations on a station a degree or two above or below the horizon and with ξ or η of the order of magnitude of 10″ the correction for a direction may amount to several tenths of a second, and under exceptional circumstances the correction may exceed a second. Since ξ and η depend on the spheroid of reference used, the computation of the triangulation would in theory be affected by a change in spheroid, but since in general practice the correction x_4, because of lack of data, is not applied at all, the actual routine of the computation is unaffected.

It seems desirable, however, that in refined work the correction x_4 should be included, especially in mountainous regions where ξ and η are likely to be large, and ζ to differ considerably from 90°, conditions both tending to make x_4 large. An unpublished trial computation by E. F. Church, of the Coast and Geodetic Survey, for triangulation of the Salt Lake base net (with estimated deflections at points between astronomic stations) failed to bring out any evident cumulative error due to the neglect of this correction. On the other hand, the discrepancy in length between two base lines in Switzerland (Aarberg in the northwest and Bellinzona in the southeast) was 86 units in the seventh decimal of the logarithm before the angles of the Bellinzona base net were corrected for the deflections of the vertical and only one unit after the angles were corrected.[8]

It seems probable also that one reason why Laplace's relation between deflections in the prime vertical as determined by longitudes and by azimuths is, in general, less closely satisfied than would be expected from our estimates of the errors of the triangulation is precisely the neglect to reduce the horizontal angles to the spheroid.[9]

We thus see that a change in the spheroid of reference (within reasonable and probable limits) has almost no effect on the triangulation as ordinarily computed. The question is, however, less simple if we suppose the spheroid to which the triangulation is fitted to be extremely erroneous, or the triangulation very extensive. Then the question becomes one of the distortion necessary to make two surfaces of different curvatures fit together at all points, and this problem can not be solved until its conditions are more fully specified. (See p. 11.) For changes of the order of magnitude we have been considering the question is not important, as has just been shown. Hayford's determination of the spheroid used the Clarke Spheroid of 1866 as a basis. Corrections were determined to this spheroid from observations covering the entire United States, but it was assumed that no second approximation was necessary in order to correct the geodetic positions for changes in the triangulation due to new values for the spherical excesses, etc. The effect of a change in the dimensions of the spheroid for any part of the United States covered by Hayford's discussion may be found by consulting the observation equations which are published by the United States Coast and Geodetic Survey in the Figure of The Earth and Isostasy from Measurements in the United States, and Supplementary Investigation in 1909 of the Figure of the Earth and Isostasy.

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[8] Ferrero: Rapport sur les triangulations, 12th Conference International Geodetic Association (held at Stuttgart, 1898), p. XXIII.

[9] Hunter: The Earth's Axes and Triangulation (Survey of India Professional Papers No. 16), p. 77. For possible observational causes of error see Primary Triangulation on the 104th Meridian and on the 39th Parallel in Colorado, Utah, and Nevada (U. S. Coast and Geodetic Survey Special Publication No. 19), p. 76.

The effect of change in the semimajor axis a on the deflection in the meridian or in the prime vertical at a point is obtained from the coefficient of the quantity called $\frac{a}{100}$ in the latitude or longitude observation equations of that point. This coefficient multiplied by 1/100 of the assumed change in a (in meters) will give the change in the deflection expressed in seconds of arc. The shape of the spheroid in this discussion is specified by the square of the eccentricity, e^2, rather than by flattening, f, and the observation equations are in terms of corrections to 10,000 e^2. But since $f = \frac{1}{2} e^2$ nearly, we have the following simple rule: To find the effect of a change of unity in the *reciprocal* of the flattening, multiply the coefficient of (10,000 e^2) by 0.23; the result is the change in the deflection in meridian or prime vertical expressed in seconds of arc.

Let us consider the effect of a change from the Clarke Spheroid of 1866 on the geographic positions of points in the United States. If it were desired to make such a change it would probably be made for the sake of international uniformity in the spheroid used, and the spheroid adopted would doubtless be one intended for a close approximation to the mean figure of the earth as a whole rather than a spheroid especially designed to fit a particular region; that is, as far as we can now judge, its semimajor axis would not be more than about 200 m. greater than that of the Clarke Spheroid, and the reciprocal of the flattening not more than three units greater. Therefore, if we double the coefficient of $\left(\frac{a}{100}\right)$ and take three-fourths of the coefficient of (10,000 e^2), we shall get estimates of the probable effect of a change to the supposed new spheroid. These coefficients are as a rule less than unity, but at Calais, Me., the coefficients are, respectively, 1.34 and 1.68 and at Sand Key, Fla.,[10] they are, respectively, 0.89 and 3.06. Suppose the changes in a and f to reinforce each other, we get in the two cases $2 \times 1.34 + 0.75 \times 1.68 = 3''.94$ and $2 \times 0.89 + 0.75 \times 3.06 = 4''.08$, respectively. That is, in the United States a change of spheroid is not likely to affect the deflections by much more than 4″. In changing spheroids there is to be considered also the change of the assumed latitude, longitude, and initial azimuth at the origin of the computation (Meade's Ranch for the United States). It is probable that in changing spheroids these quantities would be left disposable, so that the mean residual could be diminished. This quantity 4″ is just about the mean residual deflection without regard to sign, so that only in very rare cases in the United States would the outstanding deflection be doubled.

When we extend our calculations to points outside of the United States, the effects are roughly proportional to the distance from Meade's Ranch as may be seen by examining the formulas for the coefficients of $\left(\frac{a}{100}\right)$ and (10,000 e^2) given on page 76 and following of Hayford's The Figure of the Earth and Isostasy and on page 42 and following of Hayford's Supplementary Investigation in 1909 of the Figure of the Earth and Isostasy. If, therefore, the North American

[10] Calais is station No. 173 (prime vertical) on p. 19 of Hayford's Figure of the Earth and Isostasy from Measurements in the United States; Sand Key is station No. 302 (meridian) on p. 12 of Hayford's Supplementary Investigations in 1909 of the Figure of the Earth and Isostasy. Both works are published by the U. S. Coast and Geodetic Survey

Standard Datum were extended to Arctic Canada or to Mexico, the extreme change in the deflections due to changing over to the new spheroid would not be greatly increased; for a circle with its center at Meade's Ranch and a radius of 24° of great circle, which is about the distance to Calais, Me., will include Canada nearly to the Arctic Circle and Mexico and pass not far from southeast Alaska. We shall therefore be safe if we double our previous estimate and take 8″ as the maximum change of the deflection in North America due to the change from the Clarke Spheroid of 1866 to any spheroid likely to be adopted hereafter.

Geographers are not, as a rule, interested primarily in deflections. They are more apt to be concerned in the length and direction of a line from one point to another. If these points are fixed by the triangulation, the distance between them will be nearly independent of the spheroid which the triangulation is assumed to cover, unless the distance between the points is great, say 45° or more, and even in such a case, the relative change in the distance is less than the relative change in the dimensions of the spheroid. (See examples, p. 10.)

The labor of changing from one spheroid to another is considerable. All the geographic positions and azimuths must be recalculated. About 20,000 geographic positions dependent on triangulation have been published by the United States Coast and Geodetic Survey, and about 13,000 more have been computed and are awaiting publication. The United States Geological Survey and the United States Army Engineers have also published a great number of geographic positions on the North American Datum. All these would be invalidated by a change of spheroid, and the existence of two different sets of coordinates for the same point, one set on the old spheroid and one on the new, would inevitably lead to inconvenience, inconsistency, and confusion. Furthermore, it would be necessary to redraw the graticules (network of meridians and parallels) on published maps or to renumber them, giving to the meridians and parallels, as at present drawn, a number no longer a round number of degrees or degrees and minutes, but one ending in some odd number of seconds. This has been done before on some of the old Coast and Geodetic Survey charts, but it is an unpleasing and unsatisfactory expedient.

EFFECT OF CHANGE OF SPHEROID ON TRIANGULATION COVERING A LARGE AREA.

Although the changes to be made in the computation of a triangle due to a change in the spheroid used are very small for any one triangle (except occasionally for the reduction of horizontal angles to the spheroid), almost always less than 0.″01 as has just been shown, yet these effects might be cumulative to an amount which, according to Helmert (Höhere Geodäsie, vol. 1, p. 562–4) might well be appreciable in a region the size of Europe. This fact is illustrated by the examples on pages 10–13. The area of the quadrilateral in Example 1 is somewhat less than that of Europe but the closing error due to change of spheroid alone, although not large in comparison with the perimeter of the quadrilateral, is nevertheless large enough to suggest the desirability of removing or diminishing it. It does not appear, however, that the change would invalidate the argument just given,

which deals with the order of magnitude of certain corrections rather than with their exact values.

If, however, it is a question of scientific accuracy in problems of the figure of the earth or of deflections of the vertical, the question of whether it is necessary to correct for cumulative errors just mentioned will depend on the accuracy sought. The examples are meant to provide the basis for a summary judgment. It should be noted that the closing errors of the polygons in the examples increase approximately according to the *areas* of the polygons rather than according to their linear dimensions.

One way of securing greater precision would be to obtain a first approximation to the spheroid sought and then to recalculate the triangulation and the resulting deflections on the basis of the new spheroid, and finally on the basis of the new deflections to obtain a second approximation to the spheroid. The second approximation would enable corrections to be made to reduce the horizontal angles to the spheroid. A second way that would avoid the necessity of the second approximation would be to adjust the triangulation by the method of the variation of geographic positions (see U. S. C. and G. S. Spec. Pub. No. 28, Application of the Theory of Least Squares to Triangulation, pp. 91–196), after adding to each observation equation the terms necessary to allow for the variation in the elements of the spheroid.

For large areas this second method might be simplified by using selected triangulation points only, too far apart in general to be intervisible, computing the length and azimuths of the geodesic lines joining adjacent points selected from the intervening triangulation and then treating these geodesic lines as if they were directly observed. When the length and direction of such a geodesic line had been derived from the adjustment, the triangulation along that line could be adjusted to conform to the established length of the geodesic line in question.

These refinements have not been applied to determinations of the figure of the earth and of deflections of the vertical from triangulation in Europe and North America. For the smaller areas thus far involved the simpler methods used appear reasonably adequate. But as the area covered by a connected scheme of triangulation increases the time may come when more refined methods would be desirable such as the recomputation of the triangulation on the basis of the new spheroid, or the use of some form of the method of the variation of geographic positions with terms added to allow for variations in the assumed figure. When the triangulation of all North America is discussed to obtain the figure of the earth, these greater refinements would seem to be in order.

On the general subject of deflections of the vertical and of the effect of the change from one spheroid of reference to another, the following works may be consulted:

Borsch and Krüger: Die Europäische Längengradmessung in 52 grad Breite. II Heft. Publications of Prussian Geodetic Institute, 1896.

Helmert: Höhere Geodäsie, Vol. I, Chap. XII; also, Lotabweichungen, Heft I, Formeln und Tafeln sowie einige numerische Ergebnisse für Norddeutschland. Publication of Prussian Geodetic Institute, 1886.

Hunter: The Earth's Axes and Triangulation: Survey of India, Professional Paper No. 16, Dehra Dun, 1918.

Krüger: Beiträge sur Berchnung von Lotabweichungssysteme: Publication of the Prussian Geodetic Institute and the International Geodetic Association, 1898.

SUMMARY OF CONCLUSIONS.

1. The spheroids proposed during the past century as best representing the earth's mean figure have varied widely, but all indications are that Hayford's Spheroid of 1909 with correction for topography and isostatic compensation may be taken as representing our best present knowledge and it seems probable that future determinations will give nearly the same results.

2. There are various spheroids in use for geographic purposes and the demand for a change in the spheroid used for geographic purposes would naturally be for the purpose of securing international uniformity, with a spheroid representing the earth's mean figure, i. e., something like Hayford's Spheroid, as a basis.

3. It is desirable in the interests of scientific accuracy to apply the correction to reduce horizontal angles to the spheroid more extensively than has been done in the past especially in mountainous regions, where it is likely to be large.

4. For North America the change to an international spheroid from the present North American Standard Datum would probably involve changes in the deflections of the vertical of not more than 8″ in very extreme cases.

5. For ordinary geographic purposes relative directions and distances between points in North America would be little affected.

6. The labor of making the change of spheroid would be great and the advantage for purely geographic purposes practically nil; a change in spheroid and datum would invalidate published geographic positions, and the existence of old and new sets of data would cause considerable inconvenience and confusion for those making use of them.

7. In the scientific problem of the figure of the earth it would be desirable to allow for the effect of change of spheroid on the triangulation, if the area covered by a connected scheme of triangulation were as large as North America.

EXAMPLES OF THE EFFECT OF CHANGE OF SPHEROID.

The spheroids used in these examples are the Clarke Spheroid of 1866, expressed in meters, which is used in the United States, Canada, and Mexico, the Bessel Spheroid, which is widely used in Europe, and the Hayford Spheroid (1) of 1909, which is believed to be the best approximation so far obtained to the figure of the earth as a whole.[11] The elements of these spheroids are given in the table on page 1.

EXAMPLE 1.

Suppose two points A and B (fig. 1) both in latitude 30° (near or below the southern border of the United States) and 40° apart in longitude. From these points lines are run northward to points C and D on the forty-ninth parallel which either coincides with, or lies north of, the northern boundary of the United States exclusive of Alaska, except for the bit of territory near the Lake of the Woods.

[11] For the purposes of these examples the Hayford Spheroid is defined by its semimajor axis, 6,378,388 m. exactly, and its flattening, 1/297 exactly. These assumptions give a semiminor axis of 6,356,912 m., to the nearest meter, which is 3 m. greater than the value published by Hayford. From the axes as published by him the flattening comes out 1/296.96. The decision to use a flattening exactly equal to 1/297 was made by the Paris Conference of 1911 on Astronomical Constants, and it is desirable to conform to this decision in the interests both of simplicity and of uniformity. The definition of the Hayford Spheroid as here adopted has also been officially adopted in Finland. See Y. Väisälä, Tafeln für geodätische Berechnungen nach den Erddimensionen von Hayford, Veröffentlichungen des finnischen geodätischen Institutes, No. 1.

What is the effect on the distance CD of a change from the Clarke to the Bessel Spheroid?

As thus stated the problem is not defined with sufficient precision for solution. Four possible interpretations will be considered. Before considering these, let us state the distances in question for the Clarke Spheroid. The length of the arc of meridian AC or BD between the parallels of 30° and 49° is 2,109,475 m., the distance AB measured along the thirtieth parallel is 3,859,529 m., and the distance CD along the forty-ninth parallel is 2,926,965 m.

First interpretation.—We might inquire merely what are the distances between the same meridians and parallels on the Bessel Spheroid. The arc of the meridian between 30° and 49° is 2,109,286 m., or 189 m. less than on the Clarke Spheroid. These calculations were made with 7-place logarithms and the final digit given may be a unit or more in error. The arc of 40° on the thirtieth parallel is 3,858,994 m., or 535 m. shorter than on the Clarke Spheroid, and the arc of 40° on the forty-ninth parallel is 2,926,515 m., or 450 m. shorter than the corresponding arc on the Clarke Spheroid. A length of 31 m. along the meridian represents a second of latitude while 27 and 20 m. represent one second of longitude on the thirtieth and forty-ninth parallels, respectively. The differences in distances correspond to deflections in latitude of 6″ and in longitude of 20″ and 22″.

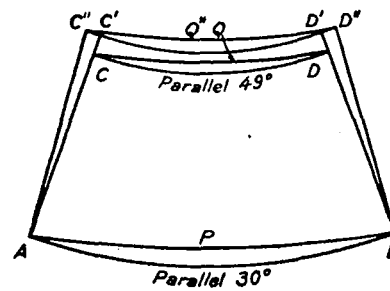

FIG. 1.

Second interpretation.—The results of the first interpretation are merely consequences of the size and shape of the two spheroids in question and have no bearing on what happens when triangulation is applied to one or the other spheroid. We can get a closer agreement with actual conditions if we suppose the distance $AB = 3{,}859{,}529$ m. to have been obtained from the Clarke Spheroid, and then laid off on the thirtieth parallel of the Bessel Spheroid. On the latter it corresponds to a difference of longitude, not of exactly 40° but of 40° 00′ 20″.0. Similarly we suppose the meridian distances $AC = BD = 2{,}109{,}475$ m., when laid off from latitude 30° northward, puts the end points C' and D', not in latitude 49° even, but in latitude 49° 00′ 06″.1. (The distances CC' and DD' are greatly exaggerated in the figure.) The distance $C'D'$ along the parallel of 49° 00′ 06″.1 is 2,926,822 m., or 143 m. less than the distance between corresponding points on the Clarke Spheroid. This is seen to be less than the discrepancy of 450 m. by the first interpretation.

Third interpretation.—We can approach still nearer to the way a chain of triangulation would actually be laid down on a spheroid by supposing AB to be connected by a geodesic line, or line by which the distance is shortest, corresponding to a portion of a great circle in a sphere (APB in fig. 1). The points A and B would correspond to vertices of a triangle, and the direction of the line like AC or BD would be given simply by the angle, PAC or PBD, P being a point on the geodesic line. The distance APB on the Clarke Spheroid is 3,839,223 m. and the angle $PAC = PBD = 79°\ 41'\ 03''.38$. Let us apply

these distances and directions to the Bessel Spheroid. A line of length 3,839,223 m. both of whose ends are on the parallel of 30° would on the Bessel Spheroid subtend a difference of longitude of 40° 00′ 20″.17 and would make at both ends angles of 79° 40′ 57″.95 with the meridian, or 5″.43 less than on the Clarke Spheroid. Now in an actual triangulation the direction of the line AC or BD would be determined simply by the angle like PAC or PBD without reference to the meridian. To transfer our triangulation from the Clarke Spheroid to the Bessel, we should take the angle 79° 41′ 03″.38 as given by the Clarke Spheriod and lay it off from AP and BP. We thus get lines AC'' and BD'' making with the meridian of the Bessel Spheroid angles $C''AC = D''AD = 5''.43$. The distances AC'' and BD'' are taken equal to AC or BD on the Clarke Spheroid, that is 2,109,475 m. We come out on practically the same parallel as in the second interpretation, namely, 49° 00′ 06″.11, the angle of 5″.43 being too small to affect the meridian distance appreciably, but the effect of this little angle is to displace the points C'' and D'' to the left and right of C' and D' by 54.7 m.; so that the distance $C''D''$ is 109 m. greater than $C'D'$. If we bear in mind that $C'D'$ as computed by the second interpretation for the Bessel Spheroid was 143 m. too small as compared with CD on the Clarke Spheroid, we see that our revised interpretation has made the distance $C''D''$ on the Bessel Spheroid more nearly the same (that is, within 34 m.) as the corresponding distance on the Clarke Spheroid. It is more consistent, however, to continue measuring distances along geodesic lines instead of along parallels. The distance CQD along a geodesic line on the Clarke Spheroid is 2,892,512 m., while the corresponding distance $C''Q''D''$ along a geodesic line on the Bessel Spheroid is 2,892,467 m. The difference of 45 m. between these two distances gives an idea of the closing errors that would result from transferring a set of triangles enclosing an area nearly equal to that of the United States from one spheroid to another.

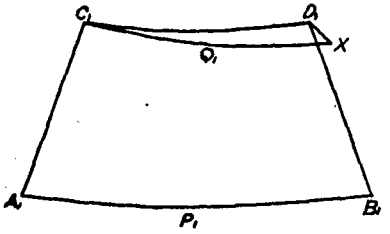

FIG. 2.

Fourth interpretation.—We can estimate the closing error due to a change of spheroid in still another way closely analagous to the third interpretation. Suppose the points A, B, C, and D fixed as before on the Clarke Spheroid. From the geographic coordinates of these points we can determine the sides and angles of the quadrilateral $ABDC$, the sides being assumed to be geodesic lines. Let us take the length AB thus determined and transfer it to the Bessel Spheroid keeping A_1 and B_1, the points corresponding to A and B, in latitude 30°. (See fig. 2.) Lay off the angles $A_1B_1D_1$ and $B_1A_1C_1$ equal to ABD and BAC and take the lengths $A_1C_1 = AC$ and $B_1D_1 = BD$. (So far this is the procedure in the third interpretation.) Lay off the angles A_1C_1X equal to ACD and make $C_1X = CD$. The corresponding procedure on the Clarke Spheroid gave us the closed quadrilateral, the point X coinciding with D. On the Bessel Spheroid this is not the case. As was found in the third interpretation, the point D is in latitude 49° 00′ 06″.11 and is 40° 00′ 25″.55 east of C. The point X however is in latitude 49° 00′ 03″.92 and is 40° 00′ 26″.88 east of C,

so X lies about 73 m. S. 22° E. of D and the quadrilateral $ABDC$ transferred from the Clarke Spheroid to the Bessel thus fails to close by 73 m.

EXAMPLE 2.

Suppose the distance from the parallel of 30° on the Clarke Spheroid of 1866 to the pole be laid off along the meridian of a Bessel Spheroid beginning at latitude 30°, where will the end of the line fall? On the Clarke Spheroid the distance is 6,681,954 m., while on the Bessel Spheroid it is 6,681,069 m., so that the Clarke distance laid off on the Bessel Spheroid would overrun the pole by 885 m. or 28″.5. (See fig. 3.) If meridian lines determined from the length on the Clarke Spheroid were laid off on the corresponding meridians of the Bessel Spheroid, instead of meeting at the pole, as on the Clarke Spheroid, the ends of the meridian lines would lie on a circle of radius 885 m. For two meridians 40° apart, as in Example 1, the two points corresponding to the Clarke pole would be 605 m. apart. If we allow for the difference in direction of 5″.43, which appears in the third interpretation of Example 1, the points A' and B' in Figure 3 would be shifted to the left and right, respectively, perpendicular to the meridian each by some 145 m., thus taking the positions A'' and B'' in figure 3, and the discrepancy of 605 m. would be reduced to about 330 m.

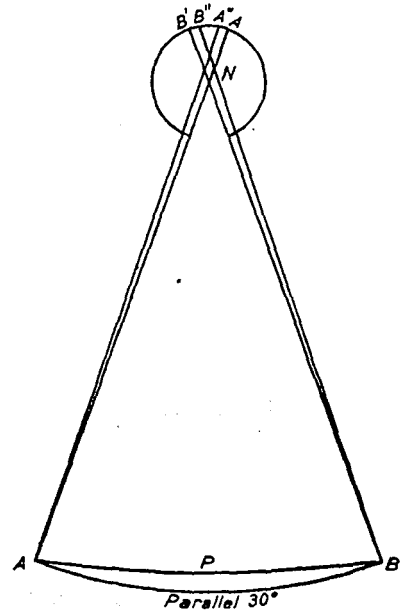

FIG. 3.

It should be noted that the difference in direction of 5″.43 arises from the different angles on the two spheroids between the meridian and the geodesic line between the two points in the same latitude (here 30°). The angle in question is 90° for all spheroids at the equator, and where the two meridians start at the equator the difference in direction does not occur and the second and third interpretations give identical results.

EXAMPLE 3.

This example is intended to simulate approximately the conditions found when a circuit of triangles was closed by a chain of triangles extending from Texas to southern California.[12] The triangulation completed up to the time when the Texas-California arc was undertaken formed three sides of a quadrilateral. The geographic positions in the triangulation forming these three sides were arbitrarily held fixed in the adjustment of the Texas-California arc and the directions in that arc were so adjusted as to close the quadrilateral.

[12] W. Bowie: The Texas-California Arc of Primary Triangulation. U. S. C. and G. S. Spec. Pub. No. 11; 1912.

All the errors accumulated in the three older sides of the quadrilateral were thus thrown into the new line. The discrepancies in latitude, longitude, and azimuth that had to be taken care of in this way were, respectively,[13] 39 m., 14 m., and 7.″5. It is therefore of interest to estimate how much of these discrepancies might reasonably be attributed to possible errors in the spheroid used; this was, of course, the Clarke Spheroid of 1866.

The geographic coordinates of the corners of the quadrilateral assumed to be equivalent to the actual figure were as follows:

	North latitude.	West longitude.
A	33° 00′	98° 00′
B	39 00	98 00
C	39 00	123 00
D	33 30	116 00

The side *AB* (see fig. 4) thus represents triangulation along the ninety-eighth meridian, the side *BC* triangulation along the thirty-

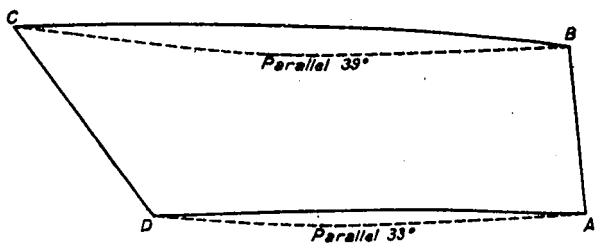

FIG. 4a.

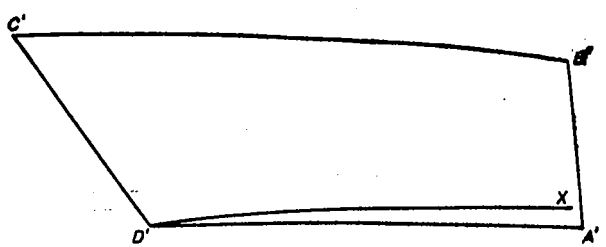

FIG. 4b.

(To correspond to the numerical example the point *X* should lie to the right of the line *A′ B′*).

ninth parallel, the side *CD* triangulation in southern California, and the side *DA* the Texas-California arc.

The sides and angles of the quadrilateral were computed from the above positions of vertices, which were first assumed to refer to the Hayford Spheroid of 1909, this being considered the best available approximation to the true spheroid. A point *A′* in latitude 33° 00′ and longitude 98° 00′ was then taken on the Clarke Spheroid of 1866 and the distance *A′B′* equal to *AB* was laid off northward on the ninety-eighth meridian. At *B′* the angle *A′B′C′* on the Clarke Spheroid was taken equal to *ABC* on the Hayford Spheroid; *B′C′* was taken equal to *BC* and so on around the quadrilateral until a line *D′X* was laid off equal to *DA*, the angle *XD′C′* being equal to

[13] Op. cit. p. 20.

ADC. The point *X* would lie near *A′* but could not be assumed to coincide with it. The difference in position between *X* and *A′* gives immediately the amounts by which the quadrilateral fails to close in latitude and longitude when transferred from one spheroid to another and a simple computation gives the correction in azimuth needed to effect closure. These failures to close are due solely to the change in spheroid.

The following are the sides and angles of the quadrilateral on the Hayford Spheroid. The sides are assumed to be geodesic lines.

Sides.		Angles.	
A B=	665,773.0 m.	*A*=	83° 09′ 31.78″
B C=	2,158,881.5	*B*=	97 56 33.99
C D=	876,002.9	*C*=	49 57 03.53
D A=	1,676,289.6	*D*=	130 45 32.15

When this quadrilateral is transferred to the Clarke Spheroid of 1866 in the manner just described the points *A′*, *B′*, *C′*, *D′*, and *X* (see fig. 4) are in the following positions:

	North latitude.	West longitude.
A′	33° 00′ 00.00″	98° 00′ 00.00″
B′	39 00 01.08	98 00 00.00
C′	39 00 00.80	123 00 02.12
D′	33 00 00.01	116 00 01.39
X	33 00 00.13	97 59 59.94

The point *X* therefore fails to coincide with the point *A′* by 0″.13 in latitude and 0″.06 in longitude, that is, by about 4 m. in the meridian and less than 2 m. perpendicular to it. The quantities are small as compared with the discrepancies of 39 m. and 14 m. developed in the adjustment. It would then appear probable that an erroneous choice of spheroid is not the principal cause of these discrepancies.

Another way of stating the failure to close is to say that the distance *A′X* is 4.3 m., which is 1 part in 1,250,000 of the total perimeter of the quadrilateral. The difference in azimuth between *D′X* and *D′A′* is less than 1″ as compared with the 7″.5 developed in the adjustment. Here again it appears unlikely that the choice of spheroid is mainly responsible for the discrepancy.

EXAMPLE 4.

This example is of the same general character as Example 3. A quadrilateral is taken to represent a circuit of triangulation in the northern part of the United States. The coordinates of the vertices are:

	North latitude.	West longitude.
A	49° 00′	123° 00′
B	39 00	123 00
C	39 00	105 00
D	49 00	103 30

If this quadrilateral is computed from the Hayford Spheroid and then transferred with angles and lengths of sides unchanged to the Clarke Spheroid, just as in Example 3, the distance between *X* and *A′* is 2.9 m., or 1 part in 1,800,000 of the perimeter of 5,209 km. The difference in azimuth is 0″.22. These discrepancies are small in comparison with the discrepancies usually disclosed in the adjustment of a circuit of triangulation as big as the one in question here.

FORMULAS FOR THE COMPUTATION OF GEODETIC POSITIONS OVER LONG LINES, DIRECT PROBLEM.

Given a point A in geographic latitude ϕ_1 and a geodesic line AB of length s issuing from A in azimuth α_1: to find the latitude of B, the back azimuth (α_2) of the line BA at B, and the differences in longitude (Δl) between B and A. The length s is supposed to be several hundred kilometers and therefore such that the approximations used in Coast and Geodetic Survey Special Publication No. 8 (Formulae and Tables for the Computation of Geodetic Positions) are no longer valid.

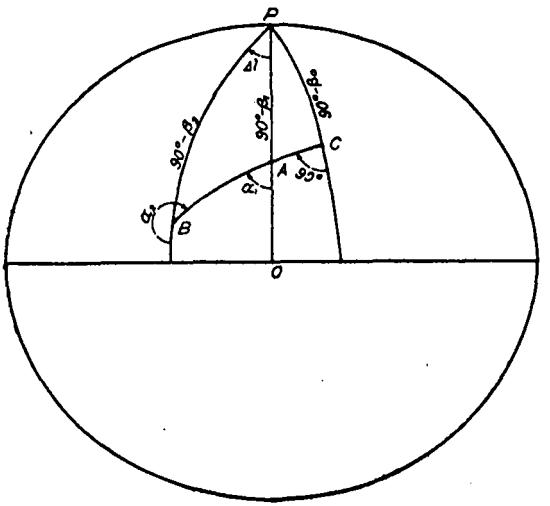

FIG. 5a.

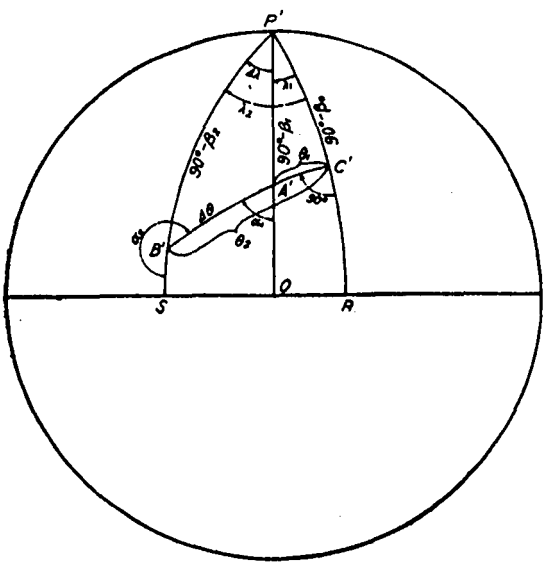

FIG. 5b.

The solution of this problem in substantially its present form was given by Bessel.[14] Among more recent works in which the problem is treated may be mentioned those of Clarke and Helmert.[15] The necessary formulas are given without proof by Albrecht[16] together with tables which are an enlargement of those given by Bessel. These tables are, of course, convenient, but do not shorten the work as much as might be expected, since the interpolations are rather troublesome and the computation necessary to replace the tables rather simple. In this statement of formulas we shall follow essentially the presentation of Helmert, who makes no use of special tables. The system of notation is similar to that of Helmert, the

[14] F. W. Bessel: Ueber die Berechnungen der geographischen Längen und Breiten aus geodätischen Vermessungen; Astronomische Nachrichten No. 86 (1825), Vol. 4.

[15] A. R. Clarke: Geodesy, Oxford, 1880, Chap. VI. F. R. Helmert, Die mathematischen und physikalischen Theorien der höheren Geodäsie, Leipzig, 1880, Vol. I, Chap. 5.

[16] Theodor Albrecht: Formeln und Hülfstafeln für geographische Ortsbestimmungen, second and later editions, Leipzig, from 1879 on. In the fourth edition (1908) the convention as to azimuth is stated, a point overlooked in earlier editions, and a typographical confusion between *spherical* and *spheroidal* is corrected.

changes being mainly for ease in printing. A comparison of the notations of Helmert, Albrecht, and this work is given on page 35.

On the spheroid, the two points A and B and the pole P (see fig. 5a), are the vertices of a *spheroidal* triangle, the sides of which are geodesic lines, a meridian being a special case of the geodesic line. It is convenient to use also an auxiliary *spherical* triangle $P'A'B'$, (see fig. 5b), in which P' corresponds to the pole and the sides $P'A'$ and $P'B'$ are taken equal to the complements of the parametric (reduced) latitudes of A and B, respectively. By a property of the geodesic line the angles $P'A'B'$ and $P'B'A'$ are then equal, respectively, to the corresponding angles PAB and PBA, but the angle APB (=spheroidal difference of longitude $=\Delta l$) is not equal to the angle $A'P'B'$ (=spherical difference of longitude $=\Delta\lambda$). However, the difference between Δl and $\Delta\lambda$ is small, since the ellipticity of the earth is small. In order to compute this difference and also the relation between the side $A'B'$ of the spherical triangle and the side $AB=s$ of the spheroidal triangle it is necessary to determine the position of the point C' in the figure and the distance $C'A'$. The point C' is that point at which the great circle $A'B'$ is perpendicular to a meridian. A convenient name for the point C' is the *vertex* of the great circle $A'B'$. The corresponding point on the spheroid, or C, where the geodesic line AB is perpendicular to the meridian may also be termed the *vertex*. There being two such points we may take for definiteness the one on the northern hemisphere regardless of the hemisphere in which A' and B' may be. If, again, in Figure 5a, we draw the meridian that is perpendicular to the geodesic line AB, extended if necessary, and call the foot of the perpendicular C, the point C corresponds to C' in the other triangle and by a property of the geodesic the reduced latitude of the vertex C on the spheroid will be equal to the latitude of C' on the sphere.

With this preliminary we proceed to a statement of the formulas. In this statement those numerical coefficients or their logarithms that are printed in italics apply only to the Clarke Spheroid of 1866 and must be modified if another spheroid is used; the numerical coefficients or logarithms printed in roman type apply to all spheroids. The unit of length is the meter. Square brackets signify that the number inclosed is the common logarithm of the coefficient in question.

FIRST STEP—COMPUTATION OF THE PARAMETRIC (REDUCED) LATITUDE OF A.

Let β_1 denote this reduced latitude. Compute

$$\tan\beta_1=\frac{b}{a}\tan\phi_1=[\mathit{9.9985252-10}]\tan\phi_1, \tag{1}$$

where b and a are the semiminor and semimajor axes of the spheroid. The computation for a change from one kind of latitude to the other may also be made (and with greater accuracy, even where tables with a small number of decimals are used) by the following formulas,[17]

$$\left.\begin{aligned}\beta_1&=\phi_1-n\sin 2\phi_1+\frac{n^2}{2}\sin 4\phi_1-\frac{n^3}{3}\sin 6\phi_1\\&=\phi_1-[\mathit{2.5443412}]\sin 2\phi_1+[\mathit{9.47323-10}]\sin 4\phi_1\\&\qquad-[\mathit{6.527-10}]\sin 6\phi_1,\end{aligned}\right\}\tag{1a}$$

[17] See O. S. Adams: Latitude Developments Connected with Geodesy and Cartography, Coast and Geodetic Special Publication No. 67, p. 15, and also p. 95 for the tables.

The quantity n denotes $\frac{a-b}{a+b}$; the numerical coefficients whose logarithms are given for the Clarke Spheroid of 1866 are expressed in seconds of arc.

SECOND STEP—TO FIND THE LATITUDE OF C' $(=\beta_0)$ AND THE DISTANCE $C'A'(=\theta_1)$.

This is merely the solution of a spherical right triangle of which the hypotenuse $A'P' = 90° - \beta_1$ and one angle $C'A'P' = \alpha_1$ are known. The following arrangement of formulas is convenient in that it provides checks for the computation and also enables the quantities sought to be determined without ambiguity as to quadrant when the triangle is taken in the general sense and its sides are not restricted to being less than 180°. The conventions are that the azimuth is reckoned from the south through the west and that west longitudes are positive.

$$\left.\begin{aligned} \sin\beta_0 \cos\theta_1 &= \sin\beta_1, \\ \sin\beta_0 \sin\theta_1 &= \cos\beta_1 \cos\alpha_1, \\ \cos\beta_0 \qquad &= \cos\beta_1 \sin\alpha_1. \end{aligned}\right\} \tag{2}$$

If the azimuth is in the second quadrant, θ_1 comes out negative, a case illustrated in Figure 6b. The formulas are, of course, perfectly general.

There is also needed the longitude of A', reckoned from the meridian of the vertex C'. Denote this by λ_1. We have then

$$\left.\begin{aligned} \cos\beta_1 \cos\lambda_1 &= \cos\theta_1 \cos\beta_0, \\ \cos\beta_1 \sin\lambda_1 &= \sin\theta_1. \end{aligned}\right\} \tag{3}$$

Formulas (2) and (3) are perfectly general and will give θ_1 in the proper quadrant if attention is paid to the signs of the other trigonometric functions. It is convenient to take values of θ_1 and λ_1 between $-180°$ and $+180°$ rather than between 0 and 360°. This is not essential however. θ_1 is positive when the arc $C'A'$ (from C' to A') is in the same direction as the arc $A'B'$ (from A' to B'). The vertex C' may be taken as always in the northern hemisphere, even though A' or B' or both may be in the southern hemisphere, so that $\sin\beta_0$ is always positive. However, when α_1 is in the third or fourth quadrant the formulas make $\cos\beta_0$ negative, so that β_0 can not be a latitude in the ordinary sense; i. e., numerically less than 90°, since $\cos\beta_0$ would then be positive. β_0 in this case is the supplement of the latitude of C', that is β_0 is measured from the equator through the pole to C', or $\beta_0 = D'P'C'$ in Figures 6c and 6d. This method of reckoning has its analogy in the practice of generalizing the formulas for computing meridian transits of stars by using declinations greater than 90° for lower transits. This method of representing β_0 makes the point D' move continuously around the equator through 360° when the azimuth (α_1) changes from 0 to 360°.

THIRD STEP—TO FIND THE ANGULAR DISTANCE $\Delta\theta = A'B'$ ON THE SPHERE CORRESPONDING TO THE LINEAR DISTANCE $AB = s$ ON THE SPHEROID.

This relation involves elliptic integrals of the second kind. The modulus depends on the latitude of the vertex C' and is so small that the elliptic integral may be readily calculated by the first few terms of a rapidly convergent series.

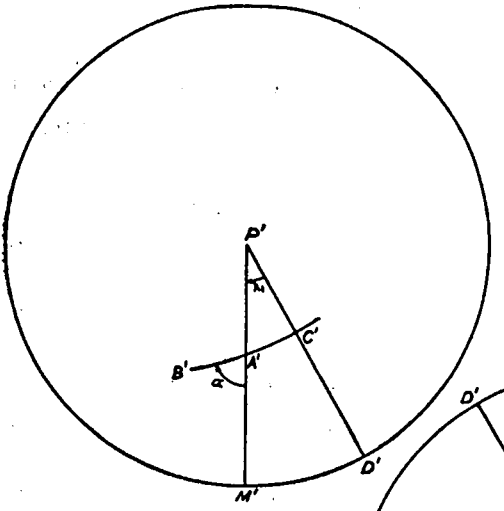

FIG. 6a.

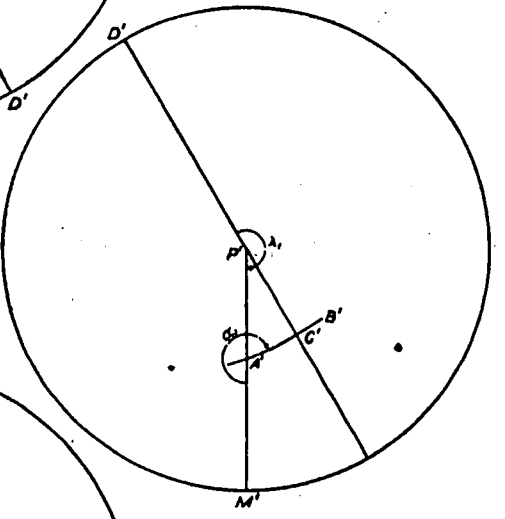

FIG. 6c.

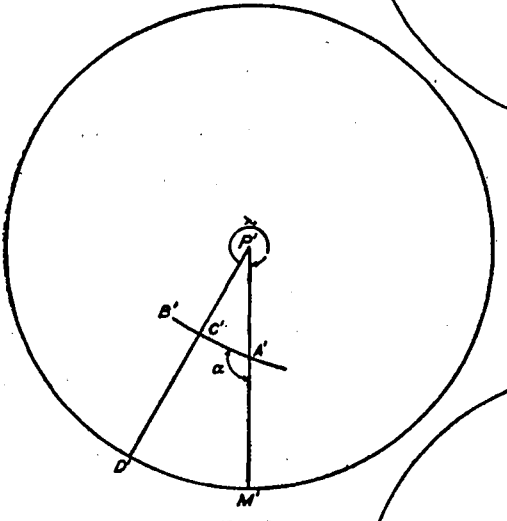

FIG. 6b.

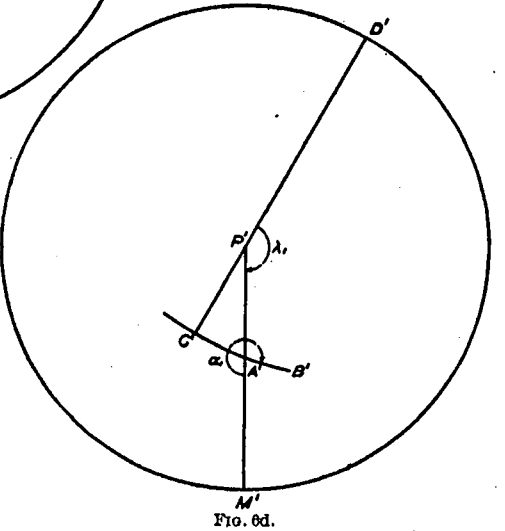

FIG. 6d.

Let δ denote, as usual, the quantity $\frac{e^2}{1-e^2}$, where e is the eccentricity of the spheroid used, let k_1 denote the modulus of the elliptic integral, and E an auxiliary quantity. Then

$$\tan E = \sqrt{\delta} \sin \beta_0 = [8.9167261 - 10] \sin \beta_0 \tag{4}$$

$$k_1 = \tan^2 \tfrac{1}{2} \mathrm{E} \tag{5}$$

$$\Delta\sigma = \frac{s\rho''}{b} \frac{1-k_1}{1+\frac{1}{4} k_1^2}. \tag{6}$$

$\Delta\sigma$ is the first approximation to $\Delta\theta$; ρ'' denotes $\frac{180 \times 60 \times 60}{\pi}$,

$$\log \rho'' = 5.3144251{,}3 \tag{7}$$

$$\log \frac{\rho''}{b} = 8.5112013{,}5 - 10, \tag{8}$$

b being in meters.

$$\log \frac{1-k_1}{1+\frac{1}{4} k_1^2} = 2 \log \cos \tfrac{1}{2} E - \tfrac{5}{4} M k_1^2 \ldots \tag{9}$$

$$\log \tfrac{5}{4} M = 6.73469, \tag{10}$$

for units of seventh decimal. M = modulus of common logarithms.

$$2\sigma_1 = 2\theta_1 + \rho'' k_1 \sin 2\theta_1, \tag{11}$$

$$2\sigma = 2\sigma_1 + \Delta\sigma. \tag{12}$$

The quantities σ_1 and σ are defined by equations (11) and (12).

$$\Delta\theta = \Delta\sigma + A \cos 2\sigma \sin \Delta\sigma + B \cos 4\sigma \sin 2\Delta\sigma \tag{13}$$

$$\left.\begin{aligned} \log A &= \log(-\rho'') + \log k_1, \\ &= 5.3144251{,}3\ n + \log k_1. \end{aligned}\right\} \tag{14}$$

$$\log B = \log(\tfrac{5}{8} \rho'' k_1 + \ldots) = 5.11031 + 2 \log k_1. \tag{15}$$

The terms of the series (13) are evidently expressed in seconds of arc.

FOURTH STEP—SOLUTION OF THE TRIANGLE $A'B'P'$ FOR THE LATITUDE, BACK AZIMUTH, AND DIFFERENCE OF LONGITUDE OF B'.

Since $\Delta\theta = A'B'$ has been found and the angle $B'A'P'$ and the side $A'P'$ are known, the remaining parts can be found, giving β_2 the parametric latitude of B' and $\Delta l = A'P'B'$ the difference of longitude. One convenient way of solving the triangle is to solve the right triangle $B'C'P'$. This triangle gives, after putting $\theta_2 = C'B' = C'A' + A'B' = \theta_1 + \Delta\theta$,

$$\left.\begin{aligned} &\sin \beta_2 = \sin \beta_0 \cos \theta_2, \\ &\cos \beta_2 \cos \alpha_2 = -\sin \beta_0 \sin \theta_2, \\ &\cos \beta_2 \sin \alpha_2 = -\cos \beta_0. \end{aligned}\right\} \tag{16}$$

If we denote by λ_2 the angle $C'P'B'$, we have

$$\left.\begin{aligned}\cos\beta_2\cos\lambda_2&=\cos\beta_0\cos\theta_2,\\ \cos\beta_2\sin\lambda_2&=\sin\theta_2.\end{aligned}\right\}\tag{17}$$

These equations give β_2, λ_2, and α_2.

For the difference of longitude $\Delta\lambda$ we have

$$\Delta\lambda=A'P'B'=C'P'B'-C'P'A'=\lambda_2-\lambda_1,\tag{18}$$

λ_1 coming from (3).

We may also attack the oblique triangle directly; or any of the formulas for the oblique triangle may be used as a check on the computation of the right triangles. Such check formulas are, for example,

$$\left.\begin{aligned}\sin\Delta\lambda&=\sin\Delta\theta\sin\alpha_1\sec\beta_2\\ &=-\sin\Delta\theta\sin\alpha_2\sec\beta_1.\end{aligned}\right\}\tag{19}$$

$$\left.\begin{aligned}\tan\tfrac{1}{2}\Delta\lambda&=\sin\tfrac{1}{2}(\beta_2-\beta_1)\sec\tfrac{1}{2}(\beta_2+\beta_1)\cot\tfrac{1}{2}(\alpha_2+\alpha_1)\\ &=\cos\tfrac{1}{2}(\beta_2-\beta_1)\operatorname{cosec}\tfrac{1}{2}(\beta_2+\beta_1)\cot\tfrac{1}{2}(\alpha_2-\alpha_1).\end{aligned}\right\}\tag{20}$$

FIFTH STEP—COMPUTATION OF THE GEODETIC LATITUDE OF *B*.

The parametric latitude of B is equal to the latitude of B', that is, to β_2. The geodetic latitude of B, or ϕ_2, is then found by the formula

$$\left.\begin{aligned}\tan\phi_2&=\frac{a}{b}\tan\beta_2\\ &=[0.0014748]\tan\beta_2,\end{aligned}\right\}\tag{21}$$

or by the series

$$\left.\begin{aligned}\phi_2&=\beta_2+n\sin 2\beta_2+\frac{n^2}{2}\sin 4\beta_2+\frac{n^3}{3}\sin 6\beta_2\ \ldots\\ &=\beta_2+[2.5443412]\sin 2\beta_2+[9.47323-10]\sin 4\beta_2\\ &\quad+[6.527-10]\sin 6\beta_2\ \ldots.\end{aligned}\right\}\tag{21a}$$

The logarithms in brackets give the terms in seconds of arc.

SIXTH STEP—COMPUTATION OF THE SPHEROIDAL DIFFERENCE IN LONGITUDE.

Although the latitudes on the auxiliary sphere are equal to the parametric latitudes on the spheroid, the relation between the differences in longitudes is less simple. Let Δl represent the difference of longitude between A and B on the spheroid; then

$$\Delta l=\Delta\lambda-\cos\beta_0\,\{A'\Delta\theta+B'\cos 2\theta\sin\Delta\theta\ \ldots\}\tag{22}$$

has the same meaning as before and

$$\left.\begin{aligned}2\theta&=\theta_1+\theta_2,\\ \theta&=\theta_1+\tfrac{1}{2}\Delta\theta.\end{aligned}\right\}\tag{23}$$

or

$$\left.\begin{aligned}\log A'&=\log f-\tfrac{1}{2}M(1-\tfrac{1}{2}f)\,k_1\\ &=7.5302093-10-[6.3360]\,k_1\end{aligned}\right\}\tag{24}$$

The term $[6.3360]\,k_1$ is expressed in units of the seventh decimal place.

$$\left.\begin{aligned}\log B'&=\log(-\tfrac{1}{4}\rho''e^2k_1)\\ &=2.5429n+\log k_1\end{aligned}\right\}\tag{25}$$

The symbol f denotes the ellipticity or flattening of the spheroid. The coefficient of k_1 in formula (24) for log A' should, in strictness, be printed in italics to show that it is for the Clarke Spheroid only, but any probable change in the flattening affects it so little that it may be considered as practically invariable. The computation of Δl completes the solution.

NUMERICAL EXAMPLE, DIRECT PROBLEM.

From a point A in geodetic latitude 55° N. there runs a geodesic line AB, 4,000 km. long, with an initial azimuth (S via W) of 50°. What is the latitude of B, the back azimuth of the geodesic line there, and the difference of longitude between A and B? Assume the Clarke Spheroid of 1866.

$$\phi_1 = 55°,$$
$$\alpha_1 = 50°,$$
$$s = 4{,}000{,}000 \text{ m.}$$

The trigonometric part of these calculations was made with Shortrede's logarithms to seven places. The hundredths of a second are, therefore, uncertain. The functions are all logarithmic.

Whenever the quantity dropped in rounding off an interpolated logarithm to seven decimal places is about half a unit of the seventh decimal place, or when two different determinations of the same logarithm give results differing by a unit of the seventh decimal place, the two alternative final figures are written in succession with a hyphen between. Thus, 9.436 4126-7 might mean either that the logarithm lay about half way between 9.436 4126 and 9.436 4127, or else that these two quantities represent two different determinations of the same logarithm.

FIRST STEP.

tan ϕ_1	0. 154 7732
constant	9. 998 5252
tan β_1	0. 153 2984
β_1	54° 54′ 30″71

This is in exact agreement (although the hundredth of a second can not be depended on) with the table on page 96 of Special Publication No. 67.

SECOND STEP.

sin α_1	9. 884 2540	cos β_1 cos α_1 = sin β_0 sin θ_1	9. 567 6473
cos β_1	9. 759 5798	sin θ_1 *	9. 614 4636
cos α_1	9. 808 0675	cos θ_1 *	9. 959 6945
cos β_1 sin α_1 = cos β_0	9. 643 8338	sin β_1 = sin β_0 cos θ_1	9. 912 8783
		tan θ_1	9. 654 7690
		θ_1	24° 18′ 16″86
		sin β_0	9. 953 1837-8

* The values of sin θ_1 and cos θ_1 are not filled out until θ_1 has been determined from its tangent. Therefore, the value of sin β_1=sin β_0 cos θ_1 may be subtracted from the value of cos β_1 cos α_1=sin β_0 sin θ_1 without having the eye distracted by intervening figures. When θ_1 has been determined and the values of sin θ_1 and cos θ_1 filled in, they are to be subtracted each from the logarithm in the adjacent line above or below, thus affording two determinations of sin β_0. A similar arrangement occurs frequently in these numerical examples.

The values of β_0 from its sine and cosine agree within a few hundredths of a second. The definitive value of β_0 may be taken from $\tan \beta_0 = \frac{\sin \beta_0}{\cos \beta_0}$ and is found to be $\beta_0 = 63° \; 52' \; 17''.30$.

$\cos \theta_1$	9. 959 6945
$\cos \beta_0$	9. 643 8338
$\cos \beta_1 \cos \lambda_1 = \cos \theta_1 \cos \beta_0$	9. 603 5283
$\cos \lambda_1$ *	9. 843 9485
$\sin \lambda_1$ *	9. 854 8839
$\cos \beta_1 \sin \lambda_1 = \sin \theta_1$	9. 614 4636
$\tan \lambda_1$	0. 010 9353
λ_1	45° 43′ 16″.55
$\cos \beta_1$	9. 759 5798-7

The agreement of $\cos \beta_1$ with the value previously used is a partial check.

THIRD STEP.

$2\theta_1 = 48° \; 36' \; 34''$

$\log \sqrt{\delta}$	8. 916 7261	$\log k_1^2$	4. 2731
$\sin \beta_0$	9. 953 1837	$\log \frac{5}{4} M$	6. 7347
$\tan E$	8. 869 9098		1. 0078
E	4° 14′ 19″.546		
$\frac{1}{2} E$	2° 07′ 09″.773	$\frac{5}{4} M k_1^2$	10(= cor. to log of first term in units of seventh decimal)
$\tan \frac{1}{2} E$	8. 568 2847		
$\log k_1$ ($= \tan^2 \frac{1}{2} E$)	7. 136 5636	$\log \rho''$	5. 314 4251
		$\log s$	6. 602 0600
		$\cos^2 \frac{1}{2} E$	9. 999 4056–10 $\Big\}\log\left(\frac{1-k_1}{1+\frac{1}{4}k_1^2}\right)$
		correction	−10
		colog b	3. 196 7762 − 10
		$\log \Delta\sigma$ ($\Delta\sigma$ in secs.)	5. 112 6659
		$\Delta\sigma$	129, 618″.18 = 36° 00′ 18″.18
$\log \rho''$	5. 3144	$\log (-\rho'')$	5. 314 4251n $\Big\} = \log A$
$\log k_1$	7. 1366	$\log k_1$	7. 136 5694
$\sin 2\theta_1$	9. 8752	$\cos 2\sigma$	8. 967 7116
		$\sin \Delta\sigma$	9. 769 2714
	2. 3262		1. 187 9775n
$\rho'' k_1 \sin 2\theta_1$	211″.9 = 3′ 31″.9	Second term of (13) =	−15. 416
$2\theta_1$	48° 36′ 33″.7		
$2\sigma_1$	48 40 05. 6		
$\Delta\sigma$	36 00 18. 2		
2σ	84 40 23. 8		
4σ	169 20 47. 6		
$\log \frac{5}{8}\rho''$	5. 1103 $\Big\} = \log B$	$\Delta\sigma$	36° 00′ 18″.18
$\log k_1^2$	4. 2731	Second term	−15. 416
$\cos 4\sigma$	9. 9924n	Third term	−0. 226
$\sin 2\Delta\sigma$	9. 9782	$\Delta\theta$	36 00 02. 54
	9. 3540n	θ_1	24 18 16. 86
Third term of (13) =	−0″.226	θ_2	60 18 19. 40
		$2\theta = \theta_1 + \theta_2$	84 36 36. 26

*See footnote, p. 22.

FOURTH STEP.

$\cos\theta_2$	9. 694 9358	$\cos\theta_2$	9. 694 9358
$\cos\beta_0$	9. 643 8338	$\sin\beta_0$	9. 953 1837
		$\sin\theta_2$	9. 938 8589
$\cos\beta_2\cos\lambda_2 = \cos\theta_2\cos\beta_0$	9. 338 7696	$\cos\beta_2\cos\alpha_2 = -\sin\beta_0\sin\theta_2$	9. 892 0426n
$\cos\lambda_2$*	9. 386 6299	$\cos\alpha_2$*	9. 939 9028n
$\sin\lambda_2$*	9. 986 7191	$\sin\alpha_2$*	9. 691 6940n
$\cos\beta_2\sin\lambda_2 = \sin\theta_2$	9. 938 8589	$\cos\beta_2\sin\alpha_2 = -\cos\beta_0$	9. 643 8338n
$\tan\lambda_2$	0. 600 0893	$\tan\alpha_2$	9. 751 7912
λ_2	75° 54′ 08″85	α_2	**209° 27′ 06″90**
$\cos\beta_2$	9. 952 1397-8	$\cos\beta_2$	9. 952 1398
λ_1	45° 43′ 16″55	$\sin\beta_2 = \sin\beta_0\cos\theta_2$	9. 648 1195
$\lambda_2 - \lambda_1 = \Delta\lambda$	**30 10 52.30**		

The values of β_2 from $\sin\beta_2$ and $\cos\beta_2$ agree as well as could be expected. The definitive value from $\tan\beta_2 = \frac{\sin\beta_2}{\cos\beta_2}$ is

$$\beta_2 = \mathbf{26° \ 24' \ 27''.26}$$

Verification from oblique triangle.

$\sin\alpha_1$	9. 884 2540	$\sin\alpha_2$	9. 691 6940
$\sec\beta_2$	0. 047 8602	$\sec\beta_1$	0. 240 4202
$\sin\Delta\theta$	9. 769 2261	$\sin\Delta\theta$	9. 769 2261
$\sin\Delta\lambda$	9. 701 3403	$\sin\Delta\lambda$	9. 701 3403

$$\Delta\lambda = \mathbf{30° \ 10' \ 52''.33}$$

$\frac{1}{2}(\beta_2+\beta_1)$	40° 39′ 28″98	$\frac{1}{2}(\alpha_2+\alpha_1)$	129° 43′ 33″45
$\frac{1}{2}(\beta_2-\beta_1)$	−14 15 01. 72	$\frac{1}{2}(\alpha_2-\alpha_1)$	79 43 33. 45
$\sin\frac{1}{2}(\beta_2-\beta_1)$	9. 391 2199n	$\cos\frac{1}{2}(\beta_2-\beta_1)$	9. 986 4263
$\cot\frac{1}{2}(\alpha_2+\alpha_1)$	9. 919 5914n	$\cot\frac{1}{2}(\alpha_2-\alpha_1)$	9. 258 3087
$\sec\frac{1}{2}(\beta_2+\beta_1)$	0. 119 9806	$\operatorname{cosec}\frac{1}{2}(\beta_2+\beta_1)$	0. 186 0568
$\tan\frac{1}{2}\Delta\lambda$	9. 430 7919	$\tan\frac{1}{2}\Delta\lambda$	9. 430 7918

$$\tfrac{1}{2}\Delta\lambda = 15° \ 05' \ 26''.15$$
$$\Delta\lambda = \mathbf{30 \ 10 \ 52.30}$$

These three values of $\Delta\lambda$ agree satisfactorily, and the final result may be taken as

$$\Delta\lambda = \mathbf{30° \ 10' \ 52''.31}$$

FIFTH STEP.

$\tan\beta_2$	9. 695 9797
$\log a/b$	0. 001 4748
$\tan\phi_2$	9. 697 4545
ϕ_2	**26° 29′ 06″58**

A computation by the series (21a) results as follows:

β_2	26° 24′ 27″.26
second term	4 39. 017
third term	. 286
fourth term	. 000
ϕ_2	26 29 06. 563,

which agrees satisfactorily with the previous result.

*See footnote, p. 22.

SIXTH STEP.

$\Delta\theta$ in secs. 129,602″54

$\log \frac{1}{2} M (1-\frac{1}{2} f)$	6. 3360	$\log f$	7. 530 2093 } $\log A'$
$\log k_1$	7. 1366	correction	−2969 }
	3. 4726	$\log \Delta\theta$ (in secs.)	5. 112 6135
correction in seventh decimal	2969	$\cos \beta_0$	9. 643 8338
			2. 286 3597
		second term of (22)	193″357 = 3′ 13″357

$\cos \beta_0$	9. 6438	$\Delta\lambda$	30° 10′ 52″31
$\log k_1$	7. 1366 } $\log B'$	second term	− 3 13. 357
constant	2. 5429 }	third term	− 0. 012
$\cos 2\theta$	8. 9728		
$\sin \Delta\theta$	9. 7692	Δl	**30° 07′ 38″94**
	8. 0653		
third term of (22)	0″012		

FORMULAS FOR THE COMPUTATION OF GEODETIC POSITIONS OVER LONG LINES, INVERSE PROBLEM.

Given the latitudes and longitudes of two points A and B on a spheriod to determine the length of the geodesic line joining them and its azimuth at A and B.

The same figures, Nos. 5a and 5b, showing the spheroidal triangle and the auxiliary spherical triangle, may be used as in the direct problem. The solution of the inverse problem is, however, indirect, requiring successive approximations. The reason for this is that the modulus of the elliptic integral, on which depends the length of the line AB and also the difference between the spherical and spheroidal differences of longitude, involves a knowledge of the latitude of the vertex C' (fig. 5b). In the present instance we know merely the sides $P'A'$ and $P'B'$, since these are the parametric colatitudes of the points A and B, but this is not enough to determine the position of C'. If we knew the angle $A'P'B'$, that is, the spherical difference of longitude, we could solve the triangle $A'P'B'$, thus determining the angle $P'A'B'$ and $P'A'C'$. With angle $P'A'C'$ and $P'A'$ known, we could solve the right triangle $P'A'C'$ and thus locate the vertex C' by its latitude β_0. The spherical difference of longitude ($\Delta\lambda$) is known to be approximately equal to the spheroidal (Δl), the difference between the two being only a few minutes of arc at most. As a first approximation we use Δl for $\Delta\lambda$, or we can get a better approximation, as hereafter explained, without a knowledge of β_0. With this approximate knowledge of $\Delta\lambda$ we get a value of β_0 and use it to obtain a better value of $\Delta\lambda$, which in turn gives a better value of β_0, and so on until two successive approximations to $\Delta\lambda$ agree within the limits of error appropriate to the number of figures used. In the following formulas the notation of the direct problem is used so far as it applies.

FIRST STEP—DETERMINATION OF THE PARAMETRIC (REDUCED) LATITUDES.

The parametric latitude β_1 is derived from the geodetic latitude φ_1 by the same formulas as in the direct problem, equations (1) or (1a) page 17, or by special tables. In a similar manner β_2 is derived from φ_2.

SECOND STEP—APPROXIMATE VALUE OF THE SPHERICAL DIFFERENCE IN LONGITUDE.

An approximate value of $\Delta\lambda$ may be found by the formula

$$\Delta\lambda = \Delta l + \rho'' f \cos\beta_1 \cos\beta_2 \sin\Delta l. \tag{26}$$

For the Clarke Spheroid of 1866 this becomes numerically

$$\Delta\lambda = \Delta l + [2.84463] \cos\beta_1 \cos\beta_2 \sin\Delta l. \tag{27}$$

A more accurate formula that will often obviate the necessity of a second approximation is due to Jordan.[18] It reads

$$\Delta\lambda = \sqrt{\frac{N}{R}}\,\Delta l\,[1 - m\,\overline{\Delta\phi}^2 - n\,\overline{\Delta l}^2]. \tag{28}$$

Here N is the radius of curvature of the prime vertical section, R the radius of curvature for a meridian section, $\Delta\phi = \phi_2 - \phi_1$ expressed in seconds of arc, Δl is to be similarly expressed, and m and n are quantities given in the table on page 34. The formula expresses $\Delta\lambda$ likewise in seconds of arc.

If the tables in U. S. Coast and Geodetic Survey Special Publication No. 8 are to be used, the formula may be written

$$\Delta\lambda = \sqrt{\frac{B}{A'}}\,\Delta l\,[1 - m\,\overline{\Delta\phi}^2 - n\,\overline{\Delta l}^2]. \tag{29}$$

N, R, A', B, m and n are to be taken out for the middle latitude $\phi = \frac{1}{2}(\phi_1 + \phi_2)$. The quantities A' and B in this instance are entirely different from the A' and B of formulas (24) and (15) and have here the meanings assigned in Special Publication No. 8. This formula gives the value of $\Delta\lambda$ within 0.″001 for values of $\Delta\phi$ and Δl less than 10°. For larger values of $\Delta\phi$ and Δl the error increases and a second approximation may be needed.

THIRD STEP—DETERMINATION OF β_0, θ_2, AND θ_1.

This may be done in more than one way. With an approximate value of $\Delta\lambda$ we find:

(*a*)
$$\sin^2 \tfrac{1}{2}\Delta\theta = \sin^2 \tfrac{1}{2}(\beta_2 - \beta_1) + \cos\beta_1 \cos\beta_2 \sin^2 \tfrac{1}{2}\Delta\lambda. \tag{30}$$

$$\cos\beta_0 = \frac{\cos\beta_1 \cos\beta_2 \sin\Delta\lambda}{\sin\Delta\theta}, \tag{31}$$

and

$$\left.\begin{aligned} \cos\theta_1 &= \frac{\sin\beta_1}{\sin\beta_0}, \\ \text{or} \\ \cos\theta_2 = \cos(\theta_1 + \Delta\theta) &= \frac{\sin\beta_2}{\sin\beta_0}. \end{aligned}\right\} \tag{32}$$

[18] W. Jordan: Neue Auflösung der geodätischen Hauptaufgabe und ihrer Umkehrung, Zeitschrift für Vermessungswesen, Vol. 12 (1883), p. 65.

For the quadrants in which the quantities are to be taken see Figure 6, page 19, and the accompanying remarks. Evidently α_2, θ_2, λ_2, and β_0 are related to β_2 and α_2 in the same way that θ_1, λ_1, and β_0 are related to β_1 and α_1.

(*b*) An alternative process gives incidentally the azimuths corresponding to the approximate value of $\Delta\lambda$ used. Gauss's equations applied to the triangle PAB give equations (33), the azimuths being reckoned from the south around through the west.

$$\left.\begin{aligned}
\sin \tfrac{1}{2}(\alpha_2+\alpha_1) \sin \tfrac{1}{2}\Delta\theta &= -\sin \tfrac{1}{2}(\beta_2-\beta_1) \cos \tfrac{1}{2}\Delta\lambda,\\
\cos \tfrac{1}{2}(\alpha_2+\alpha_1) \sin \tfrac{1}{2}\Delta\theta &= -\cos \tfrac{1}{2}(\beta_2+\beta_1) \sin \tfrac{1}{2}\Delta\lambda,\\
\sin \tfrac{1}{2}(\alpha_2-\alpha_1) \cos \tfrac{1}{2}\Delta\theta &= +\cos \tfrac{1}{2}(\beta_2-\beta_1) \cos \tfrac{1}{2}\Delta\lambda,\\
\cos \tfrac{1}{2}(\alpha_2-\alpha_1) \cos \tfrac{1}{2}\Delta\theta &= +\sin \tfrac{1}{2}(\beta_2+\beta_1) \sin \tfrac{1}{2}\Delta\lambda.
\end{aligned}\right\} \tag{33}$$

The right-hand sides of the equations being known, the values o. f $\frac{1}{2}\Delta\theta$, $\frac{1}{2}(\alpha_2+\alpha_1)$, and $\frac{1}{2}(\alpha_2-\alpha_1)$ are easily found, and thence α_2 and α_1 This form of computation has the advantage of giving the azimuths at once if the value of $\Delta\lambda$ used is sufficiently approximate. Furthermore, equations (33) will give with the use of the same set of tables more accurate numerical results than will equations (30) to (32). We have further

$$\left.\begin{aligned}
\sin \beta_0 \cos \theta_1 &= \sin \beta_1,\\
\sin \beta_0 \sin \theta_1 &= \cos \beta_1 \cos \alpha_1,\\
\cos \beta_0 &= \cos \beta_1 \sin \alpha_1,\\
\text{or}\qquad & \\
\sin \beta_0 \cos \theta_2 &= \sin \beta_2,\\
\sin \beta_0 \sin \theta_2 &= -\cos \beta_2 \cos \alpha_2,\\
\cos \beta_0 &= -\cos \beta_2 \sin \alpha_2.
\end{aligned}\right\} \tag{34}$$

FOURTH STEP—VERIFICATION OF THE VALUE OF $\Delta\lambda$.

The approximate values of β_0 and θ_1 enable the value of $\Delta\lambda$ to be calculated more accurately. The formulas are (4), (5), and (22), which are here repeated and renumbered for convenience.

$$\tan E = \sqrt{\delta} \sin \beta_0 = [8.9167261 - 10] \sin \beta_0. \tag{35}$$

$$k_1 = \tan^2 \tfrac{1}{2} E. \tag{36}$$

$$\Delta\lambda = \Delta l + \cos \beta_0 \{A'\Delta\theta + B' \cos 2\theta \sin \Delta\theta\}, \tag{37}$$

where

$$\left.\begin{aligned}
\log A' &= \log f - \tfrac{1}{2} M (1 - \tfrac{1}{2} f) k_1,\\
&= 7.5302093 - 10 - [6.3360] k_1,\\
\log B' &= \log (-\tfrac{1}{4}\rho'' e^2 k_1),\\
&= 2.5429n + \log k_1.
\end{aligned}\right\} \tag{38}$$

If the value of $\Delta\lambda$ thus found differs from the $\Delta\lambda$ previously used by more than the preassigned limit of error or by more than the probable uncertainty due to omitted figures, the computation should be repeated till the desired agreement is attained.

FIFTH STEP—COMPUTATION OF THE DISTANCE s.

When definitive values of θ, $\Delta\theta$ and k_1 have been found, the value of s is found by formulas (39) and (40) which are based on formulas (7) to (15),

$$s = b\frac{1+\frac{1}{4}k_1^2}{1-k_1}\left\{\frac{\Delta\theta}{\rho''}+k_1\cos 2\theta\sin\Delta\theta-\frac{k_1^2}{8}\cos 4\theta\sin 2\Delta\theta.\right\} \tag{39}$$

$$\left.\begin{aligned}
&\log\frac{b}{\rho''}=1.488\,7986{,}5,\ b \text{ being in meters.}\\
&\log b=6.8032\\
&\log\left(\frac{1+k_1^2}{1-k_1}\right)=2\log\sec\tfrac{1}{2}E+\tfrac{5}{4}Mk_1^2,\\
&\qquad\qquad=2\log\sec\tfrac{1}{2}E+[6.7347]\,k_1^2\\
&\log\left(-\frac{b}{8}\right)=5.900n.
\end{aligned}\right\} \tag{40}$$

The logarithm of the coefficient of k_1^2 gives the term expressed in units of the seventh place of decimals.

NUMERICAL EXAMPLE, INVERSE PROBLEM.

Given two points A and B whose latitudes are 25° and 65° and whose longitudes are 60° and 110°, respectively, on the Clarke Spheroid of 1866; find the length of the geodesic line joining them and its azimuths at A and B.

FIRST STEP.

Point A		*Point B*	
ϕ_1	25° 00′ 00″.000	ϕ_2	65° 00′ 00″.000
Red'n to parametric lat.	− 4 27. 992	Red'n to parametric lat.	− 4 28. 577
β_1	24 55 32. 008	β_2	64 55 31. 423

SECOND STEP.

$\Delta\phi=\phi_2-\phi_1$	40° 00′ 00″ 144, 000″	log m (p. 34 this pub.) log$\Delta\phi^2$	6. 1251 − 20 10. 3167
Δl	50° 00′ 00″ 180, 000″	log $m\Delta\phi^2$ $m\Delta\phi^2$	6. 4418 − 10 0. 0002766
$\phi=\frac{1}{2}(\phi_2+\phi_1)$	45° 00′ 00″		
log B } Spec. Pub. 8 log A' } arg. ϕ	8. 510 4677 8. 508 9904	log n (p. 34 this pub.) log Δl^2	5. 5234 − 20 10. 5105 − 10
log B/A'	0. 001 4773	log $n\Delta l^2$	6. 0339 − 10
log $\sqrt{B/A'}$	0. 000 7386	$n\Delta l^2$	0. 000 1081
log Δl	5. 255 2725	$1-m\Delta\phi^2-n\Delta l^2$	0. 999 6153
log $(1-m\Delta\phi^2-n\Delta l^2)$	9. 999 8329		
log $\Delta\lambda$	5. 255 8440		
$\Delta\lambda$	180,237″.0		
	50° 03′ 57″.0		
$\frac{1}{2}\Delta\lambda$	25 01 58. 5		

Since the points A and B are so far apart, it is not probable that the value found for $\Delta\lambda$ can be accepted as final.

THIRD STEP.

First alternative: Formulas (30), (31), and (32).

$\cos\beta_1$	9. 957 5384	$\frac{1}{2}(\beta_2-\beta_1)$	19° 59′ 59.″708
$\cos\beta_2$	9. 627 1589	$\log\sin\frac{1}{2}(\beta_2-\beta_1)$	9. 534 0500
$\sin^2\frac{1}{2}\Delta\lambda$	9. 252 9659	$\log\sin^2\frac{1}{2}(\beta_2-\beta_1)$	9. 068 1000
	8. 837 6632	nat. $\sin^2\frac{1}{2}(\beta_2-\beta_1)$	0. 116 97687
nat. $\cos\beta_1\cos\beta_2\sin^2\frac{1}{2}\Delta\lambda$			0. 068 81184
		nat. $\sin^2\frac{1}{2}\Delta\theta$	0. 185 78871
		$\log\sin^2\frac{1}{2}\Delta\theta$	9. 269 0193
		$\log\sin\frac{1}{2}\Delta\theta$	9. 634 5096, 5
		$\frac{1}{2}\Delta\theta$	25° 31′ 59.″08
		$\Delta\theta$	51 03 58.16
$\cos\beta_1$	9. 957 5384	$\sin\beta_1$	9. 624 7361
$\cos\beta_2$	9. 627 1589	$\sin\beta_0$	9. 966 3578
$\sin\Delta\lambda$	9. 884 6722		
$\operatorname{cosec}\Delta\theta$	0. 109 0918	$\cos\theta_1$	9. 658 3783
		θ_1	$= -62°\ 54'\ 37.''15$
$\cos\beta_0$	9. 578 4613		
β_0	67° 44′ 16.″29	$\sin\beta_2$	9. 957 0116
		$\sin\beta_0$	9. 966 3578
		$\cos(\theta_1+\Delta\theta)$	9. 990 6538
		$\theta_1+\Delta\theta$	$= -11°\ 50'\ 39.''00$

These formulas do not determine the signs of β_0, θ_1, and $\theta_1+\Delta\theta$, without ambiguity. Special criteria would be needed. On this account, even if for no other reason, the alternative formulas (33) and (34) are preferable.

Verification. $\Delta\theta=(\theta_1+\Delta\theta)-\theta_1=51°03'58.''15$. The agreement with the value of $\Delta\theta$ previously found is better than could always be obtained with seven-place logarithms.

Second alternative: Formulas (33) and (34).

$-\sin\frac{1}{2}(\beta_2-\beta_1)$	(1) 9. 534 0500n	$-\cos\frac{1}{2}(\beta_2+\beta_1)$	(2) 9. 850 0491n
$\cos\frac{1}{2}\Delta\lambda$	9. 957 1593 (3)	$\sin\frac{1}{2}\Delta\lambda$	9. 626 4830 (4)
$\cos\frac{1}{2}(\beta_2-\beta_1)$	9. 972 9860	$\sin\frac{1}{2}(\beta_2+\beta_1)$	9. 848 9194

(1) $=\sin\frac{1}{2}(\alpha_2+\alpha_1)\sin\frac{1}{2}\Delta\theta$	9. 491 2093n	(3) $=\sin\frac{1}{2}(\alpha_2-\alpha_1)\cos\frac{1}{2}\Delta\theta$	9. 930 1453
$\sin\frac{1}{2}(\alpha_2+\alpha_1)$*	9. 856 6996n	$\sin\frac{1}{2}(\alpha_2-\alpha_1)$*	9. 974 7767-8
$\cos\frac{1}{2}(\alpha_2+\alpha_1)$*	9. 842 0224-5n	$\cos\frac{1}{2}(\alpha_2-\alpha_1)$*	9. 520 0338
(2) $=\cos\frac{1}{2}(\alpha_2+\alpha_1)\sin\frac{1}{2}\Delta\theta$	9. 476 5321n	(4) $=\cos\frac{1}{2}(\alpha_2-\alpha_1)\cos\frac{1}{2}\Delta\theta$	9. 475 4024
$\tan\frac{1}{2}(\alpha_2+\alpha_1)$	0. 014 6772	$\tan\frac{1}{2}(\alpha_2-\alpha_1)$	0. 454 7429
$\frac{1}{2}(\alpha_2+\alpha_1)$	225° 58′ 04.″74	$\frac{1}{2}(\alpha_2-\alpha_1)$	70° 39′ 39.″56
$\sin\frac{1}{2}\Delta\theta$	9. 634 5097	$\cos\frac{1}{2}\Delta\theta$	9. 955 3686

* In the computation these lines can not be filled out at once. Being empty they permit (2) to be subtracted from (1) to obtain $\tan\frac{1}{2}(\alpha_2+\alpha_1)$ or (4) from (3) to obtain $\tan\frac{1}{2}(\alpha_2-\alpha_1)$ without distracting the eye by intervening figures. When $\frac{1}{2}(\alpha_2+\alpha_1)$ has been found from its tangent, $\sin\frac{1}{2}(\alpha_2+\alpha_1)$ and $\cos\frac{1}{2}(\alpha_2+\alpha_1)$ can be inserted in the vacant spaces and each subtracted from the number on the line above or below, i. e., (1) and (2), thus obtaining $\sin\frac{1}{2}\Delta\theta$ in two ways, the results of which should agree. The procedure is similar with $\sin\frac{1}{2}(\alpha_2-\alpha_1)$ and $\cos\frac{1}{2}(\alpha_2-\alpha_1)$. This is a convenient form for many of the formulas of spherical trigonometry and frequent use has been made of it in this computation.

The values of $\frac{1}{2}\Delta\theta$ from $\sin\frac{1}{2}\Delta\theta$ and $\cos\frac{1}{2}\Delta\theta$ agree within the permissible range of error. They also agree satisfactorily with the values found by the first alternative method. The value adopted may be taken from $\log\tan\frac{1}{2}\Delta\theta = \log\sin\frac{1}{2}\Delta\theta - \log\cos\frac{1}{2}\Delta\theta = 9.679\ 1411$, when $\frac{1}{2}\Delta\theta = 25°\ 31'\ 59''.09$, $\Delta\theta = 51°\ 03'\ 58''.18 = 183,838''.18 = \Delta\theta''$

$$\alpha_2 = 296°\ 37'\ 44''.30$$
$$\alpha_1 = 155\ \ 18\ \ 25.18$$

$\cos\alpha_1$	(1) 9. 958 3532n	$-\cos\alpha_2$	(3) 9. 651 4827n
$\cos\beta_1$	9. 957 5384 (2)	$\cos\beta_2$	9. 627 1589 (4)
$\sin\alpha_1$	9. 620 9229	$-\sin\alpha_2$	9. 951 3024
(1) $=\sin\beta_0\sin\theta_1$	9. 915 8916n	(3) $=\sin\beta_0\sin\theta_2$	9. 278 6416n
$\sin\theta_1$*	9. 949 5338–9n	$\sin\theta_2$*	9. 312 2838n
$\cos\theta_1$*	9. 658 3784	$\cos\theta_2$*	9. 990 6538
$\sin\beta_1 = \sin\beta_0\cos\theta_1$	9. 624 7361	$\sin\beta_2 = \sin\beta_0\cos\theta_2$	9. 957 0116
$\tan\theta_1$	0. 291 1555n	$\tan\theta_2$	9. 321 6300n
θ_1	$-62°\ 54'\ 37''.13$	θ_2	$-11°\ 50'\ 38''.94$
$\sin\beta_0$	9. 966 3578-7	$\sin\beta_0$	9. 966 3578
(2) $=\cos\beta_0$	9. 578 4613	(4) $=\cos\beta_0$	9. 578 4613

In these computations we may for definiteness take β_0 as always positive. The quadrant of θ_1 or θ_2 is then determined without ambiguity. The values of $\sin\beta_0$ and $\cos\beta_0$ are consistent within the range of error of seven-figure computation. The value of β_0 itself is not needed in subsequent work, except perhaps for the purpose of drawing a figure.

FOURTH STEP.

$2\theta_1$	$-125°49'\ 14''.26$	$\log\frac{1}{2}M(1-\frac{1}{2}f)$	6. 3360
$\Delta\theta$	51 03 58. 16	$\log k_1$	7. 1628
2θ	− 74 45 16. 10		3. 4988
Check		$\frac{1}{2}M(1-\frac{1}{2}f)k_1$	3154 (units of seventh place in log of A')
θ_1	− 62 54 37. 13		
θ_2	− 11 50 38. 94		
2θ	− 74 45 16. 07	$\log f$	7. 530 2093
		$\frac{1}{2}M(1-\frac{1}{2}f)k_1$	− 3154 } $\log A'$
		$\cos\beta_0$	9. 578 4613
		$\log\Delta\theta''$	5. 264 4357
			2. 372 7909
		First term	$=235''.934$
			$=3'\ 55''.934$
$\log\sqrt{\delta}$	8. 916 7261	$\log(-\frac{1}{4}\rho''e^2)$	2. 5429n
$\sin\beta_0$	9. 966 3578	$\log k_1$	7. 1628 } $\log B'$
		$\cos 2\theta$	9. 4200
$\tan E$	8. 883 0839	$\sin\Delta\theta$	9. 8909
E	$4°\ 22'\ 07''.736$	$\cos\beta_0$	9. 5785
$\frac{1}{2}E$	2 11 03. 868		
$\tan\frac{1}{2}E$	8. 581 4216		8. 5951n
$\log(k_1 = \tan^2\frac{1}{2}E)$	7. 162 8432	Second term	−0. 039

Δl	50° 00′ 00″.00
First term	+3 55. 934
Second term	− 0. 039
$\Delta\lambda$	50 03 55. 89+
$\frac{1}{2}\Delta\lambda$	25 01 57. 95

*See note on p. 29.

This value of $\Delta\lambda$ agrees approximately with the value found by Jordan's approximation, but not within the range of error of seven-figure computation, so formulas (33) to (38) are to be recomputed with this new value of $\Delta\lambda$.

$-\sin\frac{1}{2}(\beta_2-\beta_1)$	(1)	9.534 0500n	$-\cos\frac{1}{2}(\beta_2+\beta_1)$	(2)	9.850 0491n
$\cos\frac{1}{2}\Delta\lambda$		9.957 1598 (3)	$\sin\frac{1}{2}\Delta\lambda$		9.626 4805 (4)
$\cos\frac{1}{2}(\beta_2-\beta_1)$		9.972 9860	$\sin\frac{1}{2}(\beta_2+\beta_1)$		9.848 9194

(1) $=\sin\frac{1}{2}(\alpha_2+\alpha_1)\sin\frac{1}{2}\Delta\theta$	9.491 2098n	(3) $=\sin\frac{1}{2}(\alpha_2-\alpha_1)\cos\frac{1}{2}\Delta\theta$	9.930 1458
$\sin\frac{1}{2}(\alpha_2+\alpha_1)$*	9.856 7010-1n	$\sin\frac{1}{2}(\alpha_2-\alpha_1)$*	9.974 7771
$\cos\frac{1}{2}(\alpha_2+\alpha_1)$*	9.842 0209n	$\cos\frac{1}{2}(\alpha_2-\alpha_1)$*	9.520 0312
(2) $=\cos\frac{1}{2}(\alpha_2+\alpha_1)\sin\frac{1}{2}\Delta\theta$	9.476 5296n	(4) $=\cos\frac{1}{2}(\alpha_2-\alpha_1)\cos\frac{1}{2}\Delta\theta$	9.475 3999
$\tan\frac{1}{2}(\alpha_2+\alpha_1)$	0.014 6802	$\tan\frac{1}{2}(\alpha_2-\alpha_1)$	0.454 7459
$\frac{1}{2}(\alpha_2+\alpha_1)$	225°58′05″.45	$\frac{1}{2}(\alpha_2-\alpha_1)$	70°39′40″.00
$\sin\frac{1}{2}\Delta\theta$	9.634 5087	$\cos\frac{1}{2}\Delta\theta$	9.955 3687

The values of $\frac{1}{2}\Delta\theta$ from its sine and cosine agree within the permissible range of error. The value adopted may be taken from

$$\log\tan\tfrac{1}{2}\Delta\theta=\log\sin\tfrac{1}{2}\Delta\theta-\log\cos\tfrac{1}{2}\Delta\theta=9.679\ 1400$$
$$\tfrac{1}{2}\Delta\theta=25^\circ\ 31'\ 58''.89,\ \Delta\theta=51^\circ\ 03'\ 57''.78=183,837''.78=\Delta\theta''$$

$$\alpha_2=296^\circ\ 37'\ 45''.45$$
$$\alpha_1=155\ \ 18\ \ 25.45$$

$\cos\alpha_1$	(1)	9. 958 3534-5n	$-\cos\alpha_2$	(3)	9. 651 4875n
$\cos\beta_1$		9. 957 5384 (2)	$\cos\beta_2$		9. 627 1589 (4)
$\sin\alpha_1$		9. 620 9216	$-\sin\alpha_2$		9. 951 3012

(1) $=\sin\beta_0\sin\theta_1$	9. 915 8918-9n	(3) $=\sin\beta_0\sin\theta_2$	9. 278 6464n
$\sin\theta_1$*	9. 949 5339	$\sin\theta_2$*	9. 312 2883-4
$\cos\theta_1$*	9. 658 3782	$\cos\theta_2$*	9. 990 6536
$\sin\beta_1=\sin\beta_0\cos\theta_1$	9. 624 7361	$\sin\beta_2=\sin\beta_0\cos\theta_2$	9. 957 0116
$\tan\theta_1$	0. 291 1557-8n	$\tan\theta_2$	9. 321 6348
θ_1	$-62^\circ\ 54'\ 37''.18$	θ_2	$-11^\circ\ 50'\ 39''.40$
$\sin\beta_0$	9. 966 3579-80	$\sin\beta_0$	9. 966 3581-0
(2) $=\cos\beta_0$	9. 578 4600	(4) $=\cos\beta_0$	9. 578 4601

The values of β_0, θ_1 and θ_2 differ so little from those previously found as to lead one to expect that the value found for $\Delta\lambda$ will differ inappreciably from the one from which the computation set out, and that this approximation may therefore be considered final or nearly so.

* See note on p. 29.

$2\theta_1$	$-$ 125° 49′ 14.″36	$\log \frac{1}{2} M(1-\frac{1}{2}f)$	6.3360*
$\Delta\theta$	51 03 57.78	$\log k_1$	7.1628
2θ	$-$ 74 45 16.58		3.4988
Check		$\frac{1}{2} M(1-\frac{1}{2}f)k_1 = 3154$ (units of seventh place in log of A').	
θ_1	$-$ 62 54 37.18		
θ_2	$-$ 11 50 39.40		
2θ	$-$ 74 45 16.58	$\log f$	7.530 2093 } $\log A'$
		$\log \frac{1}{2} M(1-\frac{1}{2}f)k_1 - 3154$	}
		$\cos \beta_0$	9.578 4600
		$\log \Delta\theta''$	5.264 4348
			2.372 7887
		First term =	235.″933
			= 3′ 55.″933

$\log \sqrt{\delta}$	8.916 7261	$\log (-\frac{1}{4}\rho'' e^2)$	2.5429n
$\sin \beta_0$	9.966 3580	$\log k_1$	7.1628
		$\cos 2\theta$	9.4199
$\tan E$	8.883 0841	$\sin \Delta\theta$	9.8909
E	4° 22′ 07.″744	$\cos \beta_0$	9.5785
$\frac{1}{2} E$	2 11 03.872		
$\tan \frac{1}{2}E$	8.581 4218		8.5950n
$\log(k_1 = \tan^2 \frac{1}{2}E)$	7.162 8436	Second term	$-$ 0.″039
		Δl	50° 00′ 00.″000
		First term	3 55.933
		Second term	$-$ .039
		$\Delta\lambda$	50 03 55.89
		$\frac{1}{2}\Delta\lambda$	25 01 57.95

This value of $\Delta\lambda$ is identical with the one from which the computation set out. A divergence of one or two hundredths of a second might well have been tolerated on account of the uncertainty of final figures.

*The figures in this column are unchanged from the previous computation. This may commonly be taken for granted unless the first approximation is very rough or such unusual accuracy is sought that it can be obtained only by using logarithms of eight or more places.

FIFTH STEP.

$\frac{1}{2}E$	2° 11′ 03. 872	$\log \frac{5}{4}M$	6. 7347
$\sec \frac{1}{2}E$	0. 000 3157	$\log k_1^2$	4. 3257
			1. 0604
$\sec^2 \frac{1}{2}E$	0. 000 6314	$\frac{5}{4}M k_1^2$	11,5 in seventh place
$\frac{5}{4}M k_1^2$	11, 5*		
$\log b/\rho''$	1. 488 7986, 5	$\log k_1$	7. 16 284
$\log \Delta\theta''$	5. 264 4347, 7	$\cos 2\theta$	9. 41 988
		$\sin \Delta\theta$	9. 89 091
log first term	6. 753 8659, 7	$\sec^2 \frac{1}{2}E$	0. 00 063
First term	5, 673, 694. 8 m.	$\log b$	6. 80 322
			3. 27 748
		Second term	1894. 4 m.

4θ	149° 31′
$2\Delta\theta$	102 08
$\log\left(-\frac{b}{8}\right)$	5. 900n
$\cos 4\theta$	9. 935n
$\sin 2\Delta\theta$	9. 990
$\log k_1^2$	4. 326
	0. 151
Third term	+1. 4 m.

First term	5, 673, 694. 8 m.
Second term	+1, 894. 4
Third term	+1. 4
s	**= 5, 675, 590. 6 m.**

Recapitulating problem and results we find:

Given point A in geodetic latitude 25° N., longitude 60° W., and point B in latitude 65° N., longitude 110° W., by this computation.

Length of geodesic line $AB = 5{,}675{,}591$ m.

Azimuth at A of geodesic line $AB = 155°\ 18'\ 25''.45$.

Azimuth at B of geodesic line $BA = 296\ \ 37\ \ 45.45$.

The final figures are uncertain by one or more units.

*Eight-place logarithms have been used in this part of the computation to avoid accumulation of errors in the seventh place, but the accuracy of the preceding work hardly warrants this.

LOGARITHMS OF JORDAN'S AUXILIARY QUANTITIES, m AND n, FOR COMPUTING $\Delta\lambda$ (Clarke Spheroid of 1866).

$$m = \frac{\delta \sin^2 1''}{24} [2 - \cos 2\phi - \delta \cos^4 \phi]$$

$$n = \frac{\delta \sin^2 1''}{48} \sin^2 2\phi$$

ϕ	log m	log n	ϕ	log m	log n	ϕ	log m	log n	ϕ	log m	log n
° ′			° ′			° ′			° ′		
0 00	5.8214	−∞	18 00	5.8983	5.0618	36 00	6.0518	5.4798	54 00	6.1877	5.4798
0 30	5.8215	2.0071	18 30	5.9021	5.0823	36 30	6.0561	5.4846	54 30	6.1908	5.4747
1 00	5.8217	2.6090	19 00	5.9059	5.1020	37 00	6.0603	5.4890	55 00	6.1938	5.4693
1 30	5.8220	2.9610	19 30	5.9099	5.1211	37 30	6.0646	5.4932	55 30	9.1969	5.4637
2 00	5.8225	3.2105	20 00	5.9138	5.1395	38 00	6.0688	5.4972	56 00	6.1999	5.4577
2 30	5.8231	3.4040	20 30	5.9178	5.1572	38 30	6.0730	5.5008	56 30	6.2028	5.4514
3 00	5.8238	3.5618	21 00	5.9219	5.1744	39 00	6.0771	5.5042	57 00	6.2057	5.4448
3 30	5.8247	3.6951	21 30	5.9260	5.1909	39 30	6.0813	5.5073	57 30	6.2086	5.4379
4 00	5.8257	3.8105	22 00	5.9301	5.2069	40 00	6.0854	5.5101	58 00	6.2114	5.4307
4 30	5.8268	3.9120	22 30	5.9343	5.2223	40 30	6.0895	5.5126	58 30	6.2142	5.4231
5 00	5.8281	4.0027	23 00	5.9385	5.2372	41 00	6.0936	5.5149	59 00	6.2169	5.4152
5 30	5.8294	4.0846	23 30	5.9427	5.2516	41 30	6.0976	5.5169	59 30	6.2196	5.4070
6 00	5.8309	4.1591	24 00	5.9470	5.2655	42 00	6.1016	5.5186	60 00	6.2223	5.3984
6 30	5.8326	4.2275	24 30	5.9513	5.2789	42 30	6.1056	5.5200	60 30	6.2249	5.3895
7 00	5.8343	4.2907	25 00	5.9556	5.2919	43 00	6.1096	5.5212	61 00	6.2274	5.3802
7 30	5.8362	4.3494	25 30	5.9599	5.3044	43 30	6.1135	5.5222	61 30	6.2300	5.3705
8 00	5.8381	4.4040	26 00	5.9642	5.3164	44 00	6.1174	5.5228	62 00	6.2324	5.3605
8 30	5.8402	4.4552	26 30	5.9686	5.3281	44 30	6.1212	5.5232	62 30	6.2349	5.3501
9 00	5.8424	4.5033	27 00	5.9730	5.3393	45 00	6.1251	5.5234	63 00	6.2373	5.3393
9 30	5.8448	4.5486	27 30	5.9774	5.3501	45 30	6.1288	5.5232	63 30	6.2396	5.3281
10 00	5.8472	4.5915	28 00	5.9818	5.3605	46 00	6.1326	5.5228	64 00	6.2419	5.3164
10 30	5.8497	4.6320	28 30	5.9861	5.3705	46 30	6.1363	5.5222	64 30	6.2442	5.3044
11 00	5.8523	4.6705	29 00	5.9906	5.3802	47 00	6.1400	5.5212	65 00	6.2464	5.2919
11 30	5.8551	4.7071	29 30	5.9950	5.3895	47 30	6.1437	5.5200	65 30	6.2486	5.2789
12 00	5.8579	4.7420	30 00	5.9994	5.3984	48 00	6.1473	5.5186	66 00	6.2507	5.2655
12 30	5.8608	4.7753	30 30	6.0038	5.4070	48 30	6.1508	5.5169	66 30	6.2528	5.2516
13 00	5.8638	4.8070	31 00	6.0082	5.4152	49 00	6.1544	5.5149	67 00	6.2549	5.2372
13 30	5.8669	4.8375	31 30	6.0126	5.4231	49 30	6.1579	5.5126	67 30	6.2569	5.2223
14 00	5.8701	4.8666	32 00	6.0170	5.4307	50 00	6.1613	5.5101	68 00	6.2588	5.2069
14 30	5.8734	4.8945	32 30	6.0214	5.4379	50 30	6.1648	5.5073	68 30	6.2607	5.1909
15 00	5.8767	4.9213	33 00	6.0258	5.4448	51 00	6.1682	5.5042	69 00	6.2626	5.1744
15 30	5.8801	4.9470	33 30	6.0301	5.4514	51 30	6.1715	5.5008	69 30	6.2645	5.1572
16 00	5.8836	4.9718	34 00	6.0345	5.4577	52 00	6.1748	5.4972	70 00	6.2662	5.1395
16 30	5.8872	4.9956	34 30	6.0388	5.4637	52 30	6.1781	5.4932	70 30	6.2680	5.1211
17 00	5.8908	5.0185	35 00	6.0432	5.4693	53 00	6.1813	5.4890	71 00	6.2697	5.1020
17 30	5.8945	5.0405	35 30	6.0475	5.4747	53 30	6.1845	5.4846	71 30	6.2713	5.0823

COMPARISON OF THE NOTATION OF THIS PUBLICATION, WITH THOSE OF HELMERT, AND ALBRECHT (see p. 26).

[The notation of Jordan's formula for $\Delta\lambda$, equation (28) or (29), is not included in the comparison. The symbols N, R, B, A', m, and n there have special meanings which are explained in connection with those equations.]

Brief explanation of quantity.	Symbol used.		
	This publication.	Helmert.	Albrecht.
First point on spheroid	A	P_1	.....
Second point on spheroid	B	P_2	.....
Pole of spheroid	P	N	.....
Vertex of geodesic line on spheroid	C	P_0	.....
Vertex of great circle on sphere	C'	$\mathfrak{P}_0$	.....
First point on sphere	A'	$\mathfrak{P}_1$	.....
Second point on sphere	B'	$\mathfrak{P}_2$	.....
Pole of sphere	P'	$\mathfrak{N}$	.....
Geographic latitude of first point	ϕ_1	B_1	ϕ
Reduced latitude of first point	β_1	β_1	u
Geographic latitude of second point	ϕ_2	β_2	ϕ'
Reduced latitude of second point	β_2	β_2	u'
Reduced latitude of vertex on spheroid	β_0	β_0	$90°-m$
Difference of longitude on spheroid [1]	Δl	$L_{1.2}$	l
Difference of longitude on sphere [1]	$\Delta\lambda$	$\Delta\lambda=\lambda_2-\lambda_1$	λ
Longitude of first point reckoned from meridian of vertex (on sphere) [1]	λ_1	λ_1	.....
Longitude of second point reckoned from meridian of vertex (on sphere) [1]	λ_2	λ_2	.....
Forward azimuth at first point [2]	α_1	$\alpha_{1.2}$	A
Back azimuth at second point [2]	α_2	$\alpha_{2.1}$	A'
Distance between points along geodesic lines on spheroid	s	s	s
Arc from vertex to first point on sphere	θ_1	ϕ_1	$90°-M$
Arc from vertex to second point on sphere	θ_2	ϕ_2	$90°-M+\sigma$
Arc from first to second point on sphere	$\Delta\theta=\theta_2-\theta_1$	$\Delta\phi=\phi_2-\phi_1$	σ
Arc from vertex to point midway between first and second points on sphere	θ	ϕ	$90°-M+\frac{1}{2}\sigma$
Semimajor axis of spheroid	a	a_0	a
Semiminor axis of spheroid	b	b_0	b
Eccentricity of meridian of spheroid	e	e	e
Ellipticity of meridian of spheroid	f	n	p
Auxiliary quantity	$\delta\left(=\frac{e^2}{1-e^2}\right)$	δ	δ
Do.	$n\left(=\frac{f}{2-f}\right)$	n	n
Modulus of elliptic integrals	k_1	k_1	ϵ
Auxiliary used in computing modulus	E	E	E
Auxiliary used in computing elliptic integrals	.....	.....	$E'=\tan^{-1}\sqrt{\frac{6p}{2-p}}$
Do.	.....	.....	$\epsilon'=\tan^2\frac{1}{2}E'$
First coefficient in series for $\Delta\theta$ (this publication) or $\Delta\phi$ (Helmert) [3]	A	$\mathfrak{A}$	.....
Second coefficient in above series	B	$\mathfrak{B}$	.....
Approximate value of θ_1 or ϕ_1	σ_1	σ_1	.....
Approximate value of θ or ϕ	σ	σ	.....
First coefficient in series for difference of longitude	A'	$\mathfrak{A}'$	$2p\alpha'$
Second coefficient in above series	B'	$\mathfrak{B}'$	$-2p\beta'$
First coefficient in series for σ (Albrecht) [3]	$\frac{1+\frac{1}{4}k_1^2}{1-k_1}$	$\frac{1+\frac{1}{4}k_1^2}{1-k_1}$	α
Second coefficient in above series	.....	.....	β
First coefficient in series for difference of longitude	$\frac{A'}{2f}$	$\frac{\mathfrak{A}'}{2n}$	α'
Second coefficient in above series	$\frac{B'}{2f}$	$\frac{\mathfrak{B}'}{2n}$	β'
Modulus of common logarithms [4]	M	M	M
Radian expressed in seconds of arc	ρ''	ρ''	$\frac{1}{\sin 1''}$

[1] In this publication and in Helmert west longitudes are positive, in Albrecht negative.

[2] In this publication and in Helmert azimuths are reckoned from the south through the west, in Albrecht from the north through the east.

[3] The formulas in this case are given a different turn in this publication and in Helmert on the one hand and in Albrecht on the other, and so are not directly comparable. This use of the letters A, B, A', and B' both for points in the figure and for coefficients in the formula need cause no confusion.

[4] Not to be confounded with other sense of M in Albrecht.