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COAST AND GEODETIC SURVEY

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Special Publication No. 301

REVERSIBLE SUSCEPTIBILITY

AND THE

INDUCTION FACTOR USED IN GEOMAGNETISM

By

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ERRATA SHEET FOR SPECIAL PUBLICATION NO. 301

Possessors of Special Publication No. 301, "Reversible Susceptibility and the Induction Factor Used in Geomagnetism," are requested to make corrections by hand as listed below, for typographical errors that have been found to occur in some copies of the publication.

Page	Location on page	In place of	Read
3	Equation (4)	(illegible)	$H=-H_c-g\dot{I}_s$
6	Equations (9), (10)	I_d	$\dot{I}_d$
6	Equation following (10)	I_d	$\dot{I}_d$
15	Equation following (23)	I_d	$\dot{I}_d$
23	Lines 5, 8, 12 only	I_R	$\dot{I}_R$

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CONTENTS

ILLUSTRATIONS

TREATMENT OF UNITS AND DIMENSIONS IN THIS PUBLICATION

The functions discussed fall into two groups, namely (a) those for which the units in different systems are of different sizes, such as B, g, H, I, M, N, q, W_E, κ, and μ; and (b) those comprising abstract ratios, which remain the same regardless of the choice of units, including d, f, m, $N\mu_t\pi^w$, T, u, $\kappa/\mu_v\pi^w$, Υ/V, and ψ. Equations and verbal statements of relation are so written that their validity does not depend on the system chosen. Numerical values have the same generality insofar as they are expressed in terms of group "b" functions. When numerical values are stated for group "a" functions, this is done in terms of the unrationalized c. g. s. electromagnetic system, the one prevailing in geomagnetism.

The symbols π^v, π^w, and μ_v encountered throughout the development serve to adapt the equations to more than one system of units, and in addition the μ_v maintains dimensional coherence for the reader who does not choose to regard that parameter as a pure numeric. To somewhat shorten the expressions involving these factors, one can make use of their values in the system of his own preference, as stipulated on page 33.

To denote the magnetic condition of any given small volume of a substance, one states how great the induction would be if the substance extended throughout space in that same condition. This is termed the intrinsic induction, or (when divided by π^v) the intensity of magnetization. The actual induction at the site may be different, because of the presence of nearby or remote boundaries, inhomogeneities, and macroscopic currents. The quantity, actual minus intrinsic induction, when divided by μ_v, is called magnetizing field, alias magnetic intensity or field strength. Clearly, the field strength is chargeable to postulated pole distributions and to actual current circuits, and may be expressed in terms of them as though they were set up in vacuo.

The intensity of magnetization is treated as being dimensionally akin to induction; this much-used convention lends itself to the discussion of elongated magnets, which tend to be conservative in B when exposed to a changing medium. Susceptibility is taken as having the dimensions and magnitude of I/H. Some readers may find it helpful to substitute for this term and its symbol κ the composite expression "specific susceptibility times μ_v/π^v" where specific susceptibility is defined as $(u-1)$.

REVERSIBLE SUSCEPTIBILITY AND THE INDUCTION FACTOR USED IN GEOMAGNETISM

1. ***Background.***—The measurement of the intensity of the earth's magnetic field has afforded for over a century one of the most precise examples of geophysical measurement. This early achievement of C. F. Gauss and his contemporaries was significant not only in geophysics but also in the emerging realm of electrical technology, where it helped to meet for a long period the pressing need for a calibration technique in a variety of measurements.

Among the refinements contributing to this high precision was the recognition of the temporary change in the strength of the magnets used, due to their varying relation to the geomagnetic field during the measurements and to temperature changes (Lamont 1849). This development took place so early that the terminology and conceptual treatment became congealed without benefit of latter-day insights into the behavior of magnetic materials. One objective of the present publication is to translate these modern insights into a heightened understanding of specific environmental effects on the magnets used in geophysical work. Another is to set forth in systematic form for study and reference some empirical relations not previously so assembled, ranging from well-known geometric properties of the hyperbolic demagnetization curve to little-known interrelations governing various aspects of the magnetizing process. No new experimental data are reported, but some of the interrelations may appear in a new light. In any event, there is a need among geophysicists for such a connected account, to serve as a point of departure in a variety of instrumental problems.

A summary of some of the chief results of this study (exclusive of the appendix) was presented orally at the 1951 annual meeting of the American Geophysical Union in Washington, D. C.

It is suggested that the reader become familiar with the notation list on page 32 and refer to it occasionally during his study of this publication. Where reference is made to other authors, the bibliography on page 49 will identify the source.

2. ***Magnetization and demagnetization curves.***—For a virgin ferromagnetic specimen, the curve of magnetization will be somewhat as shown in figure 1. The character of the initial part of the curve has been intensively studied; Rayleigh (1887) found that it has a definite slope at the origin, and that it can be represented by the formula

$$I = aH + bH^2 \tag{1}$$

where I is the intensity of magnetization (magnetic moment per unit volume), H is the effective intensity of the magnetizing field, and a and b are constants. Bidwell (1911) reviews this topic and gives values of a and b for different materials, as do Weiss and Foëx (1929).

According to a relation known as Fröhlich's (sometimes as Lamont's) law, a portion of the curve lying up beyond the inflection point is approximated by the equation

$$H/I = g' + H/I', \tag{2}$$

where I'_s is the saturation value of I, and g' is a constant. This law makes a chord from the origin to a point on the curve have a slope that varies in proportion to $(I'_s - I)$ as the point moves. Though supplanted (Gokhale 1926) by an exponential function making a better fit in the saturation region, Fröhlich's law is cited here because Watson (1923) found that a similar law applies to the demagnetization curve—that is, the curve that is traced when H is caused to fall back slowly from a large positive value through zero to a reversed value H_c just sufficient to reduce I to zero. This is a portion of the major hysteresis loop, and H_c is called the *coercivity*. For this relation Watson replaced H with $(H+H_c)$ since the aforementioned chord must now spring from the intercept $H = -H_c$. In order to obtain the best fit for negative values of H (the range in which permanent magnets are worked), we discard as Watson suggests the restriction of using the true saturation value and replace I'_s with I_s, which is taken as the level of the horizontal asymptote of the mathematical function. In some cases this value may be considerably smaller than the real saturation value, as Scott (1932) observes. The equation may be written

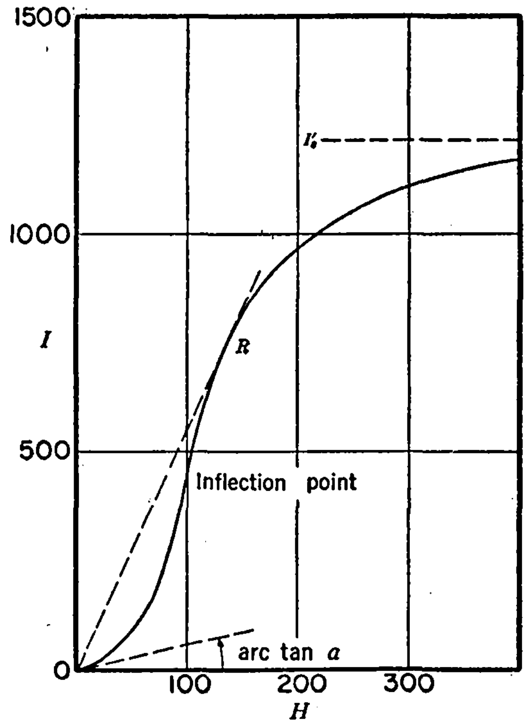

Figure 1.—Features of the normal magnetization curve, generalized.

$$HI/I_s - H + gI + H_cI/I_s - H_c = 0. \tag{3}$$

Like equation (2), this represents a rectangular or equilateral hyperbola (see left part of fig. 2).

This equation has no theoretical basis, and indeed cannot be exact in the neighborhood of the H axis, since in that vicinity the experimental curve has an inflection point where the second derivative must vanish; the hyperbola cannot meet this condition. Nevertheless, the work of many investigators has shown the hyperbolic law to fit the true curve so nearly that its gross characteristics are significant in interpreting the latter.

The vertical asymptote of equation (3) is the line

$$H = -H_c - gI_s. \tag{4}$$

Now, we have called H the effective magnetizing field, but a more specific statement is desirable. A given material is studied by inserting a specimen into a magnetic circuit, impressing a succession of values of magnetizing field on the circuit, and observing the changes in the magnetization. For consistent results, the circuit must be arranged to minimize leakage so that conditions will be essentially uniform throughout the specimen. The overall magnetizing field is apportioned to different segments of the circuit according to their relative reluctances. If the reluctance of the path external to the specimen is known from prior experiments (it should preferably be small), then the portion of the impressed magnetomotance actually effective on the specimen may be computed, likewise the corresponding field intensity, which we denote by H.

To make the effective field zero, we may adjust the impressed magnetomotance until its ratio to the total flux just equals the reluctance of the external path. If this state of the circuit is attained by continuously reducing H from a large positive value, we get for the specimen the value of I known as residual magnetization, and herein denoted by I_R; it is evaluated by setting $H=0$ in equation (3), whence

$$I_s/I_R=1+gI_s/H_c. \tag{5}$$

Now, comparing (5) with (4) we see that the abscissa of the vertical asymptote, with its minus sign dropped, bears to the coercivity H_c the ratio I_s/I_R. That is, the two coordinate axes must cut the curve and the asymptotes at distances from the origin that are in proportion. In fact, lines drawn through any point P parallel to the asymptotes of a hyperbola will cut the curve and the asymptotes at distances from P that are in proportion. If we let $p=I_R/I_s$ we may express equation (5) in the form

$$H_c=\frac{gI_R}{1-p}. \tag{6}$$

Note that the third and fourth terms of equation (3) reduce by (5) to H_cI/I_R, whence we readily obtain

$$\frac{I}{I_R}=\frac{H_c+H}{H_c+pH}, \tag{7}$$

essentially the form of equation (3) given by Scott (1932). This form has two advantages over (3), namely, it involves I only once, and it dispenses with g, using instead the more general index p, which is unaffected by a transformation such as applying a constant factor to all the ordinates or abscissae. For another useful form we solve this for H, obtaining

$$-\frac{H}{H_c}=\frac{I_R-I}{I_R-pI}. \tag{7a}$$

(In the notation used here, Watson's a becomes g/π^s, while Scott's A becomes $1/p$.)

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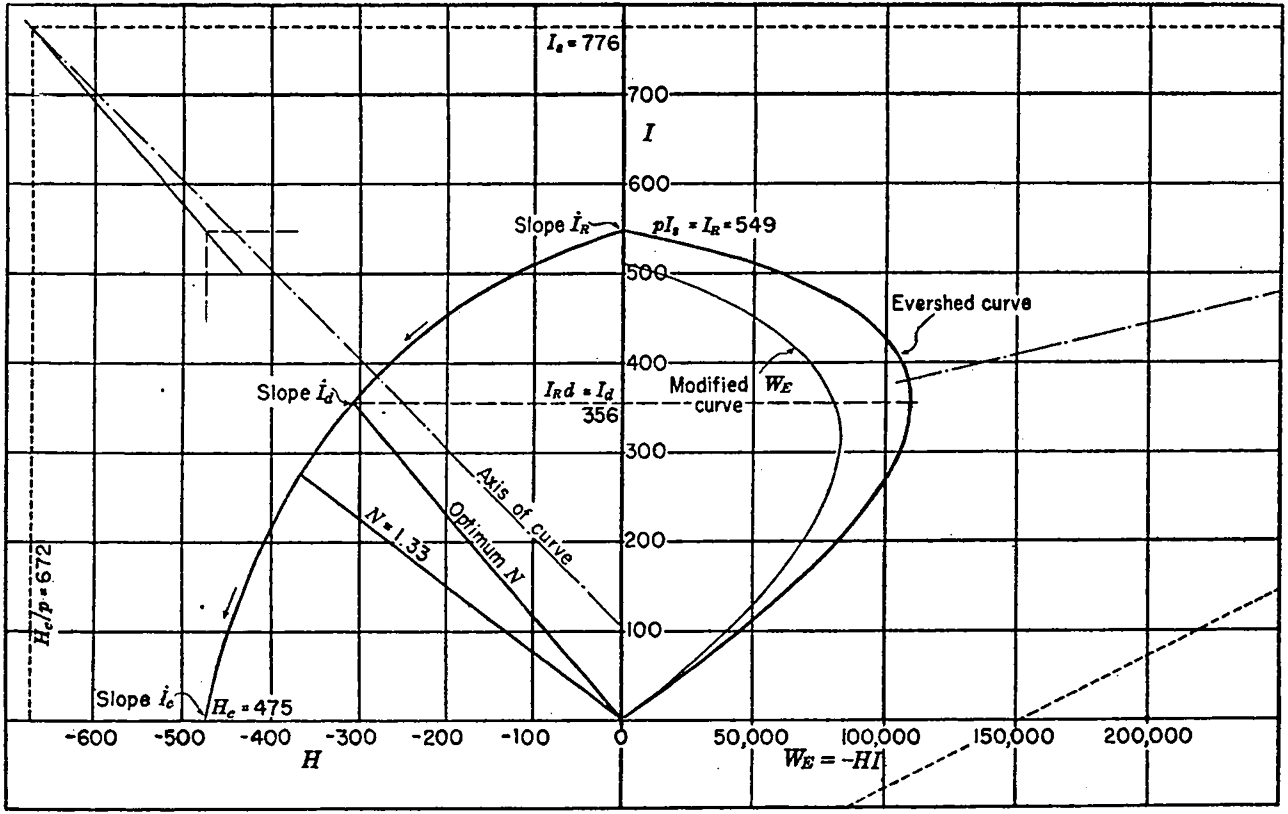

Figure 2.—**Demagnetization and energy-product curves for Alnico III.** H scale graduated in oersteds, I scale in units of 4π gauss, W_E scale in ergs/cm^3. In figs. 2—5 the demagnetization curves were drawn by means of a convenient construction due to Watson and outlined by Sanford (1927).

It is also of interest to differentiate in equation (7), obtaining

$$\frac{dI}{dH}=\frac{I_R-pI}{H_c+pH} \tag{8}$$

$$=\frac{(I_R-pI)^2}{I_RH_c(1-p)} \tag{8a}$$

$$=\frac{I_RH_c(1-p)}{(H_c+pH)^2}. \tag{8b}$$

Since the demagnetization curve can be traced in one direction only, the derivative given by equation (8) does not describe any reversible physical action, but is merely the slope of the curve at the specified point. We may take note of two special values of $\frac{dI}{dH}$, to which we assign the symbols $\dot{I}_c$ (for $I=0$) and $\dot{I}_R$ (for $H=0$). We find that

$$\dot{I}_c=1/g$$

$$=I_d/(1-p) \tag{9}$$

$$\dot{I}_R=I_d(1-p) \tag{10}$$

where $$I_d=I_R/H_c.$$

The demagnetization curve (insofar as it obeys the hyperbolic law) is symmetrical about a diagonal axis (slope -1) which goes through the intersection of the asymptotes (fig. 2). This axis is the normal to the curve at its vertex or point of greatest curvature. Now, practical curves must often be plotted with different scales for the two variables in order to bring out the pertinent relations. The validity of the properties thus far discussed is independent of such scale change, but the coordinates of the vertex will depend upon the relation of the scales. If we make one unit of ordinate correspond in scale with k unit of abscissa, the vertex will have the coordinates

$$I_v=I_s(1-g^{1/2}/k^{1/2})$$

$$H_v=g^{1/2}k^{1/2}I_v-H_c.$$

The general aspect of the curve depends chiefly on the value of kI_R/H_c. By choosing k so that this expression has a value between $(1-p)$ and $1/(1-p)$—that is, by making $1/k$ greater than $\dot{I}_R$ but less than $\dot{I}_c$—we may cause the vertex to fall in the second quadrant as in figure 2; and if we set $k=H_c/I_R$ the axis of symmetry will go through the origin.

The line from the intersection of the asymptotes through the origin will not in general be the axis of symmetry, but it will always go through the point whose projections on the axes are the intercepts of the curve. This line is the locus of the equation

$$H_cI+I_RH=0. \tag{11}$$

In figure 2 it is the line labelled "Optimum N". It is obvious that as the demagnetization curve is being traced, I/I_R falls continuously while $-H/H_c$ rises continuously. Since each of these ratios covers the range between zero and $+1$, there will be a single point at which they are equal. This point is evidently where the line given by equation (11) crosses the curve, affording a simple means of locating it graphically, as Sanford (1927) explains. It is also the point at which the slope of the curve (eq. 8) is given by $I_R/H_c = I_d$. And it can serve as a third anchor point for establishing the parameters of an experimental curve—that is, we may stipulate that the hyperbolic approximation shall be so drawn as to intersect the real curve at this point as well as at the two intercepts. There is, furthermore, a direct practical interest in this point, as will now appear.

If a ring were magnetized tangentially without leakage, its condition upon removal of the magnetizing field would be that represented by the point $H=0$, $I=I_R$; but such a magnet would have no external field. In a practical magnet there must be an air gap and this places the normal operating point somewhere to the left of the I axis. In a subsequent section we shall discuss this quantitatively in relation to specific magnet shapes. It has been shown (Evershed 1920; Watson 1923) that the energy allocated by a magnet to its external field is measured by the product of the values of $-H$ and I at which the material of the magnet is worked, and that under stated conditions this energy product shows the efficiency with which the material of the magnet is utilized to maintain the desired field.

The curve showing the relation of the energy product to I is the locus of the equation

$$W_B = \frac{IH_c(I_R - I)}{I_R - pI} \tag{12}$$

where W_B is the value of $-IH$ (see eq. 7a). Ordinarily W_B is plotted as abscissa and I as ordinate, to facilitate comparison with values on the I/H curve. Equation (12) then represents a nonrectangular hyperbola having a horizontal asymptote coincident with that of the demagnetizing curve, and an inclined one represented by the equation

$$pW_B - H_cI - H_cI_s(1-p) = 0.$$

The energy-product curve for Alnico III is shown on the right-hand side of figure 2, with portions of its asymptotes and transverse axis.

Watson showed that the maximum value of W_B for a given material is reached when the coordinates of the operating point on the demagnetizing curve are in proportion to the latter's intercepts—that is, at the point discussed above, where the curve is crossed by the line of equation (11). By sacrificing this advantage it is possible to obtain higher values of I, if desired. (The operating point of a real magnet would not remain on the curve at all, but the maximum value of W_B is nonetheless a useful criterion.)

For this optimum condition we first replace $-H/H_c$ in (7a) with I/I_R by (11). Then, using the subscript $_d$ to denote the optimum condition,

$$\frac{I_d}{I_R} = \frac{I_R - I_d}{I_R - pI_d}$$

$$p = (2 - I_R/I_d)\, I_R/I_d$$

Now let

$$d=I_d/I_R$$

Then

$$p=(2-1/d)/d$$
$$=(2d-1)/d^2 \tag{13}$$
$$1-p=(d^{-1}-1)^2$$
$$d=(1-\sqrt{1-p})/p. \tag{14}$$

If we let H_d denote the magnitude of H (i. e. the value of $-H$) for the optimum condition, we may likewise write

$$d=H_d/H_c$$

whence

$$H_c I_R d^2 = H_d I_d$$
$$=W_{E(\max.)}$$

These relations, due to Underhill (1944) and Desmond (1945), afford a means of determining p from actual experimental curves, or from published data. One might draw a distinction between d as fixed by the point on the curve whose coordinates are in proportion to the intercepts and d as defined for maximum W_E. A difference of this nature would serve as a rather sensitive test of the conformity of the experimental curve to a rectangular hyperbola; its presence would signify that the slope at the first-mentioned point was unequal to I_R/H_c. In such a case, there would likewise be discrepant versions of p, evaluated by equation (13).

Note the inherent restriction upon p and d, the former being confined to values between zero and $+1$, the latter to values between $+0.5$ and $+1$. It may also be of interest to note that the distance from the intersection of the asymptotes along the axis of symmetry to the focus of the curve, if measured in terms of the H scale, is given by

$$D_H=\frac{2}{p}\sqrt{kH_c I_R(1-p)}$$

and that D_H when multiplied by $2^{1/2}$ gives the latus rectum or when divided by $2^{1/2}$ gives the distance from the intersection of the asymptotes to the vertex, the eccentricity being $2^{1/2}$ for any rectangular hyperbola.

3. ***Fullness or convexity of the curve.***—In order to bring curves for different materials to a common basis for comparison, Sanford (1927) introduced the use of H/H_c as the abscissa and I/I_R as the ordinate. Assuming both of these functions to be plotted to the same scale, this device reduces any given curve to the form that would have been obtained by setting $k=H_c/I_R$; it makes the charted intercepts equal and causes the axis of symmetry to go through the origin. Sanford further reports that by this means a wide variety of magnet steels are found to conform fairly well to a single "master" curve for which $1/p$ was close to 1.38 (or p close to 0.725). Now, Watson in introducing the hyperbolic law made clear his intention to provide for curves that differed in convexity—that is, when plotted

by Sanford's method they would differ as to how nearly the vertex and the intersection of the asymptotes approached one another. This entails a variation in p.

Within the applicable limits, d is a single-valued function of p and vice versa. By equation (14), d^2 has the value 0.430 when $p=0.725$. Scott confirmed the uniformity of I_dH_d/I_RH_c for a number of magnet steels, giving a diagram that yields the value 0.422 for this ratio, this being a direct evaluation of d^2. Later in the same paper he concluded that the demagnetization curves that he obtained were well approximated by a master equation, which is equivalent to Sanford's but with p taken as $2^{-1/2}$. However, the corresponding value of d^2 is 0.421 (not 0.423 as stated by Scott). The index d^2 is called the "fullness factor" or the "curve factor" by Oliver (1938) and other writers. Fowle (1933) and Oliver gave 0.42 as a value that is typical of most magnet steels. Desmond (1945) quotes the same value for older materials but finds 0.58 a better value for some of the newer alloys.

In any event, the development of the newer materials has disclosed a wider variation in p and d than could be discerned from the older carbon steels, as may be seen from table 1. Most of the values of p

TABLE 1.—*Illustrative examples of data discussed in the text.* [a]

No. (see Table 2)	ρ	H_c	$4\pi I_R$	$\pi^w\mu_v N_d = \pi^w\mu_v H_c/I_R$	p	$4\pi I'_s$	$\kappa_0/\mu_v\pi^w$	$f = \kappa_0 N_d$
	g/cm³	oersteds	gauss			gauss		
16	--	5800	465	157	.74	780	.0004	.063
17	--	2650	4530	7.35	.38	--	.008	.059
18	3.8	900	1600	7.07	.52	--	.06	.424
19	7.3	785	7150	1.38	.56	--	.16	.221
20	7.1	550	6000	1.15	.49	11600	.24	.276
22	7.1	560	7350	.96	.67	--	.24	.230
23	7.0	440	7300	.76	.73	--	.24	.182
25	8.7	440	5300	1.04	.70	8600	.24	.250
27	8.4	250	10500	.299	.62	17000	.6	.180
28	--	240	9600	.314	.69	--	-----	---
29	8.3	220	9500	.291	.66	19000	.5	.146
30	7.7	65	9700	.0842	.65	--	2.4	.202
31	8.0	60	10800	.0698	.70	--	2.5	.175
32	7.8	43	10000	.0540	.70	21000	5.9	.319
1	--	52.4	7460	.0883	(.42)	17800	3.36	.297
33	--	48	8600	.0701	.73	--	-----	---
2	--	16.7	13000	.0161	(.66)	19820	5.71	.092
34	8.9	10	5000	.0251	(.28)	18000	5.5	.138
13	--	7.53	9560	.0099	(.53)	18060	12	.119
5	--	4.6	5300	.0109	(.32)	16750	14	.153
10	--	1.06	11400	.0012	(.54)	21200	17	.020
38	7.88	1.0	13000	.00097	(.61)	21500	20	.019
40	7.5	.5	12000	.000524	(.60)	20000	32	.017
41	8.60	.6	2400	.00314	(.19)	12500	68	.214
48	8.60	.05	6000	.000105	(.56)	10700	716	.075
45	7.88	.05	13600	.0000462	(.63)	21500	1989	.092
50	8.25	.04	7300	.0000689	(.44)	16500	239	.016
51	8.76	.014	2500	.0000704	(.42)	6000	3183	224

[a] To convert a value from oersteds to M. K. S. units of magnetic intensity, multiply by $1000/\pi^w$. To convert from gauss to M. K. S. units of magnetization, divide by 10,000 π^v. In either case the exponent of π is to be interpreted in terms of that system toward which the conversion is directed.

TABLE 2.—*Key to materials, to accompany table 1 and figures 7 and 8.*

No.	Name and remarks	Notes	No.	Name and remarks	Notes
1	Steel, V–121 (1% carbon).	1, 7	18	Vectolite (compressed oxides).	2, 5, 6
2	Same	7	19	New KS (18% Ni, 27% Co).	2, 5, 6
3	Common cast steel, V–122 (0.56% carbon).	1, 7	20	Mishima (13% Al, 29% Ni).	1, 2, 6
4	Same	7	21	Alnico III (=Mishima).	1, 2, 5, 6
5	Cast iron, V–118	7	22	Alnico II (6% Cu, 12.5% Co).	1, 2, 5, 6
5a	Same	2, 7	23	Alnico I (20% Ni, 5% Co).	1, 2, 5, 6
6	Silicon steel, SJ4C (0.43% Si).	7	24	Oerstet 500 (14% Al, 25% Ni).	1, 2, 6
6a	Same	2, 7	25	Magnetoflex (Cu with 24% Ni, 35% Co).	1, 2, 6
7	Silicon steel, SJ20C (1.93% Si).	7	26	Magnetoflex (Cu with 20% Ni, 20% Fe.)	1, 2, 6
7a	Same	2, 7	27	Remalloy (12% Co, 17% Mo).	1, 2, 6
8	Silicon steel, SJ50C (4.45% Si).	7	28	Cobalt steel (36% Co, 4% W, 5% Cr).	5
8a	Same	2, 7	29	Original KS (36% Co, 7% W, 3.5% Cr).	1, 6
9	Dynamo steel, V–120 (0.004% Si).	7	30	Chrome magnet steel (3% Cr, 0.4% Mn).	1, 6
9a	Same	2, 7	31	Tungsten magnet steel (5% W, 1% C).	1, 6
10	Swedish charcoal iron (0.006% Si).	7	32	Manganese steel (0.8% Mn).	6, 9
10a	Same	2, 7	33	Carbon steel (1% C)	5
11	Dynamo steel, V–117 (0.028% Si).	7	34	Cobalt (99% Co)	2, 6
11a	Same	2, 7	35	Permendur (49% Co, 2% V).	2, 6
12	Electrolytic wrought iron.	7	36	Permendur (50% Co).	2, 6
13	Electrolytic iron, plate A, 25° C.	8	37	Perminvar (45% Ni, 25% Co).	2, 6
13a	Same, 97° C	8	38	Magnetic iron (99.94% Fe).	2, 6
13b	Same, 205° C	8	39	Field iron (0.5% Si)	2, 6
13c	Same, 295° C	8	40	Transformer iron (4% Si).	2, 6
13d	Same, 400° C	8	41	Perminvar (70% Ni, 7% Co).	2, 6
13e	Same, 505° C	8	42	Mo-Perminvar (45% Ni, 25% Co, 7.5% Mo).	2, 6
13f	Same, 595° C	8	43	Permalloy (45% Ni)	2, 6
13g	Same, 655° C	8	44	Mo-Permalloy (79% Ni, 4% Mo).	2, 6
14	Electrolytic iron, plate B, −190° C.	8			
14a	Same, −120° C	8			
14b	Same, −61° C	8			
14c	Same, +23° C	8			
14d	Same, 97° C	8			
14e	Same, 195° C	8			
14f	Same, 297° C	8			
14g	Same, 392° C	8			
14h	Same, 496° C	8			
14i	Same, 550° C	8			
14j	Same, 605° C	8			
14k	Same, 655° C	8			
15	Silmanal (9% Mn, 5% Al).	5			
16	Silver alloy (8.8% Mn, 4.3% Al).	4			
17	77 Platinum-cobalt (23% Co).	1, 5, 6			

TABLE 2.—*Key to materials*—Continued

No.	Name and remarks	Notes	No.	Name and remarks	Notes
45	Magnetic iron (99.98% Fe).	3, 6	48	Permalloy (78.5% Ni).	1, 2, 6
46	Cr-Permalloy (78.5% Ni, 3.8% Cr).	2, 6	49	Sendust (9.5% Si, 5.5% Al)	6
			50	Hipernik (50% Ni)--	3, 6
47	Mumetal (74% Ni, 5% Cu).	1, 2, 6	51	1040 Alloy (71% Ni, 15% Cu, 3% Mo).	3, 6

NOTES

1. Quench-hardened.
2. Aged by annealing or baking.
3. Baked in H_2 atmosphere.
4. Data from Potter (1931).
5. Data from Fowle (1933 or 1940).
6. Data on ρ, I'_s, and κ_s from Legg (1939), also other data if no other source is cited.
7. Data from Gumlich and Rogowski (1911).
8. Data from Terry (1910).
9. Data from Sanford (1944).

there reported are based on published data on d^2. Those in parentheses, however, are estimates formed by regarding I_s as equivalent to I'_s, and are likely to be smaller than the real values of p.

Table 1 also shows values of H_c and $4\pi I_R$ that have been reported for specimens of the various materials. These parameters, however, are decidedly variable for a given composition, depending on heat treatment and other factors. Furthermore, the values of I actually encountered in magnetometer magnets are much smaller than might be supposed from these values of $4\pi I_R$. The reasons for this will later become clear.

4. ***Comparing B/H and I/H curves.***—Experimentally, an alternative procedure is to measure and plot the induction B, that is, $\pi^v I+\mu_v H$, rather than I itself. (Here π^v is the ratio of intrinsic induction to magnetization, equal to 4π in unrationalized, and 1 in rationalized, unit systems—see p. 33.) The demagnetization curve of the plotted quantity (or more conveniently of its fraction $(B)=I+\mu_v H/\pi^v$) may be regarded as an oblique hyperbola formed by vertically shearing the rectangular hyperbola that would make the best fit of the actual I/H curve. The two curves, oblique and rectangular, have the same vertical asymptote; they intersect (as do their lateral asymptotes) on the vertical coordinate axis. That straight line which if subjected to the same shearing process would coincide with the H axis is called the shearing line; it is the locus of the equation

$$\pi^v I+\mu_v H=0. \tag{15}$$

The sheared curve rises above the other one for positive, and falls below it for negative, values of H.

Following Bates (1948) we shall denote as ${}_BH_c$ the value of $-H$ required to make $(\pi^v I+\mu_v H)$ vanish—that is, the distance from the origin to the H intercept of the sheared curve. This point is the projection on the H axis of a point on the original curve, defined by the latter's intersection with the shearing line. The portion of the sheared curve to the right of this point is an expanded version (now reaching entirely across the second quadrant) of a smaller segment of the basic curve. Experimentally, ${}_BH_c$ is determined by negatively adjusting the magnetomotance to the point of zero flux, and H_c by continuing to the point where removal of the specimen would have no effect on the flux in the circuit.

If we combine equations (7) and (15) so as to eliminate I, we may then replace $-H$ with ${}_BH_c$, obtaining

$$\frac{{}_BH_c}{H_c}=(1+\pi^v I_R/H_c\mu_v)(1-\sqrt{1-4\sigma})/2p \tag{16}$$

where

$$\sigma=\frac{\pi^v p\, I_R}{\mu_v H_c(1+\pi^v I_R/H_c\mu_v)^2}.$$

Regardless of the magnitude of I_R/H_c, $4\sigma<1$. Hence we may use a series expansion of the radical in (16), obtaining

$$\frac{{}_BH_c}{H_c}=(1+\sigma+2\sigma^2+5\sigma^3+\ \ .\ \ .+\frac{(2n)!}{n!(n+1)!}\sigma^n+\ \ .\ \ .)/(1+H_c\mu_v/\pi^v I_R).$$

The disparity between ${}_BH_c$ and H_c amounts to 30 percent of H_c for the platinum-cobalt alloy listed in table 1, and about 3 percent for Alnico II, but is quite negligible for most of the older materials. From the last equation and the definition of ${}_BH_c$ it is obvious that

$$(\mu_v/\pi^v I_R+1/H_c)^{-1}\leqq {}_BH_c\leqq I_R\pi^v/\mu_v. \tag{17}$$

The upper limit was given by Hoselitz (1944), independently of the hyperbolic law. However, no such limit applies to H_c.

The foregoing comparison is the first step toward specifying one of the curves in terms of the other. To complete the process we must take into account the relation between p (as defined in terms of the I curve) and p_B (the corresponding parameter of the B curve). Now, the data which define p_B and d_B are ordinarily established by a procedure equivalent to that outlined in section 2, disregarding the obliquity of the B curve. That is, the sheared curve is simulated by a rectangular hyperbola which it intersects in three points, namely, the two intercepts and a point whose ordinate and abscissa are in the same ratio as the respective intercepts. The broken curve in figure 3 is this rectangular version of the sheared curve, the full line being the original rectangular curve corresponding to equation (7).

The equation of the broken curve is

$$\frac{(B)}{I_R}=\frac{{}_BH_c+H}{{}_BH_c+Hp_B}. \tag{18}$$

The problem is, given H_c, I_R and ${}_BH_c$, to find a relation between p_B and p such that the expression $\left[(B)-I-\frac{\mu_v H}{\pi^v}\right]$ vanishes for the point Q defined by

$$(B)=I_R d_B.$$

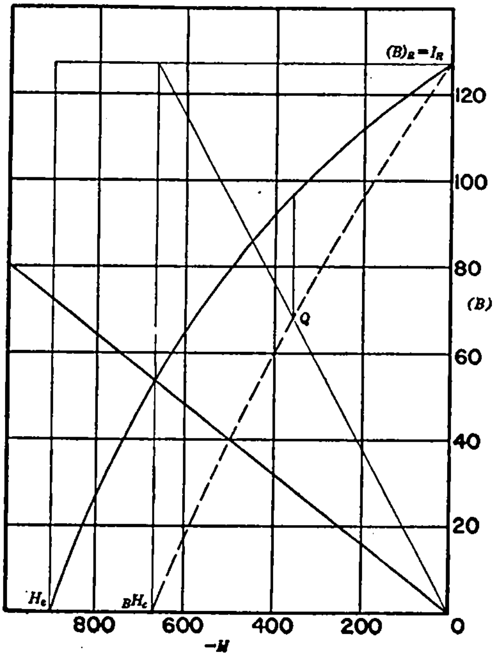

Figure 3.—H_c **and** $_BH_c$ **compared for Vectolite.** H scale graduated in oersteds, (B) scale in units of 4π gauss.

Using this condition and equations (7) and (18) we have

$$\frac{_BH_c+H}{_BH_c+p_BH}-\frac{H_c+H}{H_c+pH}-\frac{\mu_vH}{\pi^vI_R}=0;$$

and replacing H with $-_BH_cd_B$ (since point Q is where H_d for the broken curve must fall) and replacing p_B with an expression in d_B, we obtain an expression that reduces to

$$p=\frac{1-H_c/_BH_cd_B}{d_B(1+_BH_c\mu_v/\pi^vI_R)}+H_c/_BH_cd_B \tag{19}$$

$$d_B=\frac{1+\lambda-\sqrt{(1+\lambda)^2-4p\lambda}}{2p(1+_BH_c\mu_v/\pi^vI_R)} \tag{20}$$

where

$$\lambda=\frac{H_c}{_BH_c}+\frac{\mu_vH_c}{\pi^vI_R}.$$

The relation between d_B and p_B is, of course, exactly like that between d and p (see eq. 13).

Hoselitz (1944) shows that d_B cannot exceed $(1+_B\mathrm{H}_c\mu_v/\pi^v\mathrm{I}_R)^{-1}$. In consulting tabulated data in the literature it may be difficult to ascertain whether a quoted maximum energy product is derived so as to yield d or d_B. Some writers attach greater significance to the latter, but the above limitation means that when H_c is large d_B^2 cannot range far from its lower limit of 0.25, whereas d^2 is (geometrically) free to maintain a normal value irrespective of H_c. The shearing line of equation (15) being invariant, any increase of H_c will distend the basic curve so that the intersection point defining $_BH_c$ shifts to the left; at the same time, the portion of the curve intercepted between that point and the I axis is a diminished fraction of the whole, and its counterpart in the sheared version is consequently reduced in fullness as compared with the basic curve. The ultimate limit of $_BH_c$ corresponds with the point on the shearing line having the ordinate I_R, for which the basic curve would cross the shearing line horizontally. We see that the I/H curve can preserve a normal fullness under conditions so extreme as to cause the B/H curve to degenerate into virtually a straight line.

5. ***The demagnetizing factor.***—Thus far we have been mainly concerned with the description and properties of the fundamental hyperbolic demagnetization curve, which is based on the behavior of a specimen under study in a closed magnetic circuit. One must remember that a permanent magnet is not likely to be used in any such circuit. Ordinarily we may expect the operating point to be confined to a small range of values of H. As we have noted, when a magnet is removed from the test circuit H does not become zero for the magnet; rather, it assumes a negative value with a mean magnitude which, following Dubois, we shall denote by NI_n, where I_n is the mean magnetization left in the magnet (the remanent magnetization) and N is an index to the self-demagnetizing propensity of the magnet. Obviously I_n must be less than I_R. Now, consider N to be variable, as it would be for a magnet consisting of two semicircular segments hinged together on one side, or for a set of different bar magnets of uniform diameter but successively greater lengths. Disregarding nonuniformity of I and H within the magnet, N and I_n will be so related as to conform with equation (7a), with I_n replacing I and NI_n replacing $-H$. That is,

$$N=\frac{H_c(I_R/I_n-1)}{I_R-pI_n} \tag{21}$$

whence we can find the value of N corresponding to any stated point on the demagnetization curve, or vice versa. Since N is ordinarily a constant for a given magnet, there is but one point on the demagnetization curve that can describe the condition of the magnet in the absence of an applied field. To work at a different part of the curve we must alter N in some way.

The demagnetizing factor N depends chiefly upon the proportions of the magnet, and secondarily upon the shape of the demagnetization curve (the latter effect arising from the nonuniformity of H in different parts of the magnet). The value can be determined from theory for ellipsoids, since in this case I is uniform throughout the body. N is generally given for the longitudinal position. Obviously a larger value would apply to a transverse position of the specimen in relation to the applied field. (In the appendix, pp. 34–48, it is shown that the demagnetizing factor is modified slightly by the presence of a medium surrounding the magnet. We are not here concerned with this effect, which in ordinary circumstances is quite small.)

It is convenient to introduce the parameter $\mu_v \pi^w N$, which has a value not dependent upon the system of units employed. (Here π^w is unity in unrationalized, and 4π in rationalized, systems.) This parameter is of the order of 4 for globular bodies, but generally less than 1 for bar magnets; it is very small for long thin wires or for ring magnets with small air gaps. An empirical formula due to Neumann and Warmuth (1932) relates N for cylindrical rods to their length-diameter ratio m, where m is greater than 10 and the susceptibility is large. Their formula, in our notation, becomes

$$\mu_v \pi^w N m^2 = 25.26 \log m - 5.78. \tag{22}$$

This relation is discussed further by Bozorth and Chapin (1942).

Thompson and Moss (1910) suggested that the cross-sectional area might assist in comparing N for round rods with the values for other shapes. They found that for rectangular bars, N is slightly less than for cylindrical rods of the same length and cross-sectional area. This reduction may be supposed to be a function of the radius of gyration of the bar about its longitudinal axis. The latter quantity squared is given by $\frac{1}{2}r^2$ for a solid cylinder, by $\frac{1}{2}(r_1^2+r_2^2)$ for a hollow cylinder and by $\frac{1}{3}(s_1^2+s_2^2)$ for a rectangular bar, where s_1 and s_2 are the half-width and half-thickness. For bars of the same cross-sectional area these three quantities stand in the relation

$$1 : \frac{r_2^2/r_1^2+1}{r_2^2/r_1^2-1} : \frac{\pi}{6}\left(\frac{s_1}{s_2}+\frac{s_2}{s_1}\right)$$

From a study of the data presented by Thompson and Moss, N seems to vary inversely with the cube root of the radius of gyration. Although this result is based on measurements of solid bars, we shall assume (in the absence of a better rule) that to the moderate accuracy required here it holds likewise for the tubular magnets of round and octagonal shape used in magnetometers. On this basis, $\mu_v \pi^w N$ has been calculated for the "long" magnets of several of the magnetometers used by the U. S. Coast and Geodetic Survey. The values so determined range from 0.12 to 0.25. It will be noted that the cube-root rule given here allows a wide range of forms of cross section with but little change of N. Scott (1932) reports N to be approximately independent of shape when the ratio of length to cross-sectional area is held constant.

Table 1 includes a column for finding N_d, that is, the value of N that would have to be used for each material in order to realize the maximum energy product of that material. From equation (11) it is seen that

$$\begin{aligned} N_d &= H_c/I_R \\ &= 1/I_d \end{aligned} \tag{23}$$

The tabular values seem to suggest 36% cobalt steel as a material well suited for magnets of the proportions customarily used in this application, at least so far as this criterion is concerned.

One effect of the nonuniformity of H may be noted briefly. The material at the extreme ends of, say, a cylindrical magnet may drop to a point low on the demagnetization curve, but the central portion of the magnet might well remain up close to the I axis, all intermediate points on the curve being likewise represented at different places in the specimen. The mean values of I and H will depend on just how the actual operating points for different parts of the magnet are distributed along the curve. In any case, it is clear that when mean I is plotted against mean H, the point obtained will fall inside the curve rather than upon it; and if N is varied, the mean operating point will describe a curve that is less convex than the basic one determined for the same specimen in a closed magnetic circuit.

6. ***Minor loops, the reversible susceptibility, and magnetic stabilization.***—If at any point in the demagnetization curve the downward progress is arrested and the applied field is caused to retrace some of the values it has just been taken through, I does not increase as rapidly as it was decreasing but rather follows a flattened curve more nearly resembling the initial part of the magnetization curve, the part governed by Rayleigh's law (eq. 1). Upon resumption of the former progress of the applied field, this flattened curve is approximately repeated upside down until the major loop is regained, whence the rapid fall resumes. These interim changes constitute a *minor hysteresis loop* (Ewing 1892). The important role played by these minor loops in the behavior of permanent magnets is now well recognized. Figure 4 is from Sanford (1927).

For sufficiently small changes, the curvature of these minor loops may be disregarded and we may take them as straight lines, coincident for upward and downward changes. We shall here be concerned with the slope of such a line—more formally, the limiting slope of the line joining the tips of a minor loop as the loop is made indefinitely small. This slope is known as the *reversible susceptibility* (Gans 1908, 1910; Fowle 1933) and designated by the symbol κ_r. (The symbol χ_r which Gans sometimes used will here be reserved for the reversible mass susceptibility, given by κ_r/ρ where ρ is the density of the material.) The corresponding slope for a minor loop on a B vs. H diagram is the *reversible permeability*, μ_r, a quantity equal to $\mu_v+\pi^{v}\kappa_r$ as can readily be verified.

Now, a magnet with an external field is vulnerable to fortuitous fields which it may encounter, and for this reason the operating point has no security in its perch on the demagnetization curve. In fact, it cannot long remain in this condition. An external demagnetizing field will soon be encountered (if not applied intentionally) and this will cause a further downward shift along the demagnetization curve. Upon removal of this field, I does not recover its former value but instead follows a minor loop to a new operating point, again determined by N but no longer lying on the main curve. If N may be taken as independent of I under these conditions, the new point will lie on a straight line joining the previous operating point with the origin. This line has a slope equal to $-1/N$, or $(\mu_v-\pi^{v}/N)$ for the B/H curve, and is called the air-gap line. (The assumption that N is independent of I for a given dimension ratio is not accurately valid for the smallest values of N, as noted by Shuddemagen (1910). This means that those air-gap lines that lie close to the vertical axis will be curved slightly near their upper extremities.) Johnson (1939) gives a figure showing for round rods the air-gap lines corresponding with 16 different values of m, without indicating how N was calculated.

At its new operating point the magnet is stabilized with respect to any fluctuations of applied field smaller than the minor loop which brought it away from the demagnetization curve. By applying a greater stabilizing field it can be protected from larger fluctuations, at a further sacrifice of I. The operating point must be brought down to a point on the air-gap line where there is room between it and the main curve for the largest anticipated minor loop; any subsequent minor fluctuation of the field simply causes a circulation about a minor loop to the right or to the left of the mean operating point. In the words of Watson, "It is clear that the normal working point must lie on a reversible cycle, and that the length over which the cycle is truly reversible must exceed the variations which may be met with in practice."

We now see why the magnets of magnetometers are limited to rather low values of magnetization intensity. The demagnetizing factor is too large to permit I to remain close to I_R, and the stabilization so essential for consistent results depresses I still further. For the magnetometers used in Coast and Geodetic Survey work, values of I are found to be about one-fourth of the published values of I_R for similar materials.

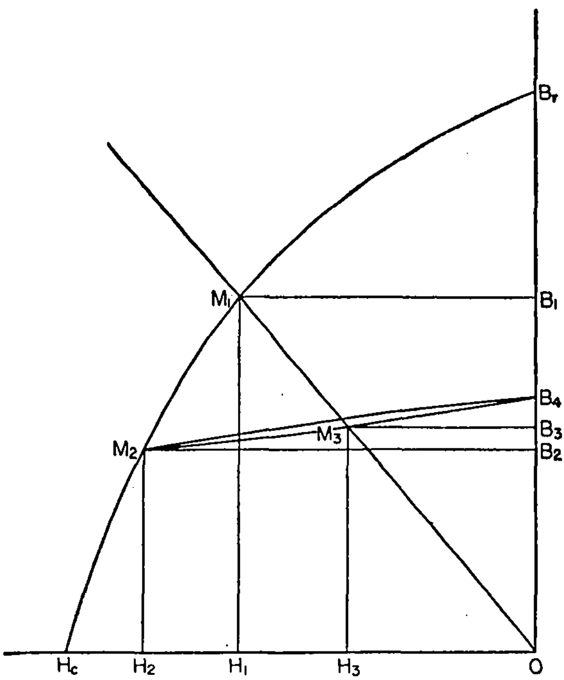

Figure 4.—Minor or recoil loop (after Sanford).

Let us now see what sort of demagnetization curve would be obtained if we studied a specimen in the open by varying the ambient field, instead of placing it in a closed circuit. Under these conditions (assuming for simplicity an ellipsoidal specimen in a vacuum) we might plot I against H_a the externally applied field, obtaining a curve such as the heavy line of figure 5. To relate such a curve to the fundamental major loop, we note that H must always equal $H_a - NI$. Then to each point on the solid curve of figure 5 there corresponds a point on the fundamental curve (light line), displaced laterally through the distance NI. In other words, the curve having H_a as the abscissa is a sheared version of the fundamental one. The vertical coordinate axis for the regular I/H curve becomes a sloping line (shown dotted) in relation to the I/H_a curve. This line is the reflection in the I axis of the air-gap line already discussed; the latter is hence sometimes called the shearing line. Note that for a given material there are many H_a curves, corresponding to different values of N, whereas the curve of I versus H is independent of N.

Alternatively, one might begin with the vertically sheared curve of figure 3; then the new shearing would be sufficient to bring the air-gap line into coincidence with the vertical axis, but the direction of displacement would be parallel to the line $\pi^{\bullet}I=\mu_v H$, rather than horizontal. The H_a intercept of the final curve would then fall somewhat to the right of the point defining ${}_BH_c$, but the analysis is quite cumbersome and need not be pursued here.

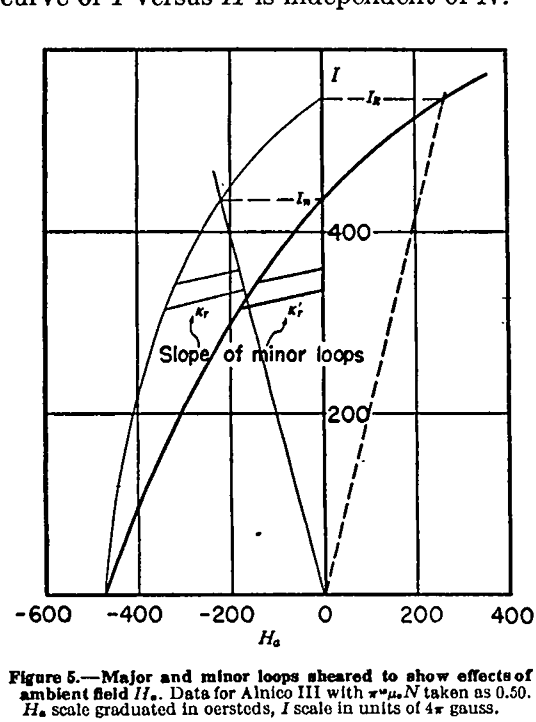

Figure 5.—Major and minor loops sheared to show effects of ambient field H_a. Data for Alnico III with $\pi^{\bullet}\mu_v N$ taken as 0.50. H_a scale graduated in oersteds, I scale in units of 4π gauss.

Not only the major loop but minor loops as well may be thus sheared, to indicate how a particular magnet is affected by fluctuations in the ambient field. The sheared minor loop is the basis of the concept of *false reversible susceptibility* κ'_r. The effective change of H (or ΔH) is less than the change in the applied field ΔH_a by the amount of the change in the term NI. That is,

$$\Delta H_a = N\Delta I + \Delta H. \tag{24}$$

Evidently we may generalize equation (24) to write

$$1/\kappa'_r = N + 1/\kappa_r \tag{25}$$

or by eliminating ΔI instead of ΔH,

$$\Delta H_a = (N\kappa_r + 1)\Delta H. \tag{26}$$

Note that if N or κ_r is sufficiently small, κ'_r may be taken as equal to κ_r (eq. 25) or κ'_0 equal to κ_0. Thus, in the measurement of weak susceptibilities by the method of Johnson and Steiner (1937) or that of Hoylman and Durbin (1944), no significant error is introduced by taking the response as proportional to susceptibility, although if either method were extended to much higher susceptibilities one would need to recognize that the procedure described actually yields κ'_0 rather than κ_0. The disparity is likely to be considerable for permanent-magnet materials.

As another incidental application of the above relations, we note that because of the practical necessity of stabilization, Evershed's energy-product curve based on the major loop (eq. 12) is not strictly pertinent for actual magnets. What is really needed is a curve of the quantity $(W_B - I\Delta H)$ plotted against I, where ΔH is the effective demagnetizing field to be stabilized against. If we are to consider ΔH_a as the corresponding applied field, which would presumably be a specified maximum, we see from (26) that the modified energy-product curve would intersect the regular one at the origin and would fall to the left of it by increasing amounts as I builds up and N decreases. The value of I for maximum energy product would be a little below I_d. The disparity would be less than that obtained by assuming ΔH equal to ΔH_a; in the latter case the curve would simply be sheared leftward so as to cross the I axis at the level at which the abcissa of the Evershed curve is $I\Delta H$. In figure 2, the lighter curve is the modified energy-product curve, taking ΔH_a constant at 100 oersteds. A somewhat different approach to this problem is given by Sanford (1944), who also points out that the effects of leakage require empirical modifications in designing magnets for most effective utilization of the steel. The foregoing remarks about the modification of the energy-product curve apply with equal force whether the curve is based on H_c or on ${}_BH_c$. It has been stated by Edwards and Hoselitz (1944) and by the authors of several subsequent papers that the curve based on ${}_BH_c$ must be used to obtain an authentic maximum energy product.

7. ***The induction factor.***—We now take up the practices in geomagnetism which were mentioned at the outset. When a magnetometer is used to determine by Gauss's method the horizontal intensity of the earth's field, one allows for the small change in the moment of the magnet due to that field by the use of an index variously known as the *Inductionsfähigkeit* (Lamont 1867), the *induction factor* (Hazard 1911), and the *induktive Kapazität* (Venske 1913). Here we use the second of these three names. The induction factor of any magnet is historically defined as the temporary change in magnetic moment which it undergoes due to unit change in the ambient field. No symbol has met with full acceptance for the induction factor. Perhaps the one most frequently seen since the time of Welsh is the Greek letter μ. In this publication Υ is adopted in order to avoid confusion with permeability, which we denote by μ as is general in the literature of physics. Furthermore, we shall attach to the increment of field the coefficient $\mu_0\pi^w$, in order that values of the ratio of induction factor to magnet volume may be the same regardless of the units chosen.

In comparison with the field in the neighborhood of a magnet, the geomagnetic field is weak, and may be treated as a small increment superimposed on the demagnetizing field corresponding to the air-gap line. Then we have

$$\begin{aligned}\mu_0\pi^w\Upsilon &= \Delta M/\Delta H_a \\ &= V\Delta I/\Delta H_a \\ &= V\kappa'_r,\end{aligned}$$

where κ'_r is the same as in equation (25) and V is the volume of the magnet. Combining this result with equation (25), we have the useful relation

$$V/\mu_v\pi^w\Upsilon = N + 1/\kappa_r \tag{27}$$

$$= N + 1/\chi_r\rho$$

whereby there is demonstrated a simple connection between induction factor and reversible susceptibility, permitting ready conversion from one to the other index, provided the values of V and N are known. Also, differentiating in equation (27) we find

$$\frac{d\Upsilon}{dN} = -\mu_v\pi^w\Upsilon^2/V. \tag{27a}$$

There are recorded in the literature almost as many methods of determining the induction factor as there are investigators who have studied the subject. Lamont's original method, however, remains the one most widely used; it is described by Hazard (1911) and others. Electromagnetic methods have been devised by Weber (1855) (see also Kohlrausch 1892), by Schmidt and Venske (Venske 1913), and by Nelson (1938). Weber used a ballistic galvanometer, while the others named impressed upon the magnet being tested a known field from a coil. Regardless of the method used, certain precautions are essential if consistent results are to be obtained. *The magnet must be stabilized* as explained in section 6, to such an extent that the applied field will not carry the operating point back to the major loop. Stabilization must be repeated for each new level of magnetization if the magnet is being tested at several points. The applied field should be within the limits for which the assumption of linearity of the minor loop is justified; this point has been discussed by Venske, and earlier by Chree (1899). Again, one should make sure the applied field is known with an accuracy commensurate with that desired in the induction factor. Another point discussed by Mascart (1899) is the usually neglected effect, in the measurement of horizontal intensity, of the component of the earth's field transverse to the axis of the magnet. The transverse induction factor is customarily much smaller than the ordinary one, on account of the larger value of N involved; by the same token the former is less sensitive than the latter to change of κ_r, hence not so effectively reduced by the use of the newer magnet alloys. The two factors appear in the torque equations with contrary signs, hence they might be equalized and their effects canceled (in theory) by suitably choosing the composition and proportions of the magnet.

There has been perennial uncertainty whether the change of moment is the same in magnitude for positive and for negative increments of applied field. Lamont found a greater effect when the field was such as to decrease the moment, but his result is of doubtful relevance, since the importance of first thoroughly stabilizing the moment was not then appreciated. Of the subsequent investigations of induction factors, those of Kohlrausch (1884) and Wild (1886) are pertinent on this matter: both conclude that with sufficient care in the measurements the difference is eliminated. Chree (1899) leaves the question open, likewise Hazard, though the latter considers the difference to be negligible in practice. It now seems clear that a persistent difference of this kind would imply an irreversible change of moment, entailing indefinite weakening of any practical magnet by reason of unavoidable minute fluctuations in the ambient field. The practical stability of the best modern magnets proves that it is possible to realize full reversibility of small changes—in other words, that (as Wild's tests indicate) there is a true reversible susceptibility, independent of the direction of the applied field, though in measuring it the minor loops must not be allowed to reach a size that would take them too near the main demagnetizing curve.

8. ***Effect of varying*** *I*.—A related question is whether and in what way κ_r and Υ are affected by differences in the intensity of magnetization. The operating point may be situated anywhere within the major loop. If it is near the descending branch and if a negative increment is applied to H, the point must shift along a minor loop so as to come *nearer* to the curve, or at least not recede from it: otherwise the major loop could not have been established as it was in the first place. Consequently, the slope of the major loop at any place in its course constitutes an upper limit on κ_r for operating points in that vicinity. But the slope of the major loop is vanishingly small at the tips, in the saturation region. Hence, κ_r must be a function of I.

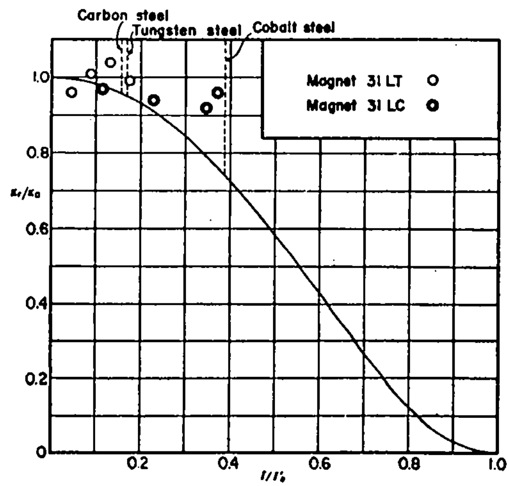

Figure 6.—Gans relation and data by Nelson.

A relation between these two quantities was noted long ago by Rayleigh (Rayleigh 1887; Ewing 1892). Williams (1913) reports that κ_r increases slightly with decrease of I, the maximum difference observed being about 8 percent. We find that the relation has been reduced to law through the investigations of Gans (1911) and Brown (1938). The relation is in the form of a set of parametric equations which may be written

$$\left.\begin{aligned} I/I'_s &= \coth\eta - 1/\eta \\ \kappa_r/\kappa_0 &= 3/\eta^2 - 3\,\mathrm{csch}^2\eta \end{aligned}\right\} \tag{28}$$

where κ_0 is the value of κ_r obtained after traversing the major loop down to a slightly negative value of I, then following a minor loop to the H axis. It is nominally the same as a in equation (1)—that is, the initial susceptibility.

The series forms of equations (28), valid for $\eta^2 < \pi^2$, give us

$$\left.\begin{aligned} I/I'_s &= \eta/3 - \eta^3/45 + 2\eta^5/945 - \ \ldots \\ \kappa_r/\kappa_0 &= 1 - \eta^2/5 + 2\eta^4/63 - \ \ldots \end{aligned}\right\} \tag{29}$$

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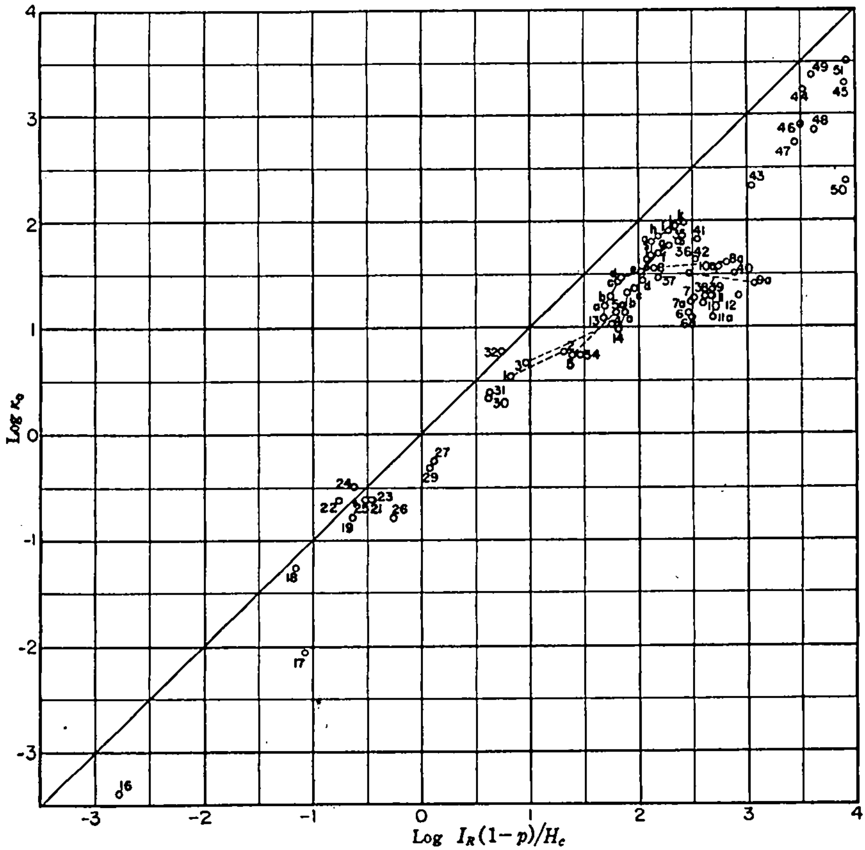

Figure 7.—Test of relationship between κ_0 and $\dot{I}_R$. Scales graduated in c. g. s. units. For key to materials see table 2, page 10.

It will be noted that as η increases without limit, I/I'_s approaches unity and κ_r/κ_0 vanishes, whereas the converse is true for η vanishing. In figure 6, the full line shows the relation between I/I'_s and κ_r/κ_0. Three typical values of I/I'_s are indicated along the upper edge of the graph, assuming $\pi^w\mu_s N=0.15$ and assuming I stabilized at 80 percent of the value on the main curve.

9. ***Relation of κ_0 to main loop.***—The Gans equations, by fixing the ratio κ_r/κ_0, indicate that κ_r is a function of I for specimens exhibiting any one hysteresis curve. This follows from the fact that κ_0 is by its nature not variable with I. However, Gans is silent as to the actual values of κ_r and κ_0, aside from their ratio. We are thus led to inquire whether there is any law by which such values might be deduced from some aspect of the hysteresis curve. Gumlich and Rogowski (1911) have noted that κ_0 bore a loose relation to maximum susceptibility (the slope of OR in figure 1) which in turn seemed to depend on the ratio I_R/H_c. More recently, Underhill (1944) gave a rule prescribing that κ_r is closely approximated by the slope of the main curve at $H=0$, given by $\dot{I}_R$ in equation (10). However, this last cannot be a precise or general law. Any rule which allows no

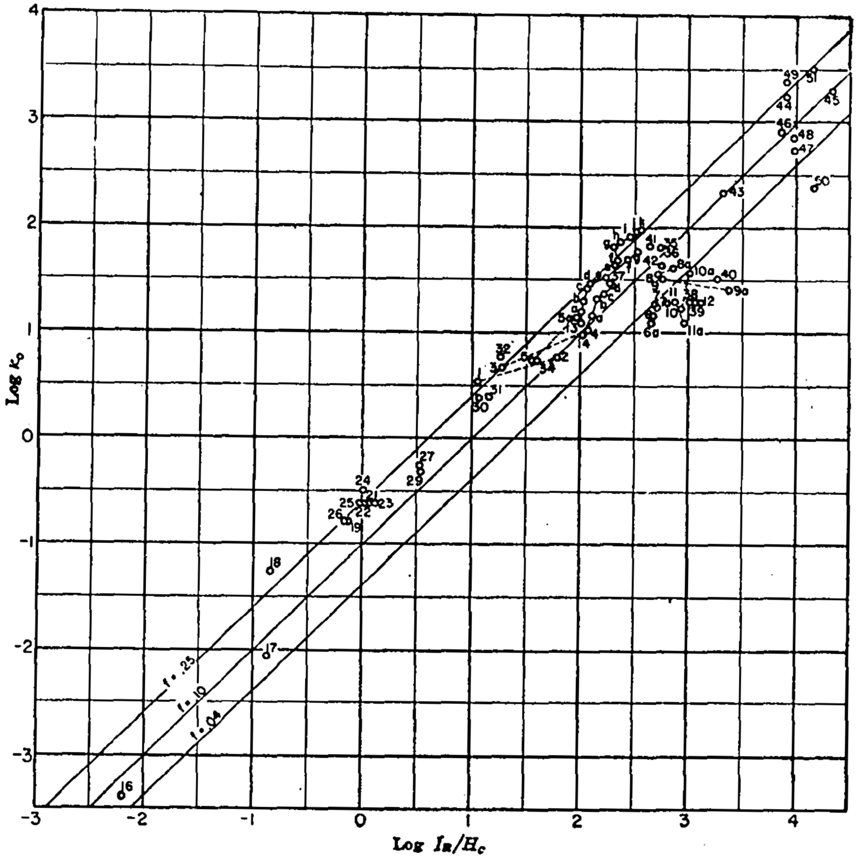

Figure 8.—Test of relationship between κ_0 and $\dot{I}_d$. Scales graduated in c. g. s. units.

variation of κ_r with I is incompatible with the inherent limitation that is set forth above and embodied in the Gans relation. In this case, the rule must fail in respect of all values of I greater than I_R, since the slope of the major loop would then be less than $\dot{I}_R$.

The possibility remains that κ_0 rather than κ_r is equal to I_R. Such a relation would be a refinement of Underhill's rule and would not conflict with the Gans relation. It will be convenient to test the matter by plotting log κ_0 against log I_R for a variety of materials, as derived from published data (fig. 7). The line shows where the points would fall if κ_0 were actually equal to $\dot{I}_R$. Though most of the points fall under the line, the 45° slope does seem to be maintained, suggesting a linear relation between I_R and κ_0. Many of the points involve a doubtful evaluation of p, as already mentioned. Noting that $\dot{I}_R=(1-p)\dot{I}_d$, it would seem that the factor $(1-p)$, seeing that it fails to make the line fit the data, might be discarded in the interest of simplicity. In figure 8, then, we plot log κ_0 directly against log $\dot{I}_d \equiv \log\,(I_R/H_c)$. The scatter of the points is not noticeably aggravated by this simplification. Now, let

$$f = \kappa_0 N_d$$

$$= \frac{\kappa_0}{I_R/H_c}. \tag{30}$$

The significance of f lies in the fact that its value appears to undergo little if any systematic change under changes of κ_0 covering a span of nearly seven orders of magnitude. Values of f for some of the materials covered by figure 8 are given in table 1. Lines are drawn in figure 8 to correspond with equation (30), assigning to f the three arbitrary values .04, .10, and .25.

It is of some interest to evaluate Υ_1, the predicted induction factor for optimum N, in the light of equation (30). If we assume the typical values $p=0.725$ and $I=0.6\ I_d$, we find that $I=0.28\ I_s$, which may be approximated as $I=0.28\ I'_s$, and from figure 6, $\kappa_r=0.86\ \kappa_0$. Under these assumptions we deduce

$$\mu_v \pi^w \Upsilon_1/V = \frac{\kappa_0}{1.16 + \kappa_0 N_d}$$

$$= \frac{\kappa_0}{1.16 + f};$$

and for the same conditions, by equation (27a),

$$\frac{d\Upsilon}{dN} = \frac{-V\kappa_0^2}{\mu_v \pi^w (1.16 + f)^2}.$$

It has been urged by C. E. Webb (Desmond 1945, discussion) that μ_r must be closely related to B_R/H_c. Equation (30) may be regarded as a modification of this concept. Webb himself confirms that μ_r is affected by shifts of the working point; we have already seen that for this reason κ_0 is preferable to κ_r and a similar distinction applies to permeability. The choice of κ_0 rather than μ_0 offers the advantage of placing the lower limit at zero instead of unity; similarly we choose H_c based on the I/H curve, called ${}_IH_c$ by some writers (Bates 1948; Stoner 1950)—not the smaller ${}_BH_c$ which we have seen cannot exceed $I_R\pi^v/\mu_v$. If $B_R/{}_BH_c$ were chosen as one variable, it too would have unity as its lower limit; whether a linear relation between this variable and μ_r would fit the data better than equation (30) could only be decided by means of more extensive data near the lower end of the curve. For large values the data would form the same pattern in either case.

Figure 8 leads to two observations. First, the various ferromagnetic materials form a long unbroken sequence extending from very large to very small values of I_R/H_c; the difference between permanent-magnet materials and such alloys as mumetal appears in this respect to be one of degree only. That is, the magnetization curve and hysteresis loop for any ferromagnetic substance may resemble that for any other with suitable changes of scale. Perhaps the extension of experimental techniques to more and more intense fields may further enlarge the list of ferromagnetics; such materials, if they have values of I_R/H_c much lower than those shown on figure 8, may well be now masquerading as paramagnetics, on account of the difficulty of polarizing them. The values of $\kappa_0/\mu_v\pi^w$ for paramagnetics generally fall in the range betwen 10^{-4} and 10^{-7}. The second point about figure 8 is the considerable variability of f, that is, the scatter of the points. Attempts to reduce the scatter by means of other variables such as κ_{max} and the Steinmetz hysteretic coefficient have proven unavailing with the meager data at hand.

A further indication of the variability of f may be found in a paper by Hornfeck and Edgar (1940) which deals largely with Alnico I. Values given there are: $H_c=430$ oersteds, $4\pi I_R=7400$ gauss, and $4\pi I_R/p=10{,}000$ gauss, agreeing fairly well with other sources (cf. item 23 of table 1). The paper cited also contains a graph (their fig. 9) showing the relation of reversible permeability to the slope of the air-gap line. When the latter is zero the ordinate of the curve represents initial permeability, and we thus deduce that $\kappa_0=0.51\ \mu_v\pi^w$, about twice the value derived from Legg; and the corresponding value of f is 0.37. Thus, f is not necessarily the same even for different specimens of the same alloy.

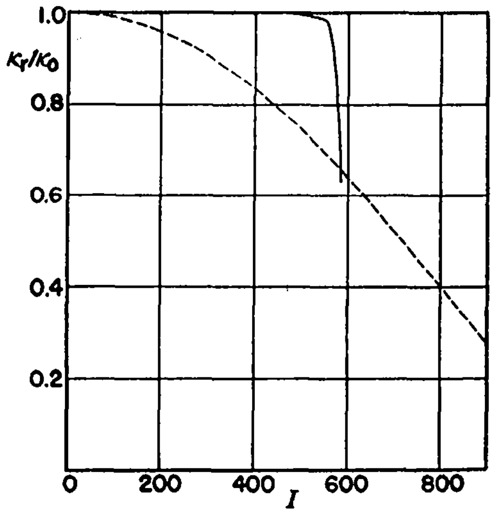

Figure 9.—Hornfeck-Edgar curve compared with Gans curve. I scale graduated in units of 4π gauss.

In figure 9 of the present publication, the cited curve has been redrawn to show κ_r/κ_0 as a function of I. When so drawn, the relation shows, for values of I a little less than I_R, a pronounced knee that is obscured in the original curve by the rapid change in $1/N$ as N approaches zero. The form of the curve here presented might be compared directly with the Gans relation if I'_s were known; alternatively, we may assume a value of I'_s that will lead to agreement at some special value of I and then make comparisons for other values of I. By taking $I'_s=1300$, we obtain agreement at $I=580$; the dotted curve of figure 9 is a replica of figure 6 under this assumption. The abrupt decrease of κ_r as I approaches I_R in the solid curve might be avoided by a slight modification of the parent curve. It has no counterpart in the Gans curve.

The latest investigation of reversible susceptibility is that of Tebble and Corner (1950). They develop some limitations on the finding of Gans (1911) cited by other workers (Weiss and Foëx 1929, Legg 1939), that κ_r is a single-valued function of I/I'_s.

10. ***Effect of I upon*** Υ.—Consider next how variations in I affect the induction factor. Kohlrausch (1892) indicated that Υ is slightly greater for unmagnetized than for magnetized bars. There is some direct experimental evidence on this point. Venske determined under varying conditions the induction factors of six magnets of the kind used in magnetometers, finding a very small increase with magnetic moment, followed by a decrease when the bars were magnetized to successively higher values. Again, Nelson (1938) made similar measurements upon two magnets, his results indicating substantially constant Υ. However, it is not clear in either case whether the experiments included the step of stabilizing the moment prior to each test. Particularly vulnerable in this respect are likely to be those measurements made in the saturation region (by Venske). With the exception of those few values, none of the measurements extended to the higher values of I/I_R (that is, values close to or exceeding unity), being confined to bar magnets with the operating point always on or near the air-gap line. Within this limitation it appears that Υ is relatively insensitive to shifts of the operating point. (Equation 27 demonstrates that Υ is less sensitive to I than is κ_r). As an additional aid in examining these results, Nelson's data have been replotted on figure 6 using equation (27) to derive κ_r from Υ, and estimating $N\mu_v\pi^w=0.15$ and $V=4$ cm^3. For magnet 31LT, $4\pi I'_s$ is estimated as 20,000 gauss.

11. ***Temperature effects.***—In discussing the behavior of ferromagnetics in weak fields where equation (1) holds, Weiss and Foëx state "The values of a and b vary markedly with thermal treatment . . . and in a very complicated fashion with the temperature, pass through an acute maximum near the Curie point and become zero at that point. As the temperature changes progressively from $-188°$ C. to the Curie point, they satisfy the relation $bc=a^n$; the values of c and n depend upon the substance and its past thermal treatment." It follows that the temperature should always be determined and reported as an essential element in any determination of Υ, κ_r, or any other quantity dependent upon κ_0. For further information on this, we turn to the work of Terry (1910) who investigated two samples of doubly purified electrolytic iron over a wide temperature range, determining the values of H_c, B_R, I'_s and κ. He found H_c to fall and κ to rise with rising temperature, whereas B_R seemed but little affected in the ordinary range.

It is difficult to extrapolate Terry's values of κ down to κ_0 with satisfactory precision. Hence κ_0 can be deduced well only for the unannealed specimens and not even for them at the highest temperatures. However, such data as could be obtained in this way have been added to figures 7 and 8. Their general trend is similar to that of the lines of constant f (equation 30), with enough disparity to make it doubtful that f is entirely independent of t, though neither do the data lend appreciable support to any suggestion that temperature differences could directly account for the scattering of the other points (in view of the wide temperature range in Terry's data).

It is, of course, well known that magnetic moment is subject to change with temperature. The fractional *decrease* of moment per degree *increase* in temperature is known in geomagnetism as the temperature coefficient of a magnet, denoted here by q. This index cannot be independent of the temperature effect mentioned above, and we shall examine a possible relation between them.

Consider the region within the major loop to be traversed by a family of curves such that each curve has a slope at every point corresponding to κ_r at that point. If in the pertinent area κ_r may be taken to be a single-valued function of I, these curves will be replicas of one another, dispersed laterally across the diagram. Consider how this system of curves may be affected by temperature, assuming a simple kind of behavior consistent with the changes usually found at ordinary temperatures. We postulate two effects—a uniform vertical expansion or contraction (coefficient α) and a lateral change such that the slope of the curves (the resultant of both effects) would be governed by the coefficient β. If the operating point is on the vertical axis it will be affected only by α, otherwise by both α and β (supposing that the point is carried by the motion of the particular curve on which it lies to a new position on the air-gap line).

As a first-order approximation, we may suppose the minor loop on which the operating point lies to be represented by a line whose equation is found by using the prescribed slope and I intercept, as follows:

$$I-I_{(R)}{}^{\natural}[1+\alpha(t-t^{\natural})]=H\kappa_r{}^{\natural}[1+\beta(t-t^{\natural})]$$

where $I_{(R)}{}^{\natural}$ is the I-intercept of the line for reference temperature $t^{\natural}$ and $\kappa_r{}^{\natural}$ is the value of κ_r at that temperature.

We may set $t=t^{\natural}$, $I=I^{\natural}$, and $H=-NI^{\natural}$ in this equation, obtaining

$$I_{(R)}{}^{\natural}=I^{\natural}(1+N\kappa_r{}^{\natural})$$

whence

$$I-I^{\natural}[1+N\kappa_r{}^{\natural}][1+\alpha(t-t^{\natural})]=H\kappa_r{}^{\natural}[1+\beta(t-t^{\natural})]$$

In order to place the operating point on the air-gap line we now replace H with $-NI$. The resulting equation (with suitable approximations) leads to the relation

$$\begin{aligned}(1-I/I^{\natural})/(t-t^{\natural}) &= \beta(1+1/N\kappa_r{}^{\natural})^{-1}-\alpha \qquad (31)\\ &= \beta N\kappa'_r{}^{\natural}-\alpha \\ &= \beta\left(1-\frac{\kappa'_r{}^{\natural}}{\kappa_r{}^{\natural}}\right)-\alpha \\ &= \frac{\beta\Upsilon}{V}\cdot N\mu_v\pi^w-\alpha.\end{aligned}$$

But the left-hand member of (31) is the expression defining q. This equation is, of course, not rigorously valid in view of the assumptions mentioned, but it may be an instructive guide in the absence of more definite information. In particular, it brings out the manner in which N and κ_r probably affect q.

Whether α and β are independent of $I^\natural$ is not clear. Even if they are, the value of $\kappa_r^\natural$ must vary with $I^\natural$ according to something like the Gans relation, so that q must decline a little with increase in moment. However, q is surely not, as has been suggested, inversely proportional to $I^\natural$; for if it were, a bar would be magnetized in zero field merely by a change in its temperature.

It appears that β is generally positive, while α seems to be less definite in sign. Supposing that α as well as β is positive, an appropriate choice of N would make q vanish. As a matter of fact, Ashworth (1898) found results indicating that suitable heat treatment and choice of N would have this effect.

The relation between κ_0 and I_R/H_c seems to be a rather loose one. Hence it will not be surprising if in some particular temperature range κ_0 rises faster than I_R/H_c, signifying that the above-mentioned family of curves representing reversible changes is compressed by rising temperature or expanded by falling temperature at a more rapid rate than the major hysteresis loop.

Now, the main loop does not affect the operating point unless they impinge on one another; if by reason of a falling temperature they do come together, any further temperature drop would be expected to shift the operating point irreversibly to a different curve of the aforementioned family. By the same token, it would appear that when a magnet is magnetically stabilized at a given temperature, its margin of safety against further irreversible change is impaired at lower temperatures. When Fleming and DeWar (1896) subjected newly magnetized bars to the temperature of liquid air, the initial effect was an irreversible drop in moment. Their conclusion was that these very low temperatures might be used to stabilize a magnet.

12. ***Effects of aging and heat treatment.***—The process of moving the operating point away from the major loop along the airgap line is one kind of stabilization. Whether accomplished magnetically or by means of temperature changes, it does not entail any permanent change in the geometry of the curves. There is, however, another kind of aging which definitely alters the shape of the major loop and the slope of the minor loops. This is a slow spontaneous drift, which may be hastened by annealing. This kind of aging is regarded by Sanford (1944) as the delayed and muted manifestation of certain structural changes that were largely suppressed by the quenching process during the making of the specimen. Examination of the pairs of points connected by dotted lines in fig. 8 confirms the usual experience that H_c/I_R is increased by quenching and reduced by annealing. (Most of the change is in H_c rather than I_R.) Venske (1914) found that for bar magnets Υ is raised by annealing, and considered that natural aging should have a similar effect. Whether f is systematically affected is not clear.

The effect of this sort of aging on I is usually (but not invariably) a decrease. Like the temperature effects already discussed (though differing from them in being irreversible), the gradual change of I may be regarded as a composite result of two coefficients—one building up κ_r and the other affecting the I intercept of a minor loop having the operating point at its lower end. The second one is usually either negative or numerically too small to overcome the first one. If it is positive, we should expect a spontaneous *increase* in moment for sufficiently small values of $N\kappa_r$, by the same reasoning followed in connection with equation (31). But it does not follow that a magnet showing spontaneous increase in moment will also have a negative value of q; for the constituent coefficients for the spontaneous changes are not necessarily in the same relation as those for the thermal fluctuations. Indeed, one instance may be cited (Howe 1943) in which there was a spontaneous increase of moment coupled with a temperature coefficient of the ordinary sense. Howe's finding that the spontaneous change was faster at high temperatures is consistent with the idea that aging and annealing are fundamentally the same process.

It is generally understood that magnetometer magnets should be safeguarded against vibration, jars, and wide changes of temperature or applied field. It is reassuring to note, however, that when an accident does occur, the outlook for continued usefulness of the magnet is good. Even with complete demagnetization, it may be expected that remagnetization and magnetic stabilization will impart the same degree of stability that had been attained before the accident. That is, the blow or other occurrence would probably have no serious effect on the shape of the curve or on the basic parameters which determine the usefulness of the magnet, though it might well change the distribution factors, requiring restandardization as for a new magnet.

13. ***Curvature of minor loops.***—Evidently the primary branch of a minor loop having the operating point at the origin is identical with the initial part of the normal magnetization curve. There appears to have been no evaluation of the curvature of minor loops when the operating point is away from the origin. However, the close relation between b and a under thermal change suggests that a similar connection would govern the curvature of minor loops in relation to κ_r.

14. ***Application.***—It appears from the preceding sections that an estimate of the induction factor can be formed with a knowledge of the eight values H_c, I_R, p, f, I'_s, N, I, and V. (It is possible to determine the first three if any three points on the demagnetization curve are given.) Knowing these eight values, we can find κ_0 by means of equation 30, then κ_r with equation 28, and finally Υ with equation 27.

Conversely, it would be of considerable interest if determinations of the induction factor were generally accompanied by statements not only of temperature but also of the volume and mass of the stripped magnet and its demagnetizing factor (or the basis for computing it), permitting the derivation of the reversible susceptibility referred to unit volume or mass (κ_r or χ_r). Such a criterion, particularly if further generalized by reduction to κ_0 or χ_0, has the cogent advantage that it is directly comparable and significant, either for different magnets of various sizes and compositions, or for the same magnet at different working points.

15. ***The induction coefficient.***—One final point remains. It has been understood from the first that the induction factor depended upon the size of the magnet. We have seen how this drawback is to be countered by converting to κ_r or χ_r (a step not proposed in the prior

literature, so far as the writer is aware). But we should not attach undue significance to the so-called *induction coefficient*,[1] a quantity equal to Υ/M which occurs in the computation of the induction factor when the method of Lamont or the related one of Nelson is employed. In these methods the initial moment serves as an implicit datum permitting the measurement of induction coefficient without calibration; this result is then multiplied by the moment of the magnet to get the induction factor. Some methods by-pass the intermediary function. Welsh, for example, added a separate deflection to find M/H_a; his equation gives the induction factor directly (Whipple 1877). Alternatively, one might use Lamont's initial equations (Hazard 1911) but calibrate the set-up with a magnet having known moment; this procedure would permit the measurement of an unmagnetized bar, for which the induction coefficient becomes infinite. In the Schmidt-Venske method, too, the induction coefficient is not involved since the absolute change of moment is the measured quantity.

The induction coefficient is seen to afford a ready-made index for comparison of differently constituted magnets as to their vulnerability to the effect of the earth's field, in geomagnetic measurements. Unlike the induction factor, it is roughly comparable for magnets of different sizes. However, we see from its inverse relation to I that it is an unreliable guide, giving no true characteristic of the material used in the magnet; it is not constant even for a particular magnet, as has been pointed out by Lamont (1867), by Hazard, by McComb (1929) and by Nelson. That is, the induction coefficient is grossly affected by change in I/I_R, whereas Υ is virtually independent of such change within the relevant range.

Acknowledgments.—This publication has been prepared with the support and encouragement of Captain Elliott B. Roberts, Chief of the Division of Geophysics, and the late Mr. Harold E. McComb, Chief of the Geomagnetism Branch, U. S. Coast and Geodetic Survey. It has been modified and clarified in several points as a result of the careful review and constructive suggestions of my colleague Mr. Louis Hurwitz. It also reflects indirectly the long-continued guidance of Dr. H. Herbert Howe, now of the U. S. National Bureau of Standards.

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[1] The literature reflects some disparity in the use of this term. We here follow Lamont, Bartels, Hazard, Venske, and Chapman in what seems to be prevailing usage. B. Stewart, and after him C. Chree, applied the same name to what is here called induction factor, whereas Kohlrausch (1892) bestowed it upon a third quantity, which we recognize as $H_a\Upsilon/M$. Kohlrausch (1884) also introduced the "inductions constant" equal to $\Upsilon/\rho V$. This would approach χ_ρ for small values of N (see eq. 27).

SUMMARY OF RECURRING NOTATION

Primary symbol	Auxiliary symbol	Significance	Where introduced
			Page
	$_a$	Reflecting the ambient or extraneous field.	17, 42
B		Magnetic induction, or flux density.	11, 34
(B)		$=B/\pi^{v}$.	11
	$_B$	Pertaining to the B/H curve, as distinguished from the I/H curve.	11
	$_c$	Coercive; pertaining to the point where the curve crosses the H axis.	3
d		$=I_d/I_R$	
		$=(H_d I_d/H_c I_R)^{1/2}$	8
d_B		$=B_d/B_R$.	12
	$_d$	Pertaining to or derived from that point of the demagnetizing curve whose coordinates are in proportion to the intercepts of the curve, or that point corresponding to the maximum of W_E.	6, 7, 15
	$_e$	External; applying to the external space about a magnet, or the gap in which the properties of the medium are active and manifested.	34
f		$=\kappa_0 N_d$.	24
H		Magnetizing force or magnetic intensity.	1, 4, 34
I		Magnetic polarization; intensity of magnetization	
		$=(B-\mu_v H)/\pi^{v}$.	1, 2, 34
I'_s		Maximum (saturation) value of I, corresponding to the horizontal asymptote of the major hysteresis loop.	3
${}'I_s$		Fictitious saturation value of I, corresponding to the asymptote of the hyperbolic approximation to the demagnetizing curve.	3
	$_i$	Internal; applying to the material of the specimen magnet.	34
M		Magnetic moment of a magnet	
		$=\int I dV$; for an ellipsoid $M=VI$.	19,36,40
m		Ratio (in a magnet having axial and polar symmetry) of the dimension along the axis of revolution to that along an equatorial diameter.	15, 38
N		Demagnetizing factor of a magnet	
		$=-H_n/I_n$.	14,16,34
	$_n$	Pertaining to that point of a curve at which a given specimen remains when all external magnetizing force is removed; used in denoting remanent magnetization.	14
p		$=I_R/I_s$	
		$=1-(d^{-1}-1)^2$	4, 8
p_B		$=1-(d_B^{-1}-1)^2$.	12
q		Temperature coefficient of a magnet	
		$=-\frac{1}{I}\cdot\frac{dI}{dt}$.	27, 28
	$_R$	Residual; pertaining to the I-intercept of a major hysteresis loop—that point at which the self-demagnetizing field has just become sufficient to annul the applied magnetizing field.	4
	$_{(R)}$	Pertaining to the I-intercept of the straight line approximating a minor loop.	27, 38
	$_r$	Reversible characteristic of material of a specimen magnet.	16,35,38

SUMMARY OF RECURRING NOTATION—Continued

Primary symbol	Auxiliary symbol	Significance	Where introduced
			Page
	s	Saturation value or a value analogous thereto (see I_s and I'_s).	3
T		See text.	36, 37
t		Temperature.	27
u		Specific permeability; ratio of permeability of a specified substance to μ_v.	34, 42
V		Volume of a specimen magnet.	19
W_E		Energy product $=-HI$.	7, 19
δ		See text.	40, 46
ζ		See text.	35
θ		See text.	40
κ		Magnetic susceptibility, I/H $=\frac{\mu_v}{\pi^v}(u-1)$	1, 16, 39
κ'		False susceptibility.	18
μ		Magnetic permeability, B/H $=u\mu_v$.	1, 11, 19, 34, 42
μ_v		$=B/H$ for a vacuum, sometimes called the permeability of space; its value is 1 in the c. g. s. electromagnetic system and $\pi^w\times10^{-7}$ in the M. K. S. systems.	
π^v		Ratio of intrinsic induction to magnetization; unity for rationalized systems, 4π for unrationalized ones.	1, 11
π^w		$4\pi/\pi^v$; unity for unrationalized systems, 4π for rationalized ones.	1, 15
ρ		Density, i. e., mass per unit volume.	16
Υ		Induction factor$=V\kappa'_r/\mu_v\pi^w$; a measure of the temporary change in magnetic moment which a magnet undergoes due to unit change in the ambient field.	19
χ		Mass susceptibility $=\kappa\rho$.	16
ψ		See text.	40
	0	Pertaining to $I=0$, or the initial part of a magnetizing curve (at the origin).	18,21,29
	$\natural$	Pertaining to reference temperature, as distinguished from actual temperature.	27
	$\flat$	Signifying that a dia- or paramagnetic ambient medium is taken into account.	34

16. ***Limiting conditions.***—This study is concerned with the effects of an isotropic ambient medium on the magnetic moment and field of a permanent magnet, and with the bearing which this topic may have on the question of what we are measuring when we make observations with the theodolite magnetometer by means of oscillations and deflections. The medium is considered to have no external boundary close enough to influence the results.

Some aspects of this topic have been considered by L. Page in a paper published in 1933. The development given here is consistent in result with that by Page, but differs therefrom in two respects, namely (a) it has a somewhat broader scope, covering for instance the case of a magnet not ideally hard, and (b) by exploiting the concept of a replica field it takes greater advantage of older developments in magnetic theory, to mitigate the mathematical complexity of the aspect here under consideration.

To permit exact results, we shall confine our attention chiefly to axially symmetrical, ellipsoidal magnets (prolate or oblate) which will be assumed to be magnetized parallel to their axes of revolution. The term "ellipsoid" will be used only in this restricted sense, unless otherwise stipulated.

The ellipsoid has a certain magnetization which is the result of its past history and which maintains a demagnetizing field in the magnet, precisely like an inverse applied field; the operating point lies in the second quadrant, somewhat within the major hysteresis loop (because of stabilization), being specified by the ordinate $I_i^\flat$ and the abscissa $H_i^\flat$. (It must be borne in mind that $H_i^\flat$ has a negative value, since the direction of the field is contrary to that of the original magnetizing field which caused the specimen to acquire the magnetization it possesses.) We use the symbol $N^\flat$ to denote the ratio $-H_i^\flat/I_i^\flat$. This is, of course, the demagnetizing factor of Dubois, here given the affix $^\flat$ because it is found to depend upon u_e as will be shown.

17. ***Basic relations and definition of*** T.—The three quantities H, B, and I at a surface point maintain definite proportions as follows:

For the exterior (fig. 10, right side),

$$\frac{B_e^\flat}{H_e^\flat}=u_e\mu_v, \tag{32}$$

$$\frac{I_e^\flat}{H_e^\flat}=\frac{\mu_v}{\pi^v}(u_e-1); \tag{33}$$

and for the interior (fig. 10, left side),

$$-\frac{B_i^\flat}{H_i^\flat}=\frac{\pi^v}{N^\flat}-\mu_v, \tag{34}$$

$$-\frac{I_i^\flat}{H_i^\flat}=1/N^\flat. \tag{35}$$

These magnitude ratios apply to the whole quantity, and equally to the normal components thereof, because our initial stipulation making I_i^b uniform and parallel to an axis of the ellipsoid implies that B_i^b and H_i^b are likewise uniform and parallel to said axis, as is well known. Of course, B_e^b and I_e^b must everywhere conform in direction with H_e^b, since u_e is assumed uniform. The six basic scalar quantities involved in these equations are the magnitudes of vectors, each of which is labeled in figure 10 with the corresponding symbol having an arrow over it. The scalars are all positive save H_i^b, for which a reversed arrow is used in the figure.

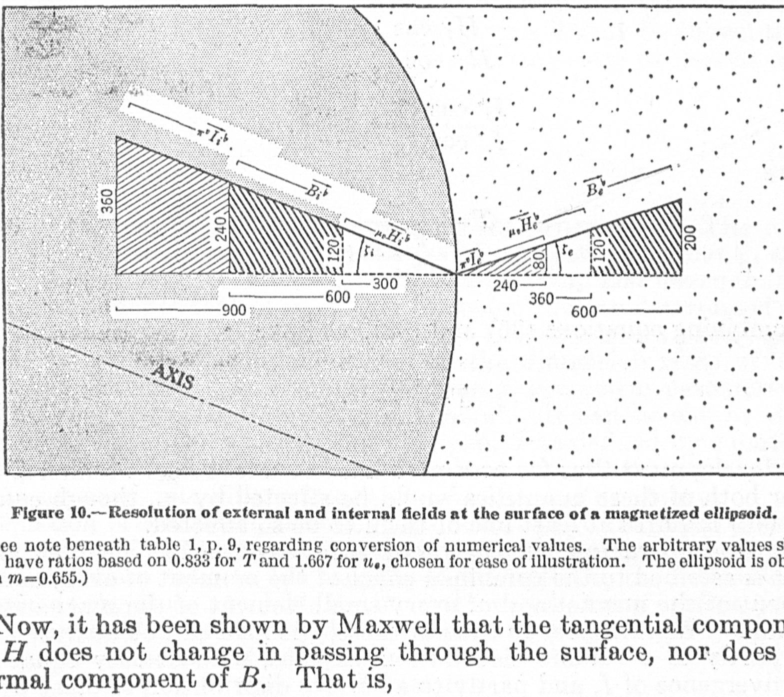

Figure 10.—Resolution of external and internal fields at the surface of a magnetized ellipsoid.

(See note beneath table 1, p. 9, regarding conversion of numerical values. The arbitrary values shown here have ratios based on 0.833 for T and 1.667 for u_e, chosen for ease of illustration. The ellipsoid is oblate, with $m=0.655$.)

Now, it has been shown by Maxwell that the tangential component of H does not change in passing through the surface, nor does the normal component of B. That is,

$$-H_i^b \sin \zeta_i = H_e^b \sin \zeta_e \tag{36}$$

$$B_i^b \cos \zeta_i = B_e^b \cos \zeta_e \tag{37}$$

where ζ_i and ζ_e are the angles which the field (B or H indifferently) makes with the normal to the surface. These four components are designated in figure 10 by means of the broken lines. Note that the lines of force pass through the surface of the magnet without crossing the normal.

In respect to the conditions within the ellipsoid we shall deal only with the reversible kind of permeability and susceptibility (p. 16). The more common sort (representing the whole value of B_i^b or of I_i^b divided by that of H_i^b) would be of scant utility here; instead of serving as an index to the property of the material, it would merely afford a distorted measure of N^b, as a moment's reflection will verify.

We have noted on page 14 that N is primarily a shape index. The basis of this effect was discerned by Maxwell (1873).

Page has shown that H_i^b is dependent on the external medium, but in a real magnet this makes I_i^b so dependent to a minute extent. In the examination of this relation, we take "magnetic moment" as meaning the magnetization integrated over the volume of the magnet; then the moment too must be influenced by the medium, as we shall see.

We can use equations (32) to (35) to replace B_i^b and B_e^b in equation (37) with expressions more useful for this study. In this way we obtain

$$\frac{-H_i^b \cos \zeta_i}{H_e^b \cos \zeta_e} = T \tag{38}$$

$$\frac{I_i^b \cos \zeta_i}{I_e^b \cos \zeta_e} = \frac{u_e + T}{u_e - 1}, \tag{39}$$

where

$$T = \frac{u_e}{\dfrac{\pi^0}{N^b \mu_v} - 1}. \tag{40}$$

Combining equations (36) and (38), we have

$$T = \frac{\tan \zeta_e}{\tan \zeta_i}. \tag{41}$$

The development thus far presented has not established whether T or N^b or both of these quantities would be affected by u_e, though equation (40) requires at least one of them to be so affected.

18. ***Configuration of the field unaltered.***—The field at any point is ascribed to the combined effect of the moment of every small element of the magnet and of every small element of the magnetized medium. With respect to each of these, the field can be analyzed as due partly to a volume distribution of poles, with density equal to the divergence of I, and partly to a surface distribution of poles with pole strength equal to the normal component of I. It has been well established that the first part must vanish in any region if (a) κ is constant, or (b) I is uniform, throughout that region. The region occupied by the medium satisfies condition (a), so that we may determine the field arising from the magnetization of the medium if we can find the surface-pole distribution which it develops at the boundary surrounding the magnet. We may think of the space occupied by the magnet as a void into which the flux emerges from the medium, developing surface poles at the boundary.

On the other hand, condition (b) is met in the magnet by reason of its shape. Consequently, the field due directly to the magnet is also representable as the effect of its surface-pole distribution alone.

Equation (39) shows that the normal component of I outside the ellipsoid, adjacent to a particular surface point, bears a definite ratio to that inside—a ratio that is independent of the location of the chosen surface point.

This means that the surface-pole distribution in the medium is a replica to a smaller magnitude of that in the magnet. Consequently, the field at any point will consist of a primary constituent due immediately to the magnet and a secondary part (due to the medium) that is directed the same as the primary part for a diamagnetic medium, or exactly opposite thereto for a paramagnetic one. The two parts will preserve a magnitude ratio that is constant in space but depends on the permeability of the medium. (The pole-strength ratio is the negative of the normal-magnetization ratio given by eq. 39, since the flux enters the medium where it leaves the magnet and vice versa.)

In other words, the configuration (that is, the geometric pattern) of the field remains unchanged when the medium changes—only the numerical magnitudes are affected. From this it may be shown that the induction at any point preserves a constant ratio to that at any other point, and specifically that

$$\frac{B_e'}{B_e}=\frac{B_i'}{B_i}. \tag{42}$$

19. ***Significance of T and some of its properties.***—The conservation of the patterns under change of u_e means also that ζ_e and ζ_i in equation (41) are not affected by change of u_e, and hence that T is likewise independent of u_e. Then its magnitude as determined for one medium must be valid for all media. In short, T is a purely geometric parameter, dependent solely on the dimension ratio of the ellipsoid. Page uses in a similar fashion a geometric parameter r which turns out to be $(1+T)^{-1}=1-N\mu_0/\pi^0$. It can be shown that for a ring magnet with a very short air gap, T as defined by equation (40) is essentially the ratio of the length of path in the gap to that in the magnet. Its independence of u_r is also readily shown.

Equation (41) has an interesting significance in geomagnetic theory. The tangent of the magnetic dip on a body of the sort we are studying bears a uniform ratio to the tangent of the latitude, the latter being defined as the angle at which the normal erected at any surface point would pierce the equatorial plane. This constant ratio is $1/T$; it is 2 for a sphere, as is well known.

Now, equation (40) may also be written

$$N'=\frac{\pi^0}{\mu_0\left(\dfrac{u_e}{T}+1\right)}, \tag{43}$$

and for a vacuum we may simply change N' to N and u_e to 1, the T requiring no change. This leads to the relation

$$\frac{N'}{N}=\frac{1+T}{u_e+T}. \tag{44}$$

Note that N' reduces for small T to $\pi^0 T/u_e\mu_0$, and for large T to π^0/μ_0. The latter result accords with the well-known statement that for a thin magnetic shell N is 4π, the maximum possible value. Note also that N' is not affected by u_r, since the latter quantity does not appear in equation (43). This equation further shows in a simple way just how N^b is affected by u_e.

Again, equation (43) yields an expression for N in vacuo which may be substituted in formulas for N given by Dubois, yielding equations that connect T with m, the dimension ratio. These equations are given

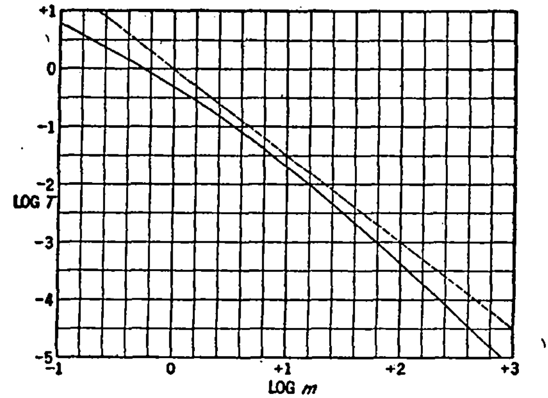

Figure 11.—Log T as a function of log m for ellipsoids. The function is governed by the following equations:

For $m<1$ (oblate ellipsoid): $\dfrac{1}{1+1/T}=\dfrac{1}{1-m^2}\left(1-\dfrac{m}{\sqrt{1-m^2}}\text{ arc cos } m\right)$.

For $m>1$ (prolate ellipsoid): $\dfrac{1}{1+1/T}=\dfrac{1}{m^2-1}\left[\dfrac{m}{\sqrt{m^2-1}}\log_e\left(m+\sqrt{m^2-1}\right)-1\right]$.

The broken line shows log $m^{-3/2}$ for comparison.

beneath figure 11, which shows log T plotted against log m. As an approximation it appears that $Tm^{3/2}$ is nearly constant, since the broken line in figure 11 representing log $m^{-3/2}$ is nearly parallel to the most useful part of the curve. Figure 12 shows $Tm^{3/2}$ plotted against log m; this bell-shaped curve may be used as a rather accurate empirical means of finding T when m is known. A computed value of $m^{-3/2}$ is merely multiplied by a value of $Tm^{3/2}$ scaled from the curve. To illustrate: If $m=15$, then log $m=1.176$, $m^{-3/2}=0.0172$, and (from fig. 12) $Tm^{3/2}=0.63$. Multiplying the last two together we find $T=0.011$.

20. ***Magnetization and magnetic moment.***—It is demonstrable that the moment will vary with u_e for any except a ring magnet. We shall develop this relation for an ellipsoid, but first we recall that any change that affects H_i^b must cause the operating point to move along a gentle slope (strictly it describes a minor loop) represented by the equation

$$I_i^b=I_{(R)}+\kappa_r H_i^b \tag{45}$$

where $I_{(R)}$ is the I intercept of the line taken to represent the minor loop. By this treatment $I_{(R)}$ is a constant that does not depend on the medium. In a sense $I_{(R)}$ is analogous to I_R, since it approximates the magnetization that would prevail if the stabilized magnet were either subjected to just enough magnetizing force to annul its self-demagnetizing field, or immersed in a medium of great permeability with no applied field.

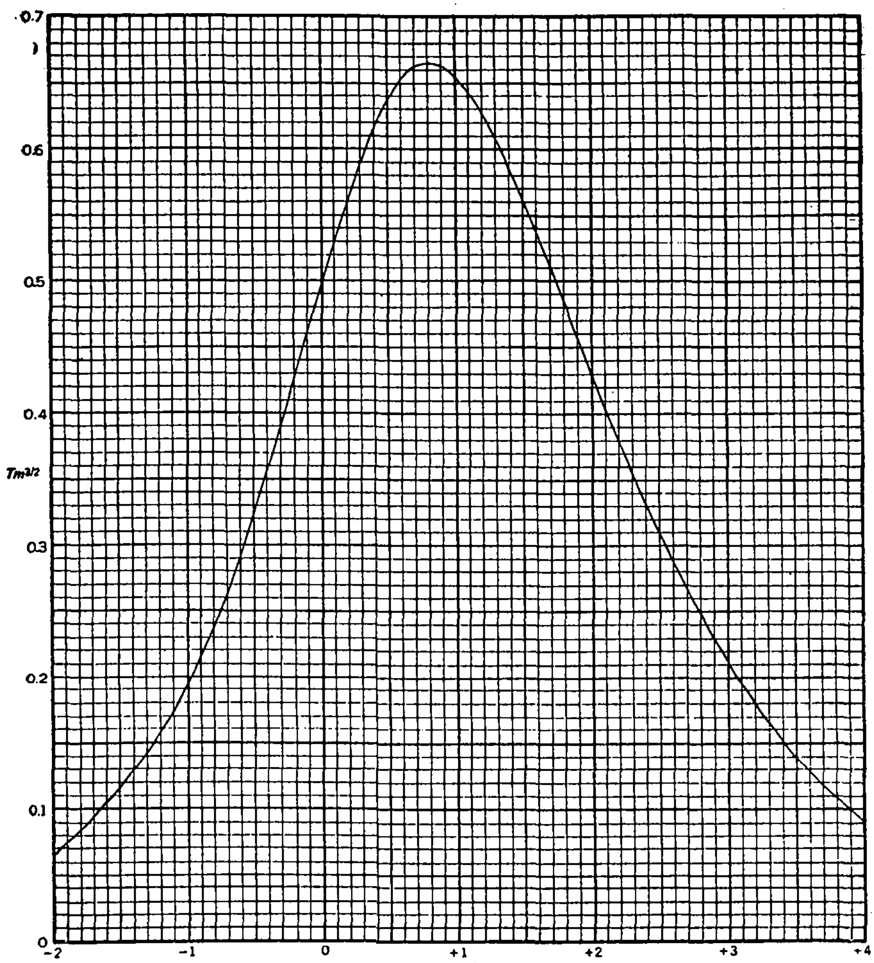

Figure 12.—Practical curve for determining T when m is known, for an ellipsoid.

In equation (45) we can replace $-H_i^b$ with $N^b I_i^b$ (see last paragraph of section 16) and then use equation (43) to remove the N^b, obtaining

$$\frac{I_i^b}{I_{(R)}}=\frac{1+T/u_e}{1+Tu_r/u_e} \tag{46}$$

and similarly for a vacuum,

$$\frac{I_i}{I_{(R)}}=\frac{1+T}{1+Tu_r}. \tag{47}$$

We have made use of the formula relating κ_r and μ_r that follows directly from definitions, applying the concept of reversible changes (see p. 16).

21. ***Effect of the medium on the field.***—We now take up the following question: Under the conventional postulates as to the field of an isolated pole, how does the field of an ellipsoid change in a changing medium, if its moment be governed by equation (46)?

In vacuo, the field of a magnet with axial and polar symmetry is described as to magnitude for any external point S by the equation

$$\pi^v H_e = \frac{M}{\mu_v r^3}(1+\delta)(3\cos^2\theta+1)^{1/2} \tag{48}$$

where M is the moment of the magnet, r is the distance of S from the center of the magnet, θ is the angle between the axis and the direction of S, and δ is a correction that depends on the shape of the magnet. This is a correction to the field, not to the moment. It provides for the departure of the field pattern from that of a dipole; in general, it is a function of r, T and θ, but in any case it vanishes for $T=\frac{1}{2}$ (a sphere) and is negligibly small for sufficiently large values of r. We shall consider that the effects of the medium are expressed separately and do not enter into δ.

We write from basic definitions

$$B_i^\flat = \pi^v I_i^\flat + H_i^\flat \mu_v$$

$$B_i = \pi^v I_i + H_i \mu_v$$

and, applying equation (42),

$$\frac{B_e^\flat}{B_e} = \frac{\pi^v I_i^\flat + H_i^\flat \mu_v}{\pi^v I_i + H_i \mu_v}$$

$$= \frac{I_i^\flat}{I_i} \cdot \frac{\pi^v - N^\flat \mu_v}{\pi^v - N \mu_v}.$$

Let

$$\psi = \frac{1+Tu_r/u_e}{1+Tu_r},$$

and

$$\psi_1 = \frac{1+T/u_e}{1+T}.$$

Then, using equations (43) and (44) to remove $N^\flat$ and N,

$$\frac{B_e^\flat}{B_e} = \frac{I_i^\flat}{I_i \psi_1},$$

and from equations (46) and (47),

$$M^\flat/M = I_i^\flat/I_i = \psi_1/\psi = \Upsilon^\flat/\Upsilon. \tag{49}$$

The last member is established by using equations (27) and (43) to express $\Upsilon^\flat$ in terms of u_e, u_r, and T, then dividing this result by the corresponding expression for vacuum. The introduction of $M^\flat$ and M comes from definitions (p. 36).

Now, our objective is to depict H_e^b in a way that will spotlight the effects of the medium. Again invoking definitions, we write

$$H_e^b/H_e = B_e^b \mu_v / B_e u_e \mu_v$$

$$H_e^b = H_e M^b / M u_e \psi_1 \tag{50}$$

$$= H_e / \psi u_e, \tag{50a}$$

and either of these forms may be combined with equation (48) to eliminate H_e; then

$$\pi^w H_e^b = \frac{M^b}{\psi_1 u_e \mu_v r^3}(1+\delta)(3\cos^2\theta+1)^{½}; \tag{51}$$

$$\pi^w H_e^b = \frac{M}{\psi u_e \mu_v r^3}(1+\delta)(3\cos^2\theta+1)^{½}. \tag{52}$$

These equations serve nicely to bring out the effects of the medium on ellipsoidal magnets. The effects are reflected in ψ (or ψ_1) and M^b. So far as ψ and ψ_1 are concerned, the effects of the medium are explicit in the defining equations, since T has been found independent of u_e. Equation (52) has ψu_e to represent the whole effect of the medium, while equation (51) shows two distinct effects. That is, the change of the magnetization of the magnet is reflected in M^b (a function of N^b and hence of u_e), whereas the influence of the polarization of the medium is covered separately by $\psi_1 u_e$. The necessity for ψ_1 is easily overlooked. It may be regarded as a correction for the absence of the medium from the space occupied by the magnet.

Of course, ψ reduces to ψ_1 for the perfectly hard magnet. Our $u_e\psi_1$ corresponds in significance with Page's γ, and examination of his result confirms that they are the same. He also derives the value for a sphere ($T=\frac{1}{2}$) which in the present notation is $u_e\psi_1=(2u_e+1)/3$.

At constant u_r and u_e, ψ is by definition a single-valued function of T; the smaller m is, the larger T, and the more closely ψ approaches $1/u_e$. On the other hand, ψ approaches 1 for small T. In other words, irrespective of u_r, for a magnetic shell the effect of ψ is to remove the u_e from equation (52), whereas for a long thin rod it has no effect. In order to make ψu_e fall midway between 1 and u_e, we must have $T=1/u_r$; and the corresponding condition for $\psi_1 u_e$ is that T must be 1. An ellipsoid satisfying the latter condition would have the proportions of a doorknob, with $m=0.55$, approximately.

Note further that, from the definition of ψ,

$$1-\psi = \frac{1-1/u_e}{1+1/Tu_r} \tag{53}$$

$$= \frac{u_e-1}{u_e}\cdot\frac{T}{T+1/u_r} \tag{53a}$$

and $(1-\psi_1)$ is represented by the same expression with u_r taken as 1. These forms are instructive in the usual situation where $T<1$. To take a typical case, let $m=15$, $T=0.01$, $u_e=1+(4\times10^{-7})$, and $u_r=15$; then

$$1-\psi=5\times10^{-8};$$

$$1-\psi_1=4\times10^{-9}.$$

To sum up, the field strength of such a magnet is modified by the medium in three distinct ways. As compared with the field in vacuo, the modifications due to air, expressed as parts in 1,000,000,000, are as follows:

400 parts decrease (direct effect for constant moment), determined by u_e;

4 parts increase (correction for absence of medium from space occupied by magnet), determined by ψ_1;

48 parts increase (correction for change of moment arising from reduction of demagnetizing field), determined by ψ/ψ_1.

Finally, we see that the presence of ψ is equivalent, for the two limiting cases of thin wire and magnetic shell, to the presence of u_e with suitable exponent, and that for the first of these limits the u_e appears in the fashion prescribed in the simple, elementary treatment.

22. ***Torque on a magnet.***—At this point our inquiry turns to the ponderomotive effect attaching to the concepts of magnetization and field strength. The torque developed upon a long, thin magnet placed transversely in a uniform, horizontal geomagnetic field H_a is of course proportional to H_a and to M^b, the integrated magnetization of the magnet. Now this relation, in consequence of the convention we have chosen for the character of I and M, does not involve μ_v; and u_e is likewise absent, conformably to the conclusion stated in the preceding paragraph. That is, the equation

$$\text{Torque}=H_aM^b \tag{54}$$

is accepted as a fundamental one, with a validity that is unimpaired by the presence of a medium so long as the transverse dimensions of the magnet are sufficiently small.[2]

As another result of the above-mentioned convention, the factor $\mu_v\pi^w$ must appear in the denominator of the expression for the torque between two thin magnets in vacuo. To allow for the presence of a medium in this expression we must furthermore insert u_e appropriately. It too goes in the denominator (again assuming long, thin magnets) since only thus is the expression capable of being reduced to the form of equation (54) by the replacement of one of the M's with its equivalent in terms of the field set up by the magnet. Then we have

$$\text{Torque}=\frac{M_f{}^bM_m{}^b}{\pi^w u_e\mu_v r^3} \tag{55}$$

where it is supposed that the magnets are placed one above the other, both directed horizontally but in azimuths differing by 90°, and that no field is present other than the fields set up by the magnets. Here

[2] Note added in proof: The elementary treatment, being found applicable for large m to the computation of the field of a magnet, is similarly adopted for the computation of torques. A more rigorous basis for this extension is available in the study cited on page 48.

f and m distinguish the two magnets, which may be regarded as "fixed" and "movable" respectively, and r is the (great) distance between them.

The effect of the magnetization of a thin wire may be regarded as the integrated effect of a great number of pole pairs that form a line sequence along the axis of the wire. Since the aggregate of any number of separate effects all having the same form is still of that form, we see that equation (55) applies to a pair of schematic magnets consisting of single pole pairs as well as to long, thin wires; consequently it is consistent with the classical expression of Coulomb's law for point poles in a medium. In our notation this law is written

$$\text{Force}=\frac{P_1P_2}{\pi^w u_e \mu_v r^2}.$$

Equation (55) is not very informative, owing to the simplifying conditions laid upon it. Our next objective will be a relation of the same sort but somewhat broader in scope—one that will apply to ellipsoids with nonvanishing transverse diameter, situated at moderate separation. The torque on each magnet (regarded as "passive") is necessarily governed by the field it experiences from the other or "active" magnet, in accordance with equation (51). Thus, the torque developed at magnet $M_m^\flat$ depends upon the field of magnet $M_f^\flat$ and hence upon $(1 + \delta_f)\ (1 + 3 \cos^2 \theta_f)^{\frac{1}{2}}/\psi_{1f}$. The same statement holds with the m's changed to f's and the f's to m's, by symmetry. The torques on the two magnets represent different aspects of a mutual interaction. There is a distinction between them, in that the viewpoint chosen determines which value of θ is operative; it must clearly be the value pertaining to the "active" rather than to the "passive" magnet. This distinction vanishes when θ is 90° for both magnets as in the situation postulated in equation (55) and here as well. Then the torque has strictly the same magnitude from both viewpoints, and the expression for this magnitude must incorporate values of $(1 + \delta)/\psi_1$ for both magnets in precisely the same way. At the same time, the whole expression must reduce to the style of equation (55) if the two values of δ vanish and those of ψ_1 become unity. It will be found that the only form satisfying all requirements is

$$\text{Torque}=\frac{(1+\delta_f)(1+\delta_m)M_f^\flat M_m^\flat}{\pi^w \psi_{1f}\psi_{1m} u_e \mu_v r^3}. \tag{56}$$

If the passive magnet be translated to a positon in the same horizontal plane with the active one, on the latter's axis (Lamont's first position), the effect of the radical containing θ is to inject the factor 2 in the numerator of equation (56). In the converse arrangement (Lamont's second position), equation (56) holds without this factor.

A similarly broadened relation is needed to supplant equation (54). Using equation (51) with $\theta = 90°$, let us replace $(1 + \delta)m^\flat/\psi_1$ for *one* of the magnets of equation (56) with an expression in $H_e^\flat$. Let the line of separation of the magnets be lengthened without change of direction, by translating that magnet to a remote position. The $H_e^\flat$ that was just now introduced still suffices to denote the effective field of that magnet at the position of the undisturbed one, but the remaining value of δ now becomes negligible by reason of the increased r. Now let us eliminate the remote magnet and replace $H_e^{\flat}$ with H_a to denote a horizontal geomagnetic field, so that the equation then expresses the torque developed by H_a upon the remaining magnet. The resulting form may readily be modified to encompass varied azimuths of the magnet. The complete expression is

$$\text{Torque} = \left[\frac{M^{\flat}}{\psi 1} - \frac{\Upsilon_t^{\flat} \mu_v \pi^w H_a \cos\phi}{\psi_{1t}}\right] H_a \sin\phi \tag{57}$$

$$= \left[\frac{M^{\flat}_1}{\psi_1} + \left(\frac{\Upsilon^{\flat}}{\psi_1} - \frac{\Upsilon_t^{\flat}}{\psi_{1t}}\right) \mu_v \pi^w H_a \cos\phi\right] H_a \sin\phi, \tag{57a}$$

where the magnet lies in azimuth ϕ relative to the field, and the subscript t identifies parameters taken *transversely* to the long axis of the magnet (see p. 20). The effect of longitudinal induction, inherent in $M^{\flat}$, is made explicit in equation (57a) by replacing the $M^{\flat}$ with $M^{\flat}_1$ plus a term in $H_a\Upsilon^{\flat}$, where $M^{\flat}_1$ is the value of $M^{\flat}$ at zero extraneous field. These equations reduce to the primitive form of equation (54) when $T=0$ and $\phi=90°$.

Two general comments are now appropriate. Equation (57), with ϕ set at 90°, shows how the customary expression connecting torque, moment, and transverse field must be qualified to allow for the presence of a permeable medium, by inserting ψ_1; for without the ψ_1 the equation is incomplete and may not be used as a means of defining magnetization.[3]

And we see now that the assumption of point poles in the classical foundation development of magnetostatic theory amounts to a restriction to line magnets, having the effect of suppressing the shape factor. The well-known disparity between the behavior of current loops and that of magnets now seems less paradoxical. For we see that (a) a magnet approximating a magnetic shell would behave like a current loop, maintaining a steady H field under a changing medium, whereas (b) a needle-shaped magnet maintains a steady B field. For intermediate forms, the varying effect of ψ must be taken into account; thus, neither of these generalizations is accurate for thick bar magnets or for ring magnets with a considerable air gap, though (b) is apt to suit the usual dimensions far more closely than (a).

23. ***Application of results.***—The foregoing results are obviously of interest in relation to the Gaussian method of measuring the geomagnetic field. We consider deflections first; in this step, a suspended magnet is deflected by a stationary one (the deflector), and the amount of the deflection is a measure of the ratio of the field of the deflector to the earth's field at the point which marks the center of the suspended magnet. For many purposes it is immaterial whether we regard the field quantities dealt with as B or H, since in

[3] Note added in proof: The quantity $M^{\flat}/\psi_1$ appearing in several important equations may be assigned a special status as the *virtual magnetic moment*, that is, the quantity which describes the strength of the magnet in a given environment, with respect to all its external manifestations. Thus, the "moment" defined through torque is the virtual moment, $M^{\flat}/\psi_1$—a composite of the volume integral of magnetization over the magnet itself ($M^{\flat}$) combined with that for the surrounding medium insofar as the medium derives its magnetization by induction from the magnet. A paper by Döring (Ann. Phys., 6 69–88, 1949) provides the basis for this point of view. Incidentally, Döring thereby adduces new grounds for upholding the definition of magnetization as having the dimensions of B (the one adopted in this publication).

any particular case the two must coincide in direction, and their magnitude ratio can be determined from the properties of the medium. The question does, however, possess some interest if we imagine the procedure conducted successively in two different media.

In deflections, we deal with the ratio of two quantities that are of the same species; thus, for Lamont's first position the basic equation may be written

$$\sin\xi = \frac{H_e^\flat}{H_a}$$

$$= \frac{B_e^\flat}{B_a}$$

$$= \frac{H_e^\flat u_e \mu_v}{B_a}$$

where ξ is the angle of deflection. In the last form we replace the numerator with an expression in M by means of equation (52); the result is

$$\pi^w r^3 \sin\xi = \frac{CM}{\psi B_a} \tag{58}$$

$$= \frac{CM^\flat}{\psi_1 B_a} \tag{59}$$

$$= \frac{CM^\flat{}_1}{\psi_1 B_a} - \frac{\pi^w C \Upsilon^\flat \sin\xi}{u_e \psi_1} \tag{60}$$

where C represents the two factors involving δ and θ in equation (51) and need not concern us here. In this relation the transverse induction factor plays no part, because whatever transverse moment is developed in the deflector has a field at the deflected magnet that coincides with the latter's axis, as a consequence of Lamont's arrangement. Equation (58) shows that if the magnet has a known moment in vacuo, the deflection it produces in another medium will depend on $\pi^w \psi B_a$. This would mean that if it were an ellipsoid in the shape of a long, thin needle, we would be measuring $\pi^w B_a$, but if it were a "shell" type (flattened) oblate ellipsoid we would be measuring $\pi^w \mu_v H_a$—assuming M to be known in either case.

By equation (57) we confirm the conclusion by Page (1935) to the effect that in deflections a value of ψ_1 for the suspended magnet would enter into the couple developed by the field of the deflector acting upon the suspended magnet. But the same effect must likewise be manifested in the opposing couple due to the action of the earth's field upon the suspended magnet, and the ψ_1 for the suspended magnet would clearly be eliminated in the equilibrium equation. The simplest way to look at this physically is to recognize that the suspended magnet merely indicates the direction of the resultant field (earth's field plus deflector field) and its own field may be ignored entirely. Thus, in deflections it is only the deflector for which we need be concerned with such effects.

In oscillations, the torque is governed by equation (57). As a variation, one might substitute M/ψ for $M^\flat/\psi_1$; and with M known, one would measure H_a with oscillations of a needle-type ellipsoid, or B_a/μ_v with oscillations of a shell-type ellipsoid. However, in actual practice we have no independent knowledge of M, but rather conduct both oscillations and deflections under the same ambient conditions, the oscillating magnet of the first step being used as the deflector of the second. The pertinent relations are expressed by equations (57) and (59), both of which involve $M^\flat/\psi_1$, but the latter expression drops out when the equations for oscillations and deflections are combined. Under this routine, then, the composite result is a measurement of $(H_aB_a\pi^w)^{\frac{1}{2}}$—that is, of $H_a(u_e\mu_v\pi^w)^{\frac{1}{2}}=B_a(\pi^w/u_e\mu_v)^{\frac{1}{2}}$. This function is not affected by the dimension ratio. In view of the relation of oscillations to equation (57) and deflections to equation (59) it seems quite proper to look upon oscillations as measuring field strength and deflections as measuring flux density.

Since $M^\flat$ is affected by longitudinal induction stemming from the ambient field, it has slightly different values in deflections and in oscillations. The difference is taken care of by the usual routine, using the longitudinal induction factor, and this routine is valid irrespective of the presence of a medium, as may be seen from the following relation:

$$\frac{M^\flat}{M^\flat_1}=1+\frac{\Delta M^\flat}{M^\flat_1}$$

$$=1-\frac{\mu_v\pi^w H_a\Upsilon^\flat \sin\xi}{M^\flat_1} \tag{61}$$

In this equation $\Upsilon^\flat/M^\flat_1$ may be changed by means of equation (49) to Υ/M_1; and with the flat signs ($^\flat$) thereby eliminated, the right-hand member represents M/M_1, thus showing that $M^\flat/M^\flat_1$ is independent of u_e. Here, as in equations (57a) and (60), the subscript 1 attached to M or to $M^\flat$ signifies the value at zero extraneous field.

The increment of torque due to the *transverse* induction effect is likewise dependent on ϕ and hence different in oscillations and deflections. The correction for this inequality depends on u_e, but it can be shown that for elongated magnets this subsidiary effect of the medium is substantially smaller than the latter's other effects through self-polarization and through change of the moment of the magnet.

24. ***Limitations on application.***—We have not examined here the effects of the correction δ in equations (48), (51), and (52). This correction takes care of the departure of the field from that of a dipole. There is, of course, no effect of δ in the oscillations, since there we are concerned not with the detailed configuration but only with the overall effect as reflected in $M^\flat/\psi_1$. But in deflections we must take δ into account. Its evaluation is awkward even for an ellipsoid. The relation of N and m cited in section 19 is closely associated with this problem, for the external and internal fields of an ellipsoid are both based on the solution of the same integral but with different lower limits, as discussed by Chrystal (p. 232) and by Abraham and Becker. A substitution is made (see Gray p. 54, or Abraham and Becker p. 144) such that the integral assumes a form

that can be integrated—specifically the one which Dwight lists as formula 152–1 on page 29. But the lower limit makes the result cumbersome for the external field.

For ellipsoids that have all three axes of different lengths, it might be possible to deduce relations by following a procedure along the lines of Page's 1933 development. For shapes other than the ellipsoid, no general calculation is feasible (except as they reduce to limiting cases of the ellipsoid). In practice, the magnet is taken as equivalent to one or more schematic magnets consisting of pole pairs with various separations. As one step beyond the simple dipole, a single pole pair is used in most of the older literature of geomagnetism and suffices for most purposes. The value of δ on this model is a rapidly converging series of which the first term (for $\theta=0$ as in Lamont's first position) is $2l^2/r^2$, l being one-half the distance between the poles (see McComb, 1952, p. 14, equations 37–38).

Schmidt has gone into the question by means of spherical harmonic analysis, by which the field of a specific magnet may be represented to any desired accuracy, as explained by Bartels (Chapman and Bartels, chapter 2).

While T has been discussed only for ellipsoids of revolution, it might be instructive to regard equation (40) as defining T for other shapes, with the understanding that $N^\flat$ refers to average values of $I_i^\flat$ and $H_i^\flat$. For ellipsoids of revolution, the effect of u_e on $N^\flat$ is not carried over to T; and for other shapes it seems possible that T would be at least less sensitive to u_e than is $N^\flat$.

The conclusion of section 18 to the effect that the geometric pattern of the field is unaltered by a change in the medium does not apply to the tubular shapes used in geomagnetic measurements, for the postulation of a zero convergence of I no longer holds, and the field of the magnet cannot be attributed exclusively to a surface-pole distribution. It might be conjectured that the change in pattern would bear a relation to the disparity between the demagnetizing factor for the shape in question and that for an ellipsoid with the same dimension ratio. It would appear that the change in pattern would become insignificant for a shape approximating an ellipsoid and also for a long thin rod or a ring magnet with very short air gap, for which the effect of the medium is swamped by the relatively high reluctance of the internal path; and both of these cases represent low values of the said disparity.

Alternatively, the effect of the medium arising through nonuniform magnetization of the magnet might be regarded as making δ a function of u_e; since δ is not particularly sensitive to m, its response to u_e would be expected to be quite small, though we cannot be sure that it is so small as to have no bearing on the results obtained in this appendix.

25. ***Summary of results.***—We have shown in this appendix how the demagnetizing factor of an ellipsoid of revolution is influenced by the surrounding medium, and how its magnetic moment and field are affected by the medium. Such a magnet in a variable medium gives rise to a field that is virtually constant in H or constant in B, in the special cases in which the magnet approximates a magnetic shell or a long, thin wire, respectively. For a prolate ellipsoid, if we correct for the effect of the medium on the moment we may say with even better approximation that the magnet's B field is independent of the

medium. What this statement neglects is a minute effect of the polarization of the medium; this correction appears not only in the field of the magnet, but also in the torque developed on the magnet through its reaction against an extraneous transverse field. Hence, the conventional rule that the torque equals the integrated magnetization of the magnet times the transverse field strength is rigorously valid only in vacuo.

Finally, we have seen that we may look upon deflections as measuring $\pi^w B$ and oscillations as measuring H; strictly speaking, the composite result is $(\pi^w BH)^{\frac{1}{2}}$. This latter result, developed specifically with regard to the ellipsoids stipulated at the outset, extends by intuitive reasoning to magnets of any shape whatever. Though one may be unable to formulate an explicit statement of the influence of the medium on the magnet's field, yet the effect, viewed as a simulated change of moment, must by the argument advanced on page 46 be the same in oscillations and deflections, hence must drop out of the final result.

NOTE ADDED IN PROOF: It has been learned that this study parallels in some respects an investigation by H. Diesselhorst (Ann. Phys., 3 11–30, 1948) in which the effects of a medium are developed by means of Maxwell stresses. Upon comparison of the two studies, with due allowance for rationalization and other differences in notation, there are readily obtained several equivalence relations, of which the following are examples (Diesselhorst symbols used on the left side, those of the present study on the right):

$$\{\mu_0\}=\pi^v\mu_v \tag{62}$$

$$\{H\}=H/\pi^v \tag{63}$$

$$\{I_H\}=\pi^v I \tag{64}$$

$$\{I_B\}=I/\mu_v \tag{65}$$

$$\{I_0\}=I_{(R)}/\mu_v \tag{66}$$

$$\begin{aligned}\{N\}&=N\mu_v/\pi^v\\&=\frac{\mu_v N^b}{\pi^v}\cdot\frac{u_e+T}{1+T}\end{aligned} \tag{67}$$

$$\{\lambda\}=1/u_e\psi \tag{68}$$

$$\{R\lambda\}=\frac{1+Tu_r}{1+T} \tag{69}$$

$$\{\lambda/\alpha\}=1+\frac{1}{Tu_r}. \tag{70}$$

Aside from the benefits of simplicity and brevity stemming from its restricted objectives, this appendix differs from Diesselhorst's analysis in several particulars, e. g., in using the ratio by which the magnetization of the magnet is affected by the presence of the medium (eq. 49). Only by replacing each M^b/ψ_1 in our equation (56) with the corresponding M/ψ can we submerge this effect, thereby changing the equation to one with the same significance as Diesselhorst's equation (42).

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