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SOME BASIC FORMALISMS IN NUMERICAL VARIATIONAL ANALYSIS

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ABSTRACT

This study aims at the theoretical development of a method of "four-dimensional analysis," namely the numerical variational analysis. The three basic types of variational formalism in the numerical variational analysis method are discussed. The basic formalisms are categorized into three areas: (1) "timewise localized" formalism, (2) formalism with strong constraint, and (3) formalism with weak constraint. Exact satisfaction of selected prognostic equations were formulated as constraints in the functionals for the first two formalisms. However, only the second formalism contains explicitly the time variation terms in the Euler equations. The third formalism is characterized by the subsidiary condition which requires that the prognostic or diagnostic equations must be approximately satisfied. The variational formalisms and the associated Euler-Lagrange equations are obtained in the form of finite-difference analogs. In this article, the filtering of each formalism and the uniqueness of solutions of the Euler equations are discussed for a limit that time and space increments (Δt and Δx) approach zero. The results from the limited case study can be applied, with some modification, for the cases where these increments are finite. In addition, a numerical method of solving the Euler equations is discussed. The discussion is facilitated, merely for the sake of simplicity, by choosing a linear advection equation as a dynamical constraint. However, the discussion can be applied to more complicated and realistic cases.

1. INTRODUCTION

The variational method, first used by the author in objective analysis (1958), seems to show some promise for optimizing observed values under certain subsidiary conditions. In the 1958 article, diagnostic equations such as the geostrophic, thermal wind, and balance equations were used as the subsidiary conditions for minimizing the variance of the difference between the observations and the analyzed fields.

Recently, the author (1969*a*, 1969*b*, 1970*a*) extended the analysis method to use prognostic equations as subsidiary conditions. The extension is demonstrated by using the simple linear advection equation and the diffusion equation as examples. It can be applied for sets of complicated prognostic equations. The variational formalism used in these articles is essentially written as

$$\delta J=\delta \sum_{\Omega} \sum_{i} \{\tilde{\alpha}_i(\varphi_i-\tilde{\varphi}_i)^2+\alpha_i(\nabla_t\varphi_i)^2\}=0 \qquad (1)$$

where δ is the variational operator, J is the functional, φ_i is the analyzed field, $\tilde{\varphi}_i$ is the observation, ∇_t is the local change in a finite-difference form, $\tilde{\alpha}_i$ and α_i are predetermined weights, and Ω is the domain in time t and space x_1, x_2, x_3. The first term in the functional is a condition used for minimizing the variance of the difference between observed and analyzed values. The second term is a simple low-pass filter in frequency. This equation is solved with dynamical constraints such as those given by the primitive equations. One of these constraints may be written as

$$\nabla_t\varphi_i=F_i(\varphi_i,\ \varphi_j,\ \nabla_{xk}\varphi_i,\ \nabla_{xk}\varphi_j) \qquad (2)$$

where F_i is a given function and ∇_{xk} represents the space derivative with respect to x_k ($k=1,2,3$). The functional (1) is in a quadratic form so that the stationary value of J becomes the minimum. Solution of φ_i is obtained by solving the Euler equations derived from eq (1) after substitution of eq (2) into eq (1). The advantage in this formalism is its mathematical simplicity, although some tedious mathematical manipulation is required in deriving and solving the Euler or Euler-Lagrange equation. The disadvantage is that the formalism is only for an instantaneous field and the functional does not describe explicitly the time variations.

The above disadvantage can be overcome easily by taking the following approaches. The first is an orthodox approach and is written as

$$\delta J=\delta \sum_{\Omega} \sum_{i} \{\tilde{\alpha}_i(\varphi_i-\tilde{\varphi}_i)^2 +\lambda_i G_i(\varphi_i,\ \varphi_j,\ \nabla_t\varphi_i,\ \nabla_{xk}\varphi_i,\ \nabla_{xk}\varphi_j)\}=0 \qquad (3)$$

where G_i represents a prognostic or diagnostic equation and λ_i is the Lagrange multiplier. (Derivation of the field equations of the atmospheric motion can be made by

finding the stationary value of the Lagrangian T (kinetic energy)$-V$ (potential energy) instead of $\tilde{\alpha}_i(\varphi_i-\tilde{\varphi}_i)^2$ under the subsidiary conditions of continuity and adiabatic processes, Sasaki 1955.) The Euler equations derived from eq (3) will include $\nabla_t\varphi_i$ and $\nabla_t\lambda_i$. This approach, however, requires a considerable amount of effort for numerically obtaining the solution of the Euler equations.

The other approach is relatively simple but seems versatile. It uses the variational formalism written in the form

$$\delta J=\delta \sum_{\Omega} \sum_{i} \{\tilde{\alpha}_i(\varphi_i-\tilde{\varphi}_i)^2+\alpha_i G_i^2\}=0 \tag{4}$$

where α_i is a predetermined weight. This approach is used by the author (1970*b*) and by Thompson (1969) for surface network data analysis. It should be noted that G is linear in eq (3) and quadratic in eq (4) and also that the coefficient of the G term is the Lagrange multiplier in eq (3) but the weight in (4). These differences result in different equations:

$$\text{from (3)} \qquad G=0 \tag{5}$$

and

$$\text{from (4)} \qquad G\doteq 0. \tag{6}$$

For convenience, eq (3) that leads to eq (5) is called the formalism with "strong constraint"; and eq (4) that results in eq (6) is called the formalism with "weak constraint." The definition of constraint is broader than usual.

Some simple low-pass filter terms such as $(\nabla_t\varphi_i)^2$ for frequency and $(\nabla_{xk}\varphi)^2$ for wave number are useful in filtering out undesired noises. These terms are optional and may be added to the functionals in eq (3) and (4).

The subsequent chapters will be devoted to discussion of the variational formalisms of three types, namely eq (1), (3), and (4). For convenience, a simple linear advection equation is used as an example. The books of Courant and Hilbert (1953, 1962) are mainly used for the mathematical aspects of this study.

2. TIMEWISE LOCALIZED FORMALISM

A brief review of the variational formalism expressed by eq (1) and (2) is given in this chapter. For simplicity, the following linear advection equation is used as the constraint (2):

$$\nabla_t\varphi+c_x\nabla_x\varphi=0 \tag{7}$$

where c_x is assumed constant and the finite-difference operators ∇_t and ∇_x are defined as

$$\nabla_t\varphi=\frac{\varphi_{n+1}-\varphi_{n-1}}{2\Delta t} \tag{8}$$

and

$$\nabla_x\varphi=\frac{\varphi_{i+1}-\varphi_{i-1}}{2\Delta x}. \tag{9}$$

The nth time level and the ith grid point are represented by the subscripts n and i, respectively; Δt, is the time increment; and Δx is the grid distance.

The variational formalism (1) is written as follows for the case of linear advection:

$$\delta J=\delta \sum_{x} \sum_{t} \{\tilde{\alpha}(\varphi-\tilde{\varphi})^2+\alpha(\nabla_t\varphi)^2\}=0. \tag{10}$$

Substitution of $\nabla_t\varphi$ from eq (7) into eq (10) results in

$$\delta J=\delta \sum_{x} \sum_{t} \{\tilde{\alpha}(\varphi-\tilde{\varphi})^2+\alpha(c_x\nabla_x\varphi)^2\}=0. \tag{11}$$

Applying the δ operator, eq (11) becomes

$$\delta J=\sum_{x} \sum_{t} \{2\tilde{\alpha}(\varphi-\tilde{\varphi})\delta\varphi+2\alpha(c_x\nabla_x\varphi)c_x\nabla_x\delta\varphi\}=0. \tag{12}$$

Further manipulation requires the following commutation equation:

$$\sum_{x} \sum_{t} \xi\nabla\eta=-\sum_{x} \sum_{t} \eta\nabla\xi \tag{13}$$

where ξ and η are arbitrary functions and ∇ represents either ∇_t or ∇_x. The proof of eq (13) is given in the author's article (1969*b*). Using eq (13), eq (12) becomes

$$\delta J=2 \sum_{x} \sum_{t} \{\tilde{\alpha}(\varphi-\tilde{\varphi})-\alpha c_x^2\nabla_x\nabla_x\varphi\}\ \delta\varphi=0 \tag{14}$$

where $\delta\varphi$ is assumed to vanish at the boundary. For eq (14) to become valid for any arbitrary value of $\delta\varphi$, $\{\ \}$ is required to vanish so that

$$\tilde{\alpha}(\varphi-\tilde{\varphi})-\alpha c_x^2\nabla_x\nabla_x\varphi=0, \tag{15}$$

which is called the Euler or Euler-Lagrange equation. This one-dimensional equation can be solved as a boundary problem.

For convenience, the range of x is taken as $-\infty<x<\infty$, and Δt and Δx are assumed to be infinitesimal. If the observed field $\tilde{\varphi}$ is expressed by a simple harmonic as

$$\tilde{\varphi}=\tilde{\Phi}\cos kx, \tag{16}$$

the particular solution is given by

$$\varphi=\frac{\tilde{\alpha}\tilde{\Phi}}{\tilde{\alpha}+\alpha c_x^2k^2}\cos kx. \tag{17}$$

The analyzed pattern is also the same harmonic wave except that the amplitude is reduced. The ratio between the analyzed and the observed values becomes

$$r\equiv\frac{\varphi}{\tilde{\varphi}}=\frac{\tilde{\alpha}}{\tilde{\alpha}+\alpha c_x^2k^2}=\frac{\tilde{\alpha}}{\tilde{\alpha}+\alpha\nu^2} \tag{18}$$

where ν is the frequency. It is seen easily from this result that the ratio r decreases monotonically as k increases. That is, the higher the frequency, the more damping

takes place, which of course is generally a desirable feature for most analysis methods. If Δx is finite, the filtering characteristics will change for high frequencies near the Nyquist frequency. The behavior of r for the case of a finite-difference analog is discussed in the author's article (1970*b*).

The above discussion can be extended easily to multidimensional problems. In the case of two dimensions, for instance, the dynamical constraint may be taken as

$$\nabla_t\varphi+c_x\nabla_x\varphi+c_y\nabla_y\varphi=0 \tag{19}$$

where c_y is the velocity component in the y direction and ∇_y is a finite-difference analog of ∂/∂_y, similar to ∇_x. The Euler equation for this case becomes

$$\tilde{\alpha}(\varphi-\tilde{\varphi})-\alpha(c_x\nabla_x+c_y\nabla_y)^2\varphi=0. \tag{20}$$

If Δx and $\Delta y\rightarrow 0$, (19) is a differential equation of the parabolic type. The characteristics are real but degenerate to one as expressed by

$$\xi=\frac{y}{c_y}-\frac{x}{c_x}. \tag{21}$$

The line orthogonal to the characteristic line ξ is not characteristic but is written as

$$\eta=\frac{y}{c_y}+\frac{x}{c_x}. \tag{22}$$

The Euler eq (20) can be rewritten in the following form by the coordinate transformation from (x, y) to (ξ,η) as

$$\tilde{\alpha}(\varphi-\tilde{\varphi})-4\alpha\nabla_\eta\nabla_\eta\varphi=0. \tag{23}$$

Therefore, it is easily seen that a unique solution can be obtained if the characteristic lines (varying ξ) densely cover the domain of consideration without leaving any empty space and if two conditions for φ are given at one or two end points of each characteristic line (fig. 1). Some similar and more detailed discussions on uniqueness of solution will appear again in the subsequent section where t behaves as y in eq (20). If we consider t as y in figures 2(a) and 2(b), the figures illustrate the essence of the above discussion.

3. VARIATIONAL FORMALISM WITH STRONG CONSTRAINT

The variational formalism (3) for a simple linear advection equation in a $(t$–$x)$ plane is written as

$$\delta J=\delta\sum_x\sum_t\{\tilde{\alpha}(\varphi-\tilde{\varphi})^2+\lambda(\nabla_t\varphi+c_x\nabla_x\varphi)\}=0 \tag{24}$$

where c_x is assumed constant. The variational operator δ is applied to φ and λ such that

$$\delta J=\sum_x\sum_t\{2\tilde{\alpha}(\varphi-\tilde{\varphi})\delta\varphi+\lambda(\nabla_t\delta\varphi+c_x\nabla_x\delta\varphi)+\delta\lambda(\nabla_t\varphi+c_x\nabla_x\varphi)\}=0. \tag{25}$$

When one uses the commutation eq (13), eq (25) becomes

$$\delta J=\sum_x\sum_t\{[2\tilde{\alpha}(\varphi-\tilde{\varphi})-(\nabla_t+c_x\nabla_x)\lambda]\,\delta\varphi+(\nabla_t+c_x\nabla_x)\varphi\delta\lambda\}$$

and

$$+\sum_x[\lambda\delta\varphi]_{t_1}^{t_2}+\sum_t[\lambda c_x\delta\varphi]_{x_1}^{x_2}. \tag{26}$$

The last two terms are concerned only with the values at the boundary. If they vanish, the Euler equations are written, setting the coefficients of the first two terms on the right-hand side of eq (26) equal to zero, that is,

$$2\tilde{\alpha}(\varphi-\tilde{\varphi})-(\nabla_t+c_x\nabla_x)\lambda=0 \tag{27}$$

and

$$(\nabla_t+c_x\nabla_x)\varphi=0. \tag{28}$$

We are interested in optimizing the initial value of φ; hence, it is desired that $\delta\varphi$ not be zero at the initial time. Satisfying this requirement is the condition

$$[\lambda]_{t=t_1}=0,\quad [\lambda]_{t=t_2}=0. \tag{29}$$

These are called natural boundary conditions. However, it is not allowable to give the value of λ on a closed boundary, as will be seen in subsequent discussion. For simplicity, we consider x_1 and x_2 to be $-\infty$ and $+\infty$, respectively. The boundary condition is then assumed as

$$[\delta\varphi]_{x_2\rightarrow\infty}=0,\quad [\delta\varphi]_{x_1\rightarrow-\infty}=0. \tag{30}$$

These boundary conditions do not violate the condition required for the uniqueness of the solution, as the characteristics with the boundary conditions (29) densely cover the domain of consideration as seen in figure 2(a).

Some general discussion will be given here, first on the uniqueness of solutions on the basis of the theory of characteristics. For simplicity, the following discussion is given for the case where $\Delta t\rightarrow 0$ and $\Delta x\rightarrow 0$. The parabolic type is characterized by a group of degenerated characteristic base curves. The curves are denoted by a set of ξ in figure 1. When the boundary conditions are given at all of the end points on the curves, a solution is uniquely determined. However, if the boundary is closed, a segment of the boundary becomes parallel to a characteristic curve in the immediate neighborhood of the segment. At least two of these segments exist on a closed boundary for the parabolic type. They are illustrated by the dotted curve in figure 1. If the boundary condition is given on these segments, the problem becomes an overspecified one.

In the present case, the characteristics are a set of lines given by $\xi=(x/c_x)-t$, as illustrated in figure 2, since

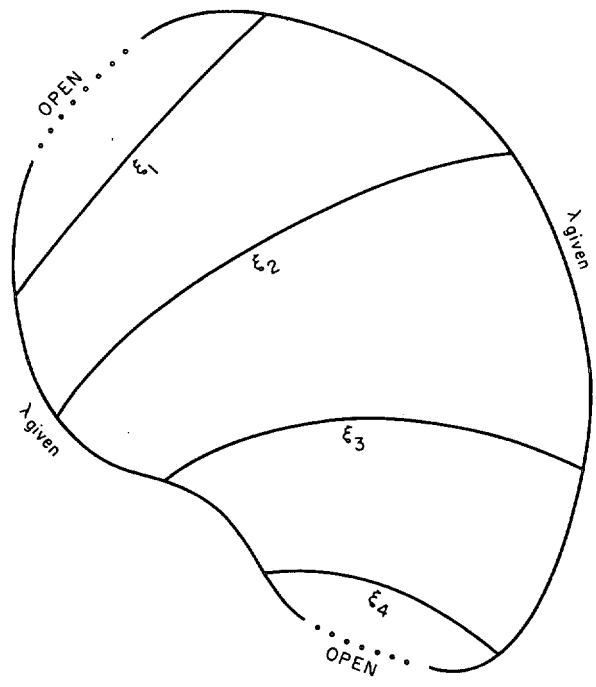

FIGURE 1.—Characteristic curves and boundary condition in parabolic type.

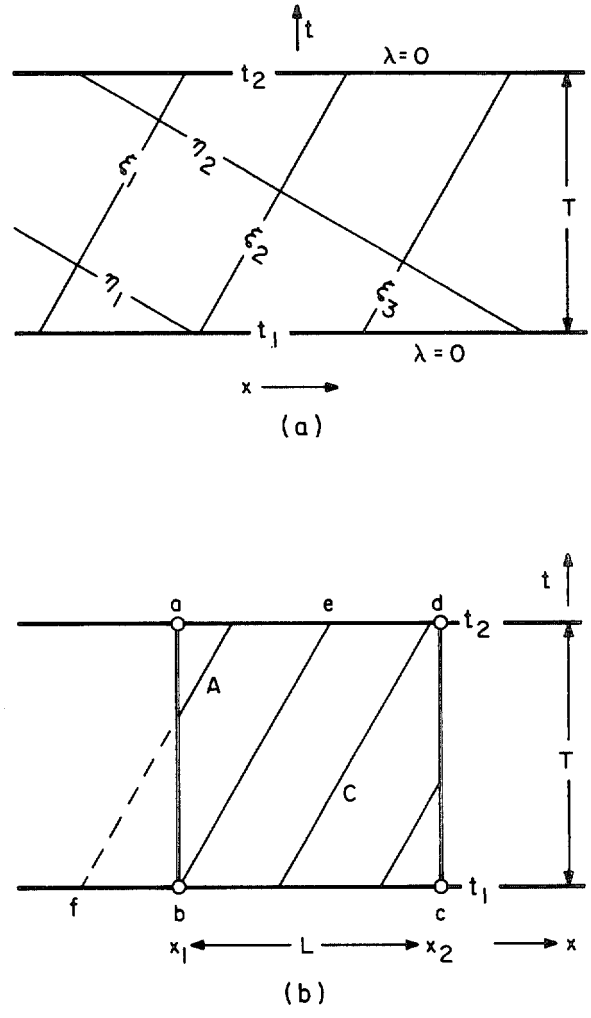

FIGURE 2.—Characteristic lines and uniqueness of solution.

c_x is a constant. The elements of the set are infinite, as the value of ξ varies infinitely. The lines normal to the characteristic lines that are given by a set of $\eta(=(x/c_x)+t)$ are also shown in figure 2. Using these new variables, ξ and η, one may write eq (27) and (28) in the form

$$2\tilde{\alpha}(\varphi-\tilde{\varphi})-2\nabla_\eta\lambda=0 \tag{31}$$

and

$$\nabla_\eta\varphi=0. \tag{32}$$

Eliminating φ from these equations, one obtains

$$\nabla_\eta^2\lambda=-\tilde{\alpha}\nabla_\eta\tilde{\varphi}. \tag{33}$$

This is a parabolic type of second-order differential equation if ∇_η is considered to be a differential instead of a finite-difference form. The characteristics are real but degenerate. The necessary and sufficient conditions for a unique solution are obtained if two conditions on λ are given on one or two end points, that is, the boundary or initial condition. This nature is similar to that for the boundary value problem of the elliptic type, but an essential difference is that the characteristics are real although degenerated. When c_x is infinite, $\xi=t$ and the characteristic line, $\xi=$constant, become parallel to the x coordinate. However, c_x is usually bound within a certain range of magnitudes, and the characteristics never become parallel to the x coordinate. Therefore, boundary conditions on t_1 and t_2 will not upset the uniqueness of solution, but those on x_1 and x_2 might. Therefore, it is probably desirable to assume a set of boundary conditions similar to those given by eq (29) and (30).

Let us consider the case where the observed field $\tilde{\varphi}$ is expressed by a simple harmonic wave

$$\tilde{\varphi}=\tilde{\Phi}\cos k(x-\tilde{c}_x t) \tag{34}$$

where k is the wave number, $\tilde{c}_x$ is the phase velocity, and $\tilde{\Phi}$ is the amplitude. Eliminating φ from eq (27) and (28), one can relate λ to $\tilde{\varphi}$ as follows:

$$\frac{1}{2\tilde{\alpha}}(\nabla_t+c_x\nabla_x)(\nabla_t+\nabla_x c_x)\lambda=-(\nabla_t+c_x\nabla_x)\tilde{\varphi}. \tag{35}$$

The general solution of eq (35) at the limit where $\Delta t\rightarrow 0$ and $\Delta x\rightarrow 0$ is

$$\lambda=\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)}\sin k(x-\tilde{c}_x t)+E(x,t)F(x-c_x t)+G(x-c_x t) \tag{36}$$

where F and G are arbitrary functions of $x-c_x t$ and E is

a function that satisfies

$$(\nabla_t+c_x\nabla_x)E=1. \tag{37}$$

For satisfying the boundary condition (29), E, F, and G are chosen in such a way that

$$\lambda=\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)}[\sin k(x-\tilde{c}_xt)-\sin k(x-c_xt)] +t[A\cos k(x-c_xt)+B\sin k(x-c_xt)] \tag{38}$$

where A and B are given by, taking $t_1=0$ and $t_2=T$,

$$A=\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)T}\sin k(\tilde{c}_x-c_x)T \tag{39}$$

and

$$B=\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)T}(1-\cos k(\tilde{c}_x-c_x)T). \tag{40}$$

Finally, φ is obtained by substituting eq (34) and (38) into (27) as

$$\varphi=\frac{\tilde{\Phi}}{k(\tilde{c}_x-c_x)T}\sin k(\tilde{c}_x-c_x)T\cos k(x-c_xt) +\frac{\tilde{\Phi}}{k(\tilde{c}_x-c_x)T}(1-\cos k(\tilde{c}_x-c_x)T)\sin k(x-c_xt). \tag{41}$$

It is interesting to analyze the behavior of eq (41) for some special cases. If $c_x\rightarrow\tilde{c}_x$, φ becomes

$$\varphi=\tilde{\Phi}\cos k(x-\tilde{c}_xt). \tag{42}$$

This means that the analyzed field is identical to the observed field $\tilde{\varphi}$. When $(\tilde{c}_x-c_x)$ and k are bound and T is taken sufficiently small, φ is given in an asymptotic form

$$\varphi=\tilde{\Phi}\cos k(x-c_xt)+\tilde{\Phi}[k(\tilde{c}_x-c_x)T+0(k^2(\tilde{c}_x-c_x)^2T^2)] \times\sin k(x-c_xt). \tag{43}$$

The first, cosine term is similar to eq (34) except that c_x instead of $\tilde{c}_x$ appears in this term. It is a result of the constraint. The sine term also appears in this case as a requirement for minimizing $(\varphi-\tilde{\varphi})^2$ as well as satisfying the dynamical constraint. This term is a correction term whose magnitude is proportional to the magnitude of the observed field $\tilde{\Phi}$, wave number k, time period T, and the difference between $\tilde{c}_x$ and c_x. The correction term is orthogonal to the observation. Also, it does not vanish at $t=0$. The term denoted by 0 in the second term is of higher order.

4. VARIATIONAL FORMALISM WITH WEAK CONSTRAINT

The basic concept of the variational formalism with weak constraint is demonstrated by the example in eq (4). Again considering a simple linear advection equation as constraint, eq (4) is written as

$$\delta J=\delta\sum_x\sum_t\{\tilde{\alpha}(\varphi-\tilde{\varphi})^2+\alpha(\nabla_t\varphi+c_x\nabla_x\varphi)^2\}=0 \tag{44}$$

where $\tilde{\alpha}$ and α are prespecified weights. The Euler equation of eq (44) is

$$\tilde{\alpha}(\varphi-\tilde{\varphi})-\alpha(\nabla_t+c_x\nabla_x)(\nabla_t+c_x\nabla_x)\varphi=0. \tag{45}$$

At the limit that Δt and Δx approach zero, eq (45) becomes the partial differential equation of parabolic type. As discussed in sections 2 and 3, the unique solution of eq (45) can be obtained when the domain of interest is covered by the characteristic lines and two conditions for φ are given at one or two ends of each characteristic line.

If the observed field $\tilde{\varphi}$ is to be given by eq (34), the particular solution of eq (45) at the limit $\Delta t\rightarrow 0$ and $\Delta x\rightarrow 0$ becomes

$$\varphi=\frac{\tilde{\alpha}\tilde{\Phi}}{\tilde{\alpha}+\alpha k^2(\tilde{c}_x-c_x)^2}\cos k(x-\tilde{c}_xt), \tag{46}$$

and the ratio between φ and $\tilde{\varphi}$ is given by

$$r=\frac{\tilde{\alpha}}{\tilde{\alpha}+\alpha k^2(\tilde{c}_x-c_x)^2}\leq 1. \tag{47}$$

A comparison of this ratio with the ratio shown in eq (18) reveals that the case of a weak constraint has less power as a low-pass filter because no filtering action will occur when c_x equals $\tilde{c}_x$. This result is similar to that of the case of strong constraint. It is, however, desirable to filter the disturbances of high wave numbers and high frequencies even if $\tilde{c}_x$ is equal to c_x. Some simple auxiliary low-pass filters that perform this service are discussed in the next sections.

5. SIMPLE LOW-PASS FILTERS

From the author's previous experiences in analysis of actual data by the variational method, it was found that adding simple low-pass filter terms to the functionals was helpful in obtaining converging solutions with less computer time. Also, it gives better results in data-sparse areas (such as over the ocean) and in noisy data areas (such as surface networks).

Some simple low-pass filters are given by the terms $\alpha_t(\nabla_t\varphi)^2$ for filtering high frequencies and $\alpha_{xk}(\nabla_{xk}\varphi)^2$ for filtering high wave numbers. The coefficients α_t and α_{xk} are simply weights, not Lagrange multipliers.

VARIATIONAL FORMALISM (10)

The functional (10) already has the low-pass filter for frequency $\alpha(\nabla_t\varphi)^2$; and as seen from eq (18), high frequencies are suppressed. It is important to give special attention to the fact that the low-pass filter $(\nabla_{xk}\varphi)^2$

changes the Euler equation from the parabolic type to the elliptic type in two-dimensional and three-dimensional cases. Adding $\alpha_{xk}(\nabla_{xk}\varphi)^2$ to the right-hand side of eq (10), the Euler equation similar to eq (20) can be derived as

$$\tilde{\alpha}(\varphi-\tilde{\varphi})-[(\alpha+\alpha_x)c_x^2\nabla_x^2+2\alpha c_x c_y\nabla_x\nabla_y+(\alpha+\alpha_y)c_y^2\nabla_y^2]\varphi=0. \quad (48)$$

At the limit that Δt, Δx, and $\Delta y\rightarrow 0$, the above equation becomes a partial differential equation of the elliptic type because the characteristic condition

$$\Delta\equiv 4\alpha^2c_x^2c_y^2-4(\alpha+\alpha_x)c_x^2(\alpha+\alpha_y)c_y^2 =-4\,[\alpha(\alpha_x+\alpha_y)+\alpha_x\alpha_y]c_x^2c_y^2<0. \quad (49)$$

This characteristic has proven to be quite helpful in obtaining a unique solution by a numerical method. Behavior of another auxiliary term $(\nabla_x u+\nabla_y v)^2$, which was used by the author (1969*a*), is also similar.

FORMALISM (24)

Addition of the low-pass filter terms enables one to write eq (24) in the form

$$\delta J=\delta\sum_x\sum_t\{\tilde{\alpha}(\varphi-\tilde{\varphi})^2+\lambda(\nabla_t\varphi+c_x\nabla_x\varphi)+\alpha_t(\nabla_t\varphi)^2+\alpha_x(\nabla_x\varphi)^2\} =0. \quad (50)$$

After using the commutation eq (13), eq (50) becomes

$$\delta J=\sum_x\sum_t\{2\tilde{\alpha}(\varphi-\tilde{\varphi})-(\nabla_t+c_x\nabla_x)\lambda+2\alpha_t\nabla_t\nabla_t\varphi +2\alpha_x\nabla_x\nabla_x\varphi]\delta\varphi+(\nabla_t+c_x\nabla_x)\varphi\delta\lambda\} +\sum_x[(\lambda+2\alpha_t\nabla_t\varphi)\delta\varphi]_{t_1}^{t_2}+\sum_t[(\lambda c_x+2\alpha_x\nabla_x\varphi)]_{x_1}^{x_2}. \quad (51)$$

The Euler equations are

$$2(\tilde{\alpha}+\alpha_t\nabla_t^2+\alpha_x\nabla_x^2)\varphi-2\tilde{\alpha}\tilde{\varphi}-(\nabla_t+c_x\nabla_x)\lambda=0 \quad (52)$$

and

$$(\nabla_t+c_x\nabla_x)\varphi=0. \quad (53)$$

From these, the equation for λ, which is the same as eq (35), is obtained as

$$(\nabla_t+c_x\nabla_x)(\nabla_t+c_x\nabla_x)\lambda=-2\tilde{\alpha}(\nabla_t+c_x\nabla_x)\tilde{\varphi}. \quad (54)$$

The boundary conditions are taken in a way similar to eq (29) and (30) as

$$[\lambda+2\alpha_t\nabla_t\varphi]_{t=0,T}=0 \quad (55)$$

and

$$[\delta\varphi]_{x\rightarrow-\infty,+\infty}=0. \quad (56)$$

The general solution of eq (54) at the limit $\Delta t\rightarrow 0$ and $\Delta x\rightarrow 0$ for the case of eq (34) is given by eq (36), namely,

$$\lambda=\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)}\sin k(x-\tilde{c}_x t) +E(x,t)F(x-c_x t)+G(x-c_x t). \quad (57)$$

Substitution of eq (57) into (52) yields

$$2(\tilde{\alpha}+\alpha_t\nabla_t^2+\alpha_x\nabla_x^2)\varphi=F(x-c_x t). \quad (58)$$

As a general form of F and G, we will consider

$$F=A\cos k(x-c_x t)+B\sin k(x-c_x t) \quad (59)$$

and

$$G=C\cos k(x-c_x t)+D\sin k(x-c_x t) \quad (60)$$

where A, B, C, and D are the constants to be determined later by the boundary conditions. The general solution of eq (58) is then given by

$$\varphi=\frac{1}{2(\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2)}[A\cos k(x-c_x t)+B\sin k(x-c_x t)]+\hat{\varphi} \quad (61)$$

where $\hat{\varphi}$ is the solution of the homogeneous part of eq (58), that is, $(\tilde{\alpha}+\alpha_t\nabla_t^2+\alpha_x\nabla_x^2)\hat{\varphi}=0$. However, $\hat{\varphi}$ does not satisfy eq (53); therefore, $\hat{\varphi}$ should vanish. Substituting eq (57) and (61) into eq (55), we get at $t=0$,

$$\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)}\sin kx+C\cos kx+D\sin kx +\frac{2\alpha_t kc_xA}{2(\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2)}\sin kx -\frac{2\alpha_t kc_xB}{2(\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2)}\cos kx=0;$$

and at $t=T$,

$$\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}-c_x)}\sin k(x-\tilde{c}_x T)+T(A\cos k(x-c_x T) +B\sin k(x-c_x T))+C\cos k(x-c_x T)+D\sin k(x-c_x T) +\frac{2\alpha_t kc_xA}{2(\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2)}\sin k(x-c_x T) -\frac{2\alpha_t kc_xB}{2(\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2)}\cos k(x-c_x T)=0.$$

Or, at $t=0$, after setting coefficients of $\cos kx$ and $\sin kx$ to be zero,

$$C-\frac{\alpha_t kc_xB}{\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2}=0 \quad (62)$$

and

$$\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)}+D+\frac{\alpha_t kc_xA}{\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2}=0; \quad (63)$$

and at $t=T$,

$$-\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)}\sin k\tilde{c}_x T+AT\cos kc_x T-BT\sin kc_x T +C\cos kc_x T-D\sin kc_x T-\frac{\alpha_t kc_x}{\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2}A\sin kc_x T -\frac{\alpha_t kc_x}{\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2}B\cos kc_x T=0 \quad (64)$$

and

$$\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)}\cos k\tilde{c}_xT+AT\sin kc_xT+BT\cos kc_xT$$
$$+C\sin kc_xT+D\cos kc_xT+\frac{\alpha_t kc_x}{(\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2)}A\cos kc_xT$$
$$-\frac{\alpha_t kc_x}{(\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2)}B\sin kc_xT=0. \quad (65)$$

The constants A, B, C, and D are determined from the above four equations. Substituting C and D obtained from eq (62) and (63) into eq (64) and (65), we get

$$T\cos kc_xT\cdot A-T\sin kc_xT\cdot B$$
$$-\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)}(\sin k\tilde{c}_xT-\sin kc_xT)=0 \quad (66)$$

and

$$T\sin kc_xT\cdot A+T\cos kc_xT\cdot B$$
$$+\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)}(\cos k\tilde{c}_xT-\cos kc_xT)=0. \quad (67)$$

From eq (66) and (67),

$$A=\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)T}\sin k(\tilde{c}_x-c_x)T \quad (68)$$

and

$$B=\frac{2\tilde{\alpha}\tilde{\Phi}}{k(\tilde{c}_x-c_x)T}[1-\cos k(\tilde{c}_x-c_x)T]. \quad (69)$$

Therefore, the analyzed field φ is obtained as

$$\varphi=\frac{\tilde{\alpha}\tilde{\Phi}}{(\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2)k(\tilde{c}_x-c_x)T}\{\sin k(\tilde{c}_x-c_x)T\cos k(x-c_xt)$$
$$+[1-\cos k(\tilde{c}_x-c_x)T]\sin k(x-c_xt)\}. \quad (70)$$

Comparing this solution with eq (41), we observe that the analyzed field with the low-pass filter differs from that without the filter by reducing the amplitude by the factor r:

$$r=\tilde{\alpha}/(\tilde{\alpha}+\alpha_t k^2c_x^2+\alpha_x k^2). \quad (71)$$

It is interesting to note that the type of differential equation at the limit Δt, $\Delta x\to 0$ remains essentially the same by addition of the low-pass filter terms.

FORMALISM (44)

Addition of the low-pass filters to eq (44) yields

$$\delta J=\delta\sum_x\sum_t\{\tilde{\alpha}(\varphi-\tilde{\varphi})^2+\alpha(\nabla_t\varphi+c_x\nabla_x\varphi)^2+\alpha_t(\nabla_t\varphi)^2$$
$$+\alpha_x(\nabla_x\varphi)^2\}=0, \quad (72)$$

and the Euler equation becomes

$$\tilde{\alpha}(\varphi-\tilde{\varphi})-(\alpha+\alpha_t)\nabla_t\nabla_t\varphi-2\alpha c_x\nabla_t\nabla_x\varphi-(\alpha+\alpha_s)c_x^2\nabla_x\nabla_x\varphi=0 \quad (73)$$

which is, unless α_t and α_x both vanish, an elliptic equation at the limit, $\Delta t\to 0$ and $\Delta x\to 0$, since the characteristic condition

$$\alpha^2c_x^2-(\alpha+\alpha_t)(\alpha+\alpha_s)c_x^2<0. \quad (74)$$

When $\tilde{\varphi}$ is given by eq (34), the analyzed field φ is obtained as

$$\varphi=\frac{\tilde{\alpha}\tilde{\Phi}}{[\tilde{\alpha}+k^2(\alpha(c_x-\tilde{c}_x)^2+\alpha_t\tilde{c}_x^2+\alpha_sc_x^2)]}\cos k(x-\tilde{c}_xt), \quad (75)$$

and the reduction factor becomes

$$r=\frac{\tilde{\alpha}}{[\tilde{\alpha}+k^2(\alpha(c_x-\tilde{c}_x)^2+\alpha_t\tilde{c}_x^2+\alpha_sc_x^2)]}. \quad (76)$$

6. A NUMERICAL METHOD

Numerical solutions of the Euler eq (48) and (73) are, in general, obtained by a method of iteration. Convergence to a solution is usually accelerated by the addition of the low-pass filters suggested in this article. Some further discussion will be given on the numerical solution of the Euler eq (27) and (28).

The Euler eq (27) and (28) may be solved by iterative uses of the marching process (proposed by Miyakoda and Moyer 1968) until the values at the end of the prediction approach those required from the boundary condition $t=T$. However, a major difficulty can arise in obtaining the unique solution of a differential equation of the parabolic type. A discontinuity can appear in the direction, η=constant, which is normal to the characteristic lines represented by ξ, as seen in figures 1 and 2. This difficulty is also seen in eq (31), (32), and (33); that is, derivatives Δ_η and Δ_η^2 are defined, but ∇_ξ and ∇_ξ^2 do not appear in the governing equations. Thus the solution is uncertain and not unique in these derivatives. The difficulty can be well demonstrated in figure 2(b). If λ is given on the boundary $\overline{abcd}$, the condition becomes overspecified. The boundary line is not unique especially at the corners a and c and can be parallel to the characteristic lines in the neighborhood of the corners. Therefore, as discussed before, λ can not be given on the entire boundary $\overline{abcd}$. If λ is only given on $\overline{bc}$ and $\overline{ad}$, the solution is uniquely determined in the subdomain C, but can not be determined in the subdomain A because one boundary condition of λ is missing on the characteristic lines which run in the domain A.

To solve the above difficulty, we now consider a practical approach that uses expansion of the solution in a series of periodical orthogonal functions such as the Fourier series. The coefficients of the series are determined from the values in the region $x_1\leq x\leq x_2$ but gives the values in the regions, $-\infty<x\leq x_1$ and $x_2\leq x<\infty$, due to the periodicity of the functions. The Fourier series expres-

sions for λ, φ, and $\tilde{\varphi}$ are written as

$$\begin{pmatrix}\lambda\\ \varphi\\ \tilde{\varphi}\end{pmatrix}=\sum_{m=-\infty}^{\infty}\begin{pmatrix}\Lambda_m\\ \Phi_m\\ \tilde{\Phi}_m\end{pmatrix}e^{ikx} \tag{77}$$

where m is an integer, the coefficients Λ_m, Φ_m, and $\tilde{\Phi}_m$ are complex variables and functions of time and k is the wave number defined as $k=2\pi m/L$. In the following discussion, Δt and Δx are finite.

Substitution of the expansion into eq (27) and (28) leads to

$$2\tilde{\alpha}(\Phi_m-\tilde{\Phi}_m)-(\nabla_t+ic_x\epsilon_x)\Lambda_m=0 \tag{78}$$

and

$$(\nabla_t+ic_x\epsilon_x)\Phi_m=0 \tag{79}$$

where $\epsilon_x=\sin k\Delta x/\Delta x$. By eliminating Φ_m from these two equations, we can derive a finite-difference equation concerning one unknown variable Λ_m as

$$(\nabla_t\nabla_t+2ic_x\epsilon_x\nabla_t-c_x^2\epsilon_x^2)\Lambda_m=-2\tilde{\alpha}(\nabla_t+ic_x\epsilon_x)\tilde{\Phi}_m\equiv\tilde{F}. \tag{80}$$

With this equation, however, it is inadequate to use a relatively simple and common method of relaxation, such as the Richardson or Liebman method or the alternating direction method, to get an iterative solution because the spectral radius of the amplification matrix exceeds one. This difficulty is due to the second term on the left-hand side of eq (80). To avoid this difficulty, we express Λ_m by a product of two new variables as

$$\Lambda_m=M_mN_m. \tag{81}$$

The following symmetric formulas are useful to further manipulate the finite-difference eq (80),

$$\nabla_t\xi\eta=\bar{\xi}\nabla_t\eta+\bar{\eta}\nabla_t\xi \tag{82}$$

where ξ and η are variables and

$$(\overline{\ \ })=[(\ \ )_{n+1}+(\ \ )_{n-1}]/2. \tag{83}$$

Proof of eq (82) is easily given as

$$\begin{aligned}\nabla_t\xi\eta&=\frac{\xi_{n+1}\eta_{n+1}-\xi_{n-1}\eta_{n-1}}{2\Delta t}\\ &=\frac{\xi_{n+1}\eta_{n+1}-\xi_{n-1}\eta_{n+1}+\xi_{n-1}\eta_{n+1}-\xi_{n-1}\eta_{n-1}}{2\Delta t}\\ &=\eta_{n+1}\nabla_t\xi+\xi_{n-1}\nabla_t\eta.\end{aligned} \tag{84}$$

Similarly,

$$\nabla_t\xi\eta=\eta_{n-1}\nabla_t\xi+\xi_{n+1}\nabla_t\eta. \tag{85}$$

Then by taking the average of eq (84) and (85), we can derive eq (82).

Substitution of eq (81) into eq (80) with the aid of eq (82) results in

$$\begin{aligned}\overline{\overline{M}}\nabla_t\nabla_tN+\overline{\overline{N}}\nabla_t\nabla_tM+2ic_x\epsilon_x\overline{M}\nabla_tN-c_x^2\epsilon_x^2MN\\ +2\nabla_t\overline{N}\nabla_t\overline{M}+2i\ c_x\epsilon_x\overline{N}\nabla_tM=\tilde{F}\end{aligned} \tag{86}$$

where the subscript m is omitted for convenience. If we can find the value of N that satisfies the relationship

$$2\nabla_t\overline{N}\nabla_t\overline{M}+2ic_x\epsilon_x\overline{N}\nabla_tM=0, \tag{87}$$

eq (86) becomes the finite-difference equation containing the unknown variable M and its second derivative $\nabla_t\nabla_tM$ without the term ∇_tM. However, the solution of eq (87) is not easily obtained due to the presence of $\nabla_t\overline{M}$ and ∇_tM, which are not identical.

To overcome this difficulty, an attempt is made to define the new finite-difference derivative

$$\nabla_t(\ \ )=\frac{(\ \ )_{n+1}-(\ \ )_n}{\Delta t}. \tag{88}$$

We can derive a formula similar to eq (82) as

$$\nabla_t\xi\eta=\hat{\xi}\nabla_t\eta+\hat{\eta}\nabla_t\xi \tag{89}$$

where $(\hat{\ \ })$ is defined as

$$(\hat{\ \ })=\frac{(\ \ )_{n+1}+(\ \ )_n}{2}. \tag{90}$$

In eq (86), $\nabla_t\nabla_t$ is replaced by $\hat{\nabla}_t\hat{\nabla}_t$, although this causes inconsistency. However, this inconsistency may not be serious because the substitution lessens the truncation error in the second derivative. The first derivative term in eq (86) remains the same. Consequently, we can rewrite eq (86) in the form

$$\begin{aligned}\hat{\hat{M}}\nabla_t\nabla_tN+\hat{\hat{N}}\nabla_t\nabla_tM+2ic_x\overline{M}\nabla_tN-c_x^2\epsilon_x^2MN\\ +2(\nabla_tN+ic_x\epsilon_x\overline{N})\nabla_tM=\tilde{F};\end{aligned} \tag{91}$$

and corresponding to eq (87), we can obtain

$$\nabla_tN+ic_x\epsilon_x\overline{N}=0 \tag{92}$$

or

$$\left(\frac{1}{2\Delta t}+i\,\frac{c_x\epsilon_x}{2}\right)N_{n+1}=\left(\frac{1}{2\Delta t}-i\,\frac{c_x\epsilon_x}{2}\right)N_{n-1}. \tag{93}$$

If we assume for simplicity that N_0 at the initial time is equal to one and N_1 is derived from

$$\left(\frac{1}{\Delta t}+i\,\frac{c_x\epsilon_x}{2}\right)N_1=\left(\frac{1}{\Delta t}-i\,\frac{c_x\epsilon_x}{2}\right)N_0, \tag{94}$$

we can obtain N_n for $n=2, 3, \ldots$ by eq (93). (This is equivalent to an implicit scheme for solving an advection equation.) Since N_n should always be a nonzero constant, we may set the following condition for Δ_t and Δx as

$$\frac{\Delta x}{\Delta t}>c_x. \tag{95}$$

Finally, eq (91) becomes

$$\hat{\hat{N}}\nabla_t\nabla_t M+\hat{\hat{M}}\nabla_t\nabla_t N+2ic_x\overline{M}\nabla_t N-c_x^2\epsilon_x^2 MN=\tilde{F}. \quad (96)$$

This equation can be solved with respect to M by using a simple relaxation method under the boundary condition

$$(M)_{n=0}=(M)_{n=N}=0 \quad (97)$$

where N is the final time level. After M is solved, Λ_m is obtained from eq (81), and the solution Φ_m can be easily calculated from eq (78). The method proposed in this section may be applied for more complicated cases, for example, using the primitive equations as constraints.

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