

# On the Use of a Material Layer Model of the Atmosphere in Numerical Prediction

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## 1. Introduction

GATES (1957) has presented an atmospheric model based on the subdivision of the atmosphere into two baroclinic layers (troposphere and stratosphere), separated by a tropopause surface which is considered material. The present paper deals with a model based altogether on the use of layers bounded by material surfaces, as an alternative to the commonly used isobaric layers. It is thus a quasi-Lagrangian model and may be considered as an application of STARR'S quasi-Lagrangian system of coordinates (1957).

The possible advantages of such a material layer model are first of all that by using vertical displacements rather than vertical velocities one may obtain a truer picture of the vertical structure of the atmosphere. As in GATES' model one might choose one of the material surfaces to coincide with the tropopause and thus avoid interpolations across the tropopause which would otherwise be necessary. Moreover, the effect of mountains may be taken into account simply by specifying the heights of the lowest material surface.

Finally it should be mentioned that the equations for the material layer model are relatively simple because there is no exchange of mass between the layers.

## 2. Equations for material surfaces

Consider a subdivision of the atmosphere into  $s$  material layers as shown in Fig. 1. The  $n$ -th layer is bounded above by a material surface number  $n-1$  and below by

surface number  $n$ . The distribution of a variable in one of these surfaces will be represented as a function of horizontal coordinates  $x$ ,  $y$  and time  $t$ .

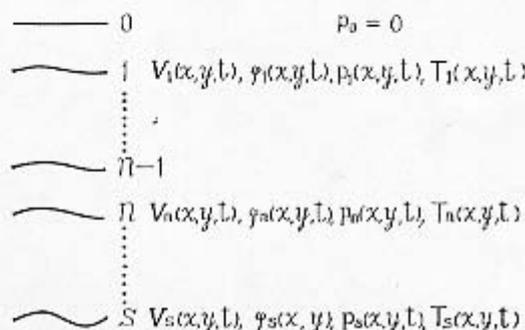


Fig. 1

Thus the distribution of velocity, geopotential, pressure and temperature in the surface number  $n$  will be designated by  $v_n(x, y, t)$ ,  $\varphi_n(x, y, t)$ ,  $p_n(x, y, t)$  and  $T_n(x, y, t)$  respectively. Since the surface is material, the individual rate of change of these variables are obtained by differentiation through  $x$ ,  $y$  and  $t$ :

$$\left. \begin{aligned} \left( \frac{Dv}{Dt} \right)_n &= \frac{\partial v_n}{\partial t} + v_n \cdot \nabla v_n, \\ \left( \frac{Dp}{Dt} \right)_n &= \frac{\partial p_n}{\partial t} + v_n \cdot \nabla p_n, \text{ etc.} \end{aligned} \right\} \quad (1)$$

where  $\nabla$  is the two-dimensional gradient operator.

With  $\alpha$  denoting specific volume, the value of the horizontal pressure force at the surface number  $n$  is seen to be  $-(\nabla \varphi_n + \alpha_n \nabla p_n)$ . The equation of motion in a point belonging to surface number  $n$

may therefore be written, when friction is ignored:

$$\frac{\partial v_n}{\partial t} + v_n \cdot \nabla v_n + k \times f v_n + \nabla \varphi_n + \alpha_n \nabla p_n = 0, \quad (2)$$

$$n=1, 2, \dots, s.$$

Conservation of potential temperature ( $\theta$ ) may be expressed by

$$\frac{\partial \theta_n}{\partial t} + v_n \cdot \nabla \theta_n = 0, \quad n=1, 2, \dots, s. \quad (3)$$

In a point where the value of  $p_n$  is known,  $\theta_n$  may be converted into  $\alpha_n$  or  $T_n$  (or *vice versa*).

To obtain  $\varphi_n$ , the following integral must be evaluated

$$\varphi_{n-1} - \varphi_n = \int_{p_{n-1}}^{p_n} \frac{RT}{p} dp. \quad (4)$$

This may be done simply by assuming that  $T$  depends linearly upon  $\ln p$  within each layer; thus

$$\varphi_{n-1} - \varphi_n = \frac{R}{2} (T_n + T_{n-1}) (\ln p_n - \ln p_{n-1}), \quad (5)$$

$$n=2, 3, \dots, s.$$

Since  $\varphi_s = \varphi_s(x, y)$  is the known geopotential of the earth's surface, eqs. (5) give all the other  $\varphi_n$  in terms of the  $T$ 's and  $p$ 's.

To complete the system of equations, the continuity equation must be integrated with respect to mass over every material layer. For the  $n$ -th layer, the integrated continuity equation becomes

$$\frac{\partial p_n}{\partial t} - \frac{\partial p_{n-1}}{\partial t} + \nabla \cdot \int_{p_{n-1}}^{p_n} v dp = 0. \quad (6)$$

Except for  $n=1$ , this equation can be made to fit into the system by approximating the integral by a trapezoidal formula:

$$\frac{\partial p_n}{\partial t} - \frac{\partial p_{n-1}}{\partial t} + \nabla \cdot \left[ \frac{1}{2} (v_n + v_{n-1})(p_n - p_{n-1}) \right] = 0, \quad n=2, 3, \dots, s \quad (7)$$

For  $n=1$ , some assumptions must be made concerning the variation of  $v$  with height

in the uppermost layer. A reasonable form of the equation seems to be

$$\frac{\partial p_1}{\partial t} + \nabla \cdot \left[ \frac{1}{2} v_1 p_1 \right] = 0. \quad (8)$$

Eqs. (2), (3), (5), (7) and (8) form a closed system.

The state of the system is given when  $v_n$ ,  $\theta_n$  and  $p_n$  are specified in each surface;  $\varphi_n$  can then be obtained from (5). The state of the system must be fully specified initially; and as in other models based on the primitive equations, the system must be balanced or at least nearly balanced initially in order to avoid spurious excitation of excessive gravity-inertia waves.

### 3. Miscellaneous remarks

*a. Static stability.* Since  $\theta$  is carried at each surface, one may say that the model takes into account variations in static stability. This is true in a sense, but it should be realized that the thickness of a layer is calculated from the temperatures at its upper and lower boundaries (eq. 5). Therefore, the effect of internal modes of vertical motion, inside a layer, upon the mean temperature of the layers is not taken into account. As a consequence, the model does not take into account the effect of the static stability of each layer separately, but only of the average static stability of adjacent layers. It is important to keep this in mind when the number of layers is decided upon.

*b. Choice of layers.* The model cannot be applied to the entire atmosphere, considered as a single layer. The next possibility is to divide the atmosphere into two layers, stratosphere and troposphere; but from what is said above, it follows that this does not work either, because the static stability of the troposphere will have no effect upon internal modes of motion inside the troposphere. In order to make sense, the model therefore requires a subdivision of the atmosphere into at least three layers. In this case, the surface number 1 may be chosen

initially to coincide with the tropopause, and the surface number 2 may for instance be chosen to coincide with the 500 mb surface initially (at later stages, it will of course deviate from the 500 mb surface).

If one can afford to use more than three layers, one has considerable freedom of choice. One may choose a number of initially isobaric surfaces, but the possibility of choosing isentropic surfaces, or the boundaries of a frontal zone should be looked into.

It may be anticipated that some of the surfaces will be strongly distorted in certain regions, notably near fronts. When this happens, it will be necessary to change over to another set of surfaces by an interpolation procedure.

*c. The horizontal pressure force* in a surface is of the form  $-(\nabla q)_n + \alpha_n \nabla p_n$ . In regions where the surface tilts strongly, each of the two terms will be much larger than their sum, so that a high degree of accuracy is necessary. However, the truncation error involved in the finite-difference approximation will affect the two terms differently and may thus disturb the balance between them. This may turn out to be a serious difficulty in the proposed scheme. It may be necessary to require that the surfaces have gentle slopes. Since the lowest surface coincides with the ground, it will be necessary to use a smoothed topography of the earth's surface. Moreover,  $\alpha_n \nabla p_n$  may also be expressed as  $RT_n \nabla \ln p_n$  or  $\theta_n \nabla H_n$  ( $H$  denoting the Exner function). The latter expression will be advantageous if the variation of  $\theta_n$  over the surface is small.

#### 4. Integrated equations for material layers

The state of the atmospheric model considered above is fully defined by specifications of the variables  $v_n$ ,  $p_n$  and  $\theta_n$  (or  $T_n$ ) in each material surface. An alternative model is obtained by extending these specifications to comprise also the total momentum and dry enthalpy of each

material layer. The equations must then be supplemented by integrated momentum and energy equations for each layer, as shown below.

The following notation will be used (with  $q$  denoting any variable):

$$\left. \begin{aligned} q_n - q_{n-1} &= q_n^* \\ \int_{p_{n-1}}^{p_n} q dp &= \bar{q}_n p_n^*, \quad \text{hence } \bar{p}_n = \frac{1}{2}(p_n + p_{n-1}). \end{aligned} \right\} \quad (9)$$

Since the integration limits are variable, integration does not commute with differentiation with respect to  $x$ ,  $y$  and  $t$ ; one has

$$\left. \begin{aligned} \nabla(\bar{q}_n p_n^*) &= (\nabla \bar{q})_n p_n^* + q_n \nabla p_n - q_{n-1} \nabla p_{n-1} \\ \frac{\partial}{\partial t}(\bar{q}_n p_n^*) &= \left(\frac{\partial \bar{q}}{\partial t}\right)_n p_n^* + q_n \frac{\partial p_n}{\partial t} - q_{n-1} \frac{\partial p_{n-1}}{\partial t} \end{aligned} \right\} \quad (10)$$

Integrating the equation of motion (in pressure coordinates) with respect to pressure over the  $n$ -th layer and using (1) and (10), one obtains

$$\begin{aligned} \frac{\partial}{\partial t}(\bar{v}_n p_n^*) + \nabla[(\bar{v}v)_n p_n^*] + k \times \int \bar{v}_n p_n^* \\ + \nabla(\bar{\varphi}_n p_n^*) - q_n \nabla p_n + q_{n-1} \nabla p_{n-1} = 0. \end{aligned} \quad (11)$$

Similarly, one finds by integrating the thermodynamic energy equation,

$$c_p \frac{\partial}{\partial t}(\bar{T}_n p_n^*) + c_p \nabla \cdot [(T\bar{v})_n p_n^*] - (\bar{\alpha}\omega)_n p_n^* = 0, \quad (12)$$

where  $\omega$  denotes  $Dp/Dt$ , and  $c_p$  the specific heat. The integrated continuity equation (6) becomes, in the present notation

$$\frac{\partial p_n}{\partial t} - \frac{\partial p_{n-1}}{\partial t} + \nabla \cdot (\bar{v}_n p_n^*) = 0, \quad (13)$$

and it is not necessary to use the approximation (7).

In order that the eqs. (11), (12) and (13) together with (2) and (3) shall form a closed system for the set of specifications

$$v_n, p_n, T_n, \bar{v}_n, \bar{T}_n,$$

it is necessary to express all terms of the equations in terms of these basic variables.

From the hydrostatic equation one finds the thickness of the  $n$ -th layer:

$$\varphi_n^* = -R \left( \frac{\bar{T}}{p} \right)_n p_n^*. \quad (14)$$

Here the value of  $(\bar{T}/p)_n$  can be approximated on the basis of an interpolation formula involving  $T_n$ ,  $T_{n-1}$  and  $\bar{T}_n$ , such as

$$\bar{\varphi}_n^* = R \bar{T}_n (\ln p)_n^* + R T_n^* \left[ 1 - \frac{\bar{p}_n}{p_n^*} (\ln p)_n^* \right] \quad (15)$$

which replaces (5).

Moreover, one finds from the hydrostatic equation

$$\bar{\varphi}_n p_n^* = R \bar{T}_n p_n^* + (\varphi p)_n^*. \quad (16)$$

This equation can be used to eliminate  $\bar{\varphi}_n$  from (11).

There remains to find expressions for the nonlinear terms. A reasonable approximation is

$$(\bar{v}v)_n = \bar{v}_n \bar{v}_n + \frac{1}{12} v_n^* v_n^*; \quad n=2, 3, \dots, s, \quad (17)$$

$$(\bar{T}v)_n = \bar{T}_n \bar{v}_n + \frac{1}{12} T_n^* v_n^*; \quad n=2, 3, \dots, s, \quad (18)$$

which is based upon linear interpolation formulas in  $p$ ; The correction for a polynomial of the second degree is very slight. In the uppermost layer ( $n=1$ ),  $v_1^*$  and  $T_1^*$  may be defined as  $2(v_1 - \bar{v}_1)$  and  $2(T_1 - \bar{T}_1)$ , respectively.

The remaining term  $(\bar{\alpha}v)_n p_n^*$  in (12) is more difficult, and it is rather important, since it incorporates the effect of the static stability inside the layer. This is the only term where  $\omega$  occurs explicitly. A reasonable approximation to this term is perhaps obtained by writing

$$\begin{aligned} - \int_{p_{n-1}}^{p_n} \omega \alpha d p &= \int_{\varphi_{n-1}}^{\varphi_n} \omega d \varphi = \frac{\omega_n + \omega_{n-1}}{2} \varphi_n^* \\ &= \frac{1}{2} \left( \frac{\partial p_n}{\partial t} + \frac{\partial p_{n-1}}{\partial t} + v_n \cdot \nabla p_n \right. \\ &\quad \left. + v_{n-1} \cdot \nabla p_{n-1} \right) \varphi_n; \quad n=2, 3, \dots, s. \quad (19) \end{aligned}$$

In the highest layer ( $n=1$ ), we may put  $\omega = \omega_1 p / p_1$ , and obtain

$$\begin{aligned} - \int_0^{p_1} \omega \alpha d p &= \frac{\omega_1}{p_1} \int p d \varphi = \omega_1 (\varphi_1 - \bar{\varphi}_1) \\ &= \left( \frac{\partial p_1}{\partial t} + v_1 \cdot \nabla p_1 \right) (\varphi_1 - \bar{\varphi}_1). \quad (20) \end{aligned}$$

On account of the interpolation formulas (15)-(20) (or similar formulas), the system (2), (3), (11), (12), and (13) forms a closed system for the independent variables  $v_n$ ,  $p_n$ ,  $T_n$ ,  $\bar{v}_n$  and  $\bar{T}_n$ . In this case a subdivision of the atmosphere into two material layers (stratosphere and troposphere) is a possible numerical model, since the static stability inside each layer is now taken into account.

The model presented has not yet been tested, and much work remains before we shall know its possible virtues and shortcomings.

## REFERENCES

- 1) GATES, W. L., 1957: A dynamical model for large-scale tropospheric and stratospheric motions. *Quart. J. R. Met. Soc.*, 83, 141-160.
- 2) STARR, V. P., 1945: A quasi-Lagrangian system of hydrodynamical equations. *J. Meteor.*, 2, 227-237.

## DISCUSSION

**Shuman:** I was lead to precisely the same formulation from considerations of computational stability. I have a stable system for computing the layer equations where you integrate superposed homogenous layers and these are quite stable and the equations are very similar, of course, to this set.

**Charney:** Have you considered what will happen when you take, say, several material layers? Over a long period of time these layers may get very close together. They will not intersect but they can become infinitesimally close.

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**A:** Yes, of course, that will happen. I am sure that one will have to choose a new system of surfaces in the middle of the calculation, by interpolation, if this is going to be applied during more than, say, one or two days. So I don't think it is very suitable for predictions for long periods.

**Mintz:** Won't these, in the adiabatic case, belong to the surfaces of constant potential temperature? And how would you describe the initial state for potential temperature surfaces which intersect the ground?

**A:** They may not be surfaces of potential temperatures. For instance, you may choose one as 500 millibar surface in the beginning, at the initial time, and then, from that it will move as a material surface. You can choose them however you like.

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