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A Convergent Method for Solving the Balance Equation

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Abstract

The non-linear balance equation is replaced by an infinite system of linear differential equations. This system renders a series of solutions which under certain conditions converge toward the solution of the balance equation.

It is shown that these conditions coincide with the conditions for ellipticity of the balance equation.

A slightly different method is shown to diverge. This is illustrated by a case where the sequence of approximate solutions is compared with the solution of the balance equation.

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1. Introduction

Past experience in the field of numerical weather prediction has made it increasingly probable that an appreciable part of the forecast errors can be attributed to the quasi-geostrophic wind assumption. In this respect the tendency for over-intensifying anticyclones ("blow-up" highs), so often displayed in the products of numerical forecasting, has particularly been under suspicion. Recent work at the Joint Numerical Weather Prediction Unit in Suitland, Md., has shown that these so-called "blow-up" highs are generally[1)] eliminated when geostrophic wind and geostrophic vorticity are replaced by a wind and a vorticity obtained from the "balance equation"[1,2]. This is a diagnostic equation arrived at by applying a two-dimensional (in a pressure-surface) divergence operator to the primitive equations of motion and in the resulting equation retaining only the divergence-free part of the wind. This implies that both wind and vorticity can be derived from a single scalar quantity, the stream-function.

As yet the difference between this type of non-geostrophic wind and the geostrophic one has not been studied in detail. It is clear, however, that the former much resembles the gradient wind. Thus, both the wind and vorticity derived from the balance equation are smaller in a region of cyclonic curvature than the corresponding geostrophic

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1) This does not apply to "blow-up" highs resulting from incorrect boundary conditions.

quantities, whereas the reverse is true in a region of anticyclonic curvature[1)] [3]. A numerical solution of the balance equation shows intersection between streamlines and isohypses in regions where the kinetic energy changes along a streamline. The cross-contour wind component is in accordance with the dynamic energy equation.

The balance equation may be written as follows:

$$f\nabla^2\psi + 2[\psi_{xx}\cdot\psi_{yy} - \psi_{xy}{}^2] + \nabla f\cdot\nabla\psi - \nabla^2\phi = 0 \tag{1}$$

where ψ is the streamfunction, f the Coriolis parameter ~~$2\Omega\sin\varphi$~~, ϕ the geopotential, and

$$\nabla^2\psi = \psi_{xx} + \psi_{yy} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \text{vorticity}$$

In addition to the fact that eq. (1) is a non-linear partial differential equation and, as such, does not directly yield to methods applicable to the solution of a linear differential equation, additional difficulties are encountered due to the hybrid character of this equation. By hybrid it is

1) It follows from equation (1) that, except for the contribution of the term $\nabla f\cdot\nabla\psi$, the vorticity $\zeta = \psi_{xx} + \psi_{yy}$ is algebraically smaller than the geostrophic vorticity whenever $\psi_{xx}\cdot\psi_{yy} - \psi_{xy}{}^2 > 0$ (elliptic points). The effect of the term $\nabla f\cdot\nabla\psi$ is to increase the vorticity in a westerly current and to decrease it in an easterly flow (in relation to the geostrophic vorticity). Moreover, a purely hyperbolic point ($\psi_{xx}\cdot\psi_{yy} - \psi_{xy}{}^2 < 0$, $\zeta = 0$) of the ψ-field does not coincide with a purely hyperbolic point of the ϕ-field.

meant that unless restrictions are imposed upon the ϕ-field, equation (1) may be elliptic in one part of the domain, parabolic in another and hyperbolic in the third. There is no theory available for solving eq. (1) in general, but at present the scale of pressure systems dealt with in numerical weather prediction is such that a slight smoothing of the analysis renders (1) elliptic over the entire map. Thus, the problem is reduced to one of finding a solution of (1) when ψ is known on a closed boundary curve. Another problem which still remains is the question of uniqueness. As we will see below, there are two solutions to eq. (1) for given boundary values, one of which has to be rejected as meteorologically irrelevant.

The criteria for ellipticity and how to distinguish between the two solutions are discussed in the second section.

In the third and last section, a method is presented by which the non-linear balance equation is replaced by a system of linear differential equations the solutions of which render a sequence of functions which under certain conditions converge toward the solution of the balance equation. It is shown that these conditions coincide with the conditions for ellipticity of the balance equation.

2. The Ellipticity Criteria

Let

$$F(r, s, t, p, q, x, y, \psi) = 0 \qquad (2)$$

be a second order differential equation, linear or non-linear, in the two independent variables x and y. Here ψ is the dependent variable and r, s, t, p, and q have the following meanings:

$$r = \psi_{xx}; \quad s = \psi_{xy}; \quad t = \psi_{yy}; \quad p = \psi_x; \quad q = \psi_y$$

The condition that eq. (2) be elliptic is

$$4 F_r \cdot F_t - F_s^2 > 0 \tag{3}$$

where the subscripts indicate differentiation, i.e., $F_r = \frac{\partial F}{\partial r}$ etc. This condition applied to equation (1) leads to the inequality

$$(f + 2\psi_{xx})(f + 2\psi_{yy}) - 4\psi_{xy}^2 > 0 \tag{3a}$$

as a condition for ellipticity of the balance equation. As discussed by Bolin [2], this implies a restriction on the second derivatives of the streamfunction ψ. Of particular interest is the fact that $f + 2\psi_{xx}$ and $f + 2\psi_{yy}$ both must have the same sign throughout the domain of integration and, as shown by Rellich [4], this implies two distinct solutions of (1) for given boundary values of ψ. For one of the solutions both these quantities are negative and for the other both are positive. Since we observe that, as a rule, the absolute vorticity $\psi_{xx} + \psi_{yy} + f$ is positive for the scale of motion presently dealt with in numerical weather prediction, the positive branch of the solution to eq. (1) is the one of interest.

In order to investigate further the conditions under which the balance equation is elliptic, the inequality (3a) is combined with eq. (1), the result of which is an ellipticity criterion of the form

$$\nabla^2\phi + \frac{f^2}{2} - \nabla f \cdot \nabla\psi > 0 \tag{3b}$$

The advantage of (3b) over (3a) is that both $\nabla^2\phi$ and f^2 are known over the entire domain of integration and we can therefore test the extent to which the balance equation is elliptic when applied to a large and medium (migrating cyclones and anticyclones) scale of atmospheric motion. The last term in (3b) implies the knowledge of the streamfunction itself but as it is small compared with the other two terms, it can, for practical purposes, be replaced by a conservative estimate or possibly neglected altogether. Experience shows that for the scales of motion mentioned above, (3b) holds true over by far the greater part of a weather map and can be imposed upon the remainder by changes in the ϕ-field comparable in size to errors in observations and analysis. Accepting the legitimacy of such changes in the analysis, we have ascertained the ellipticity of eq. (1) and can therefore solve it as a boundary value problem provided a suitable (numerical) method is available.

The choice between the meteorologically relevant and irrelevant solutions is made [2,5] by imposing the condition $\nabla^2\psi^{(n)} + f > 0$ during successive steps of a numerical procedure leading up to the solution ψ of eq. (1). The notation $\psi^{(n)}$ refers to a sequence of approximate solutions which for increasing n approach ψ. Although the inequality $\nabla^2\psi + f > 0$ is necessary it may <u>not be sufficient</u>. <u>Thus, the particular method used may produce a sequence of approximate solutions which themselves do not satisfy the ellipticity criteria (3b) and (3c) and for that reason may not converge to the true solution</u>. A method to which this statement applies will be discussed in the next section.

It is therefore suggested as safer that the conditions

$$f + 2\psi^{(n)}_{xx} > 0; \qquad f + 2\psi^{(n)}_{yy} > 0 \tag{3c}$$

together with (3a) be imposed upon the system of approximate solutions $\psi^{(0)}, \psi^{(1)}, \ldots. \psi^{(n)}$ besides initially imposing condition (3b).

A question of considerable interest, although its treatment is not within the scope of this paper, is whether the ellipticity criteria (3a), (3b) and (3c) are purely of mathematical nature or whether they also may have a physical significance. An indication that the latter may at least approximately be the case is the remarkable fact that the synoptic maps are in a close agreement with (3b). The difference between the ϕ-field as analysed and the one upon which (3b) has been imposed is, as mentioned above, well within the margin of error in analysis. It is pointed out that in the case of the stationary circular vortex, the motion of which is governed by the gradient wind equation

$$\frac{v^2}{\rho} + f\,(v - v_g) = 0 \qquad (4)$$

physical considerations lead to inequalities which resemble the ellipticity criteria. First, the requirement that the wind speed v be real renders the two inequalities

$$f + \frac{4v_g}{\rho} \geq 0; \qquad f + \frac{2v}{\rho} \geq 0 \qquad (5)$$

where v_g is the geostrophic wind speed and ρ is the radius of curvature, considered positive for cyclonic curvature and negative for anticyclonic curvature[1]. Second, the dynamic stability criterion (based on conservation of angular momentum) requires the absolute vorticity $\eta = \zeta + f$ to be positive; this may be expressed as follows:

$$f + 2\,\frac{v}{\rho} + f + 2\,\frac{\partial v}{\partial \rho} > 0 \qquad (6)$$

1) As in the case of the balance equation eq. (4) has two distinct solutions, the one of meteorological relevance being required to approach to geostrophic one for straight streamlines. The inequalities (5) apply to the latter.

This is to be compared with (3a) which for this particular case reduces to

$$(f + 2\frac{v}{\rho})(f + 2\frac{\partial v}{\partial \rho}) > 0 \qquad (7)$$

The ellipticity criterion (7) restricts the shearing vorticity $\frac{\partial v}{\partial \rho}$ more than (6) does (inequality (5) taken into account) and (5) restricts the geostrophic curvature vorticity more than required by the ellipticity criterion (3b).

These elementary considerations do not, of course, answer the question raised above, but merely suggest that a thorough investigation of the dynamic stability of a non-divergent flow may provide a physical interpretation of the ellipticity criteria.

3. An Iterative Method for Solving the Balance Equation.

In order to solve eq. (1) as a boundary value problem, we partly replace its non-linear terms by a "guess"-function $\psi^{(0)}$ as follows:

$$f\nabla^2\psi^{(1)} + \psi^{(0)}_{yy} \cdot \psi^{(1)}_{xx} + \psi^{(0)}_{xx} \cdot \psi^{(1)}_{yy} - 2\psi^{(0)}_{xy} \cdot \psi^{(1)}_{xy} + \nabla f \cdot \nabla\psi^{(1)} - \nabla^2\phi = 0 \quad (1a)$$

where $\psi^{(1)}$ is the solution to the linear differential equation (1a) satisfying the same boundary conditions as ψ. Having solved for $\psi^{(1)}$, we replace $\psi^{(0)}$ in (1a) by $\psi^{(1)}$ and replace $\psi^{(1)}$ by a third function $\psi^{(2)}$ for which we solve eq. (1a). Repetition of this procedure leads to a sequence of functions, $\psi^{(0)}$, $\psi^{(1)}$, $\psi^{(2)}$, ... $\psi^{(n)}$, which we will consider as approximate solutions to equation (1). We observe that (1a) is elliptic if

$$(f + \psi^{(n)}_{yy})(f + \psi^{(n)}_{xx}) - (\psi^{(n)}_{xy})^2 > 0 \qquad n = 0, 1, 2, \cdots \quad (8)$$

Moreover, if the above sequence of approximate solutions to (1a) does converge at all, it will converge towards the solution ψ of eq. (1). This follows from the fact that when the difference between two successive solutions of (1a), ψ^{n} and ψ^{n-1} approaches zero, eq. (1a) becomes identical with eq. (1).

For the purpose of investigating the conditions under which the series $\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots \psi^{(n)}$ does converge, we introduce the notations

$$\epsilon^{(n)} = \psi^{(n)} - \psi^{(n-1)} \quad ; \qquad \epsilon^{(n+1)} = \psi^{(n+1)} - \psi^{(n)} \tag{9}$$

and apply (1a) successively to $\psi^{(n+1)}$ and $\psi^{(n)}$. Subtraction renders the following "error" relation:

$$f\nabla^2\epsilon^{(n+1)} + \psi^{(n)}_{yy}(\epsilon^{(n+1)} + \epsilon^{(n)})_{xx} + \psi^{(n)}_{xx}(\epsilon^{(n+1)} + \epsilon^{(n)})_{yy}$$
$$- 2\,\psi^{(n)}_{xy}\,(\epsilon^{(n+1)} + \epsilon^{(n)})_{xy} + \nabla f \cdot \nabla\epsilon^{(n+1)} = 0 \tag{10}$$

It is to be observed that for all n $\epsilon^{(n)}$ is zero along the boundaries of the domain of integration which we, for the sake of simplicity, will assume to be rectangular and have the side lengths a and b. A further simplification is introduced by treating, for the time being, the quantities f, $\psi^{(n)}_{xx}$, $\psi^{(n)}_{yy}$, and $\psi^{(n)}_{xy}$ as constants which reduces the differential equation (10) to one with constant coefficients and without the term $\nabla f \cdot \nabla\epsilon^{(n+1)}$. Under these conditions we may expand both $\epsilon^{(n)}$ and $\epsilon^{(n+1)}$ in a series of sine functions:

$$\epsilon^{(n)} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} A^{(n)}_{k,m} \sin k\frac{\pi}{a} x \sin m\frac{\pi}{b} y$$
$$\epsilon^{(n+1)} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} A^{(n+1)}_{k,m} \sin k\frac{\pi}{a} x \sin m\frac{\pi}{b} y \tag{11}$$

and replace eq. (10) by the integral equation

$$f\epsilon^{n+1} = \int_0^a \int_0^b K(x,y,\xi,\eta)[\psi_{\eta\eta}^{(n)} (\epsilon^{(n+1)} + \epsilon^{(n)})_{\xi\xi} + \psi_{\xi\xi}^{(n)}(\epsilon^{(n+1)} + \epsilon^{(n)})_{\eta\eta}$$

$$- 2\psi_{\xi\eta}^{(n)}(\epsilon^{(n+1)} + \epsilon^{(n)})_{\xi\eta}] \, d\xi d\eta \tag{12}$$

where $K(x,y,\xi,\eta)$ is the Green's function for the boundary value problem $\nabla^2\epsilon^{(n+1)} = 0$, $\epsilon^{(n+1)} = 0$ on the rectangular boundaries. The Green's function K has the form

$$K(x,y,\xi,\eta) = \frac{4}{a \cdot b \cdot \pi^2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\frac{k^2}{a^2} + \frac{m^2}{b^2}} \sin \cdot k\frac{\pi}{a} x \cdot \sin m\frac{\pi}{b} y \cdot \sin k\frac{\pi}{a} \xi \cdot \sin m\frac{\pi}{b} \eta \tag{13}$$

Because of the special character of K, the method for solving (13) is straightforward. We substitute for K from (13) into (12), utilize (11) to obtain expressions for the derivatives of $(\epsilon^{(n+1)} + \epsilon^{(n)})$ and interchange summation and integration. Leaving out the details of computing the double sums and double integrals involved, we arrive at the following solution of (12):

$$f\epsilon^{(n+1)} = f \cdot \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} A_{k,m}^{(n+1)} \sin k\frac{\pi}{a} x \cdot \sin m\frac{\pi}{b} y =$$

$$= \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} [\frac{ab}{a^2m^2 + b^2k^2} (k^2\frac{b}{a} \cdot \psi_{yy}^{(n)} + m^2 \cdot \frac{a}{b}\psi_{xx}^{(n)}) + \frac{8ab \cdot \psi_{xy}^{(n)}}{\pi^2(a^2m^2 + b^2k^2)}]$$

$$(A_{k,m}^{(n+1)} + A_{k,m}^{(n)}) \sin k\frac{\pi}{a} x \sin m\frac{\pi}{b} y \tag{12a}$$

where k and m in the $\psi_{xy}^{(n)}$-term can only take on odd values; for k and m even this term is zero. Eq. (12a) provides a relation between the amplitudes $A_{k,m}^{(n+1)}$ and $A_{k,m}^{(n)}$ as follows

$$A_{k,m}^{(n+1)} = - \frac{\kappa}{1 + \kappa} \; A_{k,m}^{(n)} \tag{14}$$

where

$$f\kappa = \frac{b^2k^2}{a^2m^2 + b^2k^2} \psi_{yy}^{(n)} + \frac{a^2m^2}{a^2m^2 + b^2k^2} \psi_{xx}^{(n)} + \frac{8ab}{\pi^2(a^2m^2 + b^2k^2)} \psi_{xy}^{(n)} \tag{15}$$

This expression for κ is substituted in (14) which renders

$$A_{k,m}^{(n+1)} = - \frac{a^2m^2 \, \psi_{xx}^{(n)} + b^2k^2 \, \psi_{yy}^{(n)} + \frac{8ab}{\pi^2} \psi_{xy}^{(n)}}{a^2m^2(f + \psi_{xx}^{(n)}) + b^2k^2(f + \psi_{yy}^{(n)}) + \frac{8ab}{\pi^2} \psi_{xy}^{(n)}} \cdot A_{k,m}^{(n)} \tag{16}$$

In a region where the vorticity of the approximate streamfunction $\psi^{(n)}$ is cyclonic κ is generally positive and an arbitrary wave component of the $\epsilon^{(n+1)}$-field is smaller than the corresponding amplitude of the $\epsilon^{(n)}$-field. The negative sign of the amplitude ratio $- \kappa/1 + \kappa$ implies that the two components are 180 degrees out of phase. In the case of a negative κ (anticyclonic region) the corresponding wave components are in phase and the amplitude ratio will exceed unity when the magnitude of κ is greater than 0.5. A necessary and sufficient condition for the convergence of this method for solving eq. (1), as outlined above, is that $0.5 + \kappa > 0$, or, by virtue of (15),

$$(f + 2\psi_{xx}^{(n)}) \; a^2m^2 + (f + 2\psi_{yy}^{(n)}) \; b^2k^2 + \frac{16ab}{\pi^2} \psi_{xy}^{(n)} > 0 \tag{17}$$

We compare (17) with the inequalities (3a) and (3c) and point out, that when the latter is satisfied and applied to $\psi^{(n)}$, the two first terms in (17) are positive. Furthermore, let δ_1 and δ_2 be two positive quantities and $f + 2\psi_{xx}^{(n)} = \delta_1$ and $f + 2\psi_{yy}^{(n)} = \delta_2$, it then follows from (3a) that

$|\psi_{xy}^{(n)}| < \frac{1}{2}(\delta_1 \cdot \delta_2)^{\frac{1}{2}}$. When this is utilized (17) may be written as

$$a^2m^2 \cdot \delta_1 + b^2k^2 \cdot \delta_2 - \frac{8ab}{\pi^2}(\delta_1 \cdot \delta_2)^{\frac{1}{2}} > 0 \qquad (17a)$$

since $\psi_{xy}^{(n)}$ may be negative. The left hand side of (17a) is minimum for $m = k = 1$ and (17) and (17a) are therefore certainly satisfied if

$$a^2\delta_1 + b^2\delta_2 - \frac{8ab}{\pi^2}(\delta_1 \cdot \delta_2)^{\frac{1}{2}} = (a\delta_1^{\frac{1}{2}} - b\delta_2^{\frac{1}{2}})^2 + (2 - \frac{8}{\pi^2})a \cdot b(\delta_1 \cdot \delta_2)^{\frac{1}{2}} > 0$$

which obviously holds true since $2 > \frac{8}{\pi^2}$.

Thus the iterative method described above will furnish a set of approximate solutions, $\psi^{(0)}$, $\psi^{(1)}$, ... $\psi^{(n)}$, ... to eq. (1) which for increasing n will converge toward its solution ψ, provided all the approximate solutions satisfy the ellipticity criteria (3a) and (3c).

It follows from (16) that the rate of convergence may vary considerably from one part of the domain to another, depending on the magnitude of the second derivatives of $\psi^{(n)}$. Where these are small compared with the Coriolis parameter f the convergence is so rapid that a practically correct solution is obtained after a few iterations. The convergence is considerably slower, however, where the second derivatives are comparatively large and in particular when the relative vorticity is negative. In such areas a good first guess may be important in speeding up the numerical procedure by which the solution ψ is arrived at.

It may seem that the method described above, by which the convergence of the iterative scheme for solving the balance equation was studied, lacks in generality because it treats the coefficients f, $\psi_{xx}^{(n)}$, $\psi_{yy}^{(n)}$, and $\psi_{xy}^{(n)}$ in eq. (10) as constants. We point out, however, that the domain of integration may be divided into sub-domains small enough to allow eq. (10) to be treated as an equation with constant coefficients. In every such (rectangular) sub-domain the results presented above are therefore valid. Consequently if the theory indicates divergence in one or more of the sub-domains, the iterative method will not in general render the solution of (1) when applied to the whole domain. Similarily we expect the method to converge if the theory predicts convergence in every sub-domain. This reasoning is admittedly of heuristic nature rather than a strict proof, but similar heuristic arguments are frequently relied upon and have proven successful. One example is the application of J. von Neumann's method for deriving criteria for computational stability in connection with numerical solutions of parabolic and hyperbolic equations with variable coefficients [6]. Another is the use of convergence criteria and optimum relaxation coefficients (for instance, in numerical weather prediction) obtained for simple elliptic differential equations in the treatment of more complicated equations to which the theory does not strictly apply [7,8].

There is obviously more than one way of linearizing the balance equation (1), by initially replacing its non-linear terms by a combination of a "guess" and a function to which the resulting linear differential equation applies. An iterative method of the type described above in connection with eq. (1a) will produce a series of functions for which the convergence conditions may be investigated by the method just described. As an example, we will treat the series defined by the system of linear equations

$$f\nabla^2\psi^{(n+1)} + 2\psi_{xx}^{(n)} \cdot \psi_{yy}^{(n)} - 2(\psi_{xy}^{(n)})^2 + \nabla f \cdot \nabla\psi^{(n)} - \nabla^2\phi = 0; \; (n = 0, 1, 2 \tag{1b}$$

where the non-linear terms in eq. (1) have been replaced by an approximate solution.[1] Equation (1b) is a Poisson's equation and therefore unconditionally elliptic. In preparation for obtaining an "error" relation we observe the identity:

$$2\psi_{xx} \cdot \psi_{yy} \equiv (\nabla^2\psi)^2 - \psi_{xx}^2 - \psi_{yy}^2 \tag{18}$$

Successive application of (1b) to $\psi^{(n+1)}$ and $\psi^{(n)}$ and a subsequent subtraction render the relation

$$f\nabla^2\epsilon^{(n+1)} + (\psi^{(n)} + \psi^{(n-1)})_{yy} \cdot \epsilon_{xx}^{(n)} + (\psi^{(n)} + \psi^{(n-1)})_{xx} \cdot \epsilon_{yy}^{(n)}$$

$$- 2(\psi^{(n)} + \psi^{(n-1)})_{xy} \cdot \epsilon_{xy}^{(n)} + \nabla f \cdot \nabla\psi^{(n)} = 0 \tag{10a}$$

where $\epsilon^{(n+1)}$ and $\epsilon^{(n)}$ are defined by (9) and use has been made of the identity (18).

The method used above for solving eq. (10) is directly applicable to eq. (10a) and the detailed steps will therefore not be repeated. The result is the following relation between the amplitudes $A_{k,m}^{(n+1)}$ and $A_{k,m}^{(n)}$ of the corresponding wave components of the $\epsilon^{(n+1)}$- and $\epsilon^{(n)}$-fields:

$$A_{k,m}^{(n+1)} = -2 \cdot \frac{a^2m^2\bar{\psi}_{xx} + b^2k^2\bar{\psi}_{yy} + \frac{8ab}{\pi^2}\bar{\psi}_{xy}}{(a^2m^2 + b^2k^2) \cdot f} A_{k,m}^{(n)} \tag{16a}$$

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1) This method has been tested empirically at the Joint Numerical Weather Prediction Unit by L. P. Carstensen, who found that after a few iterations ($n = 4$ and 5) the sequence of approximate solutions did approach the true solution of (1) in most parts of the map, but clearly showed divergence in regions of strong cyclonic vorticity.

where

$$2\bar{\psi} = \psi^{(n)} + \psi^{(n-1)}$$

The condition that the amplitude $A_{k,m}^{(n)}$ decreases with increasing n may be written as

$$(f - 2|\bar{\psi}_{xx}|)\, a^2m^2 + (f - 2|\bar{\psi}_{yy}|)\, b^2k^2 - \frac{16ab}{\pi^2}\, |\bar{\psi}_{xy}| > 0 \qquad (17b)$$

For negative values of $\bar{\psi}_{xx}$, $\bar{\psi}_{yy}$, and $\bar{\psi}_{xy}$ (17b) is essentially identical with the inequality (17), whereas it is much more restrictive for positive values. The interpretation of (17b) is that in a region of strong relative vorticity of the streamfunction ψ the set of functions defined by (1b) cannot be expected to converge. This conclusion is supported by empirical evidence (see footnote p. 13). Figure 1 shows a case where the method does not lead to the solution of the balance equation but to a sequence of divergent "approximations". The charts labelled A, B, C, D, and E show the successive differences $\psi - \psi^{(n)}$, where $\psi^{(n)}$ is the solution of eq.(1b) and ψ the solution of the balance eq. The chart labelled A corresponds to n=1, B to n=2 etc. The deep low with centre near Chicago is associated with relative vorticities in excess of the Coriolis parameter, and the method of solution diverges rapidly in the central part of this low.

We point out that outside this restricted area the method is either convergent or slowly divergent (Gulf of Alaska). In accordance with theory, the differences alternate in signs in low-pressure areas and retain their signs in high-pressure areas for successive approximations.

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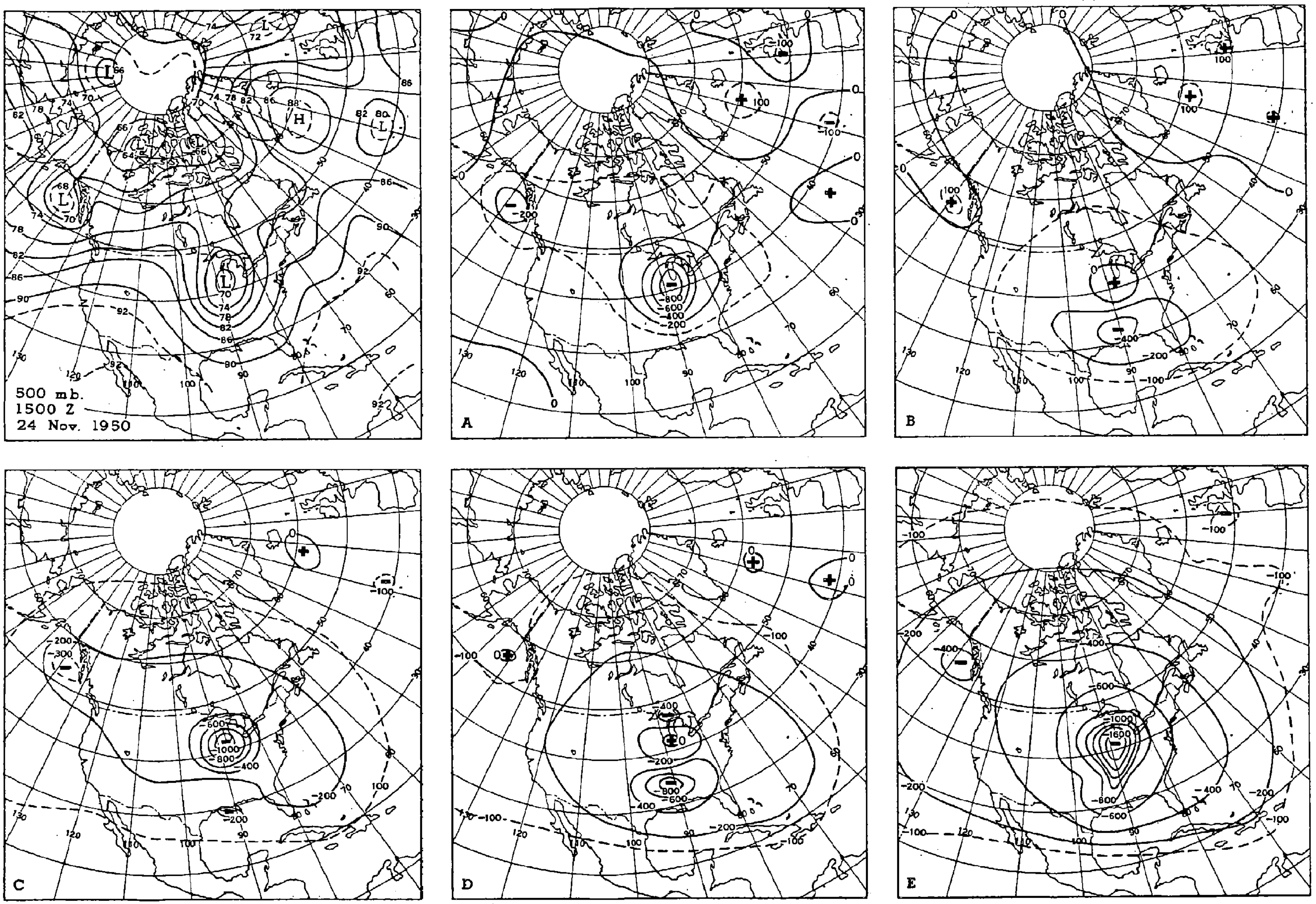

Figure 1 The charts A,B,C,D, and E show the successive differences $\psi-\psi^{(n)}$, where ψ is the solution of the balance eq. and $\psi^{(n)}$ the solution of eq. (1b), page 12. A corresponds to n=1, B to n=2, etc. The sequence of "approximate solutions" diverges.