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A Method for Solving the Balance Equation.

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Statement of the problem.

Consider the balance equation in isobaric coordinates.

$$2\psi_{xy}^2 - 2\psi_{xx}\psi_{yy} - f(\psi_{xx} + \psi_{yy}) + \phi_{xx} + \phi_{yy} = 0 \qquad (1)$$

Here ψ is the stream function, f the Coriolis parameter, and ϕ the geopotential. (1) may be derived by taking the divergence of the horizontal equations of motion, omitting terms involving vertical motion, and assuming non-divergence by expressing the horizontal velocity in terms of the stream-function, ψ. The problem touched upon in this paper is the solution of (1), treated as a boundary-value problem, for ψ when ϕ is given.

A Condition for Existence of Solutions.

A vital question, which must be answered before an attempt is made at solving (1), is whether it is fundamentally hyperbolic in nature. If it is, it cannot be treated as a boundary-value problem. This question may be attacked in ways other than that presented here (e.g., see Forsyth, p. 498), but the following development is brief and to the point in question.

We will begin by assuming a particular solution, and investigate solutions in the neighborhood of the assumed solution. In particular let us look for the infinitesimal change, $d\psi$, in ψ which corresponds to an infinitesimal change, $d\phi$, in ϕ. The variations, then, must satisfy the following equation, which is obtained by merely differentiating (1), and gathering coefficients of the variations.

$$4\psi_{xy}\, d\psi_{xy} - (2\psi_{yy}+f)\, d\psi_{xx} - (2\psi_{xx}+f)\, d\psi_{yy} + d\phi_{xx} + d\phi_{yy} = 0 \quad (2)$$

Now, the characteristic, $\underline{Ch}$, of (2) is

$$Ch = 16\,\psi_{xy}^2 - 4(2\psi_{yy}+f)(2\psi_{xx}+f)$$

which, after expanding terms and re-arranging, becomes

$$Ch = 8\left[2\psi_{xy}^2 - 2\psi_{xx}\psi_{yy} - f(\psi_{xx}+\psi_{yy})\right] - 4f^2 \quad (3)$$

After a substitution from (1) into (3), we see that the characteristic for the equation (2) in the variations is precisely

$$Ch = -8\left(\phi_{xx} + \phi_{yy} + \tfrac{1}{2} f^2\right) \quad (4)$$

Thus, unless

$$\phi_{xx} + \phi_{yy} + \tfrac{1}{2} f^2 \geq 0 \quad (5)$$

the equation (2) in the variations is hyperbolic, and cannot be treated as a boundary-value problem. The significance of this is that solutions in the neighborhood of a given solution in which condition

(5) is not satisfied, cannot be found by arbitrarily assigning boundary values and integrating. This, however, is tantamount to saying that condition (5) is necessary for solutions of (1) to exist which satisfy arbitrary boundary values of ψ.

Thus, if condition (5) is not everywhere satisfied in the geopotential field, the data must be pre-prepared by imposing (5) artificially. This could most directly be done by computing $\left(\phi_{xx} + \phi_{yy} + \frac{1}{2} f^2\right)$ in the data field, increasing to zero those values which were negative, and fitting the field of ϕ to the changed field of $\left(\phi_{xx} + \phi_{yy} + \frac{1}{2} f^2\right)$ by solving a Poisson equation using relaxation methods.

Application of Relaxation Procedures to the Non-linear Equation.

A more convenient form of (1) for the purpose of this discussion is

$$\frac{1}{2}\left(\psi_{xx} + \psi_{yy} + f\right)^2 - \left(\phi_{xx} + \phi_{yy} + \frac{1}{2} f^2\right) - 2\psi_{xy}^2 - \frac{1}{2}\left(\psi_{xx} - \psi_{yy}\right)^2 = 0 \quad (6)$$

Equation (6) is fully equivalent to (1), being merely a re-arrangement. The advantage of (6) over (1) is that, when transformed into central finite differences, the central value of ψ appears only in the first term. Consider the finite-difference transformation of (6).

$$\left(\eta_0^{\nu}\right)^2 - \left(\lambda_0^{\nu}\right)^2 - Z_0 = R_0^{\nu} \quad (7)$$

where

$$\eta_0 = \frac{1}{4}\left(\psi_2 + \psi_4 + \psi_6 + \psi_8\right) - \psi_0 + \frac{1}{4} m^2 \Delta s^2 f$$

$$r_0^2 = \frac{1}{64}(\psi_1 - \psi_3 + \psi_5 - \psi_7)^2 + \frac{1}{16}(\psi_2 - \psi_4 + \psi_6 - \psi_8)^2$$

$$z_0 = m^4 \Delta s^4 \left(\phi_{xx} + \phi_{yy} + \frac{1}{2} f^2 \right) \cdot \frac{1}{8}$$

and m is the map-scale factor and Δs is the mesh length. The subscripts refer to values at points in the mesh of the accompanying figure.

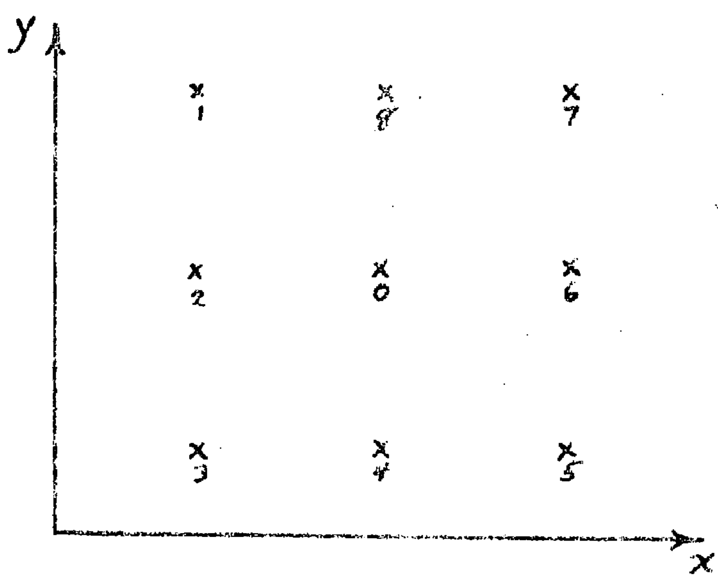

Figure. The 9-point mesh.

The superscript $(\ \ )^{\nu}$ refers to number of relaxation sweeps performed, and is attached for imminent convenience. R is the residual, and vanishes in the solution. Suppose that we attacked the numerical solution of (7) by a straightforward extension of the Liebmann process to the non-linear equation. Let us take the superscript $(\ \ )^{\nu}$, when attached to η, r, and R, to indicate precisely the values of those quantities during the ν th sweep after passing over the immediately preceding point, but before any computation is performed at the point in question. We will similarly define the superscript $(\ )^{\nu+1}$, when attached to η, r, and R, to indicate precisely their values during the νth sweep after passing over the point in question, but before any computation is done on the immediately succeeding point. Then,

$$\begin{aligned} \eta_0^{\nu+1} &= \eta_0^{\nu} - (\psi_0^{\nu+1} - \psi_0^{\nu}) \\ r_0^{\nu+1} &= r_0^{\nu} \end{aligned} \tag{8}$$

Substituting these values into (7), we have

$$(\psi_0^{\nu+1} - \psi_0^{\nu})^2 - 2\eta_0^{\nu}(\psi_0^{\nu+1} - \psi_0^{\nu}) + R_0^{\nu} - R_0^{\nu+1} = 0 \tag{9}$$

(9) is the basic relaxation equation to be solved for $(\psi_0^{\nu+1} - \psi_0^{\nu})$ at each point successively during each sweep. Its being quadratic introduces problems not otherwise encountered.

First, under what conditions are the roots of (9) complex? This is easily answered by looking at its characteristic, $\underline{Ch}$.

$$Ch = 4(\eta_0^{\nu})^2 - 4(R_0^{\nu} - R_0^{\nu+1}) \tag{10}$$

The condition for reality of the roots is

$$Ch \geq 0 \tag{11}$$

We may substitute (7) into (10), and find that the condition (11) becomes.

$$\frac{1}{4} Ch = (r_o^\nu)^2 + z_o + R_o^{\nu+1} \geq 0$$

$(r_o^\nu)^2$ is positive definite, and in view of condition (5) imposed on the data, z_o is not negative. A sufficient condition, therefore, for reality of the roots of (10) is

$$R_o^{\nu+1} \geq 0$$

In view of the fact that during the relaxation process R_o^ν may be either positive or negative, over-relaxation is not indicated. Thus, we will set

$$R_o^{\nu+1} = 0 \qquad (12)$$

It is of interest to note that with no over-relaxation, the condition (5) for the non-hyperbolic character of the differential equation (1) is a sufficient condition for non-complex roots of the basic relaxation equation (9).

Second, because a square-root routine is not built into our machines, we must devise a method for solving (9). Two fundamental classes of methods are available, table-look-up methods and iterative methods. We will here propose the iterative method indicated by

$$\chi_o^\sigma = \frac{R_o^\sigma}{2 \, \gamma_o^\sigma} \qquad (13)$$

$$\gamma_o^{\sigma+1} = \gamma_o^\sigma + \chi_o^\sigma \qquad (14)$$

$$\eta_0^{\sigma} = \eta_0^{\nu} - (\psi_0^{\sigma} - \psi_0^{\nu}) \tag{15}$$

$$R_0^{\sigma} = (\eta_0^{\sigma})^2 - (\lambda_0^{\nu})^2 - z_0 \tag{16}$$

$$\psi_0^{\nu+1} - \psi_0^{\nu} = \sum_{\sigma=1}^{\sigma=\infty} x_0^{\sigma} \tag{17}$$

It can be shown that (17) follows from (13), (14), (15), and (16), which are merely definitions. In fact, the above set of equations results from application of the Newton iteration to the radical in the roots of (9). To demonstrate the truth of (17), let us first form $x_0^{\sigma+1}$ as follows.

$$x_0^{\sigma+1} = \frac{R_0^{\sigma+1}}{2\,\eta_0^{\sigma+1}} \tag{18}$$

But, from (14) and (15),

$$\eta_0^{\sigma+1} = \eta_0^{\sigma} - x_0^{\sigma} \tag{19}$$

and from (16), (19), and (13),

$$R_0^{\sigma+1} = \left(x_0^{\sigma}\right)^2 \geqq 0 \tag{20}$$

Thus, (18) becomes

$$\frac{x_0^{\sigma+1}}{x_0^{\sigma}} = \frac{x_0^{\sigma}}{2\eta_0^{\sigma} - 2x_0^{\sigma}}$$

Or, according to (13):

$$\frac{x_0^{\sigma+1}}{x_0^{\sigma}} = \frac{1}{2} \cdot \frac{1}{\dfrac{2\left(\eta_0^{\sigma}\right)^2}{R_0^{\sigma}} - 1} \tag{21}$$

In view of (16), and (5),

$$-\infty \leq R_0^{\sigma} \leq 1$$

Therefore,

$$-\infty \leq \frac{R_0^{\sigma}}{\left(\eta_0^{\sigma}\right)^2} \leq 1 \tag{22}$$

(22) implies the following two possible ranges. Either

$$1 \leq \frac{\left(\eta_0^{\sigma}\right)^2}{R_0^{\sigma}} \leq +\infty \tag{23}$$

Or,

$$-\infty \leq \frac{(\eta_0^\sigma)^2}{R_0^\sigma} \leq 0 \tag{24}$$

Applying the ranges (23) and (24) to (21),

$$-\frac{1}{2} \leq \frac{\chi_0^{\sigma+1}}{\chi_0^\sigma} \leq +\frac{1}{2} \tag{25}$$

From (19), (13), and (16),

$$\eta_0^{\sigma+1} = \frac{(\eta_0^\sigma)^2 + (r_0^\nu)^2 + z_0}{2\,\eta_0^\sigma} \tag{26}$$

(26), (20), and (13) imply that $\chi_0^{\sigma=1}$ may be positive or negative, $\chi_0^{\sigma=2}$ may agree or not agree in sign with $\chi_0^{\sigma=1}$, but require all $\chi_0^{\sigma>2}$ to agree in sign with $\chi_0^{\sigma=2}$. Thus (25) leads to

$$\left| \sum_{\sigma=2}^{\sigma=\infty} \chi_0^\sigma \right| \leq \left|\chi_0^{\sigma=1}\right| \times \sum_{n=1}^{n=\infty} 2^{-n} = \left|\chi_0^{\sigma=1}\right| \tag{27}$$

(27) is a sufficient condition for the summation in (17) to be finite if $\chi_0^{\sigma=1}$ is finite. (25) implies that

$$\mathop{\mathrm{Limit}}_{\sigma\to\infty} \chi_0^\sigma = 0$$

which, in view of the definition (13), implies that

$$\operatorname*{Limit}_{\sigma \to \infty} R_0^{\sigma} = 0$$

which in turn means that $\psi_0^{\sigma=\infty} = \psi_0^{\nu+1}$, according to foregoing definitions. Thus, we have proven that the iterative scheme represented by (13) and (17) converges to one of the roots of (9), subject to the further condition that

$$\eta_0^{\sigma} \neq 0$$

for all σ. The last condition is realized if $\eta_0^{\sigma=1} \neq 0$, as shown by (26).

The third problem arising from the quadratic character of the basic relaxation equation (9) is that of deciding between two roots. (26) contains the solution to this third problem. It shows that unless

$$\eta_0^{\sigma=1} > 0 \tag{28}$$

the quadratic-solving process converges to the solution in which the absolute vorticity is negative, which may in all probability be rejected in view of the restriction (5) on geostrophic vorticity.

In the event that $\eta_0^{\sigma=1} < 0$, one could follow the process through to the improper root, for the proper root may easily be computed from the improper root by the following relation, where $(\psi_0^{*} - \psi_0^{\nu})$ refers to the improper root of (9).

$$\psi_0^{\nu+1} - \psi_0^{\nu} = 2\,\eta_0^{\nu} - (\psi_0^{*} - \psi_0^{\nu}) \tag{29}$$

(29) may easily be derived, after noting that according to (7), the improper root must imply vorticity equal and opposite in sign to the vorticity implied by the proper root.

A fourth problem which arises in our iteration scheme is scaling when η_0 is small in (13). We may conclude from (25) that we need only be concerned with scaling $\chi_0^{\sigma=1}$. Now,

$$\chi_0^{\sigma=1} = \frac{R_0^{\nu}}{2\,(\eta_0^{\nu})^2}\,\eta_0^{\nu} \tag{30}$$

according to (13). (10), (11), and (12) show that

$$(\eta_0^{\nu})^2 \geq R_0^{\nu}$$

Thus, if

$$R_0^{\nu} \geq 0 \tag{31}$$

then

$$|\chi_0^{\sigma=1}| \leq \frac{1}{2}\,|\eta_0^{\nu}| \tag{32}$$

Condition (31) can become satisfied by successively subtracting increments from ψ_0^{ν}, or by computing the central value of ψ_0 which corresponds to $\eta_0 = 1$, and starting the iteration from there.

Conclusion

The discussion in the preceding section is not concerned with proofs of convergence of relaxation methods applied to the particular equation at hand. Rather, it indicates a way of handling problems which arise from the quadratic nature of the equation.

As for convergence of the relaxation process, a similar procedure has been successfully applied to a similar equation (Shuman, 1955).

References

Forsyth, A. R., 1933: A Treatise on Differential Equations, 6th Ed. Macmillan and Co., Ltd., London.

Shuman, F. G., 1955: Notes on Work Done at the Institute for Advanced Study, 1953-54. (Unpublished).