BAROTROPIC DIVERGENCE AND VERY LONG ATMOSPHERIC WAVES

GEORGE P. CRESSMAN, U. S. WEATHER BUREAU
Joint Numerical Weather Prediction Unit, Suitland, Md.

[Manuscript Received June 23, 1958]

ABSTRACT

The problem of spurious retrogression of very long waves by the non-divergent barotropic forecasts is shown to be the same problem discussed extensively by Rossby, Yeh, and Bolin. This difficulty is due to the failure of the non-divergent model to allow properly for the mutual adjustment of wind and pressure fields. The equation of continuity for a homogeneous incompressible fluid with an upper free surface, proposed as a remedy by Rossby nearly 20 years ago, removes much of the difficulty. Further improvements are obtained by inclusion of a tropopause in the manner adopted by Bolin. The results of a series of 10 test forecasts are shown in verification of the function of the divergence in a barotropic model.

1. INTRODUCTION

The inability of a non-divergent barotropic forecast model to describe properly the behavior of the longest atmospheric waves has resulted in large systematic height errors of the prognostic charts issued by the Joint Numerical Weather Prediction Unit. This difficulty was correctly diagnosed by Wolff [7], who also devised a method for automatic empirical correction to the forecasts. This consisted of an enforced stabilization of hemispheric wave numbers one, two, and three, such that atmospheric waves of this size were not allowed to change either position or intensity during the forecast. The remarkable success of Wolff's method was due to the fact that these very-large-scale disturbances are in fact quasi-stationary.

However, it has been our ultimate objective to include in the forecast models proper descriptions of the physical processes which actually occur in the atmosphere, wherever possible. In looking for the appropriate physical mechanism responsible for the very long waves, one may note that their positions are relatively invariant, suggesting that geographically related factors are responsible for their formation and position. Terrain-induced vertical motion and differential surface heating appear to be likely causes.

The question of the behavior of these waves, once formed, can be considered apart from the question of their formation. Abundant discussion of this problem appears in the literature. It now appears that we have paid too little attention to the work of Rossby which bears on this question.

2. THEORETICAL BACKGROUND

In his classic paper of 1939, Rossby [4] pointed out that application of the vorticity equation to a non-divergent atmosphere did not properly take into account the adjustment of pressure and wind fields. Using a homogeneous incompressible atmosphere of thickness D, he used the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{D}\frac{dD}{dt}=0, \tag{1}$$

where u and v are the speeds in the x and y directions. This led to the introduction of a divergence into the vorticity equation, such that

$$\frac{d\eta}{dt}-\frac{\eta}{D}\frac{\partial D}{\partial t}=0, \tag{2}$$

where η is the absolute vorticity about a vertical axis, and where the geostrophic approximation is used to set $u\frac{\partial D}{\partial x}+v\frac{\partial D}{\partial y}$ equal to zero. He then arrived at a frequency equation giving the phase velocity, C, of the divergent waves, which can be written in the form

$$C=\frac{C_{ND}}{1+\frac{2.5}{N^2}} \tag{3}$$

where C_{ND} is the phase velocity of the non-divergent waves and N is the hemispheric wave number measured at about 45° N. The coefficient 2.5 applies to a homogeneous atmosphere. The above equation and the corresponding one for the group velocity of this type of wave have been thoroughly discussed by Rossby [5] and by Yeh [8]. In a footnote to his 1939 paper, Rossby [4] mentioned that an analysis of a two-layer model having an active lower layer and an inert upper layer indicated that the divergence introduced by these considerations might reach values as high as $\frac{4}{D}\frac{dD}{dt}$.

In his 1945 paper Rossby [5] again emphasized, very explicity, the necessity for allowing for the mutual adjustment of wind and pressure by means of the above type of divergence term. Phillips' [3] two-layer model can be specialized to describe Rossby's two-layer atmosphere. In the case where the motion in the lower fluid is parallel to the contours of the interface ($\mathbf{V}\cdot\nabla h=0$), and the horizontal motion in the upper fluid is negligible, Phillips' equations reduce to

$$\nabla^2\frac{\partial z}{\partial t}+\frac{f}{g}\mathbf{V}\cdot\nabla\eta-\frac{f\eta}{gh}\frac{\partial h}{\partial t}=0 \tag{4}$$

and

$$(1-\epsilon)\nabla^2\frac{\partial h}{\partial t}+\frac{f}{g}\mathbf{V}\cdot\nabla\eta-\frac{f\eta}{gh}\frac{\partial h}{\partial t}=0 \tag{5}$$

In the above equations, h is the height of the interface, z can be thought of as the height of an isobaric surface in the lower fluid, $\mathbf{V}$ and η are the velocity and absolute vorticity of the lower fluid respectively, and f and g are the Coriolis parameter and acceleration of gravity. The factor ϵ is given by the ratio of the density of the upper fluid to that of the lower fluid. From equations (4) and (5) and from the lateral boundary conditions where $\partial z/\partial t=\partial h/\partial t=0$ it is evident that

$$\nabla^2\frac{\partial z}{\partial t}+\frac{f}{g}\mathbf{V}\cdot\nabla\eta-\frac{f\eta}{(1-\epsilon)gh}\frac{\partial z}{\partial t}=0 \tag{6}$$

Equation (6) is consistent with a continuity equation for the lower fluid, equivalent to

$$\nabla\cdot\mathbf{V}+\frac{1}{K(1-\epsilon)z}\frac{\partial z}{\partial t}=0 \tag{7}$$

where $K=h/z$. In using equation (6) as a barotropic forecasting equation, it would be sufficient to use a mean value of h.

Bolin [1] made some computations with a barotropic model having a tropopause. This model was the same as those described above. He used a factor of $\frac{1}{8}$ for $K(1-\epsilon)$, without giving a description of how he arrived at such a number. His results were widely misinterpreted at the time (and in particular at the JNWP Unit), since this feature was presented as a cure for excessive anticyclogenesis. That problem has been shown by Shuman [6] to be related to the geostrophic approximation. Nevertheless, Bolin's results, while showing no improvement over the non-divergent forecasts in some respects, clearly resulted in greatly diminished height errors. It is evident now that Bolin was really controlling the very long waves, and that the reduced spread of excessive anticyclogenesis was merely a consequence of the smaller values of the group velocity obtained from his prognostic equation. At the JNWP Unit, we were aware of the problem of retrograde, very long waves only after we began to compute on a hemispheric grid and after Wolff's analysis was completed. This was partly due to the fact that the centers of height error from this source were very near the boundaries of our previously used grid and were confused with boundary errors. A suggestion by Dr. Norman Phillips led to the re-examination of the barotropic divergence.

3. THE PROGNOSTIC EQUATION

Equation (6) was reformulated in a finite-difference form as follows:

$$\left(\overline{\nabla}^2-\frac{\mu d^2\eta}{m^2\overline{\psi}}\right)\frac{\partial\psi}{\partial t}+\frac{1}{4}\mathbf{J}(\psi,\eta)-\frac{a\eta}{4}\mathbf{J}\left(\psi,\frac{p_g}{p_0}\right)=0 \tag{8}$$

In this equation, ψ is the stream function obtained from the 500-mb. height through the balance equation [6] $\overline{\nabla}^2$ and $\mathbf{J}$ are the finite-difference Laplacian and Jacobian operators, d/m is the distance on the earth between grid points, and μ is the same as $1/K\ (1-\epsilon)$. The evaluation of ϵ seems to be a doubtful matter, since the atmosphere is not composed of two homogeneous incompressible fluids. It was finally decided to use the atmosphere as an integrating machine in order to obtain a realistic value of μ. Since we interpret h as the height of the tropopause, z as the height of the 500-mb. surface and consider that

$$\frac{\mu}{z}\frac{\partial z}{\partial t}=\frac{1}{h}\frac{\partial h}{\partial t} \tag{9}$$

we can observe the relative variation of h and z in the atmosphere. An inspection of tropopause and associated 500-mb. charts reveals μ to have an approximate value of 4.*

The second Jacobian in equation (8) is the standard large-scale mountain effect, where a, the ratio of surface to 500-mb. wind speeds, is set at 0.2, p_g is the pressure at the surface of the ground, and p_0 is 1000 mb.

4. RESULTS OF COMPUTATIONS

Equation (8) was programmed for the IBM 704 computer for the 1977-point hemispheric grid used by the JNWP Unit. Computation time averages about 35 seconds per time step.

A series of 48-hour forecasts was made for varying values of μ on one initial situation, that of 0000 GMT, February 15, 1958. This case was chosen because of the very large amplitude of wave number one.

The first forecast was made with a value of $\mu=1$, corresponding to a homogeneous atmosphere having a free upper surface. The results of this are presented in figure 1. It is evident from this figure that the upper free surface model varies greatly from the non-divergent model in its treatment of wave number one. The difference is in the direction required to reduce error. Figure 2 shows the corresponding difference map for $\mu=4$. This particular 48-hour period was characterized by several of the most severe cyclogeneses observed all winter, and

*If one follows the derivation of the equivalent barotropic model as given by Charney and Eliassen [2] but retains the term $\eta\frac{\partial\omega}{\partial p}$ at the lower boundary (ω being the vertical velocity in a pressure-time coordinate system), the resulting equation is the same as equation (8) with the important exception that the factor μ is less than 1, being the ratio of the surface to the mean wind speed. In the light of the results presented here, this formulation of the equivalent barotropic model appears to underestimate the divergence necessary to stabilize the longest atmospheric waves.

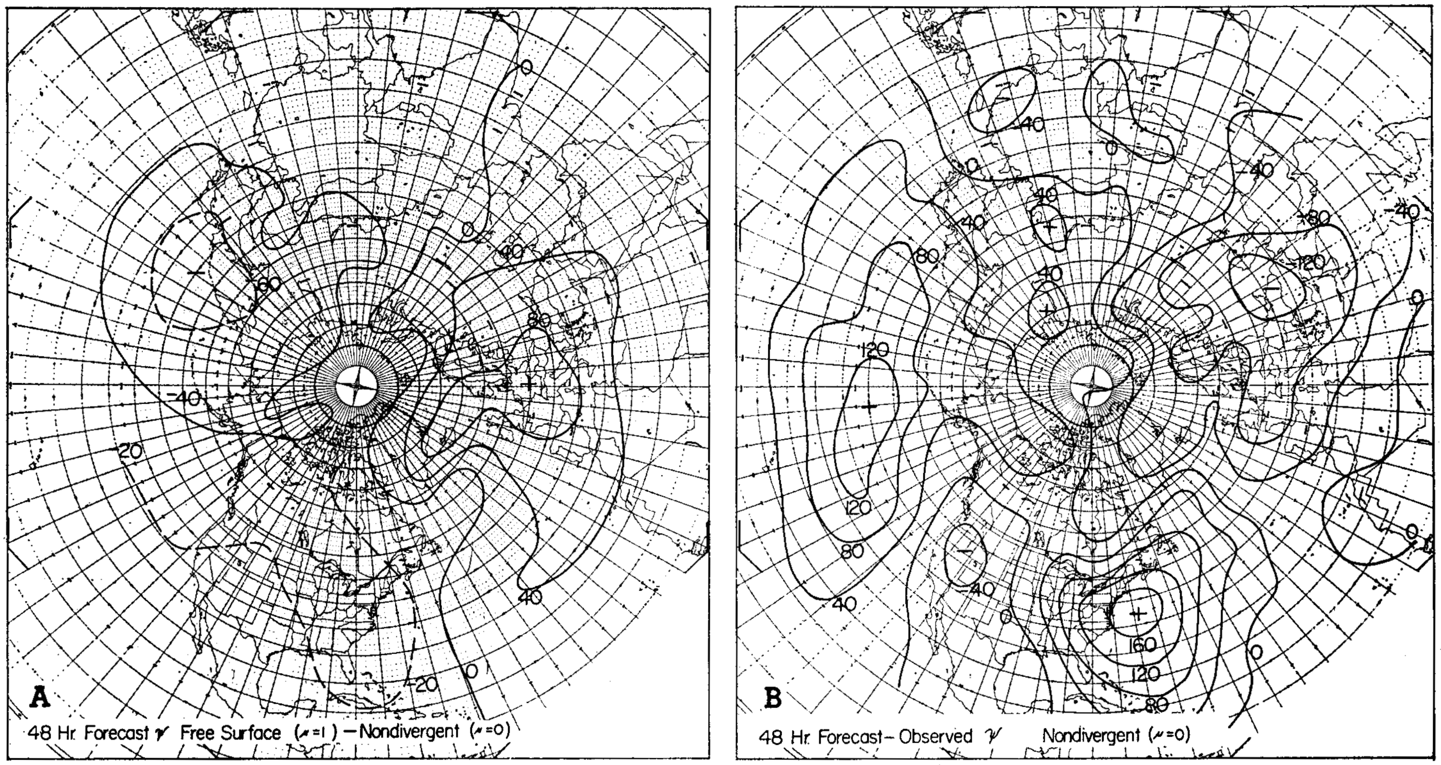

FIGURE 1.—(A) Stream function difference between 48-hour forecasts made by the barotropic model with $\mu=1$ and $\mu=0$. (B) Stream function error of the 48-hour non-divergent barotropic forecast.

it should not be expected that any barotropic forecast model could eliminate all large errors in this situation.

Fourier analyses were made by Cdr. Paul Wolff on the 48-hour stream function forecasts in order to discover the treatment of the various low-frequency components of the motion by the model. The results are presented in figure 3. The large changes of displacement with changes in μ for wave number one, compared with the minor changes for wave number five, are in good agreement with the frequency equation (3).

Figure 4 shows the errors of the 500-mb. height forecasts as a function of the coefficient μ. It is clear that the major improvement in the forecasts results from the free-surface approximation. It is also clear that little difference is made when μ is varied between values of 3 and 8.

A series of ten 48-hour forecasts was made in order to get a more representative idea of the level of accuracy of the forecasts. Comparisons were made with the JNWP Unit's operational barotropic forecast model (referred to hereafter as "O" model), which includes the very-long wave stabilization described by Wolff [7]. This was thought to be a more relevant comparison than a comparison with the unaltered non-divergent barotropic model ("N" model), since the relative levels of accuracy of the height forecasts of the O and N models were already presented by Wolff. Also, unpublished results by Wolff and Carstensen indicate a reduction of about 19 percent

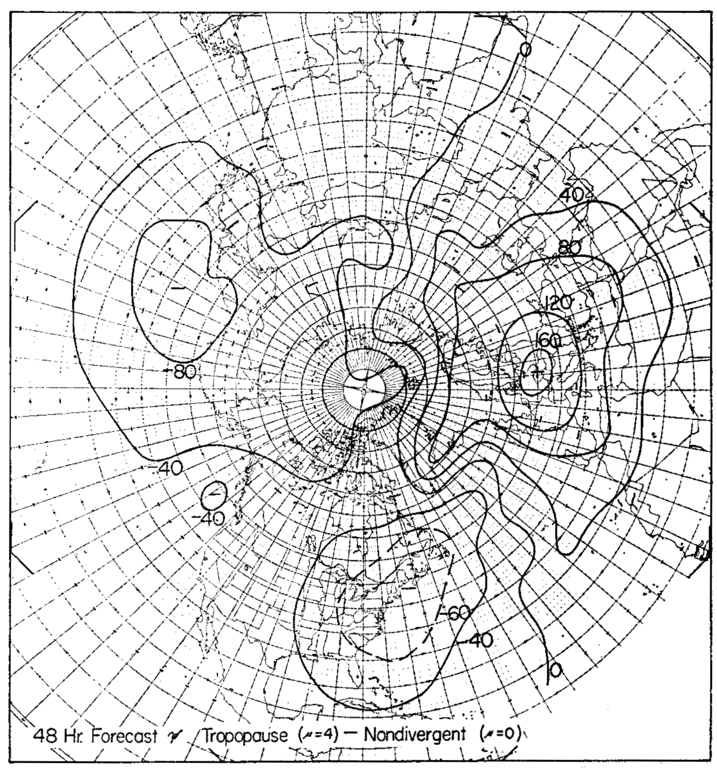

FIGURE 2.—Stream function difference between 48-hour forecasts made by the barotropic model with $\mu=4$ and $\mu=0$.

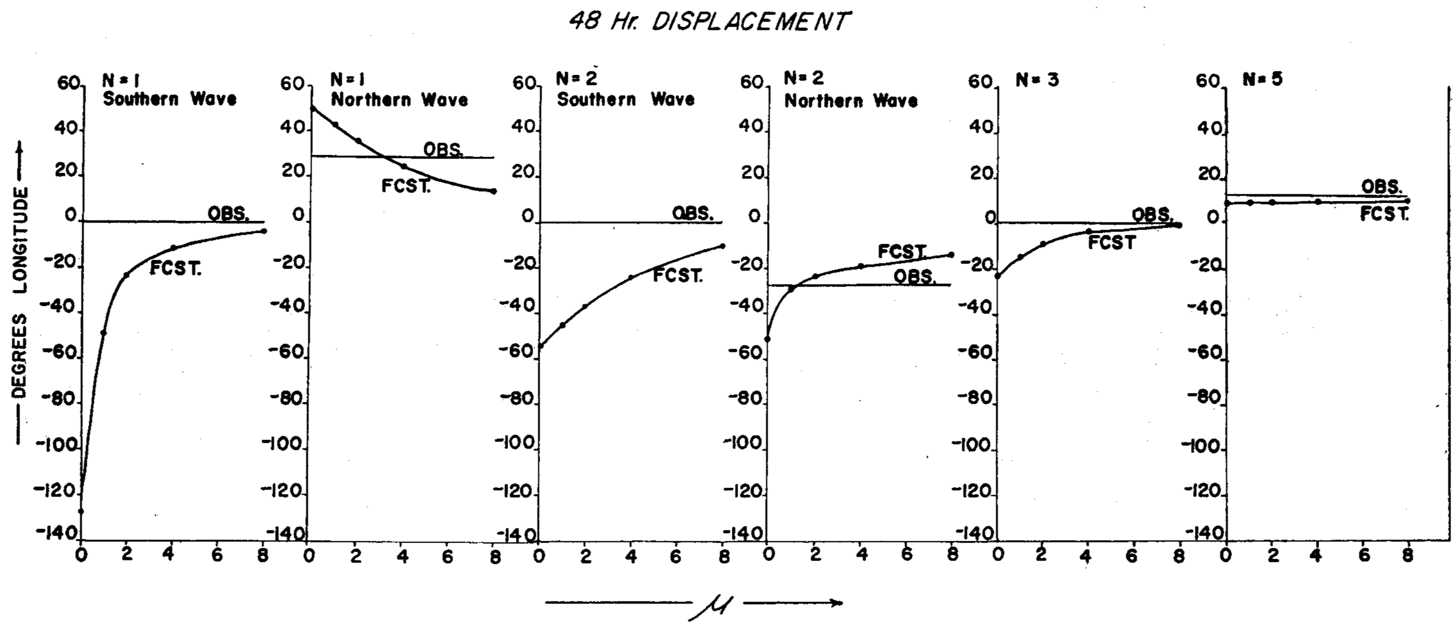

FIGURE 3.—Displacement of Fourier components of the stream function in 48-hour forecasts vs. the values of coefficient μ. Forecasts made from 0000 GMT, February 15, 1958.

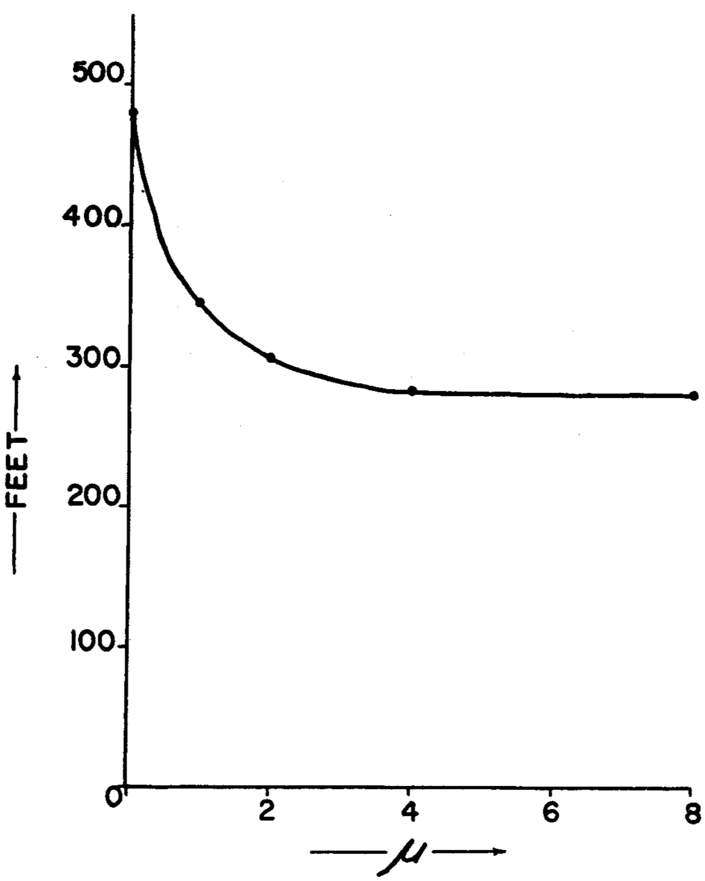

FIGURE 4.—Root-mean-square height error of the 48-hour forecast for different values of the coefficient μ. Forecast made from 0000 GMT, February 15, 1958.

in the 48-hour wind errors of a series of winter forecasts by the O model as compared with the N model. Table 1 presents the results of the ten comparative forecasts ("D" refers to the divergent barotropic model).

The average height errors for the ten forecasts show a slight but significant improvement over the O model by the D model. A correspondingly large reduction of error is obtained by the D model as compared with N model, considering the previously published results of Wolff. In comparison with the O model, the wind forecasts by the D model are not much improved. The improvement of the D model over the N model is, however, significant.

5. CONCLUSIONS

The success of forecasts made by a divergent barotropic model suggests strongly that large-scale divergences in

TABLE 1.—*Results of comparative forecasts made with the different models, D=divergent barotropic model, O=operational barotropic model, N=non-divergent barotropic model*

Initial map		RMS height error (ft.)			RMS wind error (kt.)		
Date	GMT	D	O	N	D	O	N
2/15/58	00	283	277	479	27.1	26.6	32.2
3/31/58	12	200	218	326	20.9	23.0	23.9
4/14/58	00	241	267	-------	23.0	25.3	-------
4/18/58	00	218	253	-------	22.6	22.6	-------
4/22/58	00	187	216	-------	18.9	20.5	-------
5/20/58	00	184	189	-------	19.6	19.8	-------
5/23/58	00	141	170	-------	15.4	16.6	-------
6/1/58	00	130	127	-------	16.0	15.2	-------
6/7/58	00	149	193	-------	16.9	19.0	-------
6/11/58	00	143	142	-------	16.4	16.6	-------
Average	-------	188	205	-------	19.7	20.5	-------

the atmosphere inhibit the motion of very long waves in the manner described by the model. Additional weight is given to the argument by the fact that Rossby clearly and explicitly emphasized the function of the barotropic divergence many years before numerical predictions could be made to test his conclusions.

The inclusion of a tropopause in the barotropic model, after Bolin, gives further improvement. The estimation of the proper values of the coefficient μ is a difficult matter, and is subject to some empiricism. It does not follow that a minimization of errors of the forecast heights with respect to the coefficient can lead to a correct estimation of its value, since it is possible that an overestimation might tend to compensate for correlated but physically unrelated errors. Investigations of mountain effects and surface heating with the aid of divergent forecast models may throw light on the formation of these very long waves.

It has frequently been stated that numerical forecasts cannot be considered to be in competition with other forecast systems with respect to height values (at 500 mb., for example). The above results, together with those previously published by Wolff, suggest that such statements, being based essentially on experience with the non-divergent forecast equation, and the consequently large height errors arising from misplacement of the very long waves, can no longer be considered valid. A re-evaluation of errors of height forecasts by numerical and other forecast methods appears to be desirable.

ACKNOWLEDGMENTS

The writer takes this opportunity to thank Dr. N. A. Phillips for reopening the question of barotropic divergence and Cdr. P. M. Wolff, U. S. N., for extensive assistance in verification work, particularly in connection with the Fourier analyses.

REFERENCES

1. B. Bolin, "An Improved Barotropic Model and Some Aspects of Using the Balance Equation for Three-Dimensional Flow," *Tellus*, vol. 8, No. 1, Feb. 1956, pp. 61–75.
2. J. G. Charney and A. Eliassen, "A Numerical Method for Predicting the Perturbations of the Middle Latitude Westerlies," *Tellus*, vol. 1, No. 2, May 1949, pp. 38–54.
3. N. Phillips, "A Simple Three-Dimensional Model for the Study of Large-Scale Extratropical Flow Patterns," *Journal of Meteorology*, vol. 8, No. 6, Dec. 1951, pp. 381–394.
4. C.-G. Rossby and Collaborators, "Relation Between Variations in the Intensity of the Zonal Circulation of the Atmosphere and the Displacements of the Semi-Permanent Centers of Action," *Journal of Marine Research*, vol. 2, No. 1, 1939, pp. 38–55.
5. C.-G. Rossby, "On the Propagation of Frequencies and Energy in Certain Types of Oceanic and Atmospheric Waves," *Journal of Meteorology*, vol. 2, No. 4, Dec. 1945, pp. 187–204.
6. F. G. Shuman, "Predictive Consequences of Certain Physical Inconsistencies in the Geostrophic Barotropic Model," *Monthly Weather Review*, vol. 85, No. 7, July 1957, pp. 229–234.
7. P. M. Wolff, "The Error in Numerical Forecasts Due to Retrogression of Ultra-Long Waves," *Monthly Weather Review*, vol. 86, No. 7, July 1958, pp. 245–250.
8. T.-C. Yeh, "On Energy Dispersion in the Atmosphere," *Journal of Meteorology*, vol. 6, No. 1, Feb. 1949, pp. 1–16.