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RECENT STUDIES OF COMPUTATIONAL STABILITY

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ABSTRACT

Investigations of the computational stability of finite difference formulations of the equations governing shallow water, non-linear gravitational oscillations are reported. The investigations consist of empirical analyses of the results of numerical integration of the quasi-two-dimensional and fully two-dimensional versions of the equations.

The results obtained in the quasi-two-dimensional experiments suggested the formulation of filters for use in approximating the non-linear terms. By filtering certain high frequency non-linear interactions, it is found possible to achieve relatively well-behaved long term integrations.

A number of efforts to extend this result to the fully two-dimensional equations are indicated. One procedure was found to be partially successful, but is finally rejected on the basis of the instability of its linear counterpart.

The paper concludes with a critique of possible extensions of the investigation.

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I. Introduction

During the past year, the authors have conducted a number of numerical experiments designed to provide insight into various aspects of the stability of numerical methods for the integration of hydrodynamic equations. To a considerable extent, these studies have been confined to the extension of previous work at the National Meteorological Center: Shuman (1962); Shuman and Stackpole (1968, 1969); Buck (1967); Robert, Shuman and Gerrity (1970); Gerrity and McPherson (1970). However, we have also followed up the work reported by Grammeltvedt (1969), but in a rather minimal fashion. It is the objective of this note to document our work but more significantly to indicate those avenues for additional study which appear attractive at the moment.

II. Quasi-Two Dimensional Channel

The system of equations,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0 \tag{1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = 0 \tag{1b}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0 \tag{1c}$$

may be regarded as appropriate for a single layer of homogenous, incompressible fluid constrained by a channel with walls parallel to the coordinate lines $x = 0$ and $x = P\Delta x$. The set (1) will hold provided that there is no variation in u, v or h with respect to y at the initial time, and that the walls are uniform in y. The variables are defined as

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt} \quad \text{and}$$

h the height of the surface of the fluid layer; and g is the gravitational acceleration.

The natural boundary condition for this problem is

$$u = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = P\Delta x \quad \text{for all } t$$

stating that no fluid may penetrate the channel wall.

The pair of equations (1a and 1c) stands independently of eq. (1b). These two equations are a quasi-linear, hyperbolic set whose solution may be sought by the method of characteristics. From the theory of the method of characteristics, it is known that only u or h may be specified on a boundary, not both of them. But since $u = 0$ on the boundary, it turns out that one may give the compatible condition

$$\frac{\partial h}{\partial x} = 0 \quad \text{at} \quad x = 0 \quad \text{and } x = P\Delta x$$

for all t without overspecification. It should be noted that the differential equations do not require this additional specification, but that the difference equations do.

The boundary conditions required for the solution of (1b) depend generally upon the coefficient u. In the present case, since u vanishes on the boundaries, one cannot expect any change in the value of v on the boundary. This is an ambiguous situation when difference equations are used to solve the differential equations.

There are two schemes for setting up a difference net covering the spatial region of integration. One may set gridpoints *"at the channel walls"* or one may set the wall *"between two grid points."* In the integrations reported here, we follow Shuman (1962) and place the wall between grid points.

It is clear that the two exterior points introduced in this case are needed only for convenience of computation. If the physical problem were posed without reference to walls, then the *"exterior"* points would be introduced only to permit periodicity to be imposed.

The difference grid used in this study is depicted in figure 1. All of the experiments were performed with the initial data specified by

$$u_j^n \equiv u\left(x = (j-1.5)\Delta x,\ t = n\Delta t\right)$$

$$u_j^o = \sum_{k=1}^{24} U_k \sin \frac{\pi k(j-1.5)}{24}$$

$$v_j^o = \sum_{k=0}^{23} V_k \cos \frac{\pi k(j-1.5)}{24}$$

$$h_j^o = \text{constant (7620 meters)}$$

$$g = 9.8 \text{ m sec}^{-2}$$

$$\Delta x = 381 \text{ km}$$

$$\Delta t = 600 \text{ secs.}$$

Usually, U_k = const; $k \neq 24$. Two values of the constant were run; 2.18 m $\sec^{-1}$ and 8.5 m $\sec^{-1}$. In certain experiments, U_{24} was included and given the same value as the other coefficients. Figure 2 displays the initial data for u and v in the case $V_k = U_k = 2.18$, $k = 1,23$; $V_o = 2.18$, with and without $U_{24} = 2.18$.

The finite difference notation employed is that introduced by Shuman (1962). The first experiment discussed is that using the so-called *"Advective"* form of the equations:

$$\bar{u}^t_t + u\bar{u}^x_x + g\bar{h}^x_x = 0 \tag{2a}$$

$$\bar{v}^t_t + u\bar{v}^x_x = 0 \tag{2b}$$

$$\bar{h}^t_t + u\bar{h}^x_x + h\bar{u}^x_x = 0 \tag{2c}$$

Shuman had shown (1962) that this form is weakly unstable for the system comprised of eqs. (2a and 2c). The $2\Delta x$ component ($k = 24$) was not included in the initial data in our run, but was in Shuman's experiment. Most of the experiments used a *"forward time step"* to get started. The gravity wave system (2a and 2c) behaved unstably in agreement with Shuman's result. The total energy (sum of available potential and kinetic energy) in the gravity wave grew by an order of magnitude in nine days (~1300 time steps). The kinetic energy in the v component grew by an order of magnitude in 37 days. The root mean square vorticity grew by an order of magnitude in 67 days. The computation was halted at 69 days when the variable h became negative, but the computational <u>linear</u> stability criterion must have been violated much earlier.

The next experiment used the following *"decoupled"* system for the gravity wave system of equations:

$$\begin{aligned} \bar{u}^t_t + \bar{u}^{2x}\,\bar{u}^x_x + g\bar{h}^x_x &= 0 \\ \bar{h}^t_t + \bar{u}^{2x}\,\bar{h}^x_x + \bar{h}^{2x}\,\bar{u}^x_x &= 0 \end{aligned} \tag{3}$$

while retaining the form given in (2b) for the v-equation.

The term *"decoupled"* arises from the independence of the solution of (3) on two lattices imbedded in the grid array shown in figure 1 [cf. Gerrity (1969) and Gerrity and McPherson (1970)]. However, one may show the relationship between the difference scheme in (3) and that called *"semi-momentum"* in Shuman (1962) by noting the identities:

$$\overline{\bar{u}^x u_x}^x \equiv \frac{1}{2}\,\overline{u^2}^x_x \equiv \bar{u}^{2x}\,\bar{u}^x_x \tag{4}$$

and

$$\overline{\bar{u}^x h_x}^x + \overline{\bar{h}^x u_x}^x \equiv \overline{(hu)_x}^x \equiv \bar{u}^{2x}\,\bar{h}^x_x + \bar{h}^{2x}\,\bar{u}^x_x \tag{5}$$

Our integration of the gravity wave set ought therefore to be well-behaved in view of the results of Shuman (1962) and Shuman and Stackpole (1968).

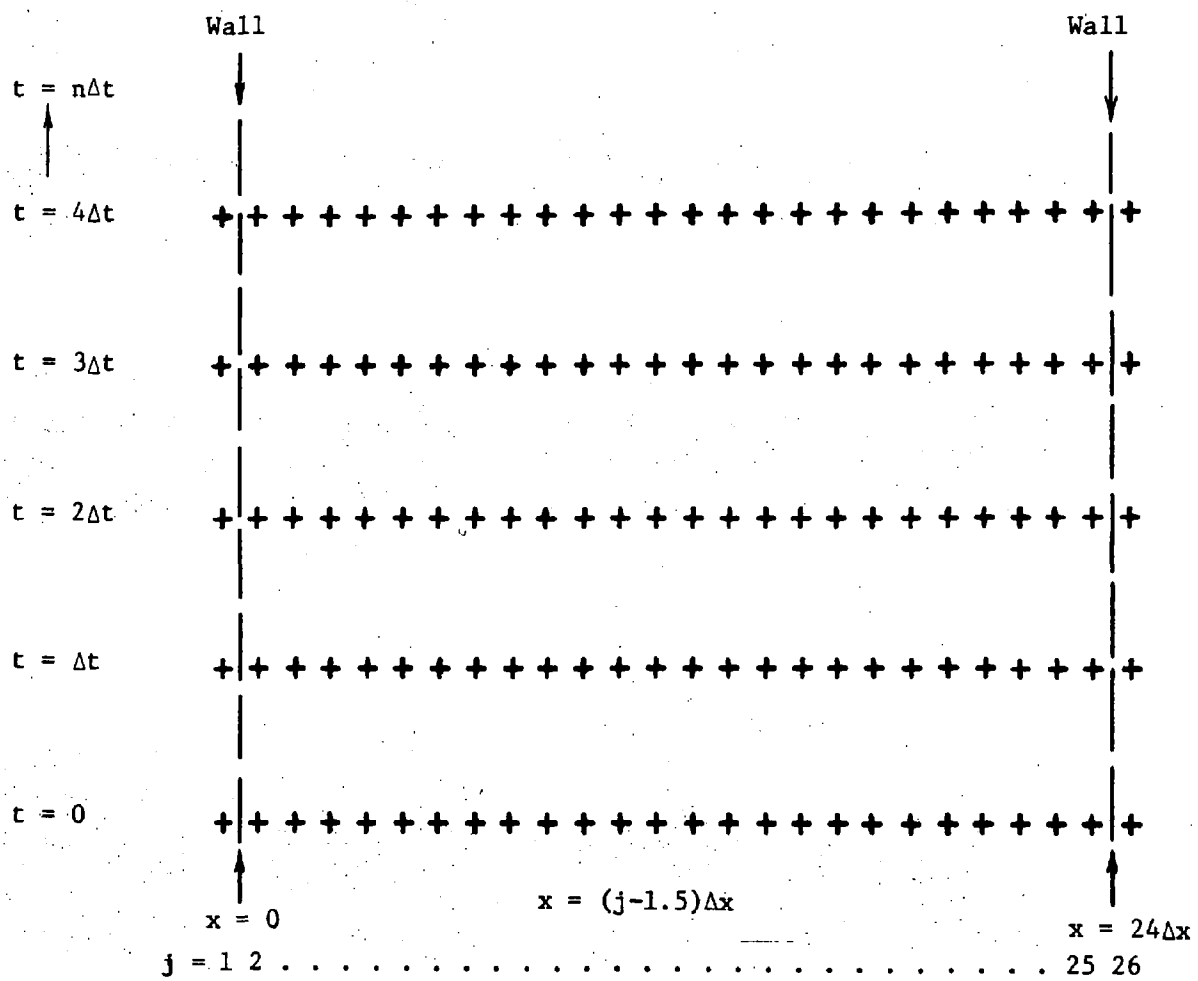

Figure 1. Schematic diagram showing region of integration in quasi-two-dimensional experiments.

The first experiment was performed without k = 24 ($2\Delta x$ wave) in the u-field. Since the difference equations are not capable of developing a $2\Delta x$ wave in u (with these boundary conditions), no energy is ever present in that wave number. The result of integrating the system for 350 days (50,000 steps) was well-behaved. It should be emphasized that a time oscillation did appear in the (h-u) solution, but it was not large. The v solution showed very small change from its initial distribution. Both energy and vorticity were well-behaved.

Using the identical system of difference equations but this time inserting $U_{24} = 2.18$ in the initial u-field, we repeated the experiment. Within five days, the kinetic energy of the v-component had grown by an order of magnitude and the root mean square vorticity grew by an order of magnitude in seven days. The gravity wave system was once more well-behaved.

Since the advective form of the v-equation had been shown to be very sensitive to the presence of a $2\Delta x$ component in the u-field, we considered several modifications of the v-equation. The gravity wave set was left in the decoupled form, eqs. 3).

We used the following *"decoupled form"* for the v-equation:

$$\bar{v}^{t}_{t} + \bar{u}^{2x}\,\bar{v}^{x}_{x} = 0 \tag{6}$$

and repeated the previous experiments.

When the "$2\Delta x$" component was excluded from the u-field, the integration was identical to that obtained with the advective form of the v-equation; that is, the system was well-behaved. But when the $2\Delta x$ component was added to the initial u-field, the decoupled v-equation became unstable. The kinetic energy of the v-wind component grew by an order of magnitude within nine days as did the root mean square vorticity in twelve days. This was slightly more stable than the result obtained with the advective form.

The similarity in the behavior of the *"advective"* and *"decoupled"* forms of the v-equation may be elucidated by noting the following relationship:

$$\bar{u}^{2x} \equiv 2\bar{u}^{xx} - u \tag{7}$$

and

$$\bar{u}^{xx} - u \equiv \frac{1}{4}(\Delta x)^2\, u_{xx}. \tag{8}$$

So,

$$\bar{u}^{2x}\,\bar{v}^{x}_{x} = u\,\bar{v}^{x}_{x} + \frac{1}{2}(\Delta x)^2\, u_{xx}\,\bar{v}^{x}_{x} \tag{9}$$

If we consider only the $2\Delta x$ component in u, $uj = (-1)^j$, where $U_{24} = 1$, and

$$\bar{u}^{2x}\,\bar{v}^{x}_{x} = [(-1)^j - 2(-1)^j]\,\bar{v}^{x}_{x} = -(-1)^j\,\bar{v}^{x}_{x} \tag{10}$$

It is, therefore, not surprising that the *"decoupled form"* behaved so similarly to the advective scheme[1].

We then tried a form of the v-equation which deliberately sets out to suppress the $2\Delta x$ component in u as it enters the v-equation. The *"semi-momentum"* form of the v-equation,

$$\bar{v}^t_t + \overline{\bar{u}^x v_x}^x = 0 \qquad (11)$$

was used. It must be noted that in contrast to the gravity wave set [cf. (4) and (5)], the semi-momentum form of the v-equation is *"coupled"* insofar as the spatial network of gridpoints is concerned.

The experiment including a $2\Delta x$ wave in the u-field was repeated using eq. (11) for v. The result was a very stable computation out to 350 days. We recall, however, that the temporal variation in u, noted earlier, was also a potential source of instability for the v-equation according to the analysis of Robert, Shuman and Gerrity.

We therefore repeated the experiment, this time excluding the $2\Delta x$ component (which cannot influence the solution for v when (11) is used) in u, but increasing the amplitude of the winds to 8.5 m sec^{-1} (i.e., we set $U_k = 8.5 = V_k$). As anticipated, the v equation became unstable once more. The kinetic energy in v grew by an order of magnitude in 85 days. The root mean square vorticity grew by an order of magnitude in 120 days. The gravity wave set remained stable through 291 days, at which point the time limit for the run was exceeded.

Now the next form used for the v-equation followed a proposal by Shuman for filtering the $2\Delta t$ oscillation associated with the leapfrog integration scheme. This filter was introduced only into the v-equation; the gravity wave set was unmodified,

$$u^* \equiv (u^n + u^{n-1})/2$$

$$\bar{v}^t_t + \overline{\overline{u^*}^x v_x}^x = 0 \qquad (12)$$

Again, the amplitudes were set to 8.5 m sec^{-1}.

---

[1]We subsequently reanalyzed this question and noted that the boundary conditions used with the *"decoupled"* difference equations were not themselves *"decoupled"*. The boundary conditions introduced in eq. (19) do preserve the decoupled lattice structure of the v-equation. Although the gravity-wave difference equations are ambiguous in form, one might also consider the use of *"decoupled"* boundary conditions for those equations. The significance of boundary conditions for stability is pointed out by Matsuno (1966) and Richtmyer and Morton (1967, cf. p. 141).

The result of this experiment was reported in Robert, Shuman and Gerrity (1970). Its stable behavior was considered to be a support for the relevance of their analysis.

Although the scheme comprised of equations (3) and (12) seemed to have overcome the instability problem, it was considered unfortunate that a decoupled version of the v-equation had not succeeded. For, if it had, one could then effect a sizable reduction in the number of spatial grid-points required for a computation with a given spatial truncation error. In effect, one could have disposed of one-half of the spatial points.

At this point, we realized that advantage could be taken of both the space and time lattice structure in order to effect the required filtering. So we introduced a new finite difference operator,

$$\tilde{u}_j^n \equiv \frac{1}{4}\left[u_{j+1}^n + 2u_j^{n-1} + u_{j-1}^n \right] = \frac{1}{2}\left[\overline{u_j^n}^{2x} + u_j^{n-1} \right] \qquad (13)$$

By use of the identies (7), (8) and (9), one may then show that (13) is equivalent to

$$\tilde{u} = u^* + \frac{(\Delta x)^2}{4} u_{xx} \qquad (14)$$

or

$$\tilde{u} = \bar{u}^{xx} - (u-u^*) \qquad (15)$$

Now $\bar{u}^{xx}$ is a spatially filtered version of u; it has no $2\Delta x$ component. And $(u-u^*)$ will contain only that portion of u which is <u>high</u> frequency in time. To make this point more clearly, let's write

$$u = (LS)(LT) + (LS)(HT) + (HS)(LT) + (HS)(HT) \qquad (16)$$

in which L means low frequency, H means high frequency, and S stands for space, T for time.

Then $\bar{u}^{xx} = (LS)(LT) + (LS)(HT)$

$u^* = (LS)(LT) + (HS)(LT)$

and $u-u^* = (LS)(HT) + (HS)(HT)$

and $$\bar{u}^{xx} - (u-u^*) = (LS)(LT) - (HS)(HT) \qquad (17)$$

Thus the u-field will contain only those components which will <u>not</u> yield instability according to Robert, Shuman and Gerrity.

In order to use the new form of the v-equation,

$$\bar{v}_t^t + \tilde{u}\,\bar{v}_x^x = 0 \qquad (18)$$

one must develop a suitable set of boundary conditions and an appropriate starting procedure. To overcome the starting problem, we set

$$v_j^1 = v_j^0.$$

The boundary values were obtained from the formulas (cf. fig. 1)

$$\begin{aligned} v_1^{n+1} &= 2v_2^n - v_1^{n-1} \\ v_{26}^{n+1} &= 2v_{25}^n - v_{26}^{n-1} \end{aligned} \tag{19}$$

For convenience in coding, we kept both space-time lattices; effectively computing two solutions for v which proceed completely independently (analogous to the gravity wave set). The gravity wave set, however, didn't involve the new filter - it was unchanged from eqs. (3).

It should be noted that the boundary conditions used for the gravity-wave set do not conform to the decoupled character of the difference equations. They are rather based on the use of two lattices. Furthermore, we note that the boundary condition used for the v-equation in both the advective and decoupled form experiments was that appropriate for the semi-momentum form of the v-equation. We shall note in subsequent remarks the significance of this fact.

An experiment using (18) and (19) was made using $U_k = V_k = 8.5$ m sec^{-1} including the $2\Delta x$ component in u. Although a large variation existed in the statistics of the run between consecutive time steps, the v-energy and root mean square vorticity did not depart by an order of magnitude from their initial values until 211 days. This is regarded as a confirmation of the prediction of relative stability.

The decoupled v-equation (6) was recently run with the boundary conditions (19). The initial data had an amplitude 8.5 m sec^{-1} and included the $2\Delta x$ component in the u field. The result was unstable within four days. This may be compared with the semi-momentum v-equation which became unstable for similar initial data only after 85 days.

Another experiment designed to elucidate the significance of boundary conditions consisted of rerunning equation (18) but with the *"coupled"* semi-momentum form of boundary condition on v. The result was unstable by 56 days as compared with the 211 day onset time when the *"decoupled"* boundary conditions (19) were used.

III. Two-Dimensional Experiments

In his (1962) paper, Shuman diagnosed the appropriate scheme for incorporating the *"Coriolis term"* in the system of equations for a single layer barotropic fluid with a free surface using the tangent plane approximation. He suggested the use of the system:

$$\bar{u}^t_t + \bar{u}^{xxyy}\,\bar{u}^{xyy}_x + \bar{v}^{xxyy}\,\bar{u}^{xxy}_y - f\bar{v}^{xxyy} + g\bar{h}^{xyy}_x = 0$$

$$\bar{v}^t_t + \bar{u}^{xxyy}\,\bar{v}^{xyy}_x + \bar{v}^{xxyy}\,\bar{v}^{xxy}_x + f\bar{u}^{xxyy} + g\bar{h}^{yxy}_y = 0 \qquad (20)$$

$$\bar{h}^t_t + (\bar{u}^{xyy}\,\bar{h}^{xyy})_x + (\bar{v}^{xxy}\,\bar{h}^{xxy})_y = 0$$

which was termed the *"filtered factor form."*

In a subsequent paper by Shuman and Vanderman (1965), another two-dimensional form of the equations was suggested,

$$\bar{u}^t_t + \overline{\bar{u}^{xy}\,\bar{u}^y_x + \bar{v}^{xy}\,\bar{u}^x_y}^{xy} - \overline{\bar{f}^{xy}\,\bar{v}^{xy}}^{xy} + g\bar{h}^{yxy}_x = 0$$

$$\bar{v}^t_t + \overline{\bar{u}^{xy}\,\bar{v}^y_x + \bar{v}^{xy}\,\bar{v}^x_y}^{xy} + \overline{\bar{f}^{xy}\,\bar{u}^{xy}}^{xy} + g\bar{h}^{xxy}_y = 0 \qquad (21)$$

$$\bar{h}^t_t + \overline{(\bar{u}^y\,\bar{h}^y)_x}^{xy} + \overline{(\bar{v}^x\,\bar{h}^x)_y}^{xy} = 0$$

This form (21) has been referred to as the *"semi-momentum form."*

The semi-momentum form is that subsequently adopted for use in the operational PE model (Shuman and Hovermale, 1968).

Very recently, yet another form of the two-dimensional equations was suggested (Shuman and Stackpole, 1969). Referred to as the *"invariant form,"* it is written,

$$\bar{u}^t_t - \overline{\zeta\bar{v}^{xy}}^{xy} - f\bar{v}^{xyxy} + \overline{[\tfrac{1}{2}(u^2 + v^2) + gh]_x}^{yxy} = 0$$

$$\bar{v}^t_t + \overline{\zeta\bar{u}^{xy}}^{xy} + f\bar{u}^{xxyy} + \overline{[\tfrac{1}{2}(u^2 + v^2) + gh]_y}^{xxy} = 0 \qquad (22)$$

$$\zeta \equiv \bar{v}^y_x - \bar{u}^x_y$$

The continuity equation is still written in the form given in (21).

The advantage of the use of the invariant form is that it provides a conservative form for the implied vorticity equation,

$$\bar{\zeta}^t_t = -\overline{[(\overline{\bar{u}^{xy}\,\zeta})^y_x + (\overline{\bar{v}^{xy}\,\zeta})^x_y]}^{xy} \qquad (23)$$

whereas the semi-momentum form admits a <u>truncation error</u> to appear in the implied vorticity equation:

$$\bar{\zeta}^t_t = -\overline{[\overline{\bar{u}^{xy}\,\zeta}]^y_x + [\overline{\bar{v}^{xy}\,\zeta}]^x_y}^{xy} + \frac{d^2}{8}\overline{\{\bar{u}^{x^2}_y - \bar{u}^{y^2}_x + \bar{v}^{x^2}_y - \bar{v}^{y^2}_x\}}^{xy}$$

in which d is the grid length, and f = 0.

The significance of this truncation error is shown in Shuman and Stackpole's paper. Using noisy data and setting $f = 0$, equations (21) were solved over a region corresponding to a large rectangular tank. Within seven days, the root mean square value of the relative vorticity was found to increase by 50% of its initial value.

When the invariant form (22) was employed, the result was markedly different. The root-mean-square vorticity in this case remained nearly constant through the seven-day period.

Although this result was positive, the extension to an integration in which $f \neq 0$ was found to be subject to a large growth in RMS divergence. A question was raised with regard to boundary conditions, but not fully resolved. A proposal was made to introduce a linear viscosity term to control both divergence and vorticity, but at different rates.

Subsequent to the successful results obtained in the stabilization of the *"quasi-two-dimensional"* equations, the present writers became involved in an effort to apply the stability hypotheses to the *"fully two-dimensional"* problem.

In the following, our experience will be recounted and put into some perspective. Our attention was focused largely upon an extension of the work reported in Shuman and Stackpole's (1969) paper, but we have also done some work following the results reported by Grammeltvedt (1969).

Review of Ideas

The basic idea advanced in the Robert, Shuman and Gerrity paper was that the interaction of high frequency components of undifferentiated coefficients with the differentiated variables could give rise to unstable growth of the numerical solutions. This suggests that the filtering of those high frequency components might yield relatively more stable computations. This certainly seemed to be validated by the test integration of the *"quasi-two-dimensional"* equations cited in Section II.

Even more interesting was the possibility of designing stable *"decoupled"* difference systems. The relative success with such a system, reported in Section II, offered encouragement to the search for a true two-dimensional analogue.

There is a tendency to confuse these two objectives - stability and decoupling. Our quasi-two-dimensional experiments clearly indicate that there is no unique relationship between them.

In Shuman and Vanderman (1965), the generalization of the one-dimensional *"semi-momentum"* difference scheme to a two-dimensional form is done in such a way as to discard the decoupled character of the one-dimensional form. They use the generalization:

$$\overline{\bar{u}^x u_x}^x \rightarrow \overline{\bar{u}^{xy} \bar{u}^y_x}^{xy}$$

$$\overline{\bar{u}^x h_x + \bar{h}^x u_x}^x \rightarrow \overline{\bar{u}^{xy} \bar{h}^y_x + \bar{h}^{xy} \bar{u}^y_x}^{xy} \tag{25}$$

But by reference to equations (4) and (5), one could equally well have used the generalization:

$$\bar{u}^{2x} \bar{u}^x_x \rightarrow \bar{u}^{2x2y} \bar{u}^{x2y}_x$$

$$\overline{hu}^x_x \rightarrow \overline{(hu)}^{x2y}_x \tag{26}$$

Scheme (26) would preserve the decoupled lattice structure available with the one-dimensional semi-momentum form, and would be compatible with the use of

$$g\bar{h}^x_x \rightarrow g\bar{h}^{x2y}_x \equiv g\bar{h}^{2y}_{2x} \tag{27}$$

for the pressure gradient.

We note, however, that our experience with the quasi-two-dimensional forms of the v-equation:

$$\bar{v}^t_t = - \overline{\bar{u}^x v_x}^x \tag{28a}$$

and

$$\bar{v}^t_t = - \bar{u}^{2x} \bar{v}^x_x \tag{28b}$$

indicated that the *"semi-momentum"* form (28a) is relatively more stable than the *"decoupled"* form (28b). On the other hand, the *"filtered-decoupled"* form of the v-equation (cf. eq. (18),

$$\bar{v}^t_t = - \tilde{u} \bar{v}^x_x \tag{28c}$$

behaved more stably than the form (28a).

Filters

The analysis of Robert, Shuman and Gerrity (1970) has not yet been extended to two dimensions. However, for purposes of designing a filter capable of suppressing high frequency components in a field, we may note that the following *"folded representation"* may be postulated to exist for an arbitrary field:

$$u \sim U_0 + U_1 e^{i\pi j} + U_2 e^{i\pi k} + U_3 e^{i\pi n} + U_4 e^{i\pi(j+k)} + U_5 e^{i\pi(j+n)} + U_6 e^{i\pi(k+n)} + U_7 e^{i\pi(j+k+n)} \tag{29}$$

j is the index of the point in the x-direction

k is the index of the point in the y-direction

n is the index of the point in the t-direction.

So if we define:

$$A = U_0 + U_1 + U_2 + U_3 + U_4 + U_5 + U_6 + U_7$$

$$B = U_0 - U_1 + U_2 + U_3 - U_4 + U_5 - U_6 - U_7$$

$$C = U_0 + U_1 - U_2 + U_3 - U_4 - U_5 + U_6 - U_7$$

$$D = U_0 - U_1 - U_2 + U_3 + U_4 - U_5 - U_6 + U_7$$

$$E = U_0 + U_1 + U_2 - U_3 + U_4 - U_5 - U_6 - U_7$$

$$F = U_0 - U_1 + U_2 - U_3 - U_4 - U_5 + U_6 + U_7$$

$$G = U_0 + U_1 - U_2 - U_3 - U_4 + U_5 - U_6 + U_7$$

$$H = U_0 - U_1 - U_2 - U_3 + U_4 + U_5 + U_6 - U_7$$

we will have the distribution in space at time $n\Delta t$, <u>n even</u>,

k ↑				
3	C	D	C	D
2	A	B	A	B
1	C	D	C	D
0	A	B	A	B
	0	1	2	3

j →

and at the next time level, n odd,

	3	G	H	G	H
	2	E	F	E	F
k ↑	1	G	H	G	H
	0	E	F	E	F
		0	1	2	3

j →

Now consider the *"filter"* $\overline{u^{n\,*}_{jk}}^{xxyy}$,

where

$$u^{n\,*}_{jk} \equiv .5[u^{n}_{jk} + u^{n-1}_{jk}] \tag{30}$$

When this filter is applied to a field, it returns a new set of values at the same points as the original field. This filter requires the existence of a coupled set of space-time lattices, and returns to each point

$$\overline{u^{n\,*}_{jk}}^{xxyy} = U_0$$

a completely filtered field.

If the differential equations are approximated by a decoupled set of finite difference equations, it is not obvious that the representation (29) is appropriate. But working on the hypothesis that it is appropriate, the following decoupled filter was designed,

$$\tilde{u}^{n}_{jk} = \frac{1}{16}[u^{n-1}_{j+1,k+1} + u^{n-1}_{j+1,k-1} + u^{n-1}_{j-1,k-1} + u^{n-1}_{j-1,k+1} + 4u^{n-1}_{jk}]$$

$$+ \frac{1}{8}[u^{n}_{j-1,k} + u^{n}_{j+1,k} + u^{n}_{j,k+1} + u^{n}_{j,k-1}] \tag{31}$$

It may be shown that this operator will return

$$\tilde{u} = U_0 + U_7$$

if it is applied to the representation (29).

Experimental Data

The majority of the experiments conducted have used as initial data the white-noise defined by

$$u_{ij} = U \sum_{r=1}^{24} \sum_{s=0}^{23} \sin\frac{\pi r(i-1.5)}{24} \cos\frac{\pi s(j-1.5)}{24}$$

$$v_{ij} = U \sum_{r=0}^{23} \sum_{s=1}^{24} \sin\frac{\pi s(j-1.5)}{24} \cos\frac{\pi r(i-1.5)}{24}$$

$$h_{ij} = H + \sum_{r=0}^{23} \sum_{s=0}^{23} H_{r,s} \cos\frac{\pi r(i-1.5)}{24} \cos\frac{\pi s(j-1.5)}{24}$$

$$x = (i-1.5)\Delta x \qquad y = (j-1.5)\Delta x$$

in which $U = 0.5$ m sec^{-1}, $H = 7620$ m, and $H_{r,s} = 0$. Also, $\Delta x = 381$ km.

Since $f = 0$ in all these experiments, one may regard the data as extended outside the boundary (which lies between grid points) by either symmetry conditions or by assuming that rigid walls bound the fluid. The initial data satisfy the conditions

$$u = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 24\Delta x$$

$$v = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 24\Delta x$$

$$\frac{\partial h}{\partial x} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 24\Delta x$$

$$\frac{\partial h}{\partial y} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 24\Delta x$$

It will be noted that the initial data possess the symmetry relations:

$$u_{ij} = v_{ji} \tag{32a}$$

$$h_{ij} = h_{ji} \tag{32b}$$

Now the differential equations are

$$\frac{\partial u}{\partial t} = - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - g\frac{\partial h}{\partial x} \tag{33}$$

$$\frac{\partial v}{\partial t} = - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - g\frac{\partial h}{\partial y} \tag{34}$$

$$\frac{\partial h}{\partial t} = - \frac{\partial}{\partial x}(uh) - \frac{\partial}{\partial y}(vh) \cdot \tag{35}$$

It may be seen that, in view of (32)

$$\frac{\partial u}{\partial t}(\xi,\eta) \equiv \frac{\partial v}{\partial t}(\eta,\xi) \tag{36a}$$

$$\frac{\partial h}{\partial t}(\xi,\eta) \equiv \frac{\partial h}{\partial t}(\eta,\xi) \tag{36b}$$

consequently the symmetry properties will be conserved.

Integral Invariants

The differential equations possess certain integral invariants over the region of integration which follow from the boundary condition.

Letting $\tilde{(\ )}$ denote an areal integral, one has

$$\tilde{h} = \text{const.} \qquad \text{mass} \tag{37}$$

$$\widetilde{h^{1-n}\,\zeta^{n}} = \text{const.} \qquad \text{potential vorticity moments} \tag{38}$$

$$\widetilde{\frac{h}{2}\,[gh + \vec{V}.\vec{V}]} = \text{const.} \qquad \text{energy} \tag{39}$$

$$\widetilde{\nabla.\vec{V}} = 0 \qquad \text{divergence} \tag{40}$$

$$\tilde{\zeta} = 0 \qquad \text{vorticity [n=1 in (38)]} \tag{41}$$

The symmetric character of the data yields

$$\zeta(\xi,\eta) = - \zeta(\eta,\xi) \tag{42}$$

from which it follows that

$$\widetilde{\zeta^{2n+1}} = 0 \tag{43}$$

The superscripts in the preceding equations are exponents not indices.

By use of the vorticity equation

$$\frac{\partial \zeta}{\partial t} = - \nabla . \zeta \vec{V} \tag{44}$$

one finds,

$$\frac{\partial \widetilde{\zeta^2}}{\partial t} = - \nabla . \widetilde{\zeta^2 \vec{V}} - \widetilde{\zeta^2 \nabla . \vec{V}} = - \widetilde{\zeta^2 \nabla . \vec{V}} \tag{45}$$

We don't find any integral constraint on $\widetilde{\zeta^2}$, nor for that matter on $\widetilde{h^2}$; thus <u>one has no a-priori reason to expect a well-behaved solution to the system of equations</u>. This is in <u>sharp</u> contrast to the case of non-divergent flow to which Fjortoft's (1953) analysis applies. In that case, there exist integral constraints on both kinetic energy and vorticity squared.

The numerical experiments must therefore be regarded as an investigation of the suitability of the finite difference scheme as a control on the destabilizing cascade of *"perturbation"* variance into the high wave numbers.

One seeks to accomplish this control without the introduction of filters which have linear counterparts. Thus one prefers to control the non-linear cascade of variance without modification of the linear oscillations. We note that one must regard the coefficients of derivatives in equations (33), (34) and (35) as constants in both space and time in the linear oscillation problem. If the coefficients have even a fixed spatial variation, the <u>existence</u> of a single valued solution cannot be asserted.

<u>Discussion of Experimental Results</u>

We had available as a benchmark, an integration (cited in Shuman and Stackpole (1969) based on the *"semi-momentum-invariant"* set of equations:

$$\begin{aligned}
\bar{u}^t_t &= \overline{\zeta \bar{v}^{xy}}^{xy} - \overline{\overline{[gh + \tfrac{1}{2}(u^2 + v^2)]}^{y}_{x}}^{xy} \\
\bar{v}^t_t &= - \overline{\zeta \bar{u}^{xy}}^{xy} - \overline{\overline{[gh + \tfrac{1}{2}(u^2 + v^2)]}^{x}_{y}}^{xy} \\
\bar{h}^t_t &= - \overline{(\bar{u}^y \bar{h}^y)_x}^{xy} - \overline{(\bar{v}^x \bar{h}^x)_y}^{xy}
\end{aligned} \tag{46}$$

In this (and the subsequent equations), the relative vorticity is calculated by the approximation,

$$\zeta = \bar{v}^y_x - \bar{u}^x_y \tag{47}$$

The equations were integrated using a ten-minute time step and the initial data outlined earlier. Each hour of forecast time (i.e., every six time steps), certain area averages were computed. These gave a record of

$\widetilde{(\nabla.\vec{V})}$	RMS divergence
$\widetilde{\zeta^2}$	RMS vorticity
$\tilde{h}$	mean height
$\widetilde{\frac{g}{2}(h-\bar{H})^2}$	available potential energy
$\widetilde{\frac{1}{2}\, h\vec{V}.\vec{V}}$	kinetic energy
$\widetilde{(\partial h/\partial t)^2}$	RMS height tendency
$\widetilde{hu}$ and $\widetilde{hv}$	momentum.

For the special form of the initial data used, one finds that

$$\widetilde{hu} = \widetilde{hv} \tag{48}$$

should hold.

The integration starts with a forward time step. The total energy (sum of kinetic and available potential energies) jumps 10% during the first hour. Thereafter, the total energy increases with oscillations by an average of 1% per day until 110 days. At that point, the growth accelerates becoming 15% per day by 113 days. Finally, the growth becomes exponentially unstable; the calculation stops at 117 days because the height becomes negative at one point.

The availability of a record of the dependent variables for the three time steps prior to the end of the run, permits us to note that the height becomes negative at a point at which there was a very large temporal oscillation.

	Time Step	16882	16883	16884
Meters	h	1564	6725	-291
m sec^{-1}	u	51	166	41
m sec^{-1}	v	147	-126	169

We also noted that the symmetry properties, which should hold for the numerical solutions, are violated. For example, hu and hv ought always to be equal, but as early as 168 hours a deviation of $1:10^5$ shows up. By 110 days, the deviation in these quantities is $1:10^2$. As the exponential growth sets in, the deviation is the same order of magnitude as the quantities themselves. It is not clear to us what significance this symptom has.

It is also pertinent to observe the behavior of the other *"statistical"* quantities. Thus, we find that RMS vorticity tends to decrease during the first 70-odd days by approximately 10%. It then undergoes an oscillation before beginning a strong growth trend about 100 days.

The RMS divergence is found to grow slowly and irregularly. It does not evidence the sudden switch to unstable growth shown by the RMS vorticity.

The mean height shows a very small decrease during the run (~.3m) but the available energy (proportional to h^2) is really the principal component of the unstable growth of the total energy discussed earlier.

We undertook the investigation of suitable stabilization schemes using the equations in a decoupled advective form. So we had the system of equations,

$$\begin{aligned} &\bar{u}^t_t + \bar{u}^{2x}\,\bar{u}^x_x + \bar{v}^{2y}\,\bar{v}^y_y + g\bar{h}^x_x = 0 \\ &\bar{v}^t_t + \bar{u}^{2x}\,\bar{v}^x_x + \bar{v}^{2y}\,\bar{v}^y_y + g\bar{h}^y_y = 0 \\ &\bar{h}^t_t + \overline{(uh)}^x_x + \overline{(vh)}^y_y = 0 \end{aligned} \tag{49}$$

This system behaved badly, running *"dry"* by three days.

We then used the *"decoupled grid"* filter (31) on the advecting velocities in the two equations of motion. The result was unsatisfactory. The equations *"ran dry"* at 30 days and the RMS vorticity grew linearly from the beginning.

Simultaneously, we tried a modification of the invariant equations in the form:

$$\begin{aligned} &\bar{u}^t_t - \overline{\zeta^*\bar{v}^{xy}}^{xy} + \overline{\bar{K}^y_x + g\bar{h}^y_x}^{xy} = 0 \\ &\bar{v}^t_t + \overline{\zeta^*\bar{u}^{xy}}^{xy} + \overline{\bar{K}^x_y + g\bar{h}^x_y}^{xy} = 0 \\ &\bar{h}^t_t + \overline{(\bar{u}^y\bar{h}^y)}^{xy}_x + \overline{(\bar{v}^x\,\bar{h}^x)}^{xy}_y = 0 \end{aligned} \tag{50}$$

in which $\zeta^* = .5\,(\zeta^n + \zeta^{n-1})$ (51)

and $K = .5\,(u^2 + v^2)$ (52)

In contrast with the *"benchmark"* experiment, the equations (50) ran without going *"dry"* through 365 days. The total energy grew, more or less linearly, by an order of magnitude. It is noteworthy, however, that both kinetic and available potential energy shared this growth.

The RMS vorticity did not vary by more than 20% during the course of the run and was quite well-behaved. The RMS divergence, on the other hand, displayed rather uniform behavior but by the end of the run had grown by a factor of four.

The symmetry of the run was observed to be violated once more as evidenced by the inequality of hu and hv. This result might be encouraging except for the following point.

The vorticity equation implicit in the system (50) is

$$\frac{\partial \zeta}{\partial t} = - \overline{[(\overline{\zeta^* \bar{u}^{xy}})_x^{\,y} + (\overline{\zeta^* \bar{v}^{xy}})_y^{\,x}]}^{xy} \tag{53}$$

If we assume no variation in y and the existence of a mean value of u, eq. (53) becomes

$$\frac{\partial \zeta}{\partial t} = - (\zeta^* U)_{2x} = - U \zeta^*_{2x} \tag{54}$$

This equation is linearly unstable because of the appearance of the "*" operator on ζ.

To test this argument, we used an experimental set-up originally designed to replicate certain of Grammeltvedt's (1969) results. The results were surprising.

The first case run (with essentially the same equations as (50) except for the introduction of the Coriolis paramenter) used as initial data, Grammeltvedt's case 3. This involves two waves superimposed on a zonal current in the initial data. One is wave number one; the other has a wavelength of $5\Delta x$. A run was made without the * operator and found to *"blow-up"* by Grammeltvedt's criterion at 16.5 days. The criterion used for defining *"blow-up"* was an increase in total available energy by 10% above its initial value. When the *-operator was introduced, the solution was calculated to 23.7 days without violation of this stability criterion.

This was an unanticipated result, but we reasoned that the really noisy behavior of the solution was unrealistic. A more pertinent test of the linear instability would be to run an experiment in which the forecast should be relatively well-behaved. Grammeltvedt's Case 1 is of that type. The initial data contain just one long wave imbedded in a zonal flow.

The two experiments (with and without the star-operator) were run with this data. One found that the star-operator resulted in a much more rapid growth of noise in the numerical solution than the run without the star. In neither case was Grammeltvedt's criterion for stability violated during 24 days of integration.

The last result[1] confirmed our suspicion that the $*$-operator was not appropriately introduced in the equations (50). But, even more significantly, these results brought home to us the fact that the stability of a computation is not a well-defined term. For linear equations with constant coefficients, Lax's theory (Richtmyer and Morton, 1967) provides a relation between computational stability and accuracy. It seems clear that this relationship is not relevant to the meteorological system of non-linear equations.

Comments on Possible Extensions of These Studies

In the course of the experiments reported in the previous sections, the authors have occasionally pondered in discomfort the severity of the test of the difference system (any difference system) represented by the initial data shown in figure 2 and the two-dimensional counterpart. The distribution of velocity represents a vigorous shock to the modelled fluid system, which is exactly its purpose. However, it is not so much the very large velocity immediately adjacent to the walls that gives rise to such professional discomfort as it is the assemblage of annoying subleties associated with the data. For example, the property of symmetry in the two-dimensional data demonstrates clearly that, rather than constituting a general set of noisy initial conditions, the data in reality is quite specialized. Moreover, the distribution depends significantly on the location of the imaginary walls which constitute the boundaries; i.e., whether the walls are positioned at or between grid points. In the former case, the 24 terms of the Fourier expansions cancel in pairs at alternate grid points, yielding a velocity field which is exactly zero on half the points.

In a series of quasi-two-dimensional experiments reported elsewhere (McPherson, 1970), it was demonstrated that the onset of instability is dependent upon the Froude number of the flow. For these experiments, the Froude number is directly related to the amplitude of the initial velocity spectrum, so that increasing Froude numbers imply increasing amplitude of the velocity spectrum. The experiments demonstrated that the onset of instability occurs earlier as the amplitude of the initial velocity spectrum increases.

With this evidence of the sensitivity of the numerical stability investigations to the initial conditions, and the highly specialized nature of the data used in these experiments, it is our contention that extensions of these studies should consider alternative specifications of noisy initial data. For example, one might generate a noisy velocity field by a random process. This would surely avoid such features as the symmetry described earlier.

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1 These and other experiments based on Grammeltvedt's paper will be documented by Polger and McPherson in the near future.

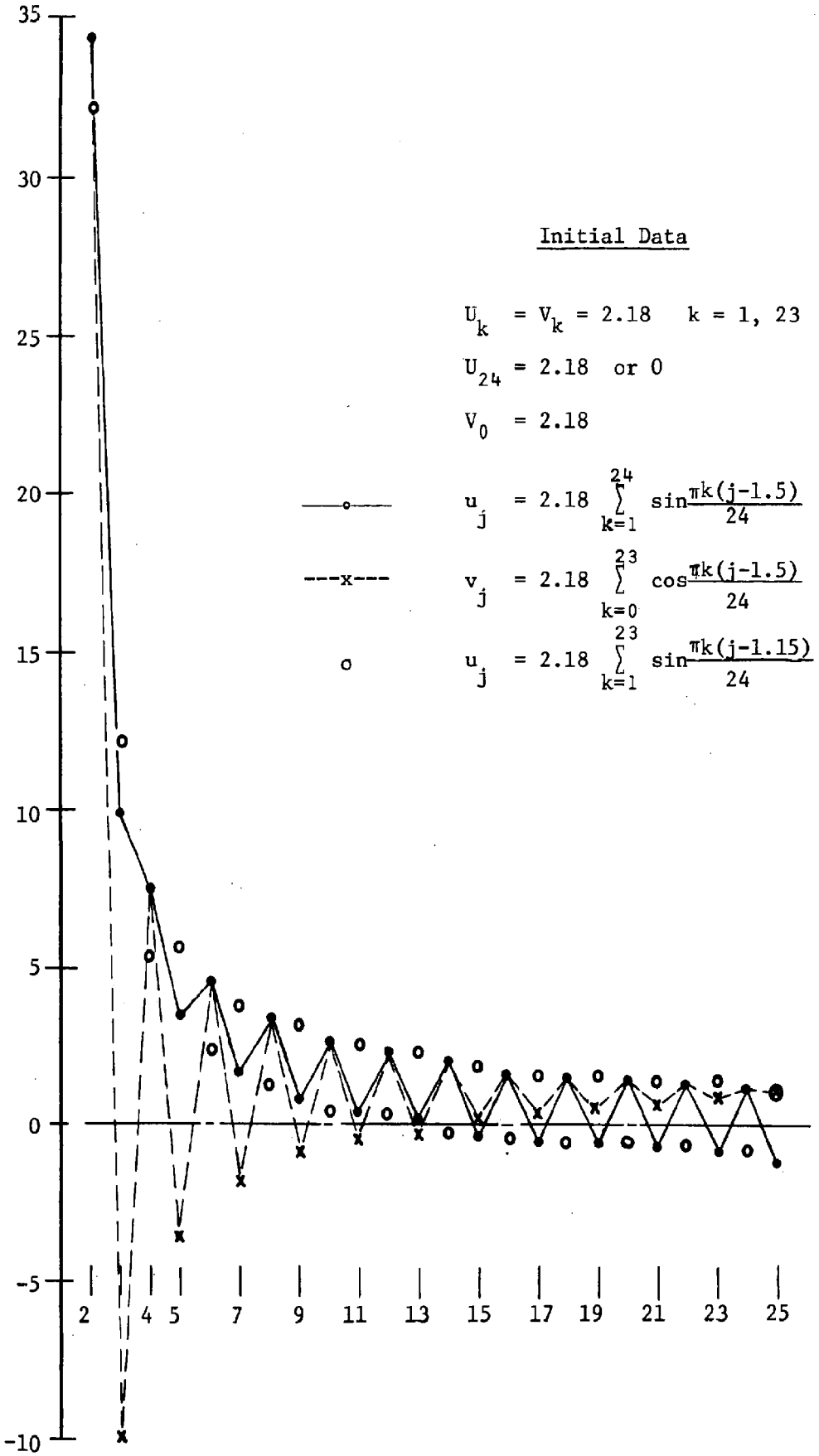

Figure 2. Graphical depiction of initial velocity components in quasi-two-dimensional experiments.

In addition to considering alternative sets of initial data, future investigations would do well to consider formulating the problem in a different coordinate system. Without exception, the experiments reported here have been conducted in cartesian geometry. Although appealingly simple, there arise fundamental questions concerning the treatment of the Coriolis parameter when rotation is allowed. In particular, the authors regard the problem of specifying appropriate boundary conditions for a rotating square tank as not solved. For this reason, and of equal importance, because there is every indication that future operational models will be formulated on a latitude-longitude grid lattice, it is recommended that future experiments be formulated in polar geometry.

As mentioned previously, interest in 'decoupled' difference system has arisen out of these stability investigations, primarily to achieve economic advantage. Further investigation[1] has indicated that substantial economies can be effected by judicious arrangement of the dependent variables, 'staggered' with respect to each other in space as well as time. Although ongoing investigations are not as yet conclusive, the results to date are sufficiently promising as to warrant serious consideration of formulating future operational models in this way. The experiments reported here, with one or two minor exceptions, have not been concerned with any grid arrangement other than the standard array of regular points. Future investigations of this type should take into account the likelihood of the incorporation of 'staggered' grids into the operational framework.

Similarly, present indications are that some type of implicit time integration technique will shortly be incorporated into the operational framework. In these experiments, however, only the explicit 'leap frog' scheme was used. Additional investigations should not only consider implicit integration methods, but alternative explicit methods such as the 'leap-frog trapezoidal' (Kurihara, 1965) which Grammeltvedt (1969) used to stabilize one of his experiments. Examinations of such explicit techniques as the 'leap-frog trapezoidal', which damps the temporal computational mode, would be most profitable in the light of the Robert-Shuman-Gerrity theory.

Concluding Remarks

The preceding account of our investigations into the problems of numerical stability has clearly raised more questions than it has provided answers; indeed, the immediately preceding section has outlined a project of considerable magnitude. Among the unanswered questions, perhaps the most intriguing has to do with the nature of the two-grid-interval wave. Innocuous in itself, without frequency (using centered

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[1] To be reported by the authors at a later date.

differences), it is potentially devastating in effect when present in a field. Accordingly, we conclude with the following tribute to the mysterious majesty of the two-grid-interval wave:

> The two-grid-interval wave may be likened to a roach
> Which doth not ruin pastry by eating,
> For his stomach is limited in size;
> But ruineth by falling into
> And tracking footprints on the pies.

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