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Joint Numerical Weather Prediction Unit

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The Reduction of Truncation Errors in Symmetrical Operators

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Consider the finite difference operator ∇^2 which corresponds to the Laplacian ∇^2 and which is given by

$$\nabla^2 \varphi_{i,j} = \frac{1}{d^2}\left[\varphi_{i,j+1} + \varphi_{i,j-1} + \varphi_{i+1,j} + \varphi_{i-1,j} - 4\varphi_{i,j}\right] \quad (1)$$

where d is the mesh length and i, j indicate the points at which φ is to be measured. Taylor's Theorem gives

$$\nabla^2\varphi = \nabla^2\varphi + d^2(\mu_1 + \mu_2) + d^4(\mu_3 + \mu_4) + \ldots \quad (2)$$

where $\mu_1, \mu_3 \ldots$ are the results of differential operators acting on φ at the point at which $\nabla^2\varphi$ is measured, and $\mu_1, \mu_3 \ldots$ are those parts of the operators which are invariant under rotation. We wish to form a finite difference operator acting on φ which will essentially eliminate μ_1 ; Thompson (1) has suggested that $\varphi_{i,j}$ be replaced by $\varphi_{i,j} - \frac{d^2}{16}\nabla^2\varphi_{i,j}$.

There is an operator corresponding to (1) which is simply formed by rotating the grid through an angle $\pi/4$ and erbending the mesh length

$$^{\times}\nabla^2\varphi_{i,j} = \frac{1}{2d^2}\left[\varphi_{i+1,j+1} + \varphi_{i-1,j+1} + \varphi_{i+1,j-1} + \varphi_{i-1,j-1} - 4\varphi_{i,j}\right] \quad (3)$$

and the equation corresponding to (2) is

$$^{\times}\nabla^2\varphi = \nabla^2\varphi + 2d^2(\mu_1 + \mu_2') + 4d^2(\mu_3 + \mu_4') + \ldots \quad (4)$$

where μ_1, μ_3 are unaltered by rotation, while μ_2' and μ_4' are different from μ_2 and μ_4. To eliminate the systematic error we take

$$2\nabla^2 - {}^{\times}\nabla^2 = \nabla^2 + \text{systematic error } O(d^4) \quad (5)$$

The new operator formed by the left hand side of (5) involves nine mesh points and hence involves the mesh values of φ over a greater area than does the operator ∇^2. Clearly we can form an operator

$$^{\times\times}\nabla^2 \varphi_{i,j} = \frac{1}{4d^2}\left[\varphi_{i,j+2} + \varphi_{i,j-2} + \varphi_{i+2,j} + \varphi_{i-2,j} - 4\varphi_{i,j}\right] \quad (6)$$

with the corresponding equation

$$^{\times\times}\nabla^2 \varphi = \nabla^2 \varphi + 4d^2(\mu_1 + \mu_2) + 16d^4(\mu_3 + \mu_4) + \ldots\ldots \quad (7)$$

and using (2), (4) and (7) we can form an operator

$$\lambda \nabla^2 + \mu\, {}^{\times}\nabla^2 + \nu\, {}^{\times\times}\nabla^2 \quad (8)$$

which eliminates also the systematic error to the order of d^4.

The values of λ, μ and ν are given by

$$\lambda + \mu + \nu = 1$$
$$\lambda + 2\mu + 4\nu = 0$$
$$\lambda + 4\mu + 16\nu = 0$$

or $\lambda = 8/3$, $\mu = -2$, $\nu = 1/3$.

The operator defined in (8) requires the values of φ at four more mesh points. We may continue adding in four more mesh points and eliminating the systematic error to higher degrees in d^2. . It is interesting to compare the meshes for the case given by Thompson (1) which uses $\varphi_{ij} - \frac{d^2}{16}\nabla^2\varphi_{i,j}$ for φ_{ij}. The meshes for $\nabla^2\varphi_{i,j}$ are

(1) Thompson

$$
\begin{array}{ccccc}
 & & -1 & & \\
 & -2 & 24 & -2 & \\
-1 & 24 & -84 & 24 & -1 \\
 & -2 & 24 & -2 & \\
 & & -1 & &
\end{array}
\qquad \times \frac{1}{16d^2}
$$

(2) $2\nabla^2 - {}^{*}\nabla^2$

$$
\begin{array}{ccc}
-1 & 4 & -1 \\
4 & -12 & 4 \\
-1 & 4 & -1
\end{array}
\qquad \times \frac{1}{2d^2}
$$

(3) $8/3\ \nabla^2 - 2\,{}^{*}\nabla^2 + 1/3\ {}^{**}\nabla^2$

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & -12 & 32 & -12 & \\
1 & 32 & -84 & 32 & 1 \\
 & -12 & 32 & -12 & \\
 & & 1 & &
\end{array}
\qquad \times \frac{1}{12d^2}
$$

All the above can be repeated for any systematic operator. In particular we have an equation exactly analagous to (2) for the finite difference Jacobian $\mathbb{J}$

$$\mathbb{J} = J + d^2(\mu_1 + \mu_2) + d^4(\mu_3 + \mu_4) + \ldots \quad (9)$$

where the u_i are the results of differential operators on φ and ψ evaluated at the point where J is evaluated. We can form J by using four points on each mesh, $(i+1, j), (i-1, j), (i, j+1), (i, j-1)$. By turning through an angle $\pi/4$ we can form ${}^{\times}\mathbb{J}$ using the points $(i+1, j+1), (i-1, j-1), (i+1, j-1), (i-1, j+1)$ and write

$$2\mathbb{J} - {}^{\times}\mathbb{J} = J + \text{systematic error } O(d^4) \tag{10}$$

Thompson (1) has suggested replacing φ and ψ by $\varphi - \frac{d^2}{8}\nabla^2\varphi$, $\psi - \frac{d^2}{8}\nabla^2\psi$ before evaluating the Jacobians and gives an example based on

$$\varphi = -Ux + A\sin\alpha y, \quad \psi = \nabla^2\varphi \tag{11}$$

He considers the Jacobian calculated from the two meshes numbered 1 and 2 in Figure 1 and takes an average. His result is

$$J \sim \frac{AU\alpha^3}{2}\sin\alpha y\left(\frac{5-\cos\alpha d}{4}\right)\left(\frac{\sin\alpha d}{\alpha d} + \frac{\sqrt{2}}{\alpha d}\sin\frac{\alpha d}{\sqrt{2}}\right) \tag{12}$$

The result of using $2\mathbb{J} - {}^{\times}\mathbb{J}$ is

$$J \sim AU\alpha^3\sin\alpha y\left(\frac{2\sqrt{2}}{\alpha d}\sin\frac{\alpha d}{\sqrt{2}} - \frac{\sin\alpha d}{\alpha d}\right) \tag{13}$$

At $\alpha d = \pi/2$ the value of the bracketed expression in (13) is 0.97, while that of $\frac{\sin\alpha d}{\alpha d}$ is $\frac{2}{\pi}$ or 0.64 and that given by Thompson's curve is 0.91.

Summarizing we may say that symmetrical part of the truncation error is successfully reduced by using the simple operators shown above.

$$
\begin{array}{ccccc}
 & & 1 & & \\
2 & & & & 2 \\
1 & & \cdot & & 1 \\
2 & & & & 2 \\
 & & 1 & &
\end{array}
$$

Figure 1.

Reference.

(1) Thompson, P.D. 1955. Tech. Mem. No. 1. J.N.W.P. Unit.