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U.S. Joint Numerical Weather Prediction Unit

A Method of Designing Finite-Difference Smoothing Operators to Meet Specifications

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The Smoothing Element

First of all, let a smoothing element be defined. The smoothing element will be the building block of more complicated smoothing operators. We shall take as the smoothing element the simplest of one-dimensional symmetrical centered finite difference operators which does not affect the mean value of a field of infinite extent, namely,

$$\theta_i^{\tau+1} = \mu\, \theta_i^{\tau} + \frac{1}{2}(1-\mu)\left(\theta_{i-1}^{\tau} + \theta_{i+1}^{\tau}\right) \qquad (1)$$

where θ is the field to be smoothed. The subscripts refer to points equally spaced in x, the independent variable, and consecutively numbered with increasing x. The superscripts $(\ )^{\tau}$ and $(\ )^{\tau+1}$ may be interpreted as indicating the unsmoothed and smoothed variable, respectively, or, as indicating results of successive smoothings; μ will be called the smoothing element index, since it completely describes a given operator of form (1).

It will be convenient to think of the dependent variable, θ, within the region of interest as consisting of the sum of cosine functions of varying amplitudes, phases, and wave numbers. Adopting this concept, we will investigate the effect of a smoothing element on individual cosine components. For example, consider the component

$$\theta_i^{\tau} = a + c \cos b(x_i - \bar{x})$$

Now,

$$\theta_{i+1}^{\tau} = a + c \cos b\Delta x \cdot \cos b(x_i - \bar{x}) + c \sin b\Delta x \cdot \sin b(x_i - \bar{x})$$

$$\theta_{i-1}^{\tau} = a + c \cos b\Delta x \cdot \cos b(x_i - \bar{x}) - c \sin b\Delta x \cdot \sin b(x_i - \bar{x})$$

where

$$\Delta x = x_{i+1} - x_i = x_i - x_{i-1}$$

If these identities are substituted into equation (1), we have, after some re-arrangements,

$$\theta_i^{\tau+1} = a + [\mu + (1-\mu) \cos b\Delta x] c \cos b(x_i - \bar{x})$$

Thus, the smoothing element (1) changes neither the wave number nor the phase, but changes the amplitude of each component by

$$\sigma^{\tau} = \frac{c^{\tau+1}}{c^{\tau}} = \mu + (1-\mu) \cos b\Delta x \qquad (2)$$

where c^{τ} and $c^{\tau+1}$ are the amplitudes of the unsmoothed and smoothed fields, respectively.

Figure 1 shows the field of μ in σ, $\cos b\Delta x$-space. It is to be noted that the smoothing element cannot be considered highly selective as to wave number. For example, if we wished to filter out of a

θ-field waves of wave-length $2\Delta x$ $(\cos b\Delta x = -1)$ by means of one smoothing element, we would reduce waves of length $10\Delta x$ $(\cos b\Delta x \simeq 0.8)$ by as much as 10% (see the line corresponding to $\mu = 0.5$).

The Design of Multi-element Operators.

It is obvious that, in order to improve on the selectivity of the 3-point smoothing element, a smoothing operator must be invented which involves more points. The problem in designing such an operator is to fit it to previously stated specifications. The approach to this problem put forth in this paper is based on the use of more than one smoothing element (1). It may be noted that applying more than one smoothing element successively, say with indices $\mu_0, \mu_1, \mu_2, \ldots, \mu_n$ results in the final ratio of smoothed amplitude to unsmoothed amplitude of

$$\begin{aligned} \Sigma &= \sigma_0 \sigma_1 \sigma_2 \ldots \sigma_n \\ &= \prod_{l=0}^{l=n} \left[\mu_l + (1-\mu_l) \cos b\Delta x \right] \end{aligned} \tag{4}$$

according to equation (3). Equation (4) may be considered the result of multiplying together various lines of σ against the argument $\cos b\Delta x$ in figure 1. Thus, in theory one may specify a single-valued curve of Σ against $\cos b\Delta x$, and express it in terms of a product of factors of form (3). One would then know precisely how to accomplish the smoothing desired. In practice, however, this would present a formidable task, and furthermore, one is not usually concerned with a precise distribution of Σ in $\cos b\Delta x$. A great deal of improvement, in terms of the desired smoothing end-product, is obtained by combining only two smoothing elements, which might be combined into one 5-point operator. In this case, Σ is a quadratic function of $\cos h\Delta x$, and the problem of design centers on two characteristics of the function. Figure 2 shows an example of the effect of a 5-point operator which was constructed from two 3-point operators. In figure 2, note how greater selectivity is obtained by repeated smoothings.

Smoothing in Two Dimensions.

Extension of the theory to two dimensions may be accomplished by smoothing in each dimension, independently of the other dimension. It can be shown that the final result is independent of the dimension in which one first smooths, and is also independent of the order in which one applies the smoothing elements.

One may adopt the view that extension of a smoothing element to two dimensions is really the application of two smoothing elements, one in each dimension, with identical indices. These two elements may then be combined into a single 9-point operator, thus,

$$\begin{aligned}\theta_0^{\tau+1} = {} & \tfrac{1}{4}(1-\mu)^2\left(\theta_1^{\tau}+\theta_3^{\tau}+\theta_5^{\tau}+\theta_7^{\tau}\right) \\ & + \tfrac{1}{2}\mu(1-\mu)\left(\theta_2^{\tau}+\theta_4^{\tau}+\theta_6^{\tau}+\theta_8^{\tau}\right) \\ & + \mu^2\,\theta_0^{\tau}\end{aligned} \tag{5}$$

The subscripts refer to mesh points in the accompanying figure, μ being the index of the two smoothing elements, one applied in each dimension.

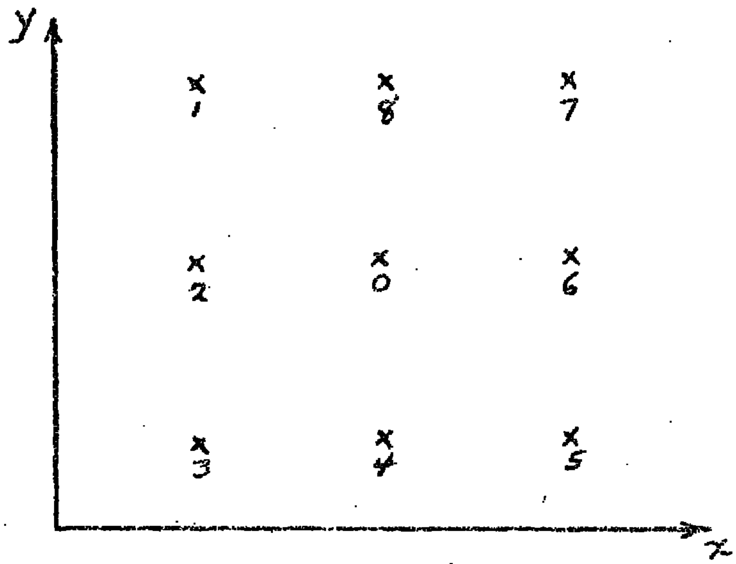

Figure. The 9-point mesh.

It is obvious that smoothing elements of two different indices leads to a 25-point operator in two dimensions. When performed on a high-speed computer of the IBM-701 type, however, two sweeps of a two-dimensional field with 9-point operators proves to be an economical compromise between the relatively large number (4) of sweeps and small number (3) of address formations per point resulting from the application of two smoothing elements in each dimension, and the small number (1) of sweeps and large number (25) of address formations per point resulting from a single application of a 25-point operator.

Example. The Construction of a Smoothing Operator.

For certain reasons, it was desired to eliminate components of wave-length about $4\Delta x$ from a field of thickness, 1000 to 800 mb, and at the same time to retain components of wave-length longer than $10\ \Delta x$. The problem arose in the prediction of the thickness field. The prediction was being carried out in such a way as to yield fields of thickness at half-hour intervals. Since the spectrum of the thickness field changed at each time-step, it was planned to smooth at each time step. It was arbitrarily decided that a change of about 10% in the longer wave-lengths due to smoothing could be tolerated in 48 time steps. One specification of the curve, therefore, was that the maximum value of Σ should be 1.002, which when raised to the 48th power is 1.107. It was decided to use smoothing elements of two different indices, so, after specifying the maximum of 1.002, we were left with 1 degree of freedom in specifying the curve of Σ against $\cos b\Delta x$ The other specification chosen was that the value of Σ at $\cos b\Delta x = 0$ (corresponding to wave-lengths of $4\Delta x$) should be .8, which when raised to the 48th power is 0.0000223. It was realized that after the operator was designed to fit these specifications, it would be necessary to inspect the curve of Σ against $\cos b\Delta x$ to see that all wave lengths longer than $10\Delta x$ would be retained essentially unchanged after 48 smoothings.

Now, according to equation (4),

$$\Sigma = \left[\mu_0 + (1-\mu_0)\xi\right] \times \left[\mu_1 + (1-\mu_1)\xi\right] \qquad (6)$$

where

$$\xi \equiv \cos b \Delta x$$

One specification is that the curve pass through $\Sigma, \xi = 0.8, 0$. Therefore μ_0 and μ_1, must satisfy the relation

$$\mu_0 \mu_1 = 0.8 \tag{7}$$

Differentiating equation (6), we find

$$\left(\frac{\partial \Sigma}{\partial \xi}\right)_x = (1-\mu_0)\left[\mu_1 + (1-\mu_1)\xi_x\right] + (1-\mu_1)\left[\mu_0 + (1-\mu_0)\xi_x\right] = 0 \tag{8}$$

where the subscript $(\ )_x$ denotes values at the maximum of Σ. If equation (8) is solved for ξ_x; and $\Sigma, \xi = 1.002, \xi_x$ are substituted into equation (6), we have a second relationship which μ_0 and μ_1 must satisfy.

$$1.002 = -\frac{(\mu_0 - \mu_1)^2}{4(1-\mu_0)(1-\mu_1)} \tag{9}$$

We now have equations (7) and (9) to solve for μ_0 and μ_1. The solution may be facilitated by noting that, if Σ_x is a specified maximum of a quadratic in ξ, then

$$\frac{1-\mu_0}{1-\mu_1} = 1 - 2\Sigma_x \pm 2\sqrt{\Sigma_x(\Sigma_x - 1)} \tag{10}$$

It should further be noted that the two branches of the right-hand side of equation (10) are identical for our purposes, since one is the inverse of the other. In our case, $\Sigma_x = 1.002$. Thus, taking the upper branch of equation (10),

$$\frac{1-\mu_0}{1-\mu_1} = -0.9144679 \tag{11}$$

Solution of equations (7) and (11) leads to two sets of μ_0, μ_1. One set, however, may be eliminated on the basis that $f_x > 1$ for that set. The other set is

$$\mu_0 = 0.52744$$
$$\mu_1 = 1.51676 \tag{12}$$

The maximum value of $\Sigma = 1.002$ for the set (12) is at $f = 0.90950$, corresponding to a wave length of 14.7 Δx.

In programming smoothing for the IBM-701 computer, two 9-point smoothing operators were used, one for each μ of the set (12). Equation (5) prescribes the procedure for computing the weights to be given each point of the 9-point mesh for each of the indices (12). Figures 3, 4, 5, and 6 show the results of smoothing the thickness field. The program as written did not smooth the boundary points, which are omitted from the figures. A better way, perhaps, to handle the boundary would be to smooth on the boundary, with a one-dimensional operator before smoothing the interior of the field. Only the four corner points would be held constant in that case.

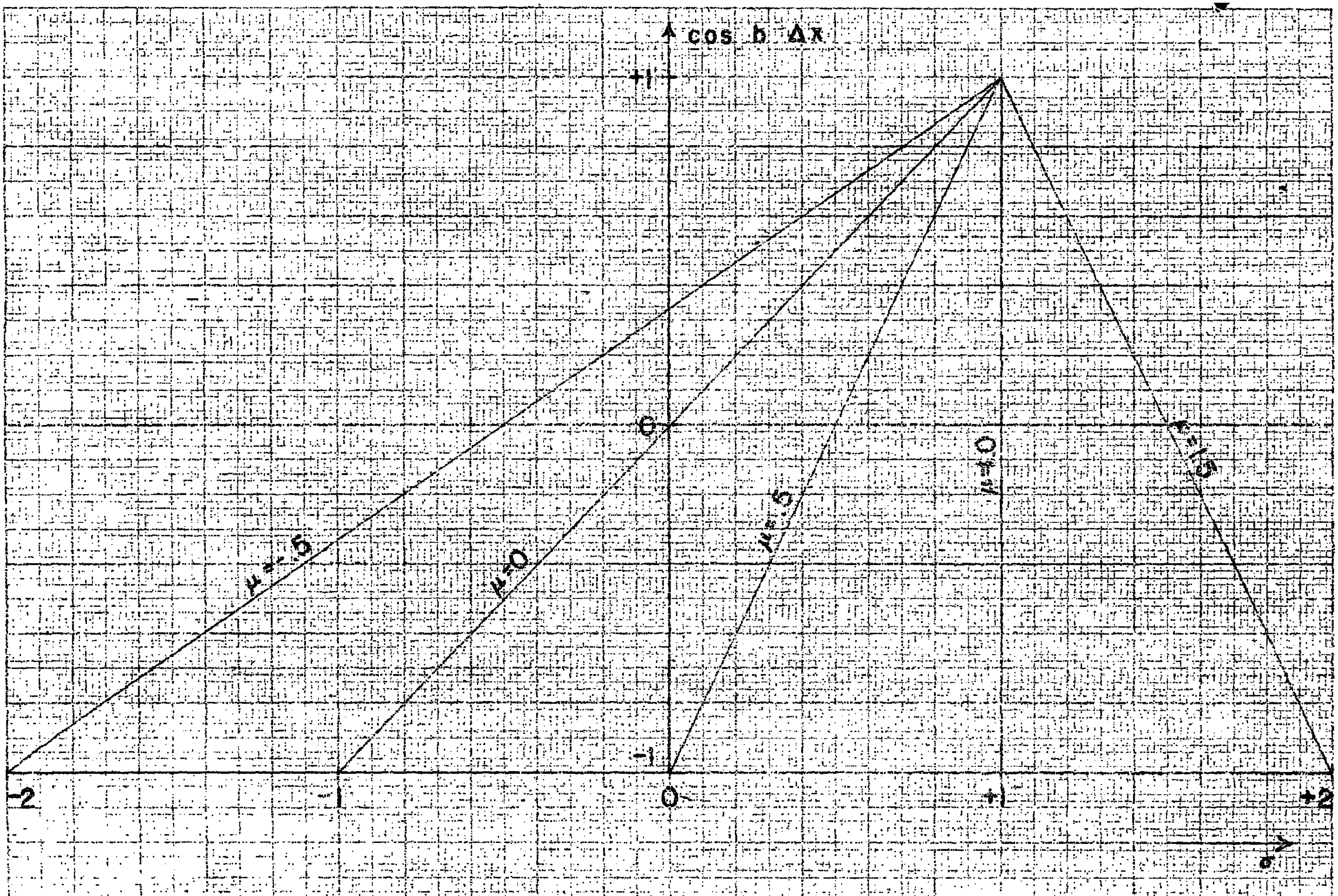

Figure 1. The field of smoothing element index in σ, $\cos b \Delta x$ - space. σ is the ratio of smoothed to unsmoothed amplitude, b is wave number, Δx is mesh length.

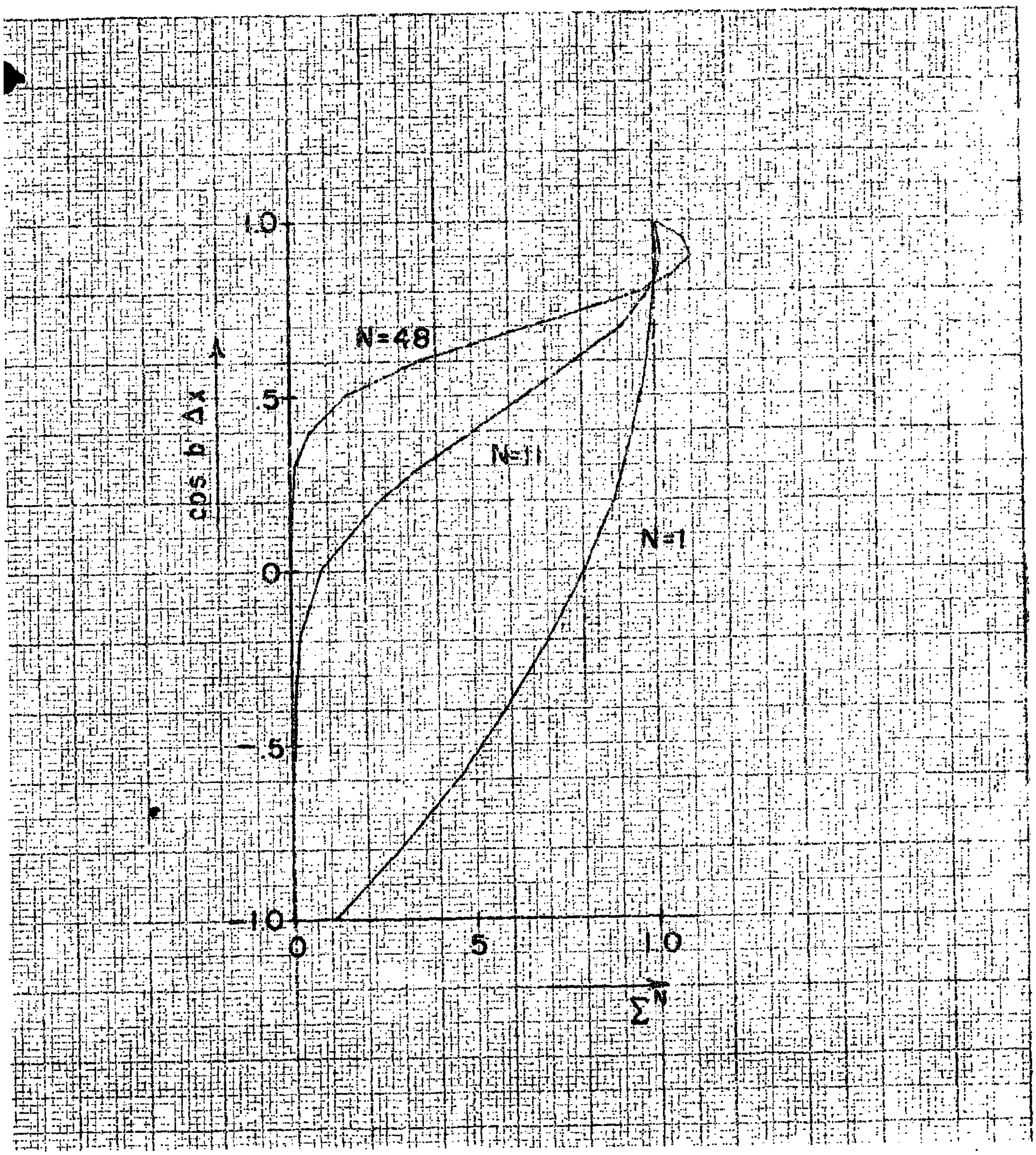

Figure 2. The ratio of smoothed to unsmoothed amplitude after N smoothings with two smoothing elements whose indices are μ_0, μ_1 = .52744, 1.51676.

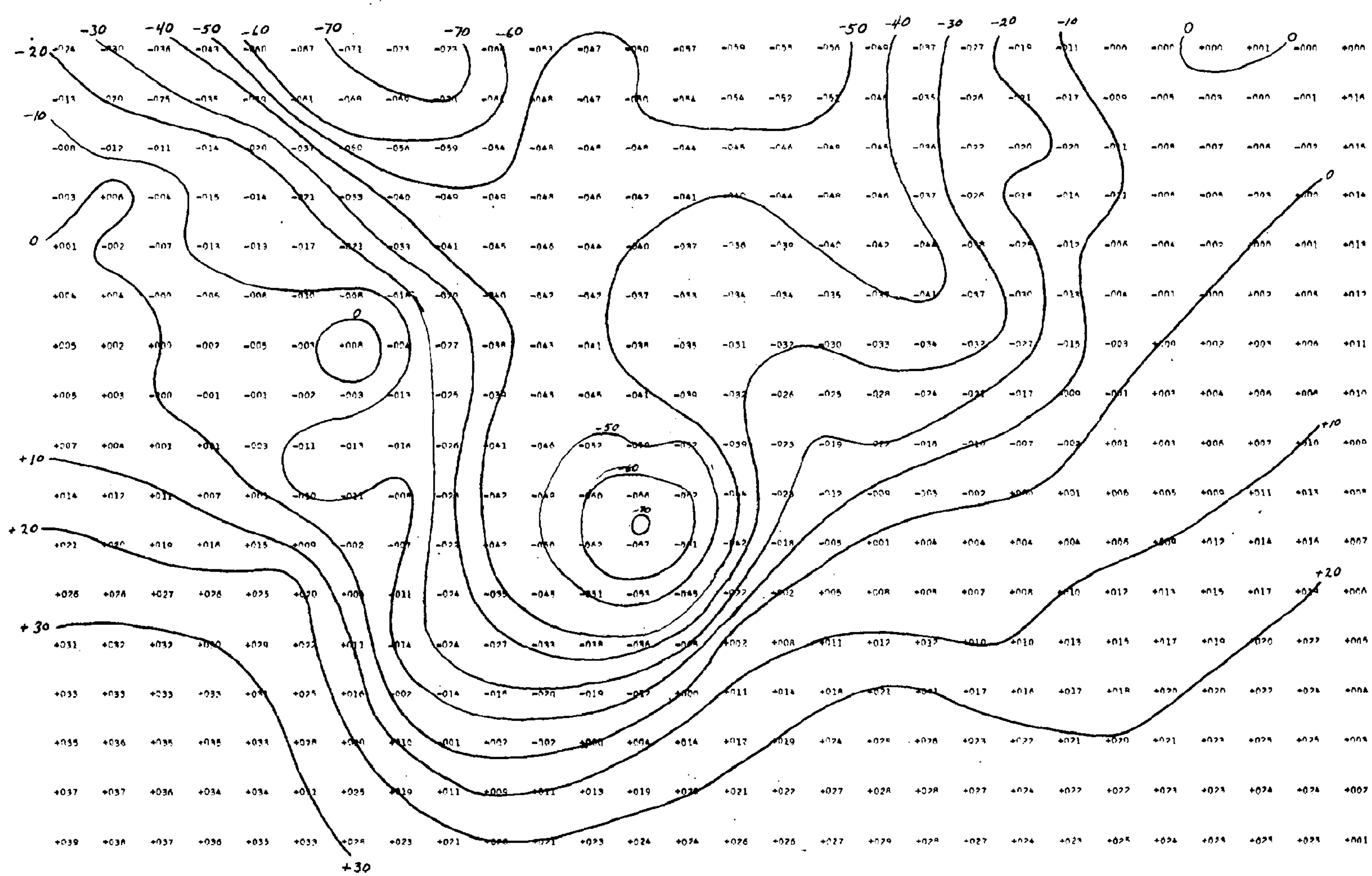

Figure 3. A thickness field, 1000 to 800 mb, unsmoothed. Unit of thickness is a dekafoot. Mesh length is 300 km: ⊢—⊣

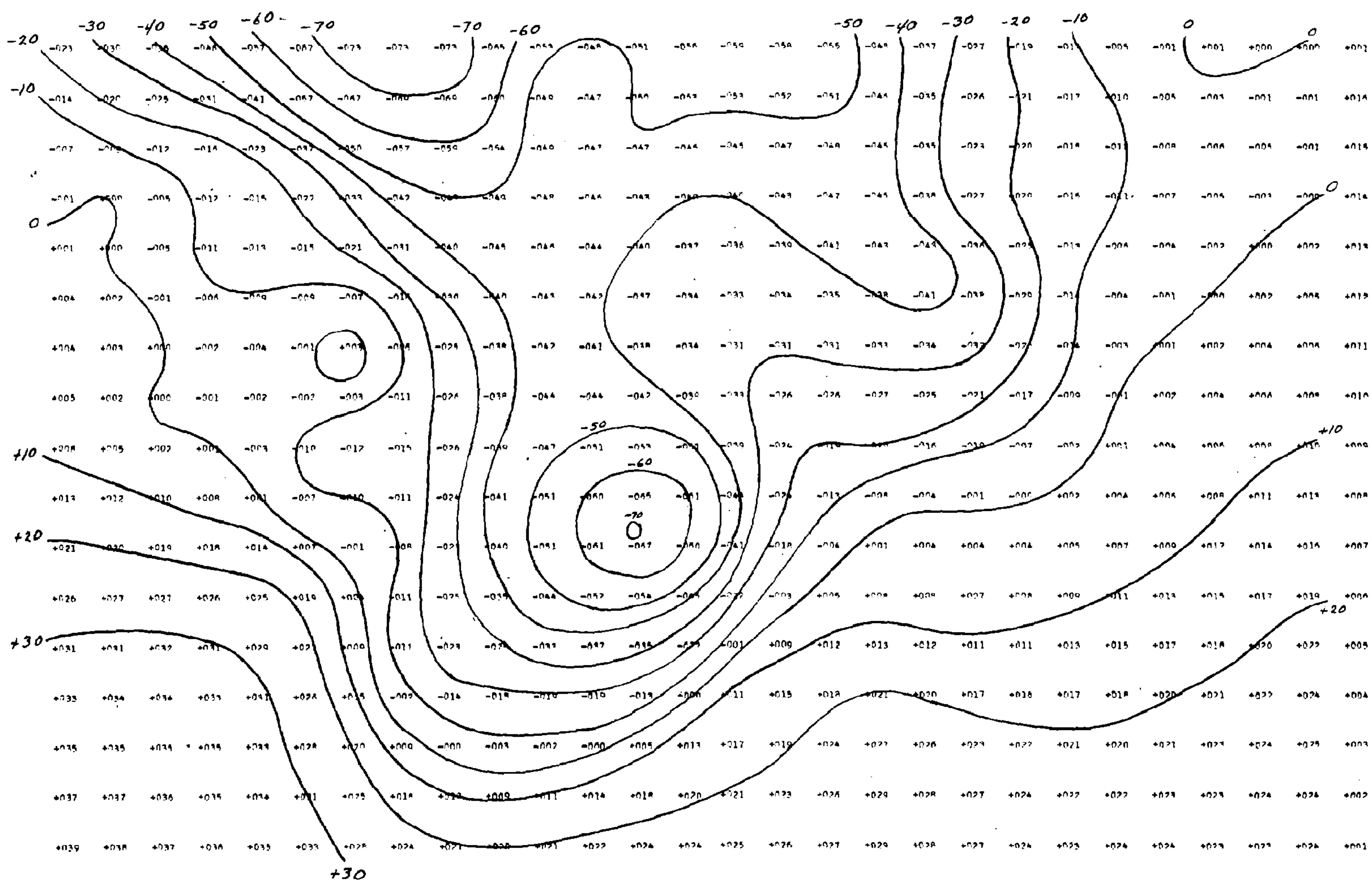

Figure 4. The thickness field of figure 3, smoothed once with smoothing elements whose indices are $\mu_0, \mu_1 = .52744,\ 1.51676.$ See the curve identified by $N = 1$ in figure 2.

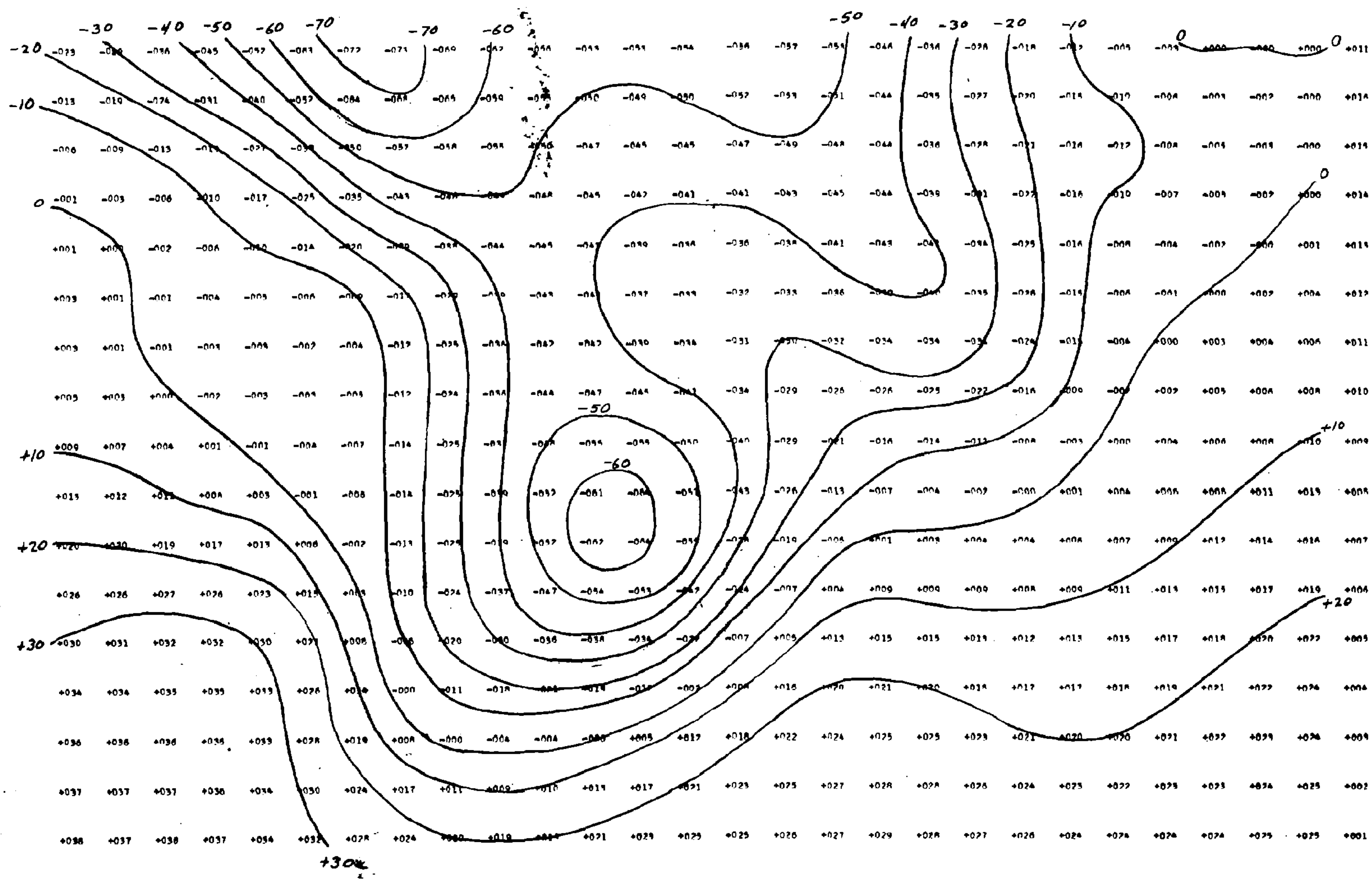

Figure 5. Figure 3 smoothed eleven times with smoothing elements whose indices are $\mu_0, \mu_1 = .52744,\ 1.51676$. See figure 2, curve identified by $N = 11$.

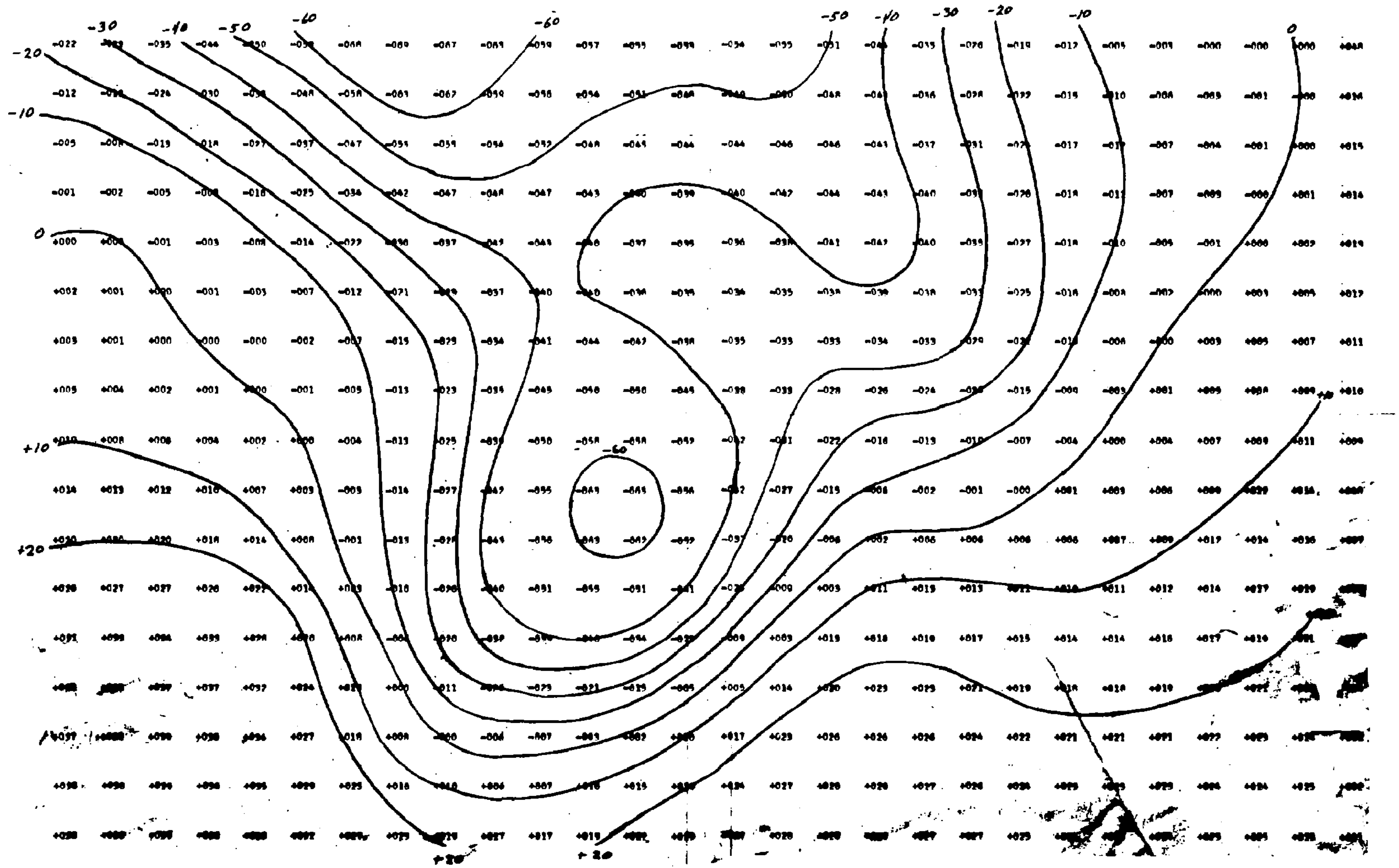

Figure 6. Figure 3 smoothed forty-eight times with smoothing elements whose indices are $\mu_0, \mu_1 = .52744, 1.51676$. See figure 2, curve identified by $N = 48$.