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Joint Numerical Weather Prediction Unit

The Analytic Solution of a Linear Partial Finite Difference Equation.

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Statement of the problem and its solution.

Consider the partial differential equation

$$\frac{\partial \theta}{\partial t} + U \frac{\partial \theta}{\partial x} = 0 \tag{1}$$

where U is a constant. Suppose that we were presented with the problem of finding the x, t -distribution of θ if we had given the x-distribution of θ at $t=0$. The solution can be immediately written as

$$\theta(x,t) = \theta(x-Ut, 0).$$

It would be of some interest to compare this solution with a solution in closed form obtained by numerical means. The latter would give us an opportunity to study the sum total of the effects of truncation errors in the estimation of derivatives with respect to both space and time. The following finite difference equation was chosen for such a study. It was formed from (1) in the way in which finite difference equations have been most commonly formed from differential equations in past numerical solutions of the meteorological equations applied to long waves and extra-tropical cyclones and anticyclones.

$$\theta(x, t+\Delta t) = \theta(x, t-\Delta t) - R[\theta(x+\Delta x, t) - \theta(x-\Delta x, t)] \tag{2}$$

In (2) Δx and Δt are the mesh lengths of a rectangular net in x and t, and

$$R \equiv \frac{U \Delta t}{\Delta x}. \tag{3}$$

It is obvious that the solution of (2) requires boundary information equivalent to two x-distributions of θ spaced in time Δt apart. If we had two such x-distributions of θ which are each cosine functions of x with identical wave numbers, the x-distribution of θ would remain a cosine function of x for all of the discrete times in a net of infinite extent. To demonstrate, let

$$\theta(x, t-\Delta t) = c_0 \cos b(x + \delta_1)$$

$$\theta(x, t) = c_2 \cos bx \ .$$

The subscripts adopted here bear the following relationship to the time coordinate.

$$\text{Subscript} = 2n = 2t \div \Delta t$$

The time of the phase change, δ, between two successive time steps has arbitrarily been set equal to the average time of the two time steps. Now,

$$\theta(x, t-\Delta t) = c_0 \cos b\delta_1 \cos bx - c_0 \sin b\delta_1 \sin bx$$

$$\theta(x+\Delta x, t) = c_2 \cos b\Delta x \cos bx - c_2 \sin b\Delta x \sin bx$$

$$\theta(x-\Delta x, t) = c_2 \cos b\Delta x \cos bx + c_2 \sin b\Delta x \sin bx$$

If these identities are substituted into (2), we obtain

$$\theta(x, t+\Delta t) = c_4 \cos b(x - \delta_3) \quad , Q.E.D.,$$

where

$$\tan b\delta_3 = \frac{2Rc_2 \sin b\Delta x - c_0 \sin b\delta_1}{c_0 \cos b\delta_1} \tag{5}$$

and

$$c_4 = \frac{c_0 \cos b\delta_1}{\cos b\delta_3} \tag{6}$$

It is of interest to note that by means of (6) a quantity can be defined which is "conservative".

$$c_{2n} c_{2n+2} \cos b\delta_{2n+1} = c_0 c_2 \cos b\delta_1 \ . \tag{7}$$

Suppose that at some point in the integration, or extrapolation into time, we arrived at x-distributions of θ at two successive time steps in which

$$c_{2n} = c_{2n+2}$$

and

$$\sin b\,\delta_{2n+1} = R \sin b\,\Delta x$$

Then, according to (5),

$$\delta_{2n+3} = \delta_{2n+1}$$

and according to (6),

$$c_{2n+4} = c_{2n+2}$$

We would then have arrived at an equilibrium state in the computation in which the solution of the finite difference equation would differ from the solution of the differential equation only in that the phase speed would be somewhat smaller than U, and the amplitude might differ from the initial amplitude. If we considered a group of cosine waves, each in the equilibrium state defined above, the group would disperse, for the equilibrium phase change depends on the wave number, b.

It will be convenient to invent two parameters, which will be called "equilibrium amplitude, c_e" and "equilibrium phase change, δ_e". These parameters are defined as follows.

$$\sin b\,\delta_e = R \sin b\,\Delta x \qquad (8)$$

$$c_e^2 \cos b\,\delta_e = c_0 c_2 \cos b\,\delta_1 \qquad (9)$$

(7) and (9) imply that

$$c_{2n}\, c_{2n+2} \cos b\,\delta_{2n+1} = c_e^2 \cos b\,\delta_e \qquad (10)$$

(10) relates two successive amplitudes and the phase change between them. From (5), (8) and (10), the following relation among two successive phase changes and the amplitude between them may be derived.

$$\tan b\,\delta_{2n-1} + \tan b\,\delta_{2n+1} = 2\left(\frac{c_{2n}}{c_\ell}\right)^2 \tan b\,\delta_\ell \,. \tag{11}$$

By squaring (11), rearranging, and squaring again, an expression can be derived in which appear only even powers of the tangents of the phase changes. The phase changes can then be conveniently eliminated by means of (10). The result is

$$\begin{aligned}(C_{2n}^2 - C_{2n+4}^2)^2 &- 8\,c_{2n+2}^2\,(c_{2n}^2 + c_{2n+4}^2)\sin^2 b\,\delta_\ell \\ &+ 16\,c_{2n+2}^4 \sin^4 b\,\delta_\ell \\ &+ 16\,c_\ell^4 \sin^2 b\,\delta_\ell \cos^2 b\,\delta_\ell = 0\,.\end{aligned} \tag{12}$$

(12) relates three successive amplitudes.

A numerical experiment was performed which indicated that

$$C_{2n}^2 = B + A\cos\beta\,(n-\bar{n}) \tag{13}$$

where B, A, β, $\bar{n}$ are constants characteristic of the particular integration being performed. The truth of (13) can be demonstrated by substituting (13) into (12). After some rearrangement the result of this substitution is

$$\begin{aligned}&A^2\left\{\begin{array}{l}-(1-\cos^2 2\beta)\\ -4\sin^2 b\,\delta_\ell\cos 2\beta\\ +4\sin^4 b\,\delta_\ell\end{array}\right\}\cos^2\beta\,(n+2-\bar{n})\\ &+4AB\sin^2 b\,\delta_\ell\left\{\begin{array}{l}-(1+\cos 2\beta)\\ +2\sin^2 b\,\delta_\ell\end{array}\right\}\cos\beta\,(n+2-\bar{n})\\ &+\left\{\begin{array}{l}A^2(1-\cos^2 2\beta)\\ -4B^2\sin^2 b\,\delta_\ell\,(1-\sin^2 b\,\delta_\ell)\\ +4\,c_\ell^4\sin^2 b\,\delta_\ell\cos^2 b\,\delta_\ell\end{array}\right\} = 0\end{aligned} \tag{14}$$

(13) is true if the coefficient of each power of $\cos\beta(n+2-\bar{n})$ in (14) is zero, for then the function (13) fits the three successive values

of C in (12), regardless of the value of n. It can easily be shown that the three coefficients in (14) are individually zero if

$$2\beta = \pi - 2b\,\delta_2 \qquad (15)$$

and $$B^2 - A^2 = c_2^4 \; .$$

One might suspect that δ_{2n+1} is also an oscillating function of n. To show this, we first find from (13) that

$$c_{2n}^2\, c_{2n+2}^2 = B^2 - A^2 + A^2\cos^2\beta + 2AB\cos\beta\,\cos\beta(n+1-\bar{n}) + A^2\cos^2\beta(n+1-\bar{n}) \; .$$

If this be solved for $\cos\beta(n+1-\bar{n})$,

$$\cos\beta(n+1-\bar{n}) = -\frac{B}{A}\cos\beta \pm \frac{1}{A}\sqrt{c_{2n}^2\,c_{2n+2}^2 - (B^2-A^2)\sin^2\beta}$$

or, according to (10) and (15),

$$\pm\tan b\,\delta_{2n+1} = \frac{B}{c_2^2}\tan b\,\delta_2 + \frac{A}{c_2^2\cos b\,\delta_2}\cos\beta(n+1-\bar{n})$$

By a substitution from this and from (13) into (11) it can be shown that the upper sign is the only correct one in each of the two preceding equations. Thus,

$$\tan b\,\delta_{2n+1} = \frac{B}{c_2^2}\tan b\,\delta_2 + \frac{A}{c_2^2\cos b\,\delta_2}\cos\beta(n+1-\bar{n}) \qquad (16)$$

β has been determined in (15), but we have yet to determine A, B and $\bar{n}$. From (13),

$$c_{2n}^2 + c_{2n+2}^2 = 2B + 2A\cos\beta\,\cos\beta(n+1-\bar{n})$$

By substituting from this for the value of $\cos\beta(n+1-\bar{n})$ in (16), and making obvious substitutions from (15), we find that

$$B = \frac{C_0^2 + C_2^2}{2\cos^2 b\,\delta_2} - C_2^2 \tan b\,\delta_2 \tan b\,\delta_1 \tag{17}$$

A is readily determined from (15) and (17). From (13),

$$\cos \beta \pi_n = \frac{C_0^2 - B}{A} \quad . \tag{18}$$

<u>Discussion</u>

Courant-Friedrichs computational instability criteria for hyperbolic equations is exhibited in the solution when extended to a general x-distribution of θ in which the spectrum extends over all wave numbers, for according to (8),

$$\sin b\,\delta_2 \leq 1$$

for all wave numbers only if

$$R \leq 1 .$$

In the numerical solutions of the meteorological forecasting problem computed in the past, small oscillations in the solution have been observed. These oscillations have a wave length of $2\Delta t$. It is suggested by the writer that these are of the same nature as the oscillation described in this simpler problem. Note that, according to (15), in the case of very small wave numbers, the wave length of the oscillation is equal to an increment of 4 in n, i.e., two time intervals. For larger wave numbers, the wave length is greater than 4. More precisely, the wave length depends on $b\,\delta_2$, which in turn depends on $\frac{\Delta x}{L}$ and R, according to (9). Here L is the wave length.

For convenience, we may rearrange (15).

$$\beta = \frac{\pi}{2}\left(1 - \frac{2}{\pi} b\,\delta_x\right) \tag{19}$$

Figure 1 shows the dependence of the "correction factor", $\frac{2}{\pi} b\,\delta_x$ on $\frac{\Delta x}{L}$ and R. Reasons are outlined in the Appendix for not extending the graph to the region where $\frac{\Delta x}{L} < \frac{1}{4}$. It may be noted that if R is near one, the "correction factor" may be considerable, even for wave lengths as long as $10\,\Delta x$.

(9) can be written

$$\sin\left(2\pi \frac{\delta_x}{U\Delta t} R \frac{\Delta x}{L}\right) = R \sin\left(2\pi \frac{\Delta x}{L}\right).$$

Thus, $\frac{\delta_x}{U\Delta t}$ can be related to R and $\frac{\Delta x}{L}$. Figure 2 shows this relationship.

In studying A and B, it is advantageous to specify how the extrapolation into time is begun, and thereby reduce the parameters to $\frac{\Delta x}{L}$ and R. Figures 3, 4, and 5 show the variation of A, B, and $A+B$ with $\frac{\Delta x}{L}$ and R, when the first extrapolation is uncentered, and is made from an observation of the x-distribution of θ at time $t_{2n} = t_2$. c_2 is, therefore, the amplitude of the wave in the initial data. It can easily be shown that such a procedure is equivalent to manufacturing an x-distribution of θ at time $t_{2n} = t_0$ by means of the following equations.

$$\tan b\,\delta_1 = R \sin b\,\Delta x$$

$$c_0 = \frac{c_2}{\cos b\,\delta_1}$$

Perhaps the most striking feature of these graphs are the maximums of both A and B at wave lengths of $4\,\Delta x$. This indicates that some benefit could be obtained by filtering out wave-lengths near $4\ \Delta x$.

The behavior of waves identified with R near one and $\frac{\Delta x}{L}$ near $\frac{1}{4}$ is reminiscent of the "peaking" which has been observed in forecasting conservative quantities in previous machine computations. In this region, the wave number of the oscillation of c is very small (Figure 1), and the amplitude is large without limit (Figure 5).

Appendix. Note on the resolving power of a finite difference net.

Consider a sine distribution of θ.

$$\theta = \beta + \alpha \sin \nu (x - \bar{x})$$
$$= \beta + \alpha (\cos \nu \bar{x} \sin \nu x - \sin \nu \bar{x} \cos \nu x)$$

$\sin \nu x$ and $\cos \nu x$ at discrete points which are equally spaced can be generated from the following identities,

$$\cos n\nu\Delta x = 2\cos(n-1)\nu\Delta x \cos \nu\Delta x - \cos(n-2)\nu\Delta x$$
$$\sin n\nu\Delta x = 2\cos(n-1)\nu\Delta x \cos \nu\Delta x - \sin(n-2)\nu\Delta x$$

where $x = n\Delta x$. Thus, at these points $\sin \nu x$ and $\cos \nu x$ can ultimately be expressed in terms of n, $\sin \nu\Delta x$ and $\cos \nu\Delta x$.

It follows that the finite difference net cannot distinguish among components whose wave numbers satisfy the relation

$$\nu_N \Delta x = N \cdot 2\pi + \nu_0 \Delta x$$

where

$$N = \pm 0, 1, 2, \ldots$$

The foregoing equation can be written

$$\frac{\Delta x}{L_N} = N + \frac{\Delta x}{L_0}$$

where L is the wave length corresponding to the wave number ν. When written in this way, we see that there is a wave length

$$|L_0| \geq 2\Delta x \quad ,$$

which the finite difference net cannot distinguish from a wave length

$$|L_N| \leq 2\Delta x \quad ,$$

for the reason that a number cannot differ from the nearest integer by more than one half.

Therefore, we may restrict discussion to components whose wave length is equal to or greater than $2\Delta x$, or to wave numbers in the following range.

$$0 \leq \nu \leq \frac{\pi}{\Delta x}$$

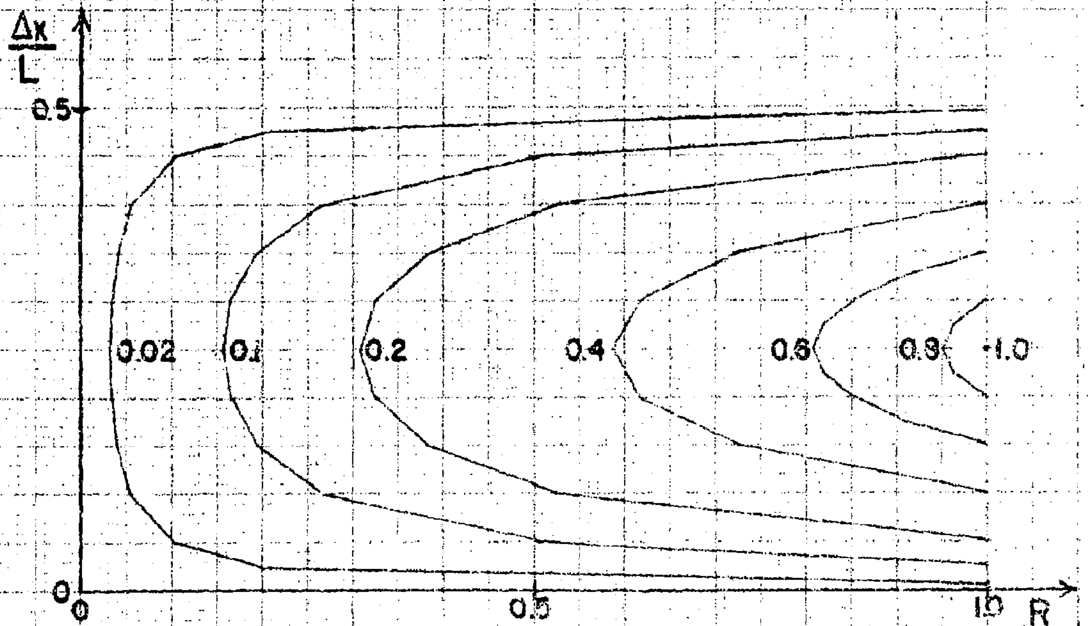

Figure 1. $\frac{2}{\pi} b \delta_e$

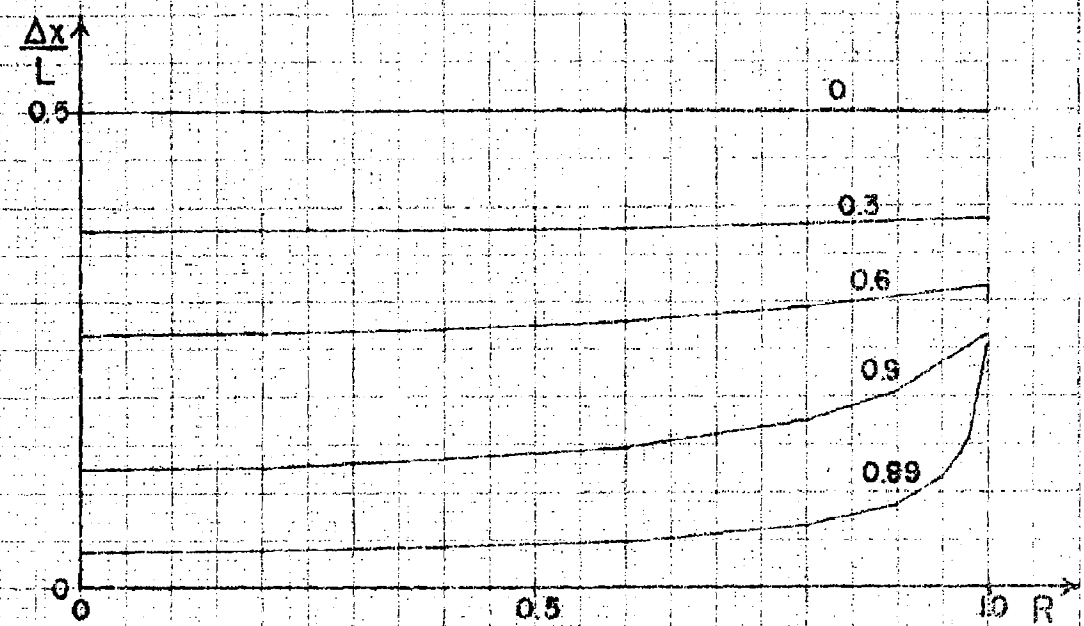

Figure 2. $\frac{\delta_e}{U \Delta t}$

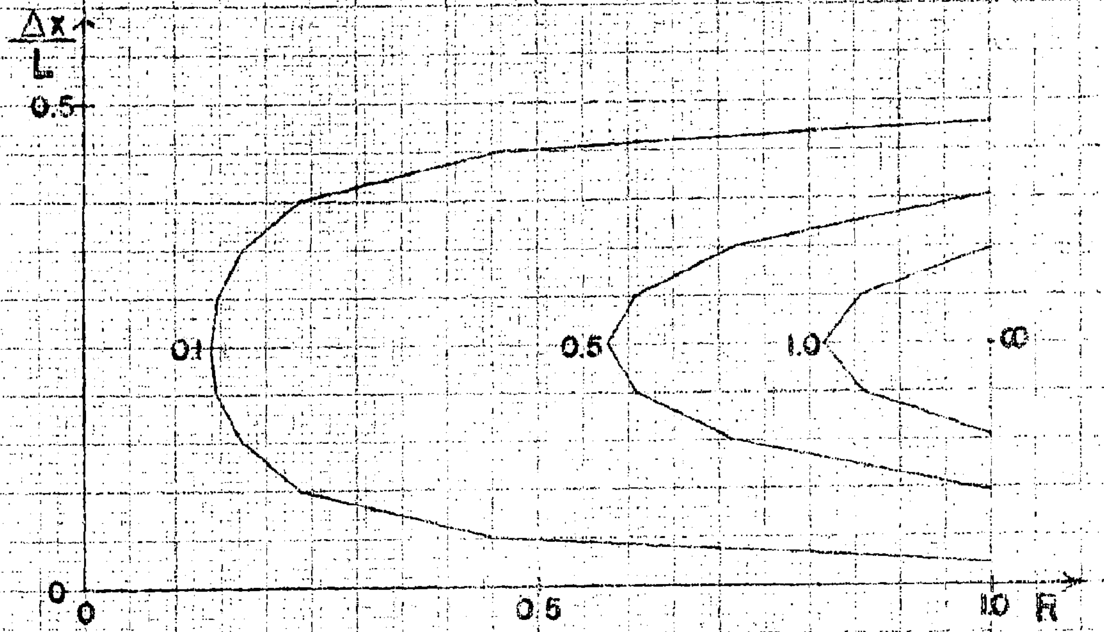

Figure 3. $\frac{\sqrt{A}}{c_2}$

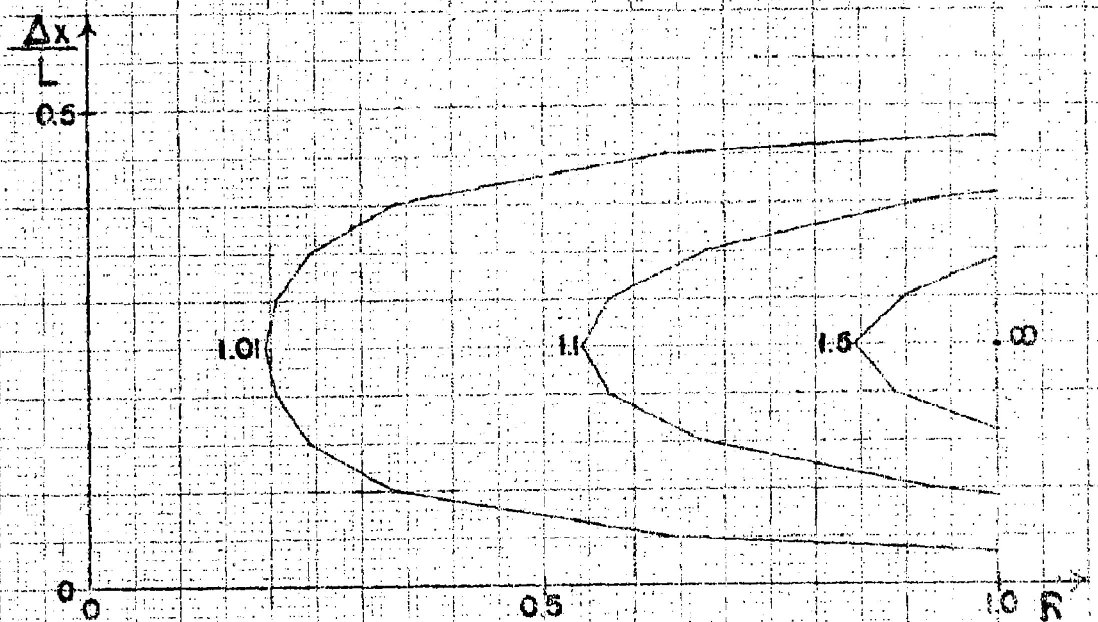

Figure 4. $\frac{\sqrt{B}}{c_2}$

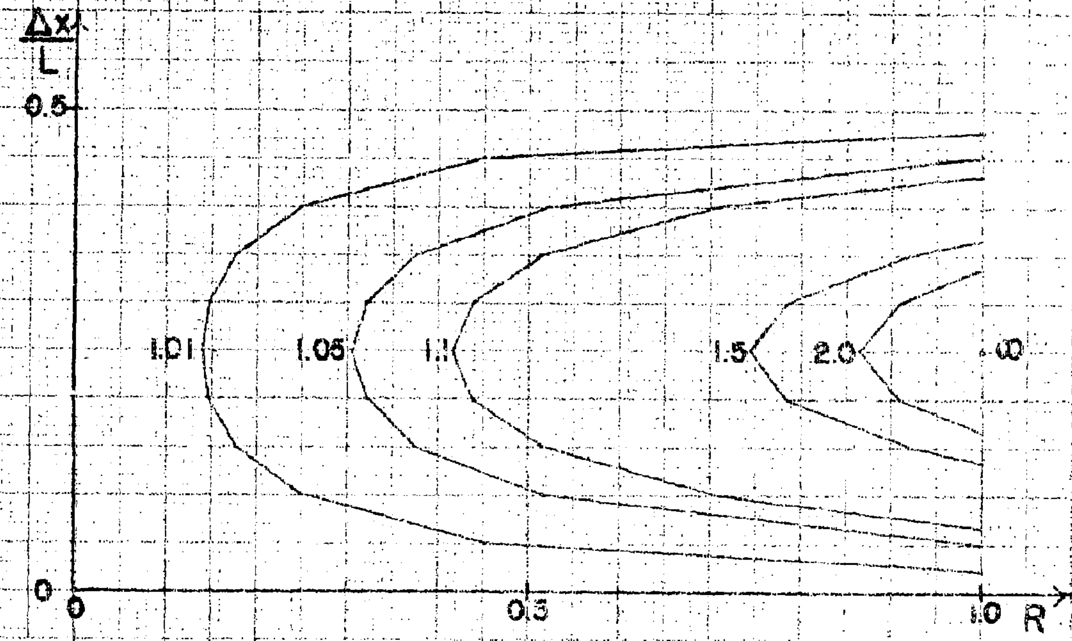

Figure 5. $\frac{\sqrt{A+B}}{c_2}$