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Joint Numerical Weather Prediction Unit

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Reduction of Truncation Errors in the Computation of Geostrophic Vorticity, the Laplacian Operator and Its Inverse

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I. Introduction

In an earlier report*, the author has discussed the systematic error in computing Jacobians by the usual method of finite-differences---i.e., that part of the truncation error which does not depend on the orientation of the finite-difference grid. By averaging the truncation error over all possible orientations of the grid, it was found that there is a residual error which tends to make the finite-difference equivalent of the Jacobian a systematic underestimate of the true Jacobian. Further, it was found that the systematic or direction-averaged truncation error can be reduced to tolerable proportions by subjecting both members of the Jacobian to a process of "inverse-averaging". Finally, we have proposed a simple scheme by which the required inverse averages can be computed rapidly and with no severe demands on high speed storage.

The purpose of this report is to apply the same general methods to the problem of identifying and reducing the systematic truncation error in computing the Laplacian of any variable by the method of finite-differences. This problem, of course, frequently arises in the computation of "geostrophic vorticity" from the geopotential at an array of regularly spaced points. This is, neglecting small variations of the Coriolis parameter,

$$\zeta_g = \frac{1}{f} \nabla^2 \phi$$

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*Tech. Memo. No. 1, JNWP Unit.

where ζ_g is the vertical component of the curl of the geostrophic wind, f is the Coriolis parameter, ∇ is the horizontal vector gradient, and ϕ is the geopotential. The usual method of computing the geostrophic vorticity ζ_g consists in replacing the derivatives that enter into $\nabla^2\phi$ by corresponding ratios of finite-differences.

$$\mathbb{V}^2\phi = \frac{1}{d^2}\Big[\phi(s+d,n) + \phi(s-d,n) + \phi(s,n+d) + \phi(s,n-d) - 4\phi(s,n)\Big]$$

in which d is the distance between adjacent points in a square grid; s and n are rectangular coordinates in horizontal planes, the positive n - axis lying to the left when the observer is facing the positive s-direction. The symbol $\mathbb{V}^2$ is used to distinguish the finite-difference equivalent of the Laplacian from the true Laplacian ∇^2. In general, $\mathbb{V}^2$ and ∇^2 are not equal, so that the usual method of computing the geostrophic vorticity invariably introduces some error. Our present concern is to isolate the systematic part of this error and to devise a means of reducing it.

II. The Nature and Effect of the Direction-Averaged Truncation Error

Following the procedure outlined in the previous report, we shall first derive an approximate expression for the truncation error corresponding to an arbitrary orientation of the grid — i.e., an arbitrary orientation of the s and n axes. Such an expression is most easily obtained by expanding $\phi(s+d,n)$, $\phi(s-d,n)$, $\phi(s,n+d)$ and $\phi(s,n-d)$ in Taylor series around (s, n). Thus

$$\phi(s+d,m) - \phi(s,m) = d\frac{\partial\phi}{\partial s} + \frac{d^2}{2}\frac{\partial^2\phi}{\partial s^2} + \frac{d^3}{6}\frac{\partial^3\phi}{\partial s^3} + \frac{d^4}{24}\frac{\partial^4\phi}{\partial s^4} + \cdots$$

$$\phi(s-d,m) - \phi(s,m) = -d\frac{\partial\phi}{\partial s} + \frac{d^2}{2}\frac{\partial^2\phi}{\partial s^2} - \frac{d^3}{6}\frac{\partial^3\phi}{\partial s^3} + \frac{d^4}{24}\frac{\partial^4\phi}{\partial s^4} - \cdots$$

$$\phi(s,m+d) - \phi(s,m) = d\frac{\partial\phi}{\partial m} + \frac{d^2}{2}\frac{\partial^2\phi}{\partial m^2} + \frac{d^3}{6}\frac{\partial^3\phi}{\partial m^3} + \frac{d^4}{24}\frac{\partial^4\phi}{\partial m^4} + \cdots$$

$$\phi(s,m-d) - \phi(s,m) = -d\frac{\partial\phi}{\partial m} + \frac{d^2}{2}\frac{\partial^2\phi}{\partial m^2} - \frac{d^3}{6}\frac{\partial^3\phi}{\partial m^3} + \frac{d^4}{24}\frac{\partial^4\phi}{\partial m^4} - \cdots$$

All derivatives in the equations above apply at the point (s, n). Adding the four equations above, dividing by d^2, and omitting terms of order higher than d^2, we find that

$$\mathbb{V}^2\phi = \nabla^2\phi + \frac{d^2}{12}\left(\frac{\partial^4\phi}{\partial s^4} + \frac{\partial^4\phi}{\partial m^4}\right) \qquad (1)$$

The second term on the righthand side of Eq. (1) is, of course, a measure of the truncation error corresponding to an arbitrary orientation of the s and n axes.

The next step is to compute the systematic or direction-averaged truncation error by averaging both sides of Eq. (1) over all possible orientations of the grid. This may be accomplished simply by letting the s - axis subtend a variable angle θ with the x - axis in a <u>fixed</u> cartesian coordinate system (x, y), and by averaging with respect to θ. Before carrying out the integration, it is first necessary to express all derivatives with respect to s and n in terms of θ and derivatives with respect to x and y. This is effected by the transformations:

$$\frac{\partial^4}{\partial s^4} = \cos^4\theta \frac{\partial^4}{\partial x^4} + 4\cos^3\theta \sin\theta \frac{\partial^4}{\partial x^3 \partial y} + 6\cos^2\theta \sin^2\theta \frac{\partial^4}{\partial x^2 \partial y^2}$$

$$+ 4\cos\theta \sin^3\theta \frac{\partial^4}{\partial x \partial y^3} + \sin^4\theta \frac{\partial^4}{\partial y^4}$$

$$\frac{\partial^4}{\partial m^4} = \sin^4\theta \frac{\partial^4}{\partial x^4} - 4\sin^3\theta \cos\theta \frac{\partial^4}{\partial x^3 \partial y} + 6\sin^2\theta \cos^2\theta \frac{\partial^4}{\partial x^2 \partial y^2}$$

$$- 4\sin\theta \cos^3\theta \frac{\partial^4}{\partial x \partial y^3} + \cos^4\theta \frac{\partial^4}{\partial y^4}$$

Finally, substituting these expressions in the second term on the right-hand side of Eq. (1), integrating with respect to θ between limits 0 and 2π, and recalling that $\nabla^2\phi$ is actually independent of the orientation of the coordinate axes,

$$\overline{\nabla^2\phi} = \nabla^2\phi + \frac{d^2}{16}\left(\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2 \partial y^2} + \frac{\partial^4\phi}{\partial y^4} \right)$$

$$= \nabla^2\phi + \frac{d^2}{16}\nabla^2\nabla^2\phi \qquad (2)$$

The subscript "a" denotes an average value, integrated over all possible orientations of the finite-difference grid. It must be emphasized that the relationship above holds for any variable ϕ. For this reason, and because $\nabla^2\phi$ generally correlates negatively with its Laplacian, Eq. (2) states that the overall effect of truncation errors is to make $\overline{\nabla^2\phi}$ a systematic underestimate of $\nabla^2\phi$.

III. A Scheme for Reducing the Truncation Error

The strategy for reducing truncation error will be laid out in such a way as to minimize the systematic or direction-averaged error, with the assurance that the remaining nonsystematic errors will not tend to accumulate. Since the overall effect of truncation error is to underestimate the amplitude of $\nabla^2\phi$, it is natural to seek some "unsmoothing" process whereby the reduction in amplitude is compensated. Such smoothing operations, in fact, have already been suggested by the author in a recent paper in Tellus (1954). The fundamental operational property of the "averaging" operators discussed in that article is:

$$\nabla^2\bar{\phi} = \alpha^2(\bar{\phi} - \phi) \tag{3}$$

where ϕ is any typical variable, α is an adjustable scale parameter, and the barred quantities are weighted area averages, defined as follows:

$$\bar{\phi}(x,y) = \frac{\alpha^2}{2\pi}\iint K_0(\alpha r)\,\phi(\xi,\eta)\,d\xi\,d\eta$$

in which K_0 is the zero-order Bessel function of the second kind with imaginary argument, $r^2 = (x-\xi)^2 + (y-\eta)^2$, and ξ and η are dummy variables corresponding to x and y. The integration is taken over the entire (ξ,η) plane. Letting $\alpha = \frac{4}{d}$, we may rewrite Eq. (3) as

$$\bar{\Phi} = \phi + \frac{d^2}{16} \nabla^2 \bar{\phi}$$

In the case of "weak averaging" —— i.e., when d is much less than the distance between adjacent maxima and minima of —— the equation above takes the approximate form

$$\bar{\bar{\phi}} = \phi + \frac{d^2}{16} \nabla^2 \phi$$

Finally, combining the equation above with Eq. (2)

$$\nabla^2 \bar{\phi} = \mathbb{V}_a^2 \phi \qquad (4)$$

This equation, like Eq. (1), is most nearly exact at the long wave-length end of the spectrum.

Eq. (4) provides the basis for a simple method of reducing truncation errors in the computation of $\nabla^2 \phi$. Since ϕ is unspecified, we may rewrite Eq. (4) as

$$\nabla^2 \phi = \mathbb{V}_a^2 \tilde{\phi} \qquad (4a)$$

where $\bar{\tilde{\phi}} = \tilde{\bar{\phi}} = \phi$. That is, $\tilde{\phi}$ is a function such that, when the averaging operator is applied to it, it yields the original function ϕ. Thus $\tilde{\phi}$ may be thought of as an "inverse average" of ϕ. Viewed in this light, Eq. (4a) states that $\nabla^2 \phi$ may be computed with a minimum of truncation error by:

(1) Computing $\tilde{\tilde{\phi}}$

(2) Computing $\nabla^2 \tilde{\tilde{\phi}}$ for a number of grid orientations, and

(3) Averaging $\nabla^2 \tilde{\tilde{\phi}}$ over the various grid orientations. With reference to Steps (2) and (3), it is probable that one or two grid orientations are adequate —— e.g., it may be sufficient to compute $\nabla^2 \tilde{\tilde{\phi}}$ by differencing once along the rows and columns of a fixed grid, and again along the diagonals of the same grid. In any event, even if $\nabla^2 \tilde{\tilde{\phi}}$ is computed for only one grid orientation, the <u>systematic</u> part of the truncation error will be removed.

The "inverse averaging" indicated under Step (1) of the procedure outlined above can be accomplished by making use of Eq. (3). Replacing ϕ by $\tilde{\tilde{\phi}}$ and letting $x = \frac{4}{d}$,

$$\tilde{\tilde{\phi}} = \phi - \frac{d^2}{16} \nabla^2 \phi$$

Thus, computing $\nabla^2 \phi$ by the usual finite-difference method,

$$\tilde{\tilde{\phi}} = \phi_0 - \frac{1}{16}\left(\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0 \right)$$

$$= \frac{1}{16}\left[4(4\phi_0 + \phi_0) - \phi_1 - \phi_2 - \phi_3 - \phi_4 \right]$$

where ϕ_0 is the value of ϕ at the point where $\tilde{\tilde{\phi}}$ is to be computed, and the subscripts 1, 2, 3, and 4 refer to the values of ϕ at the four surrounding points. Since the formula above involves multiplications by powers of 2 only, $\tilde{\tilde{\phi}}$ may be computed by any binary machine through the execution of only the orders "reset accumulator and add", "add", "subtract", "shift accumulator left", "shift accumu-

lator right" and "store accumulator" —— without the use of erasible memory for storing intermediate results.

IV. The Reduction of Truncation Error in the Solution of the Poisson Equation.

The results of the previous section can be applied to the problem of reducing the systematic truncation error in the solution of the equation

$$\nabla^2 \phi = F(x, y) \tag{5}$$

where $F(x, y)$ is a known function. The usual method of solution consists in replacing $\nabla^2 \phi$ by $\mathbb{V}^2 \phi$ and solving the resulting finite-difference equation by a purely numerical method (e.g., the relaxation method) —— a practice that introduces serious error if the difference between $\nabla^2 \phi$ and $\mathbb{V}^2 \phi$ is systematic. Since we are concerned primarily with the reduction of systematic errors, we may omit the subscript in Eq. (4a). This equation, when combined with Eq. (5), then gives

$$\mathbb{V}^2 \tilde{\tilde{\phi}} = F(x, y)$$

This equation states that the method of solution outlined above does not yield ϕ, but actually yields $\tilde{\tilde{\phi}}$. Accordingly, a more nearly correct value of ϕ may be obtained by applying the averaging operator to $\tilde{\tilde{\phi}}$, the solution of the usual finite-difference form of the Poisson equation. Formally, at least, the averaging operation re-

quires integration over the entire plane; however, the weighting function K_0 diminishes very rapidly beyond a distance d away from the point where the average applies, so that the area over which the average is taken is in reality quite small.

As one would expect, the truncation errors in the computation of the Laplacian and its exact inverse cancel, a fact that is verified by the mutually inverse relation between the operators $(\widetilde{\ })$ and $(\overline{\ })$. In practice, of course, the inversion of the Laplacian operator is not performed immediately after the Laplacian is computed, but after the distribution of $\nabla^2\phi$ has been extrapolated in time, whence the cancellation of errors is not complete. Nevertheless, provided the time intervals over which the extrapolation is carried out are small enough, the truncation errors in computing the Laplacian and its inverse do tend to cancel, in which case it may be unnecessary to reduce the errors in each operation by the methods proposed here.

Reference

Thompson, P.D., 1954: Prognostic Equations for the Mean Motions of Simple Fluid Systems and Their Relation to the Theory of Large-Scale Atmospheric Turbulence, Tellus, Vol. 6, No. 2, pp. 150-164.