Technical Memorandum No. 5

Joint Numerical Weather Prediction Unit

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An Iterative Method for Solving an Elliptic Equation With Variable Coefficients And Its Convergence Criterion

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no. 5

8 March 1955

National Oceanic and Atmospheric Administration

U.S. Joint Numerical Weather Prediction Unit

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April 13, 2004

I. Introduction

One of the key equations in the theory of large scale nongeostrophic motions of the atmosphere is of the form:

$$(\nabla^2 - \nu^2)\,\nabla^2\phi = \nu^2\,(\nabla\phi + \mathbb{V}_0)\cdot\nabla Z \qquad (1)$$

where ∇ is the horizontal vector derivative, ν is a constant measuring the atmosphere's static stability, ϕ is the velocity potential for the irrotational part of the horizontal motion, $\mathbb{V}_0$ is the nondivergent part of the horizontal velocity, and Z is the ratio of the absolute vorticity to the Coriolis parameter (the vertical component of the earth's vorticity).

In the development of methods for predicting the large-scale nongeostrophic motions, Eq. (1) is regarded as a linear equation in which ϕ is the (unknown) dependent variable, and $\mathbb{V}_0$ and Z are known functions of the horizontal coordinates. The mathematical problem, to state it briefly, is to construct a function ϕ that satisfies Eq. (1) everywhere in the interior of a closed region, and which yields prescribed values of ϕ and $\nabla^2\phi$ on the boundary of the region.

Owing to the fact that the coefficients of the first order terms in Eq. (1) vary from point to point in an unprescribed manner, it is generally impossible to solve Eq. (1) in closed finite form. Accordingly, it is necessary to resort to approximate methods, of which the most straightforward is probably the relaxation method. The most direct application of the relaxation method, however, suffers from certain disadvantages. Aside from the inconvenience of referring to a variable relaxation pattern, there is no absolute assurance (short of mathematical proof) that the sequence of approximations generated by the relaxation procedure converges. Moreover, since the coef-

ficients of Eq. (1) are variable, it is difficult to demonstrate that the convergence of this sequence is a mathematical certainty. Of course, the mere absence of a proof of convergence should not necessarily be taken as grounds for rejecting relaxation methods, for it may simply reflect our ineptness at producing rigorous justification for a method that is otherwise of obvious value. Nevertheless, it would be preferable to develope a method for generating a sequence of approximations whose convergence is assured, particularly so when the procedure would not normally be repeated enough times to establish reasonable belief in its convergence.

2. A Method of Successive Approximations

An alternative method for solving Eq. (1) is suggested by the fact that the lowest order terms are the only ones whose coefficients are variable, whereas the general character of the problem and its solution is probably governed by the terms of higher order. Briefly described, the method is this:

1) Beginning with an arbitrarily chosen "guess" at the values of ϕ over the closed region, compute the right hand side of Eq. (1).

2) Regarding the righthand side of Eq. (1) as known, solve Eq. (1) for ϕ, subject to appropriate boundary conditions on ϕ and $\nabla^2\phi$. It is known that the relaxation method generates a convergent sequence in this case.

3) Substitute the result of the previous step into the righthand side of Eq. (1) and return to Step (2).

This cyclic procedure is repeated until the difference between the last two approximations to ϕ is less than the error that can be tolerated. The last approximation is taken as the solution to Eq. (1). The remaining questions are whether or not the sequence of approximations generated by this procedure

converges toward some definite function and, if so, whether or not the limit is the solution of Eq. (1).

In mathematical terms, the sequence of approximations $\phi_0, \phi_1, \phi_2, \cdots \phi_m$ generated by this method are defined by the following equations:

$$
\begin{aligned}
L(\phi_1) &= \nu^2(\nabla\phi_0 + \mathbb{V}_0)\cdot\nabla Z \\
L(\phi_2) &= \nu^2(\nabla\phi_1 + \mathbb{V}_0)\cdot\nabla Z \\
&\vdots \\
L(\phi_m) &= \nu^2(\nabla\phi_{m-1} + \mathbb{V}_0)\cdot\nabla Z
\end{aligned}
$$

where L( ) is the linear operator $(\nabla^2 - \nu^2)\nabla^2$. A more useful system of equations is obtained by subtracting the ith equation in the array above from the (i + 1)st equation. Thus,

$$
\begin{aligned}
L(\Phi_1) &= \nu^2\nabla\Phi_0\cdot\nabla Z \\
L(\Phi_2) &= \nu^2\nabla\Phi_1\cdot\nabla Z \\
&\vdots \\
L(\Phi_m) &= \nu^2\nabla\Phi_{m-1}\cdot\nabla Z
\end{aligned}
\tag{2}
$$

in which $\Phi_i = \phi_{i+1} - \phi_i$. The question of convergence is clearly reducible to asking whether or not Φ_m approaches zero as m becomes infinite. Moreover, if the difference Φ_m between successive approximations ϕ_m and ϕ_{m+1} is zero, then

$$
\begin{aligned}
L(\phi_{m+1}) &= \nu^2(\nabla\phi_m + \mathbb{V}_0)\cdot\nabla Z \\
&= \nu^2(\nabla\phi_{m+1} + \mathbb{V}_0)\cdot\nabla Z
\end{aligned}
$$

This equation simply states that ϕ_{n+1} (or ϕ_n) is the solution of Eq. (1). In summary, the guarantee that this method is feasible hinges entirely on being able to show that Φ_n approaches zero as n becomes infinite.

3. <u>The Convergence Criterion</u>

We shall prove that Φ_n does, in fact, vanish as n becomes larger and larger, by showing that

$$\iint \nabla \Phi_n \cdot \nabla \Phi_n \, dx\,dy < k \iint \nabla \Phi_{n-1} \cdot \nabla \Phi_{n-1} \, dx\,dy \qquad (3)$$

for all n, where the integration is to be carried out over the closed region, x and y are rectangular coordinates, and k is a positive constant less than unity. If this can be shown, then it follows that

$$\lim_{n \to \infty} \iint \nabla \Phi_n \cdot \nabla \Phi_n \, dx\,dy = 0$$

The condition above can be met if and only if $\lim_{n \to \infty} \Phi_n$ has the same value at all points in the closed region. In particular, since Φ_n must vanish on the boundary, $\lim_{n \to \infty} \Phi_n$ is zero everywhere.

To simplify the derivation of Eq.(3) from Eq. (2), we shall extend the boundary of the closed region out to infinity, and will require that Φ_n vanish at infinity.* In this case, we may make use of the inversion methods discussed by the author in a recent article in Tellus (1954). The power of these techniques stems from the properties of a linear integral operator M( )— which, applied to a typical variable ψ, is defined as:

$$M(\psi) = \frac{\nu^2}{2\pi} \iint K_0(\nu r)\, \psi(\xi, \eta)\, d\xi\, d\eta .$$

*It will be noted that the solution of Eq. (1) for an infinite region is not exactly the problem that was originally set. On the other hand, one would expect that the condition for convergence over an infinite region would be, if anything, stronger than the convergence criterion for a finite region.

where K_0 is the zero-order Bessel function of the second kind with imaginary argument; ξ and η are dummy variables corresponding to x and y; and $r^2 = (x-\xi)^2 + (y-\eta)^2$. It will be noted that M( ) is essentially a normalized weighted average, for M(1) = 1. The fundamental property of the operator M( ) is:

$$\nabla^2 M(\psi) = M(\nabla^2\psi) = \nu^2\left[M(\psi) - \psi\right]$$

Applying the operator M( ) to a representative member of the array of Eqs. (2), we find that

$$M\left[(\nabla^2 - \nu^2)\nabla^2\Phi_m\right] = \nu^2\left[\nabla^2 M(\Phi_m) - \nabla^2\Phi_m\right] - \nu^2\nabla^2 M(\Phi_m)$$

$$= \nu^2 M(\nabla\Phi_{m-1}\cdot\nabla Z)$$

whence

$$\nabla^2\Phi_m = -M(\nabla\Phi_{m-1}\cdot\nabla Z) \equiv -Q \qquad (4)$$

Moreover, applying M( ) to the equation above p successive times, we may generate the following array of equations:

$$\nu^2\left[M(\Phi_m) - \Phi_m\right] = -M(Q)$$

$$\nu^2\left[M^2(\Phi_m) - M(\Phi_m)\right] = -M^2(Q)$$

$$\vdots \qquad\qquad \vdots$$

$$\nu^2\left[M^p(\Phi_m) - M^{p-1}(\Phi_m)\right] = -M^p(Q)$$

We next add together all the members of this array, noting that the first term on the lefthand side of the ith equation just cancels the second term of the (i + 1)st. Thus,

$$\nu^2\left[M^p(\Phi_m) - \Phi_m\right] = -\sum_{i=1}^{p} M^i(Q)$$

Since M( ) is simply a normalized average with positive weighting, $M^p(\Phi_m)$ must approach some definite limit as p becomes infinite. In fact, because Φ_m vanishes on the boundary at infinity, $M^p(\Phi_m)$ approaches zero as p is made larger and larger. Letting p become infinite, then

$$\Phi_m = \frac{1}{\nu^2} \sum_{i=1}^{\infty} M^i(Q) \tag{5}$$

For the present, it is assumed (and is later shown) that the summation in Eq. (5) converges. Multiplying Eqs. (4) and (5), and averaging over the entire (x,y) plane,

$$l.i.m. \iint \Phi_m \nabla^2 \Phi_m \, dx\,dy = l.i.m. \iint -\frac{1}{\nu^2} Q \sum_{i=1}^{\infty} M^i(Q) \, dx\,dy \tag{6}$$

The symbol " l .i.m." is the limit-in-the mean——the limit of the area average as the periphery of the area is extended out to infinity. The left hand side of Eq. (6) may be written in more convenient form by noting that

$$l.i.m. \iint \Phi_m \nabla^2 \Phi_m \, dx\,dy = l.i.m. \iint (\nabla \cdot \Phi_m \nabla \Phi_m - \nabla \Phi_m \cdot \nabla \Phi_m) \, dx\,dy$$

The first integral on the righthand side of this equation can be transformed by Gauss' Theorem into a line integral taken around the boundary at infinity. Since Φ_m is zero on the boundary, that integral vanishes. Thus, introducing this result into Eq. (6),

$$l.i.m. \iint \nabla \Phi_m \cdot \nabla \Phi_m \, dx\,dy = l.i.m. \iint \frac{1}{\nu^2} Q \sum_{i=1}^{\infty} M^i(Q) \, dx\,dy \tag{7}$$

It remains to show that the righthand side of Eq. (7) is, under predeterminable conditions, less than

$$l.i.m. \iint k \nabla \Phi_{m-1} \cdot \nabla \Phi_{m-1} \, dx\,dy$$

By introducing the original definition of the operator M( ), and inverting the order of successive integrations, it is easily shown that

$$\sum_{i=1}^{\infty} M^{i}(Q) = \iint Q(\xi,\eta)\, H(x-\xi, y-\eta)\, d\xi\, d\eta \tag{8}$$

where

$$H(x-\xi, y-\eta) = \sum_{i=1}^{\infty} G_i(x-\xi, y-\eta)$$

and the G_i are given by the recursion formula

$$G_i = \iint G_1(\xi-X, \eta-Y)\, G_{i-1}(X-x, Y-y)\, dX\, dY$$

$$G_1(x-\xi, y-\eta) = \frac{\nu^2}{2\pi} K_0(\nu r)$$

Substituting from Eq. (8) into Eq. (7),

$$\text{l.i.m.} \iint \nabla\Phi_m \cdot \nabla\Phi_m\, dx\, dy = \text{l.i.m.} \iint \frac{1}{\nu^2} Q(x,y) \iint Q(\xi,\eta)\, H(x-\xi, y-\eta)\, d\xi\, d\eta\, dx\, dy \tag{9}$$

The righthand side of this equation may be rewritten as

$$\text{l.i.m.} \iiiint \frac{1}{\nu^2} Q(x,y)\, Q(\xi,\eta)\, H(x-\xi, y-\eta)\, d\xi\, d\eta\, dx\, dy$$

$$= \frac{1}{\nu^2} \iint H(s,\tau) \left[\text{l.i.m.} \iint Q(x,y)\, Q(x-s, y-\tau)\, dx\, dy \right] ds\, d\tau$$

Now, the expression in square brackets will be recognized as the auto-correlation function, which attains its greatest value when s and τ are zero. Thus, since H is always positive, Eq. (9) implies that

$$\text{l.i.m.} \iint \nabla\Phi_m \cdot \nabla\Phi_m\, dx\, dy < \frac{1}{\nu^2} \iint H(s,\tau) \left[\text{l.i.m.} \iint Q^2(x,y)\, dx\, dy \right] ds\, d\tau$$

$$< \left[\frac{1}{\nu^2} \iint H(s,\tau)\, ds\, d\tau \right] \text{l.i.m.} \iint Q^2(x,y)\, dx\, dy$$

Finally, noting that H is symmetrical around the point (o, o),

$$\text{l.i.m.} \iint \nabla\Phi_m \cdot \nabla\Phi_m \, dx\,dy < \frac{\bar{H}}{r^2} \, \text{l.i.m.} \iint Q^2(x,y) \, dx\,dy \tag{10}$$

where

$$\bar{H} = 2\pi \int_0^\infty \rho H(\rho) \, d\rho \qquad\qquad \rho^2 = s^2 + r^2$$

Since the singularity of H at $\rho = 0$ is logarithmic, and H vanishes exponentially at infinity, $\bar{H}$ converges.

The final link in the proof is provided by the peculiar nature of Q —— which, by definition, is

$$Q = M(\nabla\Phi_{m-1} \cdot \nabla Z)$$

Since M( ) is a normalized average with positive weighting, the Schwarz inequality states that

$$Q^2 < M\left[(\nabla\Phi_{m-1} \cdot \nabla Z)^2\right] \tag{11}$$

Now, according to the rule for expanding vector triple products,

$$\nabla\Phi_{m-1} \cdot \left[\nabla Z(\nabla\Phi_{m-1} \cdot \nabla Z)\right] = (\nabla\Phi_{m-1} \cdot \nabla\Phi_{m-1})(\nabla Z \cdot \nabla Z)$$

$$- (\nabla\Phi_{m-1} \times \nabla Z) \cdot (\nabla\Phi_{m-1} \times \nabla Z)$$

This result, taken together with Eq. (11), implies that

$$Q^2 < M\left[(\nabla\Phi_{m-1} \cdot \nabla\Phi_{m-1})(\nabla Z \cdot \nabla Z)\right]$$

Thus, since the intensities of the gradients of Φ_{m-1} and Z are essentially uncorrelated, Eq. (10) may be written as

$$\text{l.i.m.} \iint \nabla\Phi_m \cdot \nabla\Phi_m \, dx\,dy < \frac{\bar{H}}{\nu^2} \text{ l.i.m.} \iint M(\nabla\Phi_{m-1} \cdot \nabla\Phi_{m-1}) \times M(\nabla Z \cdot \nabla Z) \, dx\,dy$$

$$< \frac{\bar{H}\, M(\nabla Z \cdot \nabla Z)_{max}}{\nu^2} \left(\text{l.i.m.} \iint M(\nabla\Phi_{m-1} \cdot \nabla\Phi_{m-1}) \, dx\,d\right.$$

Finally, noting that the integral of an averaged function is equal to the integral of the unaveraged function,

$$\text{l.i.m.} \iint \nabla\Phi_m \cdot \nabla\Phi_m \, dx\,dy < \lambda \text{ l.i.m.} \iint \nabla\Phi_{m-1} \cdot \nabla\Phi_{m-1} \, dx\,dy \qquad (12)$$

where $$\lambda = \frac{\bar{H}\, M(\nabla Z \cdot \nabla Z)_{max}}{\nu^2}$$

Eq. (12) is equivalent to Eq. (3) for an infinite region of integration.

As pointed out earlier, the condition that λ be less than unity guarantees that the sequence of approximations generated by this iterative procedure will converge to the solution of Eq. (1). In general, since $\bar{H}$ is a universal constant*, convergence is guaranteed if the maximum mean square gradient of absolute vorticity does not exceed a certain critical value, which is independent of the solution of Eq. (1). Although the convergence criterion is conditional, it is probable that the critical value of λ is somewhat greater than unity, since Eq. (12) was arrived at through a long chain of

*By introducing nondimensional variables (formed by multiplying all lengths by ν) into the definitions of H and $\bar{H}$, it is easily shown that $\bar{H}$ does not depend on ν .

rather strong inequalities. The fact that the iteration procedure outlined here will lead to a rapidly convergent sequence under well-defined conditions makes it feasible for purposes of numerical computation.

Reference

Thompson, P. D., 1954: Prognostic Equation for the Mean Motions of Simple Fluid Systems and Their Relation to the Theory of Large Scale Atmospheric Turbulence. Tellus, Vol. 6, No. 2, pp. 150-164.