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A Theory of Large-Scale Disturbances in Nongeostrophic Flow

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April 13, 2004

Abstract

The primary aim of this paper is to derive a complete system of modified hydrodynamical equations, which govern a large class of "meteorological" motions of the atmosphere, but which cannot generate solutions corresponding to sound and gravity waves. The latter have no significant effect on the behavior of the large-scale disturbances; they do, however, create serious computational difficulties (through the mere possibility of their mathematical existence), and cannot be resolved by measurements at widely separated points.

By a detailed analysis of the motions of a linear system, it has been found that the omission of the total derivative of the horizontal divergence from the divergence equation is necessary to exclude gravity wave solutions. In effect, this modification diverts all of the transformed internal and potential energy into the wave energy of the large-scale disturbances. In slightly stronger form, this type of modification is also <u>sufficient</u> to exclude gravity wave solutions from the general nonlinear equations, and is essentially a generalization of the "filtering approximations" introduced by Charney (1948).

The scheme proposed for integrating the modified nonlinear equations is an iterative method, whereby a conditionally convergent sequence of approximate solutions can be generated from a known initial distribution of pressure. This method is applied to the solution of a system of equations which are one order of approximation higher than the equations of quasi-geostrophic motion, and provides the basis for a practicable method of numerical forecasting.

I. Introduction.

Apart from the lack of sufficiently accurate, detailed or representative observations of the atmosphere's meteorological state, the principal obstacles to numerical integration of the general hydrodynamical equations in their primitive form are that:

(1) The time increments required to maintain computational stability are so small that the usual method of repeated extrapolation is prohibitive[1], from the standpoints of both accumulated roundoff error and sheer volume of calculation, and

(2) The atmosphere is so close to a state of mechanical equilibrium that its departures from equilibrium cannot be measured directly with significant accuracy; hence, the rate at which the state changes cannot be computed accurately from the primitive equations.

The first of these difficulties stems from the fact that errors are amplified by the finite-difference method (to the point of obscuring all features of meteorological interest), unless the increment of time over which the state is extrapolated is less than the time it would take for the "fastest wave" to traverse the distance between adjacent points in the finite-difference grid[2]. It is important to realize that this phenomenon (computational instability) has nothing to do with the existence or nonexistence of particular types of wave motion

[1] E.g., the method proposed by Richardson (1922)
[2] This is essentially the condition stated by Courant, Friedrichs and Lewy (1928), and later extended to a wider variety of hyperbolic difference equations by Charney, Fjortoft and von Neumann (1950).

in physical actuality; the relevant question is whether or not the hydrodynamical equations are capable of generating classes of solutions corresponding to the various types of wave motion. The speed of the "fastest wave" is the greatest phase speed admitted by the equations. Thus, since the general hydrodynamical equations possess solutions corresponding to sound waves, numerical integration of the general equations requires that the time increments be at least as small as the time it would take a sound wave to travel from one observing point to the next. This would demand that the time increments be less than one percent of the typical period of large-scale atmospheric disturbances.

The second difficulty is manifested in the fact that the winds are very nearly geostrophic——i.e., the acceleration is almost invariably a small resultant of individually large Coriolis and pressure gradient forces per unit mass. Accordingly, in order to compute the acceleration (or the rate at which the velocity field changes) with ten percent accuracy, the pressure gradient and velocity must be measured with one or two percent accuracy. Moreover, even if meteorological observations were accurate enough, it is doubtful that they would be representative of the macroscopic state of the atmosphere within comparable limits. This difficulty alone is sufficient to guarantee the failure of finite-difference methods for integrating the hydrodynamical equations in their primitive form.

It must be emphasized that difficulties of type (1), arising from the fact that the general hydrodynamical equations have solutions corresponding to high speed sound and internal gravity waves, are very

real and have disastrous computational consequences. It is inconceivable, on the other hand, that the mere existence of sound and gravity waves has any significant effect on the course of meteorological events, for the large-scale disturbances travel much too slowly to be sound or gravity waves themselves, and their energy is observed to be much greater than that of sound and gravity waves. It is natural to ask, therefore, if the hydrodynamical equations can be modified in some way such that the solutions corresponding to sound and gravity waves are excluded, but the solutions corresponding to the large-scale slowly-moving disturbances are left essentially intact. If so, the maximum time increment permitted by the condition for computational stability would depend only on the speed of the large-scale disturbances, and would be in better accord with the time resolution necessary to trace their progress over periods of a day or so.

This question was first treated systematically and in general terms by Charney (1948), who discovered that the introduction of the conditions for geostrophic and hydrostatic equilibrium into an incomplete system of derivative equations has the effect of excluding or "filtering out" the solutions corresponding to sound and gravity waves[3]. In particular, he showed that universally introducing the hydrostatic approximation filters out the sound waves, and that replacing the true wind by the geostrophic wind in the potential vorticity and thermody-

[3]Although the general question of filtering had not yet been raised, Rossby's (1939) early studies of nondivergent barotropic flow led to a crude type of filtering. Since there is no transformation of potential to kinetic energy in such a system, it cannot propagate gravity waves; similarly, compressional (sound) waves cannot exist in an incompressible medium. It follows, then, that Rossby's equations for nondivergent barotropic flow could not generate solutions corresponding to sound and gravity waves.

namic energy equations completely excludes the gravity waves. The resulting pair of modified derivative equations involves essentially only two dependent variables (pressure and the vertical air speed) and can be integrated by finite-difference methods.

It is also worth noting that Charney's procedure for excluding the undesirable types of motion simultaneously evades difficulties of type (2). This stems from the fact that the prognostic element of Charney's theory is contained in the vorticity equation, which cannot reflect the almost exact balance between the Coriolis and pressure gradient forces—simply because the pressure gradient force can exert no torque. Thus, Charney's modified equations for quasi-geostrophic motion are demonstrably free of the major mathematical difficulties inherent in numerical integration of the general hydrodynamical equations. The remaining question is whether or not the modified equations adequately describe the behavior of the large-scale disturbances.

During the past few years, the predictive value of the theory of quasi-geostrophic motion has been tested in a few specially selected cases by Charney, Fjortoft, and von Neumann (1950), Charney and Phillips (1953), and in a wide variety of situations by Staff Members of the University of Stockholm (1952 and 1954), Bushby and Sinds (1954), and by Thompson and Gates (1955). In the main, the results of these tests show that the theory accounts for most of the day-to-day change in the pressure and flow patterns in the middle troposphere, and that it provides the basis for a moderately successful numerical method for predicting the pressure distribution. In a large number and variety of cases, however, the theory of quasi-geostrophic motion has failed to

account even for the movement of fully developed and well defined maxima (or minima) of potential vorticity——a quantity that is conserved under general conditions of adiabatic, nonviscous flow. In such cases, the errors are probably due in large measure to the fact that the potential vorticity pattern is carried along by the actual wind, which sometimes departs considerably from the geostrophic wind, and to the truncation errors that are inherent in any finite-difference method. The latter, however, have the systematic effect of retarding the movement of potential vorticity maxima, whereas it is frequently observed that they are predicted to move too rapidly. In some instances, moreover, the errors in the predicted behavior of the vorticity maximum are directly traceable to subgeostrophic winds in a sharp trough.

These facts alone suggest that the equations of quasi-geostrophic motion are too drastic a modification of the general hydrodynamical equations——i.e., that this modification not only excludes solutions corresponding to sound and gravity waves, but also severely alters the solutions corresponding to the large-scale disturbances. It is a matter of some interest, therefore, to inquire if the modifications introduced by Charney are <u>necessary</u> to exclude gravity waves[4]; they are, of course, sufficient to do so, but may prove more than sufficient. If the latter is the case, the way is open to a more general theory of large-scale atmospheric disturbances.

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4) Although the condition for hydrostatic equilibrium may not be <u>necessary</u> to exclude sound waves, it appears unlikely that this approximation is responsible for any serious distortion of the solutions corresponding to the large-scale disturbances. In what follows, we shall retain Charney's use of the hydrostatic approximation as a device for excluding sound waves from the outset.

The object of this paper is to reexamine the filtering problem, with a view to finding what modifications of the general hydrodynamical equations are both necessary and sufficient to exclude solutions corresponding to gravity waves. To gain some insight into the general problem, we shall deal first with the motions of a simple linear system—— simple enough that its behavior can be analyzed in detail, but general enough that it is capable of propagating both the long "planetary" and gravity waves. In this case, the modification of the general linear equations which is both necessary and sufficient to exclude gravity waves is the omission of the total derivative of horizontal divergence from the divergence equation —— i.e., the equation obtained by applying the divergence operator to the horizontal vector equation of motion. This approximation is most nearly valid when the speed of the large-scale disturbances (relative to the medium) is much less than that of internal gravity waves, and is essentially a statement that we are primarily interested in the large-scale slowly-moving disturbances, rather than in small-scale phenomena. Interpreted in physical terms, the filtering approximation guarantees that whatever potential and internal energy is transformed must go into the energy of the rotational quasi-nondivergent motions (large-scale disturbances) and none into the divergent quasi-irrotational motions (gravity waves). In effect, this modification uncouples the mechanism by which energy passes back and forth between its potential and kinetic forms, which is, indeed, the mechanism by which gravity waves are propagated.

The analogous modification of the general nonlinear equations

for adiabatic nonviscous flow is necessary a fortiori and we shall show that, with one further modification, it is sufficient a posteriori. The modified derivative equations comprise a complete system of seven equations involving seven dependent variables. Six of these are diagnostic equations, relating the vertical speed, the stream function (for the nondivergent part of the motion), and the velocity potential (for the irrotational part) to a given distribution of pressure; the seventh is the prognostic equation, which regenerates the pressure distribution at a later time. Together with known boundary and initial conditions, this system of equations determines the state of the atmosphere at all times, and thus constitutes a mathematically complete theory of large-scale disturbances in nongeostrophic flow. This theory embraces the theory of quasi-geostrophic motion as a special case.

Finally, we shall propose a numerical method for solving the modified equations, beginning with the initial distribution of pressure. This is an iterative method whereby one can generate a rapidly convergent sequence of approximations to the nongeostrophic wind and horizontal divergence. Since the modified equations are otherwise free of difficulties of types (1) and (2), the proposed techniques of numerical integration provide the basis for a practicable and rather general method of numerical weather prediction. A scheme for solving the second order approximation to the system of exact modified equations is given as an illustration of methods that are feasible with existing computing equipment.

2. The Filtering Approximation for a Linear System

In order to see precisely what feature of the hydrodynamical

equations it is that permits them to generate solutions corresponding to gravity waves, we shall first study the nongeostrophic motions of a linear baroclinic system, whose behavior is capable of complete mathematical analysis and which exhibits the essential aspects of each type of motion. Specifically, we shall consider wave perturbations superposed on a westerly current of air in which the wind speed U varies linearly with pressure, but is constant in each isobaric surface. Moreover, to facilitate comparisons between the phase speeds of different types of waves, they will be constrained to travel along the same axis, simply by requiring that the velocity perturbation be independent of the north-south coordinate.

Since we shall further confine attention to adiabatic nonviscous flow, a physically possible equilibrium state of this system is one in which there is no motion upward or northward, and in which the geostrophic wind is everywhere equal to the zonal wind U. Thus, denoting the variables that characterize the undisturbed equilibrium state by the subscript zero,

$$u_0 = U(p) = -\frac{g}{f}\frac{\partial}{\partial y} z_0(y, p)$$

$$v_0 = 0 \qquad\qquad \omega_0 = 0$$

where u and v are the horizontal components of velocity in the eastward and northward directions, respectively; x and y are the corresponding horizontal coordinates; ω is the total derivative of pressure; and z is the height above sea level. Differentiations with re-

spect to x and y (and time t) are carried out with pressure held fixed, the pressure p being regarded as the vertical coordinate. As usual, g is the gravitational acceleration and f is the Coriolis parameter. Unless otherwise indicated, the notation in these and all succeeding expressions is standard. In the perturbed state,

$$u = u(x, p, t) \qquad v = v(x, p, t) \qquad \omega = \omega(x, p, t)$$

Subject to these constraints, the perturbation equations that govern the motions of the system are:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial v}{\partial x} + \beta v + f\frac{\partial u}{\partial x} = 0 \tag{1}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{dU}{dp}\frac{\partial \omega}{\partial x} - f\frac{\partial v}{\partial x} + g\frac{\partial^2 z}{\partial x^2} = 0 \tag{2}$$

$$g\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial z}{\partial p} - f\frac{dU}{dp}v + \frac{\sigma^2}{p^2}\omega = 0 \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial p} = 0 \tag{4}$$

in which $\beta = df/dy$; $\sigma^2 = (R^2T^2/g\theta)\,\partial\theta/\partial z$; R is the gas constant; T and θ are the absolute temperature and potential temperature representative of the undisturbed state. The

subscript M in the first term of Eq. (2) is merely a way of identifying that term, and of distinguishing the operator $(\partial/\partial t + U\,\partial/\partial x)_M$ from other operators of the same form. Eq. (1) will be recognized as the vorticity equation for this particular system, obtained by applying the horizontal curl operator to the horizontal vector equation of motion. Eq. (2) is the "divergence equation", formed by applying the horizontal divergence operator to the horizontal vector equation of motion. Eq. (3) is essentially the thermodynamic energy equation for adiabatic flow, and Eq. (4) is the continuity equation. The vertical equation of motion has been replaced by the condition for hydrostatic equilibrium——which, after Charney, has been introduced freely into all the equations above as a means of excluding sound waves from the very beginning. These four equations involve only four dependent variables, and thus comprise a complete system. Together with the conditions that ω vanish at a pressure of zero millibars and take on specified values at sea level, they completely determine the types of wave motion that can exist in this system.

In order that the development of the linear equations parallel that of the general nonlinear equations, the elimination of variables will be carried out rather indirectly and, for purposes of later comparison, in some detail. We begin by solving the continuity equation for the horizontal divergence $\partial u/\partial x$, and substituting the result into the vorticity equation (1) and the divergence equation (2).

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial v}{\partial x} + \beta v - f\frac{\partial \omega}{\partial p} = 0 \qquad (5)$$

$$-\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)_M \frac{\partial \omega}{\partial p} + \frac{dU}{dp}\frac{\partial \omega}{\partial x} - f\frac{\partial v}{\partial x} + g\frac{\partial^2 z}{\partial x^2} = 0 \qquad (6)$$

The divergence equation (6) is next solved for the vorticity $\partial v/\partial x$, and the result substituted into the vorticity equation (5). At the same time differentiating the vorticity equation (5) with respect to p,

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left[g\frac{\partial^3 z}{\partial x^2\partial p}-\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)_M\frac{\partial^2\omega}{\partial p^2}\right]+f\frac{dU}{dp}\frac{\partial^2 v}{\partial x^2}$$

$$+f\beta\frac{\partial v}{\partial p}-f^2\frac{\partial^2\omega}{\partial p^2}=0 \qquad (7)$$

Finally, we eliminate z by applying the Laplacian operator (which, in this case, reduces to $\partial^2/\partial x^2$) to the adiabatic equation (3), and by substituting into the vorticity equation (7). Thus, rearranging terms and factoring,

$$\frac{\sigma^2}{p^2}\frac{\partial^2\omega}{\partial x^2}+\left[f^2+\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)_M\right]\frac{\partial^2\omega}{\partial p^2}$$

$$=2f\frac{dU}{dp}\frac{\partial^2 v}{\partial x^2}+f\beta\frac{\partial v}{\partial p} \qquad (8)$$

The equation above is to be regarded as a means of computing the unknown distribution of ω from a given velocity field. Reduced to essentials, the right hand side of Eq. (8) represents the sum of the advection of absolute vorticity with the thermal wind and the Laplacian of the temperature advection.

Together with the vorticity equation (5), Eq. (8) completes a system of two equations involving the unknowns ω and v. To simplify the solution of this system, we shall apply Eq. (8) at a central isobaric surface $p = \bar{p}$, where the pressure is about half the sea level pressure; Eq. (5) will be applied at two isobaric surfaces, the

pressure lying midway between $\bar{p}$ and zero in one of them and midway between $\bar{p}$ and sea level pressure in the other. Thus, replacing all vertical derivatives by the corresponding ratios of finite differences, and denoting conditions at the highest and lowest surfaces by the subscripts "one" and "two", respectively, we may rewrite Eqs. (5) and (8) as

$$\left(\frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x}\right) \frac{\partial v_1}{\partial x} + \beta v_1 - \frac{f\omega}{\bar{p}} = 0 \tag{9}$$

$$\left(\frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial x}\right) \frac{\partial v_2}{\partial x} + \beta v_2 + \frac{f\omega}{\bar{p}} = 0 \tag{10}$$

$$s^2 \frac{\partial^2 \omega}{\partial x^2} - \left[f^2 + \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)_M \right] \omega = \frac{\bar{p} f}{2} \left[\Delta U \frac{\partial^2}{\partial x^2} (v_2 + v_1) + \beta (v_2 - v_1) \right] . \tag{11}$$

where $s^2 = \sigma^2/2$, $\Delta U = U_2 - U_1$, and the variables without subscripts apply at the central surface. Since we are primarily concerned with the modes of free oscillation, we have set $\omega = 0$ at sea level.[5] We now seek solutions of the form:

5) Nonzero values of ω at sea level introduce a stationary forced oscillation which, because the system is linear, is independent of the free oscillations.

$$v_1 = V_1 e^{i\alpha(x-ct)} \qquad v_2 = V_2 e^{i\alpha(x-ct)} \qquad \omega = \Omega e^{i\alpha(x-ct)}$$

where V_1, V_2 and Ω are amplitude factors, α is the wave number of a sinusoidal perturbation, and c is its phase speed (possibly complex). Substitution of these expressions into Eqs. (9), (10) and (11) leads to a system of three linear algebraic equations involving the three unknown amplitude factors V_1, V_2 and Ω. Since the latter system of equations is homogeneous, the condition that there exist a nonzero solution is simply that the determinant of the coefficients vanish. After considerable manipulation, this condition may be written as

$$\left\{\alpha^2(c-U)+\beta\right\}\left\{\alpha^2(c-U)+\beta\frac{\alpha^2\left[s^2-(U-c)(U-c)_M\right]}{f^2+\alpha^2\left[s^2-(U-c)(U-c)_M\right]}\right\} + \frac{\alpha^4(\Delta U)^2}{4}\,\frac{f^2-\alpha^2\left[s^2-(U-c)(U-c)_M\right]}{f^2+\alpha^2\left[s^2-(U-c)(U-c)_M\right]} = 0 \tag{12}$$

in which $(U-c)_M$ corresponds to the operator $\left(\partial/\partial t + U\,\partial/\partial x\right)_M$. The equation above will be recognized as the frequency equation—— which determines the admissible phase speeds of perturbations with wave number α and which, accordingly, determines the types of wave motion that can exist in this system. Formally at least, there are four possible types of "waves", each corresponding to one of the four roots of Eq. (12) and all of which must be present in order to satisfy general initial conditions.

Approximate roots of Eq. (12) may be found by noting that, under ordinary conditions, $s \gg \beta/\alpha^2$. For example, let us tentatively assume that there exists a pair of roots such that $(U-c)^2 \ll s^2$. Thus, omitting the second terms in the square brackets in Eq. (12) and solving for $(c-U)$,

$$c - U = -\frac{\beta}{2\alpha^2}\left(\frac{\mu^2+2\alpha^2}{\mu^2+\alpha^2}\right) \pm \delta$$

$$\delta = \left[\frac{\beta^2}{4\alpha^4}\left(\frac{\mu^2}{\mu^2+\alpha^2}\right)^2 - \frac{(\Delta U)^2}{4}\left(\frac{\mu^2-\alpha^2}{\mu^2+\alpha^2}\right)\right]^{\frac{1}{2}} \tag{13}$$

where $\mu = f/s$. For reasonable values of ΔU, the modulus of $(c-U)$ is of the order of β/α^2, so that our original assumption is justified a posteriori and Eq. (13) is, in fact, a good approximation to two of the roots of Eq. (12). For real initial conditions, the conjugate complex roots given by Eq. (13) represent a single slowly-moving wave, stable if the vertical wind shear ΔU falls below the critical value $2\beta/\mu^2$ $(\delta > 0)$, and unstable over an increasingly broad band of wave numbers α as ΔU increases beyond the critical value $(\delta < 0)$. When ΔU vanishes, Eq. (13) reduces to the Rossby trough formula and, under less restricted conditions, is generically the same as the frequency equations for large-scale perturbations in quasi-geostrophic flow that have been derived by Phillips (1951), Fjortoft (1950) and by Thompson (1953). The type of solution given by Eq. (13) thus corresponds to the large-scale disturbances; it is this type that we wish to retain as nearly intact as possible.

Approximate values of the remaining two roots of Eq. (12) may be

may be calculated by noting not only that $s \gg \beta/\alpha^2$, but also that $s \gg |\Delta U|$. Tentatively assuming that the remaining roots lie in the neighborhood of $c - U = \pm s$, we find that Eq. (12) has the two approximate real roots:

$$c - U = \pm \left(s^2 + \frac{f^2}{\alpha^2} \right)^{\frac{1}{2}} \tag{14}$$

Since $s \gg f/\alpha$, our original supposition regarding the nature of the roots is justified after the fact, and Eq. (14) does give a good approximation to the two remaining roots of Eq. (12). These solutions, it will be recognized, correspond to two independent trains of internal gravity waves, one travelling upstream, the other downstream. In their pure form, they travel at speeds $c - U = \pm s$; in the present case, their speeds are very slightly modified by the effects of the earth's rotation.

The difficulty in satisfying the condition for computational stability arises from the fact that, even if the initial conditions were carefully adjusted to exclude gravity waves at the beginning, random errors would admit them at some later stage. Accordingly, $|U| + s$ must be taken as the speed of the "fastest wave", <u>if the equations can generate solutions corresponding to gravity waves</u>. If they cannot, the speed of the "fastest wave" is of the order of $|U - \beta/\alpha^2|$, which is much less than $|U| + s$. Our next concern is to see how Eqs. (1)-(4) must be modified in order to exclude solutions of the type given in Eq. (14), but leaving those of Eq. (13) essentially untouched.

Inspecting Eq. (12), we see that the condition which is necessary and sufficient for the nonexistence of solutions of the type given in Eq. (14) is that $(U-c)(U-c)_M$ vanish. But it is necessary for the existence of solutions of the type given in Eq. (13) that $(U-c)$ not vanish. Thus, the necessary and sufficient condition for the exclusion of gravity waves only is that $(U-c)_M$ vanish. By tracing the operator $(\partial/\partial t + U\,\partial/\partial x)_M$ back through the derivation of Eq. (12), it is easily seen that the latter condition is tantamount to requiring that the first term in Eq. (2) be omitted. In this case, therefore, the modification of the hydrodynamical equations which is both necessary and sufficient to filter out the solutions corresponding to internal gravity waves is the omission of the total derivative of the horizontal divergence from the divergence equation. This is the generalized filtering approximation.

Since the effect of introducing this approximation is the same as that of setting $(U-c)_M = 0$ in Eq. (12), Eq. (13) is exact for the filtered motion[6]. As indicated earlier, the solution given by Eq. (13) is a good approximation to the exact solution of Eq. (12) which corresponds to the large-scale slowly-moving disturbances, provided the speed of the large-scale disturbances is much less than that of internal gravity waves. Accordingly, since the latter proviso is almost unvariably fulfilled, and because the energy of the gravity waves is a small fraction of the total wave energy, the solutions of meteorologi-

6) It should be noted that Eq. (13) is identical with the frequency equation that results from the introduction of the geostrophic approximation into Eqs. (3) and (5). This result is circumstantial, however, for it stems from the fact that the equilibrium state of this particular linear system is one of geostrophic balance. It does not imply that the two types of filtering approximations are equivalent in the general case.

cal interest are not seriously altered by the filtering approximation.

A more revealing indication of the essential validity of the filtering approximation is contained in an equation which relates changes in the "wave energy" of this linear system to changes in its internal and potential energy. This relationship is derived from the unmodified equations by multiplying the vorticity equation (1) by the vorticity $\partial v/\partial x$, multiplying the divergence equation (2) by the divergence $\partial u/\partial x$, and adding them together. Noting that U and β do not depend on x, we may write the result in the form

$$\frac{\partial}{\partial t}\frac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]+\frac{\partial}{\partial x}\frac{U}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]+\frac{\partial}{\partial x}\left(\frac{\beta v^2}{2}\right)$$

$$+\frac{dU}{dp}\frac{\partial \omega}{\partial x}\frac{\partial u}{\partial x}+g\frac{\partial^2 z}{\partial x^2}\frac{\partial u}{\partial x}=0$$

The equation above is next integrated with respect to x over one full wavelength of a sinusoidal perturbation with wave number α, and with respect to p from a pressure of zero to a constant pressure near sea level. Thus, since all quantities are periodic in the x-direction,

$$\frac{\partial}{\partial t}\int_0^L\int_0^{p_0}\frac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]dp\,dx+\int_0^L\int_0^{p_0}\frac{dU}{dp}\frac{\partial \omega}{\partial x}\frac{\partial u}{\partial x}\,dp\,dx$$

$$-g\alpha^2\int_0^L\int_0^{p_0}z\frac{\partial u}{\partial x}\,dp\,dx=0 \qquad (15)$$

where $L = 2\pi/\alpha$ is the wavelength of the perturbations, and p_0 is a constant pressure near sea level. By making use of Eq. (4), the second

term in Eq. (15) may be rewritten as

$$\int_0^L \int_0^{p_0} \left[U \frac{\partial \omega}{\partial x} \frac{\partial^2 \omega}{\partial p^2} + \frac{\partial}{\partial x} \frac{U}{2} \left(\frac{\partial \omega}{\partial p} \right)^2 - \frac{\partial}{\partial p} \left(U \frac{\partial \omega}{\partial x} \frac{\partial \omega}{\partial p} \right) \right] dp\, dx$$

As before, we require that ω vanish at $p = 0$ and $p = p_0$, whence the integrals of both the second and third terms of the integrand in this expression vanish. Moreover, we shall suppose that ω has a sinusoidal vertical profile with $\partial^2 \omega / \partial p^2 = -2\omega / \bar{p}^2$, so that the remaining integral reduces to

$$-\frac{1}{\bar{p}^2} \int_0^{p_0} \int_0^L \frac{\partial}{\partial x} \left(U \omega^2 \right) dx\, dp$$

This integral also vanishes. The third term in Eq. (15) may be combined with Eq. (4) and integrated by parts, to give it the form:

$$\alpha^2 \int_0^L \int_0^{p_0} \left(\frac{1}{\rho} \frac{dp}{dt} \right) dp\, dx$$

in which ρ is the density. Since the flow is adiabatic, this expression can be rewritten as

$$\alpha^2 \int_0^L \int_0^{p_0} \left(C_p \frac{dT}{dt} \right) dp\, dx$$

where T is the absolute temperature, and C_p is the specific heat at constant pressure. For perturbations of the type under consideration,

the integral above reduces to

$$\alpha^2 \int_0^L \int_0^{p_0} c_p \left[\frac{\partial T}{\partial t} + \frac{\partial}{\partial x}(Tu) + v \frac{dT_0}{dy} + \frac{\partial}{\partial p}(T\omega) \right] dp\, dx$$

The integrals of all terms in this integrand except the first vanish. Thus, introducing the hydrostatic approximation and integrating by parts, we may write the third term of Eq. (15) in the form:

$$\alpha^2 \frac{\partial}{\partial t} \int_0^L \int_0^{p_0} (c_v T + gz)\, dp\, dx$$

in which c_v is the specific heat at constant volume. Finally, substituting this result into Eq. (15),

$$\frac{\partial}{\partial t} \int_0^L \int_0^{p_0} \left[\frac{1}{2\alpha^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2\alpha^2}\left(\frac{\partial u}{\partial x}\right)^2 + (c_v T + gz) \right] dp\, dx = 0 \quad (16)$$

The quantity $(c_v T + gz)$ will be recognized as the sum of the internal and potential energy per unit mass, and the first two terms of the integrand in Eq. (16) may be thought of as the "total wave energy" per unit mass. Thus, Eq. (16) simply states that any change in the internal and potential energy of this linear system must be exactly balanced by an opposing change in its total wave energy.

Retracing our way through the derivation of Eq. (16), we see that the omission of the first term in Eq.(2) has the effect of removing the second term of the integrand in Eq. (16). In other words, the filter-

ing approximation insures that whatever internal and potential energy is transformed goes into the energy of the rotational nondivergent part of the motion, and that none goes into the divergent irrotational part[7]. In interpreting this result, it is convenient to think of gravity waves as a characteristically divergent quasi-irrotational motion, and to regard the large-scale disturbances as a characteristically rotational quasi-nondivergent motion —— each existing independently of the other. This, of course, is strictly true only of their pure forms; nevertheless, owing to the looseness of the coupling between them (due to the presence of the vorticity $\partial v/\partial x$ in the divergence equation and the divergence $\partial u/\partial x$ in the vorticity equation), the two types of wave motion closely approach their pure forms. With this interpretation, Eq. (16) implies that <u>the filtering approximation diverts all of the transformed energy into the energy of the large-scale disturbances</u>. The essential reason for this is apparent when one realizes that the mechanism by which gravity waves are propagated is a continuous reversible transformation of energy back and forth between the potential energy and the kinetic energy of the vertical motions. Since the latter has been ruled out as a possible recipient of the transformed energy, gravity waves cannot exist.

These features of the filtering approximation have direct bearing on one of the difficulties of type 2, for they guarantee that the energy of the <u>divergent</u> part of the motion will remain small, even if the change in energy implied by the <u>general</u> equations (with inaccurate or incomplete initial conditions) is fantastically out of proportion with

7) According to Helmholtz' theorem, the motion can always be resolved into two such components.

the observed amplitudes of internal gravity waves.

Thus far, this section has dealt only with the filtering approximation for a baroclinic linear system, which, although it is general enough to maintain both gravity waves and large-scale disturbances, lacks some aspects of nonlinear systems. Since the motions of this linear system are a subclass of the motions of the general nonlinear system, it follows that the filtering approximation for the linear system is necessary *a fortiori* to exclude gravity waves in the general case. The remaining question is one of sufficiency in the general nonlinear case, to which the foregoing analysis provides no clear answer. It is important, therefore, to see in detail why the filtering approximation is sufficient to exclude gravity waves from the linear system.

The essential fact is that the introduction of the filtering approximation into Eq. (2) results in the removal of all time derivatives from Eq. (8), which, together with the vorticity equation (5), comprises a complete system of equations. Thus Eq. (8) becomes a *diagnostic* equation, relating the unknown distribution of ω to a known velocity field; the *prognostic* element enters only into the vorticity equation (5). Since the latter equation contains time derivatives of no higher than the first order, it cannot propagate wave-solutions in two directions at once. Consequently, it cannot generate solutions corresponding to gravity waves, simply because gravity waves travel simultaneously in both directions. In effect, the filtering approximation makes sure that the distribution of divergence implied by Eq. (8) is that associated with the large-scale disturbances, rather than the divergence characteristic of gravity waves. Initial values of

the latter depend directly on wind measurements, and cannot be computed with significant accuracy.

Following the reasoning outlined above, we shall let the form of a more general ω -equation serve as a guide in choosing filtering approximations that are sufficient to exclude gravity waves from the nonlinear system.

3. The Nonlinear Equations for the Filtered Motion

Our next concern is to introduce analogous filtering approximations into the general nonlinear equations. Following the pattern of development outlined in the foregoing section, we begin by replacing the primitive hydrodynamical equations by an equivalent system of derivative equations, obtainable by applying the horizontal divergence and curl operators to the horizontal vector equation of motion. For simplicity, we shall consider the adiabatic flow of a nonviscous ideal gas, so that

$$\frac{\partial \zeta}{\partial t} + \mathbf{V}\cdot\nabla\eta + \eta\,\nabla\cdot\mathbf{V} = 0 \tag{17}$$

$$\frac{d}{dt}(\nabla\cdot\mathbf{V}) + (\nabla\cdot\mathbf{V})^2 + 2J(u,v) + \beta v - f\zeta + \nabla^2\Psi = 0 \tag{18}$$

$$\frac{d}{dt}\left(\frac{\partial p}{\partial \theta}\right) + \frac{\partial p}{\partial \theta}\nabla\cdot\mathbf{V} = 0 \tag{19}$$

$$\frac{\partial \Psi}{\partial \theta} = \frac{1}{\kappa} p^{\kappa} = \frac{C_p T}{\theta} \equiv P \tag{20}$$

where $\mathbb{V}$ is the horizontal projection of the velocity vector; $\zeta = \partial v/\partial x - \partial u/\partial y$ is the vorticity; ∇ is the horizontal vector gradient; $\eta = \zeta + f$; the symbol J is the horizontal Jacobian; $\Psi = gz + C_p T$ is the Montgomery stream function; θ is the potential temperature; and $\kappa = R/C_p$. All differentiations with respect to the horizontal coordinates and time are carried out with the <u>potential temperature</u> θ held fixed, θ being regarded as the vertical coordinate. Eqs. (17)-(20) are the vorticity, divergence, continuity and hydrostatic equations, respectively. As before, the latter has been introduced freely into the other three equations, as a means of excluding solutions corresponding to sound waves. It should be noted that Eqs. (17)-(20) comprise a complete system of four equations involving the four dependent variables u, v, p, and Ψ , without referring explicitly to the thermodynamic energy equation. The condition for adiabatic flow is embodied in the statement that $d/dt = \partial/\partial t + \mathbb{V}\cdot\nabla$, and is used as an independent equation only to locate the moving coordinate surfaces.

Following the procedure analogous to the derivation of Eq. (5), we next eliminate the horizontal divergence between the vorticity equation (17) and the continuity equation (19) with the result that

$$\frac{\partial \zeta}{\partial t} + \mathbf{v}\cdot\nabla\eta - q\,\frac{\partial \mathbf{v}}{\partial \theta}\cdot\nabla p + q\,\frac{\partial \omega}{\partial \theta} = 0 \tag{21}$$

in which the "potential vorticity" $q = -\eta\,\partial\theta/\partial p$. The equation that corresponds to Eq. (6) is obtained by introducing the necessary condition for the exclusion of gravity waves into Eq. (18) —— i.e., by omitting from that equation the vector invariant which corresponds to $(\partial/\partial t + U\,\partial/\partial x)_M$. Moreover, since it has been shown that the effect of this filtering approximation is to divert all of the transformed energy into the rotational part of the motion, it is only consistent to omit the second term of Eq. (18) as well. Thus, the nonlinear equation corresponding to Eq. (6) is

$$f\zeta = \nabla^2\Psi + 2J(v,u) + \beta u \tag{22}$$

The equation analogous to Eq. (7) is derived by solving Eq. (22) for ζ , substituting the result for ζ in the first term of Eq. (21), and by differentiating Eq. (21) with respect to θ .

$$\nabla^2\frac{\partial}{\partial t}\left(\frac{\partial\Psi}{\partial\theta}\right) + 2\frac{\partial^2}{\partial t\,\partial\theta}J(v,u) + \beta\frac{\partial^2 u}{\partial t\,\partial\theta} + f\frac{\partial}{\partial\theta}(\mathbf{v}\cdot\nabla\eta)$$
$$- f\frac{\partial}{\partial\theta}\left(q\frac{\partial\mathbf{v}}{\partial\theta}\cdot\nabla p\right) + f\frac{\partial}{\partial\theta}\left(q\frac{\partial\omega}{\partial\theta}\right) = 0 \tag{23}$$

Continuing parallel to the derivation of Eq. (8), we now eliminate Ψ from the first term of Eq. (23) by making use of Eq. (20) and the adia-

batic equation. Finally, letting $dP/dt = \Omega$ and noting that $\omega = p^{1-k}\Omega$, we may rewrite Eq. (23) in the form:

$$\nabla^2\Omega + f\frac{\partial}{\partial\theta}\left[q\frac{\partial}{\partial\theta}\left(p^{1-k}\Omega\right)\right] + 2\frac{\partial^2}{\partial t\,\partial\theta}J(v,u) + q\frac{\partial^2 u}{\partial t\,\partial\theta}$$

$$= \nabla^2(\mathbf{V}\cdot\nabla P) - f\frac{\partial}{\partial\theta}(\mathbf{V}\cdot\nabla\eta) + f\frac{\partial}{\partial\theta}\left(q\frac{\partial\mathbf{V}}{\partial\theta}\cdot\nabla p\right) \qquad (24)$$

This equation corresponds to Eq. (8), the ω -equation for the linear system.

It was pointed out earlier that the reason why the filtering approximation is <u>sufficient</u> in the linear case is that the modified ω -equation contains no time derivatives. Consequently, since the third and fourth terms on the lefthand side of Eq. (24) do involve time derivatives, it appears that the filtering approximations so far introduced are not <u>sufficient</u> to exclude the irrelevant types of motion in the general case. For this reason (and because they merely represent a correction to the geostrophic vorticity), the third and fourth terms on the lefthand side of Eq. (24) will be omitted. As will be seen later, this approximation is introduced only for purposes of computing Ω and a rather small correction to the nondivergent part of the wind. With this modification, Eq. (24) becomes a <u>diagnostic</u> equation, in which Ω is regarded as the unknown variable, and u, v, and p are regarded as known. In mathematical terms, Eq. (24) is a nonhomogeneous linear equation of elliptic type (for $q > 0$). Together with the conditions that Ω vanish at the top of the atmosphere and take on

known values at the ground surface[8], this equation determines the value of Ω at each point (x, y, θ). Its solution for any given distribution of u, v, and p can be computed by relaxation or matrix methods.

The essential results of this analysis can be summarized in most convenient form by resolving the wind $\mathbb{V}$ into a divergent irrotational part and a nondivergent rotational part. I.e., we let

$$\mathbb{V} = \nabla\phi + \mathbb{K} \times \nabla\psi \tag{25}$$

where $\mathbb{K}$ is a unit vector directed vertically upward, ϕ is the velocity potential for the irrotational part of the motion, and ψ is the stream function for the nondivergent part. Thus, introducing the variables ϕ and ψ, we may rewrite Eqs.(21), (24), (19), (22), and (20) as:

$$\nabla^2\left(\frac{\partial\psi}{\partial t}\right) + \mathbb{V}\cdot\nabla\eta - g\frac{\partial\mathbb{V}}{\partial\theta}\cdot\nabla p + g\frac{\partial}{\partial\theta}\left(p^{1-\kappa}\Omega\right) = 0 \tag{26}$$

8) With good approximation, the change in P experienced by a particle moving over the ground surface is that due to its vertical displacement. Thus, denoting conditions at the ground by the subscript zero, and letting h be the height of the terrain,

$$\Omega_0 = -\frac{g}{\theta_0}\,\mathbb{V}_0\cdot\nabla h$$

$$\nabla^2\Omega + f\frac{\partial}{\partial\theta}\left[q\frac{\partial}{\partial\theta}\left(p^{1-\kappa}\Omega\right)\right] = \nabla^2(\mathbf{v}\cdot\nabla p) - f\frac{\partial}{\partial\theta}(\mathbf{v}\cdot\nabla\eta) + f\frac{\partial}{\partial\theta}\left(q\frac{\partial\mathbf{v}}{\partial\theta}\cdot\nabla p\right) \tag{27}$$

$$\nabla^2\phi = -\frac{\partial}{\partial p}\left(p^{1-\kappa}\Omega\right) + \frac{\partial\mathbf{v}}{\partial p}\cdot\nabla p \tag{28}$$

$$f\nabla^2\psi = \nabla^2\Psi + 2J(v,u) + \beta u \tag{29}$$

$$\frac{\partial\Psi}{\partial\theta} = \frac{1}{\kappa}p^{\kappa} \tag{30}$$

The five equations above are the modified nonlinear equations for the "filtered" motion. Combined with the vector equation (25), they comprise a complete system of seven equations involving the seven dependent variables u, v, ϕ, ψ, p, Ω, and Ψ. Together with initial values of Ψ and the previously stated boundary conditions on Ω, these equations determine the state of the atmosphere at all times, and thus constitute a <u>mathematically</u> complete theory of large-scale disturbances.

It is important to realize that all of the Eqs. (25)-(30) are <u>diagnostic</u> in character, except Eq. (26). Since the latter contains

time derivatives of only the first order, the modified equations cannot generate solutions corresponding to gravity waves. Moreover, because the prognostic equation (26) is a vorticity equation and cannot reflect the almost complete balance between the Coriolis and pressure gradient forces, the computed changes in ψ (and all other variables) are not small differences between individually large terms of the same sign. In short, the system of equations (25)-(30) is not subject to the defects of the primitive equations discussed at some length earlier.

The modified equations (25)-(30) reduce to the equations for quasi-geostrophic flow in a special case —— namely, when $\nabla\psi$ is set equal to $\nabla\Psi/f$, and Φ is set equal to zero in Eqs. (26) and (27). The resulting equations, combined with Eq. (30), are exactly equivalent to the equations derived by Charney (1948) through the introduction of the geostrophic approximation into the potential vorticity and thermodynamic energy equations. Thus, since the equations of quasi-geostrophic flow do not refer explicitly to the continuity equation (28) or the divergence equation (29), they are demonstrably less general than the modified Eqs. (25)-(30). That the latter are quite general may be seen from the fact that only two approximations have been introduced into the equations of adiabatic nonviscous flow in hydrostatic equilibrium —— one of which is necessary to exclude gravity waves, but is not generally sufficient, and the other of which makes sure that gravity waves are excluded in all cases. The form of the prognostic equation, for instance, is left unaltered by the filtering approximations. It is suggested, therefore, that Eqs. (25)-(30) are essentially the

equations which govern the meteorological behavior of the atmosphere on a large scale and over short periods of time.

4. Numerical Integration of the Modified Nonlinear Equations.

The problem of integrating the modified Eqs. (25)-(30) resolves itself naturally into two subproblems. The first of these —— the diagnostic problem —— is that of computing the initial distributions of the variables u, v, ϕ, p, Ω, and ψ (or Ψ) from a known initial distribution of Ψ (or ψ). The second —— the prognostic problem —— is that of predicting the distribution of ψ from an inferred knowledge of the initial distributions of u, v, ϕ, ψ, p, Ω, and Ψ. Since Eqs. (25)-(30) are highly nonlinear, the solution of both these problems must be carried out by approximate numerical methods.

The scheme that is proposed for solving the diagnostic problem is a method of successive approximations, whereby a rapidly convergent sequence of approximate solutions can be generated from the known initial conditions. In outline, the basic steps of the procedure for computing u, v, ϕ, ψ, p, and Ω from Ψ are as follows:

(1) Compute the initial distribution of p from the initial distribution of Ψ, using Eq. (30).

(2) Setting $\nabla\psi = \nabla\Psi/f$ and $\dot{\phi} = 0$, and using Eq. (25), compute q and the entire right hand side of Eq. (27). Solve Eq. (27) for Ω, subject to the conditions that Ω vanish at an infinite height and at the lateral boundaries and take on values at the ground surface which are consistent with the slope of the terrain.

(3) Substitute the values of Ω obtained from the previous step into Eq. (28), and set $\nabla\psi = \nabla\Psi/f$. Solve Eq. (28) for ϕ,

taking $\phi = 0$ at the lateral boundaries.

(4) Substitute the values of ϕ obtained from the previous step into Eq. (29). Solve Eq. (29) for ψ, taking $\psi = \Psi / f$ on the lateral boundaries.

(5) Repeat steps (2)-(4), replacing the next to last approximations to ψ and ϕ by the results of steps (3) and (4) in the last repetition.

This scheme defines a cyclic series of computations, which is to be continued until the last approximation (to $\mathbf{V} \cdot \nabla \eta$, say) differs from the next to last approximation by less than a previously assigned maximum error. It has been shown by Thompson (1955) that sequences of approximations generated by this method converge under ordinary conditions, when the maximum space-averaged gradients of η and p fall below certain critical values[9]. Moreover, since the initial approximation in this sequence is simply the geostrophic approximation, it is probable that the sequence converges very rapidly.

The end results of the calculations described above are the initial distributions of ϕ, ψ, and Ω. Together with Eq. (25) and the previously computed initial distribution of p, this is sufficient to compute all of the terms of Eq. (26) except the first. Accordingly, Eq. (26) provides a means of computing $\partial \psi / \partial t$, and of extrapolating from the initial distribution of ψ to predict the distribution of

9) The fact that it is possible to compute the initial distributions of u, v, ϕ, ψ, p, and Ω from a known initial distribution of Ψ implies that the <u>meteorological</u> behavior of the atmosphere is determined by the initial distribution of pressure alone. The complexity of the computation also reveals that the connection between the pressure and wind fields is not a simple one, but depends on a complicated relationship between the kinematic, dynamic, thermodynamic, and hydrostatic properties of the flow.

ψ at a time slightly later than the initial instant. The remaining problem —— that of regenerating the distributions of ϕ, Ω, and p from the predicted distribution of ψ —— may be solved by altering the order of the steps in the diagnostic procedure outlined earlier, as follows:

(1) Introduce the predicted values of ψ into Eq. (29), and set $\phi = 0$. Solve Eq. (29) for Ψ, taking $\Psi = f\psi$ on the lateral boundaries.

(2) Substitute the result of the previous step into Eq. (30) and compute p.

(3) Substitute the predicted values of ψ and the result of the previous step into Eq. (27), and set $\phi = 0$. Solve Eq. (27) for Ω, subject to the conditions that Ω vanish at an infinite height and at the lateral boundaries and take on known values at the ground surface.

(4) Substitute the results of steps (2) and (3) into Eq. (28) and solve for ϕ, taking $\phi = 0$ on the lateral boundaries.

(5) Repeat steps (1)-(4), replacing the next to last approximations by the results of steps (1)-(4) in the last repetition.

The essential products of this calculation are the distributions of ϕ, Ω and p. Together with the predicted distribution of ψ from which they were inferred, these results enable us to recompute $\partial\psi/\partial t$ from Eq. (26) and to extrapolate ψ from its predicted distribution. Accordingly, this process of alternate diagnosis and extrapolation can be repeated indefinitely, to build up the predicted distributions of all variables over any interval of time after (or before) an in-

stent when the distribution of Ψ was known.

The procedure of numerical integration described above consists of well known numerical operations (e.g., solution of Poisson or Helmholtz equations in two and three dimensions) and, coupled with high speed automatic computing techniques, provides the basis for a practicable method of numerical weather prediction. Owing to the generality of Eqs. (25)-(30), this scheme might be expected to prove superior to present methods, based on the special theory of quasi-geostrophic motions.

5. An Approximate Method of Wind, Vertical Motion, and Pressure Forecasting.

Very exact numerical solutions of the diagnostic problem, computed by the iterative method outlined in the foregoing section, would require an enormous expenditure of time and effort —— more, certainly, than is justified by the limitations inherent in any finite-difference method, or by the quality of meteorological data. For this reason, and because the theory of quasi-geostrophic motion is generally a good first approximation to the present theory of nongeostrophic motion, it is probably not necessary to repeat the basic cycle of the diagnostic procedure more than one or two times. Moreover, in order to compare methods derived from the theory of quasi-geostrophic motion with those founded on a theory of nongeostrophic motion, both should have the same physical basis —— i.e., they should be based on a set of equations that is common to both, prior to the introduction of the two

different types of filtering approximations. It is of some interest, therefore, to develope a method (based on the nongeostrophic equations) that is comparable with current methods of numerical forecasting, from the standpoints of both physical content and general order of complexity.

Virtually all of the extant methods of numerical prediction are based on approximate forms of the equations for adiabatic nonviscous flow in hydrostatic equilibrium. The dynamical element of these methods, for example, is usually contained in a horizontal equation of motion that does not reflect local accelerations due to vertical displacements (relative to the isobaric surfaces). It is also customary to "linearize" the equations, to the extent of replacing the absolute vorticity and static stability by their values in a motionless Standard Atmosphere, wherever they enter undifferentiated in the original vorticity and thermodynamic energy equations. With such approximations, the equations corresponding to Eqs. (25)-(30) are:

$$\mathbb{V} = \nabla\phi + \mathbb{k} \times \nabla\psi \tag{31}$$

$$\frac{\partial}{\partial t}\nabla^2\psi + \mathbb{V}\cdot\nabla\eta - f\frac{\partial\omega}{\partial p} = 0 \tag{32}$$

$$\frac{\sigma^2}{p^2}\nabla^2\omega + f^2\frac{\partial^2\omega}{\partial p^2} = f\frac{\partial}{\partial p}(\mathbb{V}\cdot\nabla\eta) - \nabla^2\left(\mathbb{V}\cdot\nabla\frac{\partial\Phi}{\partial p}\right) \tag{33}$$

$$\nabla^2\phi = -\frac{\partial\omega}{\partial p} \tag{34}$$

$$f\nabla^2\psi = \nabla^2\Phi + 2J(u, v) + \beta v \tag{35}$$

where $\Phi = gz$ is the geopotential. Otherwise, the notation is that of preceding sections, except that differentiations with respect to the horizontal coordinates and time are now carried out with pressure held fixed and the pressure is regarded as the vertical coordinate. The equations above comprise a complete system of six scalar equations involving the variables u, v, ϕ, ψ, ω, and Φ (the hydrostatic equation has been used to eliminate the absolute temperature from the adiabatic equation). Together with a given initial distribution of Φ and the conditions that ω vanish at $p = 0$ and take on known values at the ground surface, they determine the state of this idealized atmosphere at all times.

It was pointed out earlier that the theory expressed in Eqs. (31)-(35) reduces to the theory of quasi-geostrophic motion if the geostrophic approximation is introduced into Eqs. (32) and (33) ---- i.e., if $\nabla\psi$ is set equal to $\nabla\Phi/f$ and ϕ is set equal to zero. Our pres-

ent purpose is to derive from the "exact" nongeostrophic equations (31)-(35) a set of equations that are one order of approximation higher than the equations for quasi-geostrophic motion. This can be done simply by performing the first cycle of the iterative diagnostic procedure outlined in section 4. Carrying out this process schematically, we obtain the following system of approximate equations:

$$\mathbb{V} = \nabla\phi + \mathbb{k} \times \nabla\psi \qquad (\eta = \nabla^2\psi + f) \tag{31a}$$

$$\nabla^2 \frac{\partial\psi}{\partial t} + \mathbb{V}\cdot\nabla\eta - f\frac{\partial\omega}{\partial p} = 0 \tag{32a}$$

$$\frac{\sigma^2}{p^2}\nabla^2\omega + f^2\frac{\partial^2\omega}{\partial p^2} = \frac{\partial}{\partial p} J\left(\Phi, \frac{1}{f}\nabla^2\Phi + f\right) - \frac{1}{f}\nabla^2 J\left(\Phi, \frac{\partial\Phi}{\partial p}\right) \tag{33a}$$

or

$$= f\frac{\partial}{\partial p} J\left(\psi, \nabla^2\psi + f\right) - \nabla^2 J\left(\psi, \frac{\partial\Phi}{\partial p}\right) \tag{33b}$$

$$\nabla^2\phi = -\frac{\partial\omega}{\partial p} \tag{34a}$$

$$f\nabla^2\psi = \nabla^2\Phi + 2J\left(\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x}, \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}\right) + \beta\left(\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}\right) \tag{35a}$$

Again, it should be emphasized that the geostrophic approximation has not been introduced into the prognostic vorticity equation (32a).

The procedure for predicting the distributions of all variables from a given initial distribution of Φ is as follows:

(1) Compute the right hand side of Eq. (33a) from the initial values of Φ. Solve Eq. (33a) for ω, subject to the conditions that ω vanish at $p = 0$ and at the lateral boundaries and take on prescribed values at the ground surface[10].

(2) Substitute the result of the previous step into Eq. (34a). Solve Eq. (34a) for ϕ, taking $\phi = 0$ on the lateral boundaries.

(3) Substitute the initial values of Φ and the result of the previous step into Eq. (35a). Solve Eq. (35a) for ψ, taking $\psi = \Phi / f$ on the lateral boundaries.

(4) Using Eq. (31a), compute W and η from the results of steps (2) and (3).

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10) Eq. (33a) or (33b) becomes a nonhomogeneous Helmholtz equation under the transformations $p = p_0 e^{-z}$, $\omega = W(x, y, z) e^{-z/2}$, $x = x'\sigma / f$ and $y = y'\sigma / f$. The latter is of the same type as a well known equation for the three-dimensional distribution of potential due to a given distribution of oscillating charges, with prescribed potential on an infinite plane, and may be solved by the method of images. There are, in fact, certain advantages in expressing the solutions of Eqs. (33a) and (33b) in integral form and evaluating the integrals by <u>finite sums</u> —— rather than solving the <u>finite-difference</u> form of the equations by relaxation methods. This is particularly true when the nondimensional increments of pressure are much smaller than the nondimensional increments of horizontal distance.

(5) Substitute the results of steps (1) and (4) into Eq. (32a). Solve Eq. (32a) for $\partial\psi/\partial t$, taking $\partial\psi/\partial t = 0$ on the lateral boundaries.

(6) Extrapolate ψ over a short time interval, using the results of steps (3) and (5).

(7) Substitute the predicted values of ψ from step (6) into Eq. (35a) and set $\phi = 0$. Solve Eq. (35a) for Φ, taking $\Phi = -f\psi$ on the lateral boundaries.

(8) Substitute the predicted values of ψ and the results of the previous step into Eq. (33b). Solve Eq. (33b) for ω, noting that the values of ω at the ground surface change with time.

(9) Substitute the results of the previous step into Eq. (34a). Solve Eq. (34a) for ϕ, as before.

(10) Using Eq. (31a), compute V and η from the results of steps (6) and (9).

(11) Substitute the results of steps (8) and (10) into Eq. (32a). Solve Eq. (32a) for $\partial\psi/\partial t$, as before.

(12) Extrapolate ψ over a short time interval, using the results of steps (6) and (11).

(13) Repeat steps (7)-(12), changing the number of the step referred to in step (7) to read (12).

This cyclic procedure is to be repeated until the aggregate of short time intervals, over which the distribution of ψ is extrapolated, adds up to the desired period of the forecast. The final results

are the predicted distributions of ϕ , ψ , ω , and Φ . The predicted winds may be calculated directly from ϕ and ψ , and the predicted temperature may be inferred from Φ by making use of the hydrostatic equation. The measure of vertical motion that is most directly connected with the thermodynamics of condensation and precipitation is, of course, ω itself. In brief, the method outlined above is well suited to predicting the general setting or environment in which localized influences operate to produce "weather" in the usual sense.

In conclusion, it is worth noting that the computations required to produce a one day forecast by this procedure are not much more involved than those required by present methods of numerical forecasting, and can be carried out in a matter of hours by any of several high-speed automatic machines now in existence. The application of methods based on this theory of nongeostrophic motion is under study at the Joint Numerical Weather Prediction Unit at the present time, and will be the subject of later reports.

6. Summary

A resumé of the main results and conclusions is given in the last three paragraphs of the Introduction.

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