Uncertainty of Initial State
As a Factor in the Predictability
of Large Scale Atmospheric Flow Patterns

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Abstract

This article deals with the predictability of the atmosphere or, more exactly, with the gradual growth of "inherent" errors of prediction, due to errors in an initial state that is reconstructed from measurements at a finite number of points. By investigating the initial time-derivatives of the error arising from random analysis error, it is found that the increase of the RMS (root-mean-square) wind error in predictions over periods of a few days depends on:

1) the period of the forecast
2) the initial RMS vector wind error
3) the difference between the characteristic scale of the initial error field and the scale of fluctuations in the true initial flow pattern.
4) the area average of the vertical wind shear between 250 and 750 mb.
5) the RMS vector deviation of the wind at about 500 mb from its area average, and
6) the average static stability of the atmosphere.

The joint effect of these various factors is given explicitly by a single equation, relating the increase of inherent error to the statistical properties of the initial error field and true initial flow pattern.

In many winter situations, and for initial error fields whose scale is typical of the present observational network, the inherent RMS vector wind error may double its initial value after two days, and rise to the error of sheer guessing in about a week. Doubling the overall density of regular reporting stations would virtually eliminate the increase of inherent error in forecasts over a few days. It is also found that zonally-averaged wind fields are inherently more predictable than unaveraged wind fields, at

least in cases of predominantly barotropic flow.

The results outlined above are interpreted in the terms of a variety of practical and administrative problems, in each of which a dominant factor is the predictability of the atmosphere. Examples are the problems of estimating limits of confidence in forecasts, deciding on the most economical density and distribution of regular reporting stations, fixing the maximum range beyond which detailed "forecasts" have lost essentially all predictive and economic value and, finally, that of establishing the point of rapidly diminishing returns in the development of more complicated and costly methods of prediction.

I. Introduction

One of the most far-reaching questions in meteorology concerns the "predictability" of the atmosphere --- i.e., not merely the extent to which its behavior is predicted in practice, but the extent to which it is possible to predict it with a theoretically complete knowledge of the physical laws that govern it. On the purely scientific side, it is an important point of doctrine to know whether or not our uncertainty as to the atmosphere's future state is accounted for by economic (or, ultimately, human) incapacity to observe and compute, or whether it is essential and due to some irreducible minimum of indeterminacy that lies beyond human limitation.

On the practical side, common experience indicates that incomplete data coverage leads to a rapid and widespread decay of predictability over periods of more than a day or two, and it is generally granted that increasing the density of regular reporting stations would result in a qualitative increase in predictability and in the economic value of prediction. The cost of (say) doubling the density of upper air stations, however, would be enormous --- certainly on the order of hundreds of million U. S. dollars. Thus, the question of increasing data density is an economic problem of national or international proportions, and it is hardly surprising that national weather services in the past have been reluctant to double their budgets for a problematical return in the accuracy of forecasts. Until one can quantitatively predict the predictability that would result from increased data coverage, it is impossible to establish the point of rapidly diminishing returns, beyond which further outlay would be unprofitable. That, of course, is the real problem.

Another related and equally mundane question is whether or not it is useless to try to predict the state of the atmosphere in detail over periods longer than several days. It is often asserted, but not definitely known that the predictability of the atmosphere decays so rapidly that a week-long "prediction" is no better than a sheer guess. Whatever is actually the case, it is important to establish the maximum range of predictability, beyond which the details of the state are essentially unpredictable. In this same connection, Namias (1947) and others have suggested that certain statistical characterizations of the state of the atmosphere might be inherently more predictable than the details of its state. Again, the practical question will remain unresolved until it is definitely known whether or not such statistics are in fact more predictable, and (if so) how much more predictable, and whether or not the possible gain in predictability is more than overbalanced by the attendant loss of detail or information content of the prediction.

Still another problem is that of estimating the probable error or limits of confidence in predictions, a question that bears directly on the gradual growth of error toward the level of complete "unpredictability", and one which is frequently brought up as an objection to any deterministic approach to prediction. That such objections are valid is sometimes admitted even by dynamic meteorologists (who have borne the brunt of criticism); they have not, however, been too ready to inject a jarring note of probability into what is otherwise a pleasingly deterministic theory --- a course that is nevertheless inevitable if the existence of error is fully recognized.

Between the extremes of theory and practice lies the problem of detecting when the development of more and more general, complicated, and costly methods of dynamical prediction, and more and more refined computing techniques will not produce a significant increase in the overall accuracy of predictions. It is conceivable, for example, that we are already approaching the point where the total error of prediction is comparable in magnitude to the error due to uncertainties of initial state alone. If and when this is the case, it would be pointless to channel precious scientific manpower into such narrow avenues of purely technological development, or to spend additional funds for more powerful and costly computing facilities. At that point, the only hope of improving the quality of prediction will lie in reducing the uncertainty arising from the one remaining controllable source --- namely, the uncertainty inherent in ignorance of the actual state of the atmosphere at any given time.

The problems outlined above are certainly not a complete catalog, but they are sufficient to indicate that a great number and variety of problems --- theoretical, practical, and administrative --- bear more or less directly on the question of predictability. These same problems will be reviewed later in the light of subsequent results.

Turning to more general questions of predictability, it should be noted that, if one accepts the universal applicability of physical law, one must also concede that the future state of the atmosphere is (in a certain sense) determinate and absolutely predictable. That is, the principles of conservation of momentum, mass, and energy imply that the chains of events following two identical states of the atmosphere are also identical, provided equal amounts of heat energy are absorbed at equivalent times and places. This is, in fact, one of the fundamental tenets of the recently developed methodology of numerical weather prediction. The distinction between this view and the pragmatic view of predictability is simply that we cannot really say when two states are identical.

For the sake of illustration, let us adopt the plausible hypothesis that the behavior of the atmosphere is actually described by the hydrodynamical (Navier-Stokes) equations, which simply express the general conservation laws in mathematical form, and which have been successfully applied in the analysis of a wide variety of hydrodynamical phenomena. Now, it is possible to integrate those equations by approximate numerical methods with a mathematical error that is less than any preassigned value, no matter how small. Imagine, for example, that we first reconstruct the initial state of the atmosphere by drawing isopleths (of isobaric height, say) on a very large chart for contour intervals of ten feet or one foot, or whatever the sharpness of one's pencil allows. The initial height values will be interpolated to the nearest foot or inch at a regular network (or grid) of points spaced 100 kilometers or perhaps even 10 kilometers apart. Still in imagination, we next integrate the hydrodynamical equations by the finite-difference method, starting with the interpolated initial values at the points of this very fine grid, and approximating derivatives by differences between values at neighboring points in the grid. The significant fact is that Leray (1931) has shown that, by making the grid finer and finer, one can make the truncation errors of such an integration arbitrarily small. Moreover, by carrying more and more significant digits in computing machines of greater and greater storage capacity, it is possible to make roundoff errors as small as desired, and so to remove the one remaining type of purely mathematical error.

This is not to say, however, that solutions of the general hydrodynamical equations computed in this fashion would agree with the future state of the atmosphere to within comparably narrow limits. Even if meteorological data contained no instrument, reading, roundoff, or transmission errors, and even if the analysts' isopleths fitted the available data perfectly, a reconstruction of the initial state from data at a finite number of points will not be the same as the true state. At various points it will differ from the true state by an amount that depends on distance from the point in question to the nearest reporting station and which, on the average, depends on the distance between reporting stations.

A typical distribution of the difference between reconstructed and true isobaric height fields is shown in Fig. 1 (c). This pattern was obtained by subtracting a hypothetical "true" height field, illustrated in Fig. 1 (a), from the reconstructed height field shown in Fig. 1 (b). The latter was constructed as objectively as possible by linear and quadratic interpolation between the circled points, where the "true" height values were given. It will be noted that the distribution of the difference between the true and reconstructed height fields, hereafter called the "analysis error", displays a cellular appearance. On reflection, one sees that this structure is typical, because the lines of zero error must pass through the circled points in this case and, in actuality, must pass close to the points where data are given.* Thus, the distribution of the analysis error has a characteristic scale or "grain-size", closely related to the distance between observing stations. As indicated earlier, the analysis error also has a characteristic magnitude, which generally increases as the distance between stations is increased.

Another important property of the analysis error stems from the fact that the large-scale transient weather disturbances are, on the average, located at random with respect to the observing stations, and vice versa.

* In general, the analysis error is greater in average magnitude than the errors of measurement.

Thus, since the "phase" (e.g., zero lines) of the pattern of analysis error is essentially fixed by the locations of the stations, the analysis error tends to be uncorrelated with the true state and, in this sense, may be regarded as a random variable.

Returning to the main thread of argument, let us now suppose that the general hydrodynamical equations are integrated in the manner outlined earlier, but starting with two different initial states. In one case, we shall begin with the true state and, in the other, with the reconstructed state, consisting of the true state plus a field of analysis error that is random with respect to the true state. In the first case, the prediction must be correct by hypothesis. In the second, the prediction is incorrect initially, and will generally differ from the correct one at all later times. The amount by which it differs at various times after some specified initial instant is, of course, a measure of unpredictability or error. In fact, according to our present view, the only element of uncertainty which cannot be removed by bigger and faster computing machines is that due to the analysis error. Thus, the difference between two solutions of the hydrodynamical equations, starting with two initial states that differ by a random error field, is a measure of the essential unpredictability of the atmosphere. In particular, if the average magnitude of such differences at some later time approaches the average error of guessing, the atmosphere has become essentially unpredictable beyond that time. The remaining problem is to find out how, on the average, the unpredictability grows over an increasing period of time after an arbitrarily specified initial instant, and on what statistical properties of initial error and true state the growth rate depends.

The problem just posed, with certain idealizations to be introduced later, is substantially the question that will be subjected to mathematical analysis and examined in some detail in the remainder of this article. To state the results of this study in rather general terms, it will be found possible to express the area-averaged RMS* vector wind error at any time as a function of:

1) the initial RMS vector wind error.

2) the period of the forecast.

3) a statistical measure of the characteristic size of transient disturbances in large-scale atmospheric flow patterns.

4) a statistical measure of the scale of the "analysis error" --- which, in turn, depends on the distance between adjacent observing stations.

5) the RMS vector deviation of the wind at about 500 mb from its area average.

6) the RMS vector deviation of the vertical wind shear between 250 and 750 mb from its area average.

7) the area average of the vertical wind shear between 250 and 750 mb, and

8) the average static stability of the atmosphere.

That these are important (if not the dominant) factors in the predictability of the atmosphere is consistent not only with the results of numerous studies of atmospheric instability, but with intuition and general experience.

As might be expected, the growth rate of MS error is proportional to the initial MS analysis error, and is very nearly proportional to the MS

* In what follows, "RMS" is used as an abbreviation for the "root-mean-square" value taken over a very large area. Similarly MS stands for "mean-square"

vector deviation of the wind from its average --- a measure of the rate of interaction between error and true state. The growth of MS error also depends critically on the relative scales of the initial analysis error and the macroscopic fluctuations in the true flow pattern, the error increasing rapidly if the distance between adjacent observing stations is appreciably greater than the characteristic half-wavelength of the true disturbances, and even diminishing if it is appreciably less. In general, the increase in the MS vector wind error over periods of a few days is proportional to the square of the period.

The results just described apply qualitatively to both quasi-barotropic and baroclinic initial states. In cases of strongly baroclinic flow, however, the effects outlined above are considerably modified by thermodynamic processes, the growth of RMS error increasing markedly with stronger vertical wind shear (horizontal temperature gradient) and smaller static stability. For initial error fields whose scale is representative of the existing observational network over the western half of the northern hemisphere, and under conditions of only moderately strong average vertical wind shear, the MS vector wind error arising from analysis error only may increase by 25 percent of its initial value after one day, double its initial value after two days, and so on.

It is, indeed, found that the zonally-averaged wind field is inherently more predictable than the unaveraged wind field, at least in the case of nondivergent barotropic flow. This result stems from the fact that the percentage increase in MS error of the average wind after a specified period is proportional to the fourth power of the period, rather than the second power. Thus, since the base of the exponent (a nondimensional measure of time) is about the same in either case, the growth of error in the averaged wind is initially slower than in the unaveraged wind. Ultimately, this difference is due to the fact that zonal averages are not contaminated <u>initially</u> by random analysis error.

From even this bare summary, it may be seen that the results outlined above have direct bearing on several aspects of predictability. In sections III - IX, we shall discuss the main results of the mathematical analysis in more detail, and interpret them in the terms of the practical and administrative problems raised at the beginning of this introduction. As much as possible, the discussion will be kept free of sustained mathematical argument, referring only to necessary definitions and final results. The mathematical development will be relegated to three appendices. The first will deal with the predictability of nondivergent barotropic flow --- an exercise that contains all the essential mathematical ingredients of an analysis of more general and complicated problems, and which is a valuable preliminary to the study of baroclinic flows. The second is merely an outline of the key points in the analysis of predictability in a simple (but fairly general) type of baroclinic flow, emphasizing the analogy to the case of barotropic flow. The last is a very brief analysis of the predictability of zonally-averaged barotropic flow.

II. <u>Problem and General Approach</u>

The problem that will be dealt with here falls short of the very general question posed earlier in several noteworthy, but probably not crucial respects. In the first place, the equations we shall integrate are not the hydrodynamical equations in their most general form, but the equations for a mathematical idealization or "model" of the atmosphere. This model is very similar to the familiar quasi-geostrophic model originally developed by Charney (1948), which has been tested extensively and applied to routine numerical weather prediction in the past few years, and is even more closely related to the "quasi-nondivergent" model proposed by Kuo (1956).

The virtue of the quasi-nondivergent model, aside from its evident mathematical simplicity, is that it reproduces the behavior of the very large-scale slowly-moving disturbances in the true atmosphere, without exhibiting all of the latter's meteorologically irrelevant behavior --- the existence of which would be implied by the general hydrodynamical equations. It has been shown (on theoretical grounds) by Charney (1948) and others, and fairly well verified by numerical experiment that the approximations of this simple model exclude solutions corresponding to sound and gravity waves, but leave those corresponding to the large-scale weather disturbances essentially intact. It is true, of course, that those approximations also exclude convective instability, boundary-layer turbulence, and other small-scale phenomena that bear on the predictability of the atmosphere to some extent. It is hardly conceivable, however, that these effects could increase its predictability. This suggests that the estimated predictability of the quasi-nondivergent model should be taken as an upper limit on the predictability of the true atmosphere.

We shall deal, specifically, with two variants of the quasi-nondivergent model. One of these is a two-level model whose state is characterized by the streamfunctions at the 250 and 750 mb surfaces, and whose equations are easily derived by approximating vertical derivatives by centered finite-differences taken over 500 mb intervals between 0 and 500 mb, 250 and 750 mb, and 500 and 1000 mb. The other, a degenerate form of the quasi-nondivergent model, is simply the well-known nondivergent barotropic model. Its state shall be characterized by the streamfunction at the 500 mb surface.

Let us now consider two different initial states of the model. One, the true state, shall be characterized by the streamfunction ψ_o; the other, the state reconstructed from data at a finite number of points, by the streamfunction $(\psi_o + R_o)$. The function R_o is the analysis error. It will be left unspecified in detail, but will be assumed random with respect to ψ_o, and will take on the statistical properties of real analysis error. For simplicity, it will also be assumed that R_o vanishes at all times around the edges of a very large region A, with the assurance that this restriction cannot much affect average conditions over the whole of the region. We next imagine that the equations for the quasi-nondivergent model have been integrated by the usual method of extrapolating over successive short intervals of time. In one case, we start integrating from the true initial state ψ_o, to obtain the "correct" state ψ at time t. In the other case, we begin with the reconstructed initial state $(\psi_o + R_o)$, and incorrectly predict the state $(\psi + R)$ at time t. Thus, at each point and at time t, we commit an error R.

Now, consider a statistical measure of error E, defined as the MS value of the gradient of R taken over the whole of the area A. Owing to the relation between wind and streamfunction, E is the MS vector wind error. We choose this particular measure of error partly because we are primarily interested in predicting the wind (gradient of streamfunction), rather than the absolute size of the streamfunction, and partly because the form of the equations implies that the absolute size of the streamfunction is physically irrelevant and, for that matter, unpredictable.

The question is this: How does the MS vector wind error E vary with time? On what statistical properties of the initial state ψ_o does the growth of E depend, and in what way? How does the growth of E depend on the statistical properties of R_o --- e.g., average distance between observing stations, and RMS analysis error?

The most direct approach to these problems, of course, would be to carry out comparative numerical integrations of the equations for the quasi-nondivergent model. That is, one numerical prediction made from a reconstructed initial state over a region where upper air data are actually quite dense could be compared with another made from an initial state independently reconstructed from a fraction of the actual data. It might even be sufficient to compare forecasts made from reconstructed initial states with forecasts made from the same initial states, but to which random initial error fields (with the correct statistical properties) have been added. There are, however, two disadvantages in this approach. The first and most obvious is that the difference between pairs of comparative forecasts would be due partially to roundoff and truncation errors, and would not isolate the essential unpredictability due to analysis error. The second, and probably more important disadvantage is that the error E would not be expressed as a function of the relevant statistical parameters, but as a collection of numerical values. Thus, unless one carried out an enormous number of integrations, it would be extremely unlikely that he would discover the correct combination of factors on which the error depends. It seems desirable, therefore, to devise some analytic approach to the problem, if only as a supplement to numerical experiment.

The procedure that will be followed in the detailed mathematical analysis of the problem is this: We begin by calculating the first time-derivative of E, which is related, by definition, to the first time-derivative of the error R at each point. The latter is then expressed in terms of space-derivatives only by making use of the equations for the quasi-nondivergent model. Similarly, we form the second time-derivative of the error E, and again eliminate the first time-derivative of R by substitution from the equations for the model. In this way, by successive differentiation and substitution from the equations for the quasi-nondivergent model, all the time-derivatives of E at any time can be expressed in terms of the values of ψ and R current at that time.

The next step in the basic procedure is to evaluate the time-derivatives of the error E at the initial instant, taking advantage of the fact that R is initially random. Owing to the latter property, the initial values of the time-derivatives of E take on relatively simple integral forms that can be expressed approximately in terms of the MS gradients of the initial streamfunction ψ_o and analysis error R_o and the autocorrelation functions for ψ_o and R_o. The final step is simply to represent the error E at any time t as a Taylor series in ascending powers of t, in which the coefficients are the time-derivatives of E at the initial instant ($t = 0$). This series explicitly relates the statistical measure of error E to the statistical properties of the initial wind and error fields, as well as to the period of the forecast.

III. The Predictability of Barotropic Flow

The result of applying the procedure outlined above in the case of nondivergent barotropic flow is most concisely expressed in a formula for the percentage change in the MS vector wind error E over a given interval of time t. With certain approximations of integration, discussed more fully in Appendix I,

$$\frac{E - E_o}{E_o} = (M^2 - m^2)\ \overline{\nabla\Psi \cdot \nabla\Psi}\ t^2\ F(M,m) \qquad (1)$$

The function Ψ is related to the initial streamfunction ψ_o as follows:

$$\psi_o = \Psi - Uy$$

where U is the zonal component of the average wind, taken over the entire area A, and y is the coordinate distance toward the north. Thus, since the meridional component of the average wind must vanish, $\nabla\Psi\cdot\nabla\Psi$ is the square of the vector deviation of the wind from its average. The bar placed above a quantity denotes its area average, taken over the entire region A. The quantity M is an inverse measure of the scale of fluctuations in the true initial flow pattern, defined as

$$M^2 = \frac{\overline{\nabla\Psi\cdot\nabla\Psi}}{\overline{\Psi^2}}$$

Similarly, m is an inverse measure of the scale of the initial error field, defined as

$$m^2 = \frac{\overline{\nabla R_o\cdot\nabla R_o}}{\overline{R_o^{\,2}}}$$

The function F(M,m) takes on either of two forms, depending on whether m exceeds M, or vice versa.

$$F(M,m) = \frac{5}{72}\begin{cases} \dfrac{M^2}{m^2} & \text{if } m > M \\[2ex] \dfrac{m^2}{M^2} & \text{if } m < M \end{cases}$$

As might be expected, the change in error $(E - E_o)$ is proportional to the initial error E_o. It is also proportional to the MS value of the vector deviation of the wind from its average, a measure of the rate at which the various interactions and exchange processes operate.

The most startling and significant implication of Eq. (1) is that the growth of error is evidently very sensitive to the difference in the scales of the initial error field and the fluctuations in the true initial flow pattern. If, for example, the scale of the initial error field were appreciably greater than that of the true fluctuations, the error would increase rapidly and the predictability of the winds would correspondingly decrease. Conversely, if the scale of the initial error field were appreciably less than that of the true fluctuations, the error might actually decrease for a while. On the face of it, this result seems paradoxical. It is, however, consistent with the results of independent studies of the stability of barotropic flow made by Lorenz (1953) and Thompson (1957), both of which indicate that the kinetic energy of perturbations decreases if their scale is less than that of meridional fluctuations in the speed of a nonuniform zonal current, and increases if it is greater. One may, in fact, interpret R_o as a real perturbation instead of a random initial error, and regard Eq. (1) as a generalized criterion for the stability of nonequilibrium flows with respect to random perturbations of finite amplitude.

Evidence of the effectiveness of this mechanism of barotropic stability is contained in the fact that wind fields predicted by solving the barotropic vorticity equation grow noticeably "smoother" and more "zonal" as the forecast period is increased --- smoother than can be accounted for by the artificial averaging process that is applied intermittently to reduce small-scale truncation error. In general, the mechanism of baroclinic instability (by which the actual fluctuations in the initial flow pattern were originally produced) tends to amplify components whose scale lies in the range of "barotropic damping". Thus, since the method of prediction does not distinguish between analysis error and real fluctuations of comparable or slightly greater scale, the phenomenon described above may be interpreted as a gradual transfer of kinetic energy from disturbances of small scale to those of larger scale through the mechanism of barotropic stability. Indeed, the recent work of Phillips (1956) indicates that this is the principal mechanism by which the kinetic energy of the westerlies is maintained.

In summary, there is strong reason to believe that the effect of scale on the predictability of a truly barotropic atmosphere would be a very real one. Although it is peculiar to barotropic flow, it is also one of the dominant effects in strongly baroclinic flows.

One never observes, of course, that the errors of barotropic forecasts decrease as the period is increased. A part of this might be attributable to truncation error. As suggested earlier, however, a more important reason for the invariable growth of error probably lies in the fact that baroclinic flows are unstable with respect to perturbations of smaller scale than are barotropic flows --- or, more obviously, the fact that the atmosphere is not really barotropic. Thus, in order to come to any definite conclusions about the predictability of the atmosphere, we must investigate the growth of error in predictions based on a baroclinic model.

Before proceeding to the discussion of more general results, it should be pointed out that Eq. (1) and all subsequent expressions for the error growth contain only the first few terms of the complete Taylor expansion, and are probably not valid for periods longer than a few days. The error cannot, for example, grow indefinitely, since its maximum value is limited by the total initial kinetic energy. Likewise, it cannot <u>decrease</u> indefinitely, for it must always remain positive. In particular, Eq. (1) is correct only to within terms involving the fourth power of the period. In that case, the first time-derivative of E vanishes at $t = 0$, as do all derivatives of odd order. The latter stems from the randomness of the <u>initial</u> error field --- which, together with considerations of reversibility, implies that forecasts and "hindcasts" over periods of equal length should be equally in error.

IV. The Predictability of Baroclinic Flow

We now take up the results of applying the procedure outlined in Section II in the case of the two-level baroclinic model. With approximations that are more fully discussed in Appendix II, the percentage change in overall error during a period of length t takes the form:

$$\frac{E^* - E^*_o}{E_o} = t^2 \left\{ F(M,m) \left[(M^2 - m^2) \left(\frac{2m^2 + \mu^2}{m^2 + \mu^2}\right)\left(\overline{\nabla\Psi^* \cdot \nabla\Psi^*} + \overline{\nabla\Psi' \cdot \nabla\Psi'}\right) + \frac{m^2\mu^2}{m^2 + \mu^2}\left(\overline{\nabla\Psi' \cdot \nabla\Psi'} - \overline{\nabla\Psi^* \cdot \nabla\Psi^*}\right)\right] + \frac{3\mu^2}{8}\left[\frac{\mu^2(2m^2 - M^2)}{m^2(m^2 + \mu^2)}\,\overline{\nabla\Psi' \cdot \nabla\Psi'} + \frac{2\mu^2}{m^2 + \mu^2}\,U'^2\right] \right\} \qquad (2)$$

Here, for mathematical simplicity, the statistical measure of error E* was taken to be

$$E^* = \frac{1}{2}\left(\overline{\nabla R_1 \cdot \nabla R_1} + \overline{\nabla R_2 \cdot \nabla R_2}\right) + \mu^2\left(\overline{R_1^2} + \overline{R_2^2}\right)$$

in which the subscripts 1 and 2 refer to conditions at the 250 and 750 mb surfaces, respectively, and μ^2 is a positive constant. Since E* is still positive definite, its effectiveness as a measure of predictability is unimpaired. The quantities M, m and F(M,m) are as defined in Section III, and are assumed to have the same values at 750 mb as they do at 250 mb. The constant μ, which is essentially a measure of temperature lapse-rate, is closely related to the wave number of maximum baroclinic instability, and is defined as

$$\mu^2 = 2f^2 \left(\frac{R^2T^2}{g\theta} \frac{\partial\theta}{\partial z}\right)^{-1}$$

where f is the Coriolis parameter; R, the gas constant; g, the gravitational acceleration; T and θ are representative values of the absolute temperature and potential temperature, respectively. The variables Ψ^* and Ψ' are related to the initial streamfunctions ψ_1 (at 250 mb) and ψ_2 (at 750 mb) as follows:

$$\Psi^* = \frac{\Psi_2 + \Psi_1}{2} \qquad \Psi' = \frac{\Psi_2 - \Psi_1}{2}$$

$$\psi_1 = \Psi_1 - U_1 y \qquad \psi_2 = \Psi_2 - U_2 y$$

where U_1 and U_2 are the zonal components of the area-averaged wind at 250 and 750 mb, respectively. Thus, $\overline{\nabla\Psi^* \cdot \nabla\Psi^*}$ may be thought of as the MS vector deviation of the <u>vertically averaged</u> wind from its <u>area average</u>. Similarly, $\overline{\nabla\Psi' \cdot \nabla\Psi'}$ is to be interpreted as the MS vector deviation of <u>half</u> the vertical wind shear between 250 and 750 mb from its area average. Finally, U' is simply half of the zonal component of the area-averaged vertical wind shear between 250 and 750 mb.

According to Eq. (2), as in the case of barotropic flow, the growth rate of error is proportional to the initial value of the MS vector wind error. Moreover, since the term whose coefficient is $(M^2 - m^2)$ is one of the dominant terms on the righthand side of Eq. (2), the growth rate is roughly proportional to the MS vector deviation of the vertically-averaged wind from its area average. For the same reason, it is evident that a general decay of predictability in baroclinic flow is favored by an initial error field whose scale is large relative to the scale of initial disturbances in the true flow pattern, and discouraged by the reverse.

The results described above, of course, apply equally well in the case of nondivergent barotropic flow. In the present case, however, they are considerably modified by the thermodynamic processes operative in baroclinic flow. It should be noted that the term in Eq. (2) involving U' is the only one that does not depend on deviations from purely zonal flow and is, therefore, frequently dominant. It invariably increases the rate of error growth, by an amount that is proportional to the square of the average vertical wind shear and inversely proportional to the square of the average static stability. In other words, the atmosphere tends to be unpredictable in situations of strong horizontal temperature gradient and near-adiabatic lapse-rate --- a result that is probably not very surprising to the practicing forecaster.

The joint effect of scale and vertical wind shear is illustrated in Figure 2, on which $(E^* - E^*_o)/E_o$ after one day is plotted as a function of average vertical wind shear and the ratio of error scale λ to the scale L of fluctuations in the true initial flow pattern. The vertical shear is expressed in units of the RMS vector deviation of the vertically averaged wind from its area average. The nondimensional numbers $t^2M^2\ \overline{\nabla\psi^* \cdot \nabla\psi^*}$ and μ^2/M^2 were assigned fixed values of 5 and 0.56, respectively --- both of which are fairly representive of actual conditions in the atmosphere. The ratio of $\overline{\nabla\psi^* \cdot \nabla\psi^*}$ to $\overline{\nabla\psi' \cdot \nabla\psi'}$ was set equal to 4.

Figure 2 shows that the percentage growth of error in the case of no average vertical wind shear is much like that in barotropic flow, the error increasing when the scale of the initial error field is greater than that of the true fluctuations in the initial flow pattern, and vice versa. As the vertical shear increases, however, the rate of error growth for initial error fields of any fixed scale also increases. Simultaneously, the region of scale and shear over which the growth rate is positive includes initial error fields of smaller and smaller scale. This general result is in accord with studies of baroclinic instability made by Fjörtoft (1950), Phillips (1951), Eady (1952), and Thompson (1953), all of whom find that increased vertical wind shear and decreased static stability tend to destabilize baroclinic flows, and also points up the close connection between the problem of predictability and the stability problem.

The seriousness of these results can be most quickly appreciated by estimating the ratio of the scale of actual fields of initial error to the scale of fluctuations in atmospheric flow patterns. Now, the average distance between regular radiosonde, rawinsonde, and dropsonde reports from land-stations, fixed ship positions, and weather reconnaissance aircraft is about 700 miles --- when taken over North America, the Caribbean Sea, the North Atlantic and most of the Pacific, and weighted according to the area that each report represents.* Identifying the average distance between reports with the characteristic half wavelength of the initial error field, and taking the average <u>half</u> wavelength of fluctuations in the true flow pattern to be about 1000 miles, we see that the existing circumstances lie somewhere near the dashed line on Figure 2. Thus, when the relative wind shear falls below 0.9, the <u>inherent</u> predictability of the atmosphere is quite high. (This is not to say, of course, that forecasts in such situations will be correct, for they are actually subject to truncation errors,

* All reports obviously cannot be given equal weight, as one can easily see in the extreme case when all reports are clustered around one point.

boundary errors, and defects of the physical model). On the other hand, there are many winter weather situations in which the relative wind shear exceeds 1.0, in which case E* may increase by as much as .5 E_o after one day, 2.0 E_o after two days, 4.5 E_o after three days, and so on. Accordingly, if the initial MS vector wind error E_o is a substantial fraction of the MS vector wind, the error E* rapidly approaches the level of complete unpredictability.

V. The Maximum Range of Predictability

Having gained a general idea of the rate at which initial analysis error contaminates a forecast, we are now in a position to make a crude estimate of the maximum time-range of predictability, beyond which the atmosphere is essentially unpredictable. Before doing so, however, one must first decide what he means by "unpredictable". Now, it is certainly in accord with the common-sense meaning of the word to regard the occurrence or non-occurrence of an event as "unpredictable", if any prediction is no better than a guess. Thus, we may shift the burden of definition to that of defining a standard method of "guessing" and a standard error of guessing, against which the errors of "predictions" can be judged. We shall then say that the atmosphere is unpredictable beyond the time when the inherent errors of prediction approach the error of guessing.

One of the most obvious methods of "guessing" is simply to select values of wind speed for each point at random, from a population of values whose frequency distribution is that of a representative sample. A meteorologist, of course, would not guess in this fashion, but would insure that his "guess" varied smoothly from point to point. This does not affect the error of random selection, however, for such modifications are equivalent to selecting homogeneous subpopulations from a collection of subpopulations whose frequency distribution is the same as for individual values. It is easily shown that the MS vector wind error of this type of guessing is exactly twice the MS vector deviation of the wind from its average.

On a little reflection, one sees that the procedure described above is not a good method of guessing, simply because always placing one's bet on the average wind would result in a MS vector wind error that is exactly half that of random selection. Accordingly, the RMS vector wind error of guessing will be taken to be the RMS vector deviation of the wind from its area average. It will be found convenient to express the RMS error of guessing in units of the RMS analysis error. In these units, the RMS error of guessing is of the order of 4 over most of the western half of the northern hemisphere.

Let us now suppose that $(E^* - E_o^*)/E_o$ after <u>one</u> day is of the order of .50, a figure that is made plausible by the results of Section IV. Thus, since μ^2/M^2 is approximately .5, E^* is approximately $2E$, so that

$$E \simeq E_o \left(1 + \frac{t^2}{4}\right) \quad \text{for t in days}$$

The question is now reduced to asking how big t must be in order that E (the MS vector wind error of prediction) equal the MS error of guessing --- which, in units of E_o, is 16. The period τ in question is evidently

$$\tau \simeq \sqrt{60} \text{ days} \simeq 7.7 \text{ days}$$

According to this estimate, the atmosphere is <u>essentially</u> unpredictable beyond a period of about a week.

The result stated above should not be taken quite literally, since the MS vector wind error of prediction undoubtedly does not continue to increase like the square of the period beyond two or three days. Offsetting this, however, are the facts that we have considered only the _inherent_ error due to uncertainty of initial state and have disregarded the various other types of error and, second, that the quasi-nondivergent model is probably more predictable than the true atmosphere.

These estimates, although they are admittedly and necessarily rather crude, must be of the correct order of magnitude and are consistent with general experience. Verification studies carried out by Brier (1944) and others are in unanimous agreement that the forecaster loses virtually _all_ ability to predict the details of a weather situation for periods longer than about five days. The significance of the present results does not, however, lie in merely explaining the _observed_ decay of accuracy, but in being able to predict what the decay of predictability would be in circumstances that are altogether different from the present one --- e.g., if the data density were drastically increased or reduced.

Another point deserving emphasis is that the foregoing results apply to the predictability of flow patterns _in natural detail_, or to meteorological events that depend on the phases of individual disturbances --- as, for instance, rain or overcast skies on a particular day. These results do not necessarily apply to the predictability of space- or time-averaged flow patterns. As mentioned earlier, it has been suggested by Namias (1947) that averaged wind fields might be inherently more predictable than detailed flow patterns, and might be predicted over longer periods. In the next section, we shall discuss the predictability of zonally-averaged flow patterns in a case where direct comparisons are possible, namely, in nondivergent barotropic flow.

VI. The Predictability of Zonally-Averaged Flow

In an earlier paper, Thompson (1957) has developed a method for predicting the zonally-averaged flow of a nondivergent barotropic fluid. This enables us to compare the predictability of zonally-averaged flow with that of unaveraged flow in a case that resembles the actual situation enough to warrant a few tentative conclusions. Making use of techniques similar to those outlined in Section II, and introducing approximations discussed more fully in Appendix III, we find that

$$E \simeq [\tfrac{1}{4} (\overline{M}^2 - 2m^2) E_o t^2]^2 \frac{1}{W}\int_0^W U^2 dy \qquad (3)$$

in which m, E_o, and t are as defined in Section III, and W is the entire width of the flow. The quantity U is the difference between the zonally-averaged eastward component of the wind at a particular latitude and its average taken over all latitudes; E is now the MS error in U taken over all latitudes. The constant $\overline{M}$ is the "characteristic wave number" of meridional fluctuations in U.

According to Eq. (3), the statistical measure of error (when normalized with respect to variations in the quantity to be predicted) is equal to the fourth power of a nondimensional measure of the forecast period. Referring back to Eq. (1), we see that the corresponding normalized error in predictions of the unaveraged flow can be put in a very similar form, the non-dimensional measure of the forecast period being about the same, but raised to the second power. This implies that the zonally-averaged flow is more predictable than the unaveraged flow for periods during which the nondimensional measure of time is less than unity. Superficially, at least, it also implies that the zonally-averaged flow becomes less predictable than the unaveraged flow over longer periods, a conclusion that should not be taken seriously in view of the obvious limitations of truncated Taylor expansions over finite intervals. It is hardly conceivable that a quantity which is more predictable over short periods should be less predictable over long periods.

The greater predictability of zonally-averaged flow --- which stems directly from the fact that the second order time-derivative of the error vanishes initially --- is due to an essential difference between the initial states of averaged and unaveraged flows. The reason is simply that the averaged flow is not contaminated initially by random analysis error. Although it is certainly not conclusive, this result suggests that any average which is not sensitive to analysis error is more predictable than the unaveraged flow pattern. At the same time, it must be pointed out that the possible increase of predictability brought about by averaging is acquired at the price of losing "information content" --- in the usual sense that the prediction of averages tells less about the unexpected or abnormal event.

VII. Predictability as a Measure of Confidence in Forecasts

Another question that was raised in the introduction and which needs little elaboration is that of estimating the probable error of predictions. Although Eq. (2) says nothing about the total error to be expected from approximations of all types, it does provide a means of calculating the inherent error due to the uncertainty of initial state, and a means of rating forecasts according to the degree of confidence we should place in them. In general, of course, the probable error does not depend only on the initial error field, whose statistics are fixed by the character of the observational network, but also depends on conditions of scale, vertical wind shear, and static stability in the particular meteorological situation in question. Thus, the inherent error should (and can) be computed daily, and issued in conjunction with the forecast itself.

By this time, it should be abundantly clear that the theory described here is a combination of the dynamical and probabilistic approaches to prediction. The probabilistic element is introduced directly into the dynamical equations by specifying only the statistics of the analysis error; this, in fact, is all that is known about an essentially random error field. It is not claimed that this is the only approach, or even the best approach. It is merely a first step toward reconciling a deterministic approach with the meteorological facts of life.

VIII. The Point of Rapidly Diminishing Returns in the Development of Numerical Forecasting Methods.

Our next concern is to compare the total error of predictions with the inherent error due to uncertainty of initial state alone. Experience over the past few years indicates that the RMS vector wind error of numerical predictions over 36 hours is about 50% of the RMS vector deviation of the wind from its area average. Judging from the rough estimates made in Section V, the inherent RMS error after 36 hours is about 30% of the same standard. The difference between the total RMS error and the inherent RMS error --- roughly 20% of the RMS vector deviation of the wind from its average --- must be due to the combined effects of truncation error, roundoff error, boundary error, and shortcomings of the physical model. A part of this error, conceivably, might also be due to some fundamental indeterminacy in the behavior of the atmosphere.

The result stated above is subject to two rather important interpretations. In the first place, errors of numerical method and defects of the present physical models are quite large enough to account for the residual error not accounted for by the inherent error, and are controllable within economic limitations. In short, there is nothing in these results to indicate that the atmosphere is unpredictable in any fundamental sense --- i.e., that there is any mysterious principle of uncertainty operating, other than the very real one discussed at length in this article.

It is also interesting to note that the inherent RMS error after 36 hours is about the same size as the residual RMS error, both being about one quarter of the RMS deviation of the wind from its average. This indicates that it is certainly worth trying to halve the residual error through the use of higher order finite-difference approximations, special smoothing techniques, hemispheric finite-difference grids, more reference levels, and more realistic physical models. It is also evident that even further efforts in this direction would at most result in something like a 25% reduction in a total RMS error that would still be about 40% of the RMS deviation of the wind from its average, and that such further gains in accuracy would be bought at enormous cost. In summary, the development of better and more complicated numerical prediction methods has not yet reached the point of rapidly diminishing returns, but will have approached it when the RMS error due to "controllable" sources has been about halved. At that point, the only significant gain in actual predictability is to be made by reducing the inherent error --- that is, by increasing the density of regular reporting stations. It is also clear that the time to prepare for such drastic action has already come.

IX. The Effect of Increasing the Density of Observing Stations.

It has been stressed several times earlier that the growth of the inherent error --- due to errors in reconstructing the initial state from data at a finite number of points --- depends crucially on the scale of the initial error field and, ultimately, on the spacing of the upper air observing stations. We have shown, further, that initial analysis error over the oceans and other regions of poor data coverage leads to a widespread and rapid growth of error --- so rapid that the detailed flow of the atmosphere becomes completely and essentially unpredictable after periods of about a week. This situation, as has long been recognized, can be remedied effectively only by increasing the density of reporting stations over regions where the density is now quite low. The remaining questions are these: What return in increased predictability can be expected from increasing the overall density of reporting stations, and how does this compare with the corresponding outlay of funds? Where is the point of rapidly diminishing return per outlay? How should new stations be located in effecting an increase of overall station density?

The facts bearing on these questions are summarized in Figure 3, which shows a graph of the cost of maintaining a uniformly dense network of stations (expressed in units of the cost of maintaining a uniformly dense network, whose density is equivalent to the overall density of the present network) plotted against the percentage increase of the <u>inherent</u> RMS vector wind error after two days for a relative shear of 1.2. For simplicity, it was assumed that the cost of maintenance per observation is the same for all stations, fixed or moving, regardless of type. This assumption would be fairly true, for example, of land stations and installations on ships moving along their normal courses. Our present situation is estimated to lie near the circled point on the curve.

Inspecting Figure 3, we see that doubling the cost and overall density of observing stations would all but eliminate the increase of inherent error. Thus, although it would not completely eliminate the initial error, it would reduce the total RMS vector wind error in 48 hour predictions to something on the order of 45% of the RMS deviation of the wind from its average, rather than the existing error of about 60%. Moreover, according to our present estimates, trebling the cost and overall density of the observational network would produce no significant reduction of inherent error, beyond that attainable by doubling the present outlay. Thus, the point of rapidly diminishing returns lies somewhere around double the present cost.

In actuality, of course, the density of observing stations is far from uniform. Thus, the optimum strategy in effecting an overall increase in density is to establish stations in the regions where the density is lowest up to the density level of the surrounding regions, and then to increase the density over a more inclusive region up to the density level of areas of good coverage. This is simply because the cost of producing an effective increase of overall density in a nonuniform network is less than the cost of producing the same increase of density in a uniform network. In fact, if economy were a strong factor, and if the aims were to produce uniform forecasting accuracy over very large areas, it would be desirable to replace some land stations in regions where they are extremely dense by stations aboard moving ships.

From considerations of symmetry alone, the optimum distribution of observing stations is very nearly one of uniform density, slightly modified by the varying degree of meteorological activity over different regions and by the fact that maximum forecasting accuracy should be attained in regions of dense population. Experience with numerical prediction over the past two years has shown that two and three day forecasts for the United States alone are strongly dependent on the initial state of the atmosphere over more than half a hemisphere. Accordingly, the fact that few people live or travel in the mid-Pacific is not a strong argument for concentrating observing stations in the United States.

It is hoped that the results described in the foregoing sections of this article, or more refined estimates based on a similar theory of predictability, can be used as a common basis for decision and action in dealing with a problem that, by its very nature, is truly international. It is also to be hoped that these rather crude estimates of predictability will not be accepted at face value, but will provoke more complete studies of the empirical factors that enter into this important problem.

X. The Effects of Heat Sources and Viscosity

The estimates of predictability presented earlier do not include the effects of viscous momentum transfer and nonadiabatic heating. Thus, although these processes are probably not important factors in the predictability of atmospheric flow patterns over short periods of time, it is still a matter of interest to speculate about their general effect. By and large, the net effect of internal viscosity is to reduce the variance of relative vorticity, and to obliterate gradually any trace of previously existing circulation centers. Consequently, the state of a viscous fluid depends more on the dynamical properties of the system and the external influences acting on it than it does on an initial state at some time in the remote past. It is hard to imagine, therefore, that the effect of viscosity is to increase the predictability of the flow, except in a sense to be discussed later.

Similarly, it is difficult to conceive that nonadiabatic heating acts in such a way as to increase the predictability of the detailed flow pattern over short periods. Apart from the fact that the distribution of heat sources is not precisely known, it is improbable that the existence of convective instability on any scale is conducive to accurate prediction. It is also true, of course, that nonadiabatic heating has a decisive effect on long period trends in (say) monthly averages --- as, for instance, the change of seasons. It is certainly not necessary to know the detailed state of the atmosphere in the middle of the summer in order to predict the coming of winter. This is due to the fact that long-period averages depend very little on the detailed state of the atmosphere at some time in the far distant past (partly through the dissipative action of viscosity), but are positively controlled by the cumulative effect of energy input. It may turn out, in fact, that averages whose variations depend primarily on the external influences exerted on a quasi-linear system are more predictable over monthlong periods than the detailed state is over a week or so.

XI. Summary of Results

A brief review of the main results is given in the last five paragraphs of the introduction.

Appendix I - The Predictability of Nondivergent Barotropic Flow

We begin with the definition of the mean square error E.

$$E = \frac{1}{A}\int_A \nabla R \cdot \nabla R \quad dA$$

where A is the area of a very large region (also designated by A), dA is an element of A, and R is the error in the streamfunction at various points and at time t. All notation is standard, unless specified otherwise. Differentiating E with respect to time, we have

$$\frac{\partial E}{\partial t} = \frac{2}{A}\int_A \nabla R \cdot \nabla \frac{\partial R}{\partial t} \, dA$$

$$= \frac{2}{A}\int_A \left(\nabla \cdot R\nabla \frac{\partial R}{\partial t} - R \nabla^2 \frac{\partial R}{\partial t}\right) dA$$

A part of the integral above may be transformed by applying Gauss' theorem, as follows:

$$\frac{\partial E}{\partial t} = \frac{2}{A}\oint_C R \frac{\partial}{\partial n}\left(\frac{\partial R}{\partial t}\right) dC - \frac{2}{A}\int_A R \nabla^2 \frac{\partial R}{\partial t} \, dA$$

in which C is the path around the boundary of A, dC is an element of C, and a derivative with respect to n is the component of the vector gradient normal (outward) to C. Thus, since R vanishes on C by hypothesis,

$$\frac{\partial E}{\partial t} = -\frac{2}{A}\int_A R \nabla^2 \frac{\partial R}{\partial t} \, dA \qquad (4)$$

Now, the equation governing the evolution of the flow pattern from an initially <u>incorrect</u> wind field is

$$\nabla^2 \frac{\partial}{\partial t} (\psi + R) + J(\psi + R, \nabla^2\psi + \nabla^2 R) + \beta \frac{\partial}{\partial x} (\psi + R) = 0$$

whereas the equation for the correct flow pattern is

$$\nabla^2 \frac{\partial\psi}{\partial t} + J(\psi, \nabla^2\psi) + \beta \frac{\partial\psi}{\partial x} = 0 \tag{5}$$

β is, as usual, the Rossby parameter and x is the eastward coordinate. Subtracting the equation above from the one preceding it,

$$\nabla^2 \frac{\partial R}{\partial t} = - [J (R, \nabla^2\psi + \nabla^2 R) + J(\psi, \nabla^2 R) + \beta \frac{\partial R}{\partial x}] \equiv - N \tag{6}$$

We next eliminate $\nabla^2 \frac{\partial R}{\partial t}$ between Eqs. (4) and (6), with the result that

$$\frac{\partial E}{\partial t} = \frac{2}{A}\int_A R\ J(\psi, \nabla^2 R)\ dA + \frac{1}{A}\int_A J(R^2, \nabla^2\psi + \nabla^2 R)\ dA \tag{7}$$

$$+ \frac{1}{A}\int_A \beta\ \frac{\partial R^2}{\partial x}\ dA$$

By applying Stokes' theorem, the second integral on the right-hand side of Eq. (7) can be transformed into a line integral around C, whose integrand contains R as a factor; thus, since R vanishes on C, that integral vanishes. For similar reasons, the third integral on the right-hand side of Eq. (7) vanishes. At any time t, therefore,

$$\frac{\partial E}{\partial t} = \frac{2}{A}\int_A R\ J(\psi, \nabla^2 R)\ dA$$

$$\equiv \frac{2}{A}\int_A \mathbb{k} \cdot \nabla\psi \times R\nabla(\nabla^2 R)\ dA \tag{8}$$

where $\mathbb{k}$ is a unit vector, directed vertically upward. At t = 0, ψ and R are uncorrelated by hypothesis, so that the right-hand side of Eq. (8)

vanishes <u>initially</u> --- i.e., designating conditions at $t = 0$ by the subscript zero,

$$\left(\frac{\partial E}{\partial t}\right)_0 = 0 \tag{9}$$

We now differentiate Eq. (8) with respect to time. At any time,

$$\frac{\partial^2 E}{\partial t^2} = \frac{2}{A}\int_A \mathbb{k} \cdot \nabla\psi \times \left[\frac{\partial R}{\partial t} \nabla (\nabla^2 R) + R\nabla \left(\nabla^2 \frac{\partial R}{\partial t}\right)\right] dA$$

$$+ \frac{2}{A}\int_A \mathbb{k} \cdot \nabla \left(\frac{\partial \psi}{\partial t}\right) \times R\nabla (\nabla^2 R)\, dA \tag{10}$$

At $t = 0$, however, the second integral on the right-hand side of Eq. (10) vanishes, since $\frac{\partial \psi}{\partial t}$ is given in terms of ψ <u>alone</u> by Eq. (5), and because ψ and R are uncorrelated <u>initially</u>. Initially, therefore,

$$\frac{\partial^2 E}{\partial t^2} = \frac{2}{A}\int_A \mathbb{k} \cdot \nabla\psi \times \left[\frac{\partial R}{\partial t} \nabla (\nabla^2 R) - \left(\nabla^2 \frac{\partial R}{\partial t}\right) \nabla R\right] dA$$

$$+ \frac{2}{A}\int_A J\left(\psi, R\nabla^2 \frac{\partial R}{\partial t}\right) dA \tag{11}$$

By applying Stokes' theorem, the second integral on the right-hand side of Eq. (11) can be transformed into a line integral around C, whose integrand contains a factor of R; thus, since R vanishes on C, that integral also vanishes.

The next step is to eliminate the time derivatives of R from Eq. (11) by resubstitution from Eq. (6). We have

$$\nabla^2 \frac{\partial R}{\partial t} = - N$$

from which

$$\frac{\partial R}{\partial t} = \iint_A G(\xi,\eta)\ N(x+\xi,\ y+\eta)\ d\xi d\eta \qquad (12)$$

where G is the Green's function corresponding to the Laplace operator, for $G = 0$ on C; x and y are rectangular coordinates in a horizontal plane; ξ and η are variables of integration corresponding to x and y. As implicitly assumed in Eq. (12), the Green's function for a large region A is very nearly independent of x and y at almost all points of the region. Similarly, letting $\nabla^2 R = Z$,

$$\nabla^2(\nabla R) \equiv \nabla(\nabla^2 R) = \nabla Z$$

$$\nabla R = -\iint_A G(\xi,\eta)\ \nabla Z(x+\xi,\ y+\eta)\ d\xi d\eta$$

Thus, substituting these results into Eq. (11),

$$\frac{\partial^2 E}{\partial t^2} = \frac{2}{A}\iint_A \mathbb{k}\cdot\nabla\psi\ (x,y) \times [\nabla Z\ (x,y)\iint_A G(\xi,\eta)\ N(x+\xi,\ y+\eta)\ d\xi d\eta$$

$$- N(x,y)\iint_A G(\xi,\eta)\ \nabla Z(x+\xi,\ y+\eta)\ d\xi d\eta]\ dxdy$$

From now on, it is to be understood that this equation applies only at $t = 0$, and that the right-hand side is to be evaluated from the <u>initial</u> fields of ψ and R. We now invert the order of integration in the equation above, with the result that

$$\frac{\partial^2 E}{\partial t^2} = 2\iint_A G(\xi,\eta)\ X\ (\xi,\eta)\ d\xi d\eta \qquad \text{where} \qquad (13)$$

$$X(\xi,\eta) = \frac{1}{A}\iint_A \mathbb{k}\cdot\nabla\psi(x,y) \times [N(x+\xi, y+\eta)\,\nabla Z(x,y)$$

$$- N(x,y)\,\nabla Z(x+\xi, y+\eta)]\,dxdy$$

Finally, introducing the definitions of N and Z,

$$X(\xi,\eta) = \frac{1}{A}\iint_A \{J(\psi,\nabla^2 R)\,[J_d(R,\nabla^2\psi + \nabla^2 R) + J_d(\psi,\nabla^2 R) + \beta\frac{\partial R_d}{\partial x}]$$

$$- J(\psi,\nabla^2 R_d)\,[J(R,\nabla^2\psi + \nabla^2 R) + J(\psi,\nabla^2 R) + \beta\frac{\partial R}{\partial x}]\}\,dxdy$$

where quantities without subscript apply at the variable point (x,y) and those with subscript d apply at the point $(x+\xi, y+\eta)$.

Owing to the randomness of R and the fact that ψ and R are initially uncorrelated, certain terms in the integrand above vanish in the mean. Others, however, do not vanish. In fact, the integral of one such term

$$\frac{1}{A}\iint_A J(\psi,\nabla^2 R)\,J_d(\psi,\nabla^2 R)\,dxdy$$

is the autocorrelation function for the advection of initial vorticity error with the true initial wind. Similarly, $J(\psi,\nabla^2 R)$ is correlated with $J_d(R,\nabla^2\psi)$, simply because R (which oscillates around zero) is highly correlated with $\nabla^2 R$, and ψ is correlated with $\nabla^2\psi$. As a result of such considerations, X may be written as

$$X(\xi,\eta) = \frac{1}{A}\iint_A [J(\bar{\Psi},\nabla^2 R)\,J_d(R,\nabla^2\bar{\Psi}) + J(\bar{\Psi},\nabla^2 R)\,J_d(\bar{\Psi},\nabla^2 R)$$

$$- J(\bar{\Psi},\nabla^2 R_d)\,J(R,\nabla^2\bar{\Psi}) - J(\bar{\Psi},\nabla^2 R_d)\,J(\bar{\Psi},\nabla^2 R)]\,dxdy \qquad (14)$$

in which $\psi_o = \bar{\Psi} - Uy$, U is the zonal component of the average wind over A, and y is the northward coordinate. Since both Ψ and R oscillate around zero, they are both highly correlated with their Laplacians. For purposes of approximating X, therefore, we write

$$\nabla^2 R = - m^2 R \qquad\qquad \nabla^2 \Psi = - M^2 \Psi$$

where m and M are inverse measures of scale, defined as

$$m^2 = - \frac{\overline{R\nabla^2 R}}{\overline{R^2}} \qquad\qquad M^2 = - \frac{\overline{\Psi\nabla^2 \Psi}}{\overline{\Psi^2}}$$

and the bar above a quantity denotes its area average over A. Introducing these approximations into Eq. (14), we find that

$$X(\xi,\eta) \simeq m^2(M^2 - m^2) \, [\overline{J(\Psi,R) \; J(\Psi,R_d)} - \overline{J(\Psi,R) \; J_d(\Psi,R)}] \tag{15}$$

This equation makes it clear that X is a kind of autocorrelation function.

The next problem is to express X in terms of the autocorrelation functions for the initial fields of Ψ and R. Let us first consider the quantity

$$\overline{J(\Psi,R) \; J(\Psi,R_d)} = \overline{|\nabla\Psi| \; |\nabla\Psi| \quad |\nabla R| \; |\nabla R_d| \quad \sin\alpha \sin\alpha_d}$$

in which α is the angle between $\nabla\Psi(x,y)$ and $\nabla R(x,y)$, and α_d is the angle between $\nabla\Psi(x,y)$ and $\nabla R(x + \xi, y + \eta)$. Now, it is clear that the decrease of X, as ξ and η become larger and larger, is not due to any systematic decrease in the correlation between $|\nabla R|$ and $|\nabla R_d|$, but to a decrease in the correlation between α and α_d. Approximately, then

$$\overline{J(\Psi,R) \; J(\Psi,R_d)} = \overline{\nabla\Psi \cdot \nabla\Psi} \quad \overline{\nabla R \cdot \nabla R} \quad \overline{\sin\alpha \sin\alpha_d}$$

Moreover, assuming that all angles α are equally probable in the mean,

$$\overline{\sin\alpha\,\sin\alpha_d} \simeq \frac{1}{2}\,\overline{X}_R$$

where $\overline{X}_R$ is the normalized autocorrelation function for the initial field of R. Following a similar line of reasoning, we find that

$$\overline{J(\Psi,R)\;J_d\;(\Psi,R)} \simeq \frac{1}{2}\,\overline{\nabla\Psi\cdot\nabla\Psi}\;\;\overline{\nabla R\cdot\nabla R}\;\;\overline{X}_R\,\overline{X}_\Psi$$

in which $\overline{X}_\Psi$ is the normalized autocorrelation function for the initial Ψ field. Introducing these estimates into Eq. (15), and substituting the resulting expression for $X(\xi,\eta)$ into Eq. (13), we have

$$\frac{\partial^2 E}{\partial t^2} = m^2(M^2 - m^2)\;\overline{\nabla\Psi\cdot\nabla\Psi}\;\;\overline{\nabla R\cdot\nabla R}\iint_A G(\xi,\eta)\;(\overline{X}_R - \overline{X}_R\,\overline{X}_\Psi)\;d\xi d\eta \qquad (16)$$

This formula expresses the second time-derivative of E at $t = 0$ in terms of statistical properties of the initial fields of Ψ and R --- namely, the characteristic scales of those fields, their mean-square gradients, and their autocorrelation functions. The Green's function G is analytic, and can be computed with any desired degree of accuracy.

The most direct procedure, of course, would be to determine the autocorrelations for Ψ and R empirically. It is desirable, on the other hand, to express $\overline{X}_\Psi$ and $\overline{X}_R$ explicitly in terms of the scale parameters m and M. Accordingly, for purposes of estimate, we shall assign them simple analytic forms. Provided the statistics of the Ψ and R fields are isotropic, the second derivatives of the autocorrelation functions at $(\xi,\eta) = (0,0)$ are

$$\frac{d^2\overline{X}_\Psi}{dr^2} = -\frac{M^2}{2} \qquad\qquad \frac{d^2\overline{X}_R}{dr^2} = -\frac{m^2}{2}$$

where $r^2 = \xi^2 + \eta^2$. We now assume that the equations above hold from $r = 0$

to the points where the autocorrelation functions vanish. That is,

$$\bar{\chi}_{\Psi} = \begin{cases} 1 - \frac{M^2 r^2}{4} & \text{when } r < \frac{2}{M} \\ 0 & \text{when } r > \frac{2}{M} \end{cases}$$

$$\bar{\chi}_{R} = \begin{cases} 1 - \frac{m^2 r^2}{4} & \text{when } r < \frac{2}{m} \\ 0 & \text{when } r > \frac{2}{m} \end{cases}$$

To a degree consistent with earlier approximations of G, these expressions lead to:

$$\iint_A G(\xi,\eta)(\bar{\chi}_R - \bar{\chi}_R\bar{\chi}_{\Psi})\, d\xi d\eta \simeq \frac{5}{36} \begin{cases} \frac{M^2}{m^4} & \text{if } m > M \\ \frac{1}{M^2} & \text{if } m < M \end{cases}$$

Substitution of this result in Eq. (16) gives

$$\left(\frac{\partial^2 E}{\partial t^2}\right)_o = 2(M^2 - m^2)\, \overline{\nabla\Psi_o \cdot \nabla\Psi_o}\;\; \overline{\nabla R_o \cdot \nabla R_o}\;\; F(m,M)$$

where

$$F(m,M) = \frac{5}{72} \begin{cases} \frac{M^2}{m^2} & \text{if } m > M \\ \frac{m^2}{M^2} & \text{if } m < M \end{cases}$$

Finally, expanding E in a Taylor series around $t = 0$, and recalling that $(\partial E/\partial t)_o$ vanishes,

$$E - E_o = \frac{t^2}{2!}\left(\frac{\partial^2 E}{\partial t^2}\right)_o + \frac{t^3}{3!}\left(\frac{\partial^3 E}{\partial t^3}\right)_o + \cdots$$

$$\simeq E_o(M^2 - m^2)\, \overline{\nabla\Psi_o \cdot \nabla\Psi_o}\; t^2\, F(m,M)$$

From considerations of reversibility, the third derivative of E at $t = 0$ (and all higher derivatives of odd order) must vanish. The equation above is Eq. (1) of Section III. Together with Eq. (8), the result above implies that the error cannot remain random with respect to ψ; although initially random, the error field is organized through nonlinear interactions to produce an increase or decrease of error as the period t is increased.

Appendix II - The Predictability of Baroclinic Flow

The procedure for calculating the initial time-derivatives of mean-square error in the case of the two-level quasi-nondivergent model is exactly analogous to the pattern of development described in some detail in Appendix I, and will be reproduced only in outline. As before, we begin by considering a statistical measure of error E.

$$E = \frac{1}{A}\int_A \frac{1}{2}\left(\nabla R_1 \cdot \nabla R_1 + \nabla R_2 \cdot \nabla R_2\right) dA$$

$$= \frac{1}{A}\int_A \left(\nabla R^* \cdot \nabla R^* + \nabla R' \cdot \nabla R'\right) dA$$

in which R_1 and R_2 are the errors in the streamfunctions at 250 and 750 mb, respectively, and

$$R^* = \frac{R_2 + R_1}{2} \qquad\qquad R' = \frac{R_2 - R_1}{2}$$

Unless otherwise indicated, all "starred" and "primed" variables shall bear this same relationship to the corresponding variables at 250 and 750 mb. The equation analogous to Eq. (4) is

$$\frac{\partial E}{\partial t} = -\frac{2}{A}\int_A \left(R^*\nabla^2 \frac{\partial R^*}{\partial t} + R'\nabla^2 \frac{\partial R'}{\partial t}\right) dA \tag{4a}$$

In this case, the equations by which the time-derivatives of R^* and R' are eliminated from $\partial E/\partial t$ and higher derivatives of E are

$$\nabla^2 \frac{\partial R^*}{\partial t} = - [J(R^*, \nabla^2\psi^* + \nabla^2 R^*) + J(\psi^*, \nabla^2 R^*)$$

$$+ J(R', \nabla^2\psi' + \nabla^2 R') + J(\psi', \nabla^2 R') + \beta \frac{\partial R^*}{\partial x}] \doteq - N^* \qquad (6a)$$

$$\nabla^2 \frac{\partial R'}{\partial t} = - [J(R^*, \nabla^2\psi' + \nabla^2 R') + J(\psi^*, \nabla^2 R')$$

$$+ J(R', \nabla^2\psi^* + \nabla^2 R^*) + J(\psi', \nabla^2 R^*) + \beta \frac{\partial R'}{\partial x}$$

$$- \mu^2 \frac{\partial R'}{\partial t} - \mu^2 J(\psi^* + R^*, R') - \mu^2 J(R^*, \psi')] \doteq - N' \qquad (6b)$$

Substituting from Eqs. (6a) and (6b) into Eq. (4a), and introducing the fact that R^* and R' are assumed to vanish on C,

$$\frac{\partial E^*}{\partial t} = \frac{2}{A}\int_A [R^* J(\psi^*, Z^*) + R^* J(\psi', Z') + R' J(\psi^*, Z')$$

$$+ R' J(\psi', Z^*) - \mu^2 R' J(R^*, \psi')] \, dA \qquad (8a)$$

where

$$E^* = \frac{1}{A}\int_A (\nabla R^* \cdot \nabla R^* + \nabla R' \cdot \nabla R' + \mu^2 R'^2) \, dA \qquad \text{and}$$

$$Z^* = \nabla^2 R^* \qquad\qquad Z' = \nabla^2 R'$$

As before, the initial randomness of R^* and R' requires that $\partial E^*/\partial t$ vanish at $t = 0$. Differentiating Eq. (8a) with respect to time, and again taking advantage of the facts that R^* and R' are initially random and vanish on C, we obtain an equation analogous to Eq. (11).

$$\frac{\partial^2 E^*}{\partial t^2} = \frac{2}{A}\int_A \{ \mathbb{k}\cdot\nabla\psi^* \times [\frac{\partial R^*}{\partial t}\nabla Z^* - \frac{\partial Z^*}{\partial t}\nabla R^* + \frac{\partial R'}{\partial t}\nabla Z' - \frac{\partial Z'}{\partial t}\nabla R']$$

$$+ \mathbb{k}\cdot\nabla\psi' \times [\frac{\partial R^*}{\partial t}\nabla Z' - \frac{\partial Z'}{\partial t}\nabla R^* + \frac{\partial R'}{\partial t}\nabla Z^* - \frac{\partial Z^*}{\partial t}\nabla R']$$

$$- \mu^2 \frac{\partial R'}{\partial t} J(R^*, \psi') + \mu^2 \frac{\partial R^*}{\partial t} J(R', \psi')\} \, dA \qquad (11a)$$

Time-derivatives of R^* and R' are again eliminated by recourse to Eqs. (6a) and (6b), with the result that

$$\frac{\partial^2 E^*}{\partial t^2} = 2 \iint_A G(\xi,\eta)\, X(\xi,\eta)\, d\xi d\eta \qquad \text{where now} \qquad (13a)$$

$$X(\xi,\eta) = \frac{1}{A}\iint_A [\, \mathbb{k}\cdot\nabla\psi^* \times (N_d^* \nabla Z^* - N^* \nabla Z_d^*$$

$$+ N_d' \nabla Z' - N' \nabla Z_d')$$

$$+ \mathbb{k}\cdot\nabla\psi' \times (N_d^* \nabla Z' - N' \nabla Z_d^*$$

$$+ N_d' \nabla Z^* - N^* \nabla Z_d')$$

$$- \mu^2 N_d' J(R^*, \psi') + \mu^2 N_d^* J(R', \psi')]\, dxdy$$

We next introduce the definitions of N^*, N', Z^* and Z', and make further use of the initial randomness of R^* and R' to put X in a form similar to that of Eq. (15).

$$\chi(\xi,\eta) = m^2(M^2 - m^2)\left(\frac{2m^2 + \mu^2}{m^2 + \mu^2}\right) [\overline{J(\Psi^*, R_d)\, J(\Psi^*, R)} + \overline{J(\Psi', R_d)\, J(\Psi', R)}$$

$$- \overline{J(\Psi^*, R)\, J_d(\Psi^*, R)} - \overline{J(\Psi', R)\, J_d(\Psi', R)}]$$

$$+ \frac{m^4\mu^2}{m^2 + \mu^2} [\overline{J(\Psi', R_d)\, J(\Psi', R)} - \overline{J(\Psi^*, R_d)\, J(\Psi^*, R)}$$

$$- \overline{J(\Psi', R)\, J_d(\Psi', R)} + \overline{J(\Psi^*, R)\, J_d(\Psi^*, R)}]$$

$$+ \frac{\mu^4(2m^2 - M^2)}{m^2 + \mu^2} \overline{J(\Psi', R)\, J_d(\Psi', R)} + \frac{\mu^4 m^2}{m^2 + \mu^2} U'^2\, \overline{\nabla R \cdot \nabla R_d} \qquad (15a)$$

in which $\psi^* = \Psi^* - U^*y$ and $\psi' = \Psi' - U'y$; U^* is the zonal component of the area average of the vertically-averaged wind, and U' is the zonal component of the area average of half the vertical wind shear between 250 and 750 mb. For simplicity (although it is not necessary), it has been assumed that the statistics of R^* and R' are the same. Following procedures similar to those by which Eq. (16) was derived from Eq. (15), we arrive at a formula that expresses the initial value of the second time-derivative of E^* in terms of the statistical properties of the initial error and wind fields:

$$\left(\frac{\partial^2 E^*}{\partial t^2}\right)_0 = \overline{\nabla R \cdot \nabla R}\ \{[m^2(M^2 - m^2)\left(\frac{2m^2 + \mu^2}{m^2 + \mu^2}\right)(\overline{\nabla\Psi^* \cdot \nabla\Psi^*} + \overline{\nabla\Psi' \cdot \nabla\Psi'})$$

$$+ \frac{m^4\mu^2}{m^2 + \mu^2}(\overline{\nabla\Psi' \cdot \nabla\Psi'} - \overline{\nabla\Psi^* \cdot \nabla\Psi^*})]\iint_A G(\xi,\eta)(\overline{X}_R - \overline{X}_R\overline{X}_\Psi)\, d\xi d\eta$$

$$+ \frac{\mu^4(2m^2 - M^2)}{m^2 + \mu^2}\overline{\nabla\Psi' \cdot \nabla\Psi'} \iint_A G(\xi,\eta)\ \overline{X}_R\overline{X}_\Psi\, d\xi d\eta$$

$$+ \frac{2\mu^4 m^2}{m^2 + \mu^2} U'^2 \iint_A G(\xi,\eta)\ \overline{X}_R\, d\xi d\eta\} \qquad (16a)$$

Finally, substitution of the approximate autocorrelation functions described in Appendix I, and expansion of E* in a Taylor series around $t = 0$ leads to Eq. (2) of Section IV.

Appendix III - The Predictability of Zonally Averaged Barotropic Flow

We now consider the MS error E in predictions of the zonal component $\bar{u}$ of the zonally averaged wind, starting with reconstructed initial wind fields in which R_u and R_v are the analysis errors in the eastward and northward wind components u and v.

$$E = \frac{1}{W}\int_0^W R^2 dy$$

where R is the error in $\bar{u}$ at various latitudes, and W is the entire width of the flow. Differentiating E successively with respect to time,

$$\frac{\partial E}{\partial t} = \frac{2}{W}\int_0^W R\,\frac{\partial R}{\partial t}\,dy$$

$$\frac{\partial^2 E}{\partial t^2} = \frac{2}{W}\int_0^W [R\,\frac{\partial^2 R}{\partial t^2} + (\frac{\partial R}{\partial t})^2]\,dy$$

$$\frac{\partial^3 E}{\partial t^3} = \frac{2}{W}\int_0^W [R\,\frac{\partial^3 R}{\partial t^3} + 3\,\frac{\partial R}{\partial t}\,\frac{\partial^2 R}{\partial t^2}]\,dy$$

$$\frac{\partial^4 E}{\partial t^4} = \frac{2}{W}\int_0^W [R\,\frac{\partial^4 R}{\partial t^4} + 4\,\frac{\partial R}{\partial t}\,\frac{\partial^3 R}{\partial t^3} + 3\,(\frac{\partial^2 R}{\partial t^2})^2]\,dy \qquad (17)$$

Initially, R vanishes, because the analysis error R_u is random and oscillates around zero. Moreover, the first derivative of $\bar{u}$ is given by

$$\frac{\partial \bar{u}}{\partial t} = -\frac{\partial}{\partial y}\,\overline{uv}$$

in which a "bar" above a quantity now (and henceforth) denotes its <u>zonal</u>

average. The equation above implies that, <u>initially</u>

$$\frac{\partial R}{\partial t} = - \frac{\partial}{\partial y} (\overline{vR_u} + \overline{uR_v} + \overline{R_u R_v})$$

Since the analysis errors R_u and R_v are assumed to be uncorrelated with the true initial wind field, and are themselves independent, $\partial R/\partial t$ vanishes <u>initially</u> at all latitudes. Together with Eq. (17), these results imply that

$$(\frac{\partial E}{\partial t})_o = (\frac{\partial^2 E}{\partial t^2})_o = (\frac{\partial^3 E}{\partial t^3})_o = 0$$

$$(\frac{\partial^4 E}{\partial t^4})_o = \frac{6}{W}\int_o^W (\frac{\partial^2 R}{\partial t^2})_o^2 \ dy$$

Thus, expanding E in a Taylor series around $t = 0$,

$$E \simeq \frac{t^4}{4W}\int_o^W (\frac{\partial^2 R}{\partial t^2})_o^2 \ dy \tag{18}$$

From this formula, it follows immediately that the MS error in the zonally-averaged wind increases like the <u>fourth</u> power of the period.

It remains to estimate the second time-derivative of the error R for any particular latitude at $t = 0$. This can be done most simply by making use of Eq. (21) of Thompson (1957), in an approximate form justified later in that article. At any latitude,

$$\frac{\partial^2 \overline{u}}{\partial t^2} \simeq \overline{v^2} \frac{\partial^2 U}{\partial y^2} - 2 \ \overline{v\nabla^2 v} \ U \tag{19}$$

in which

$$U = \overline{u} - \frac{1}{W}\int_o^W \overline{u} \ dy$$

Now, since R vanishes <u>initially</u>, Eq. (19) implies that

$$\left(\frac{\partial^2 R}{\partial t^2}\right)_o = \frac{\partial^2 U}{\partial y^2}\,[\overline{(v + R_v)^2} - \overline{v^2}] - 2U\,[\overline{(v + R_v)(\nabla^2 v + \nabla^2 R_v)} - \overline{v\nabla^2 v}]$$

Thus, because R_v is random with respect to v,

$$\left(\frac{\partial^2 R}{\partial t^2}\right)_o = \overline{R_v^2}\,\frac{\partial^2 U}{\partial y^2} - 2\overline{R_v \nabla^2 R_v}\,U$$

For purposes of estimate, we now introduce a characteristic wave-number m of the initial error field, and a characteristic wave-number $\overline{M}$ of meridional fluctuations in the mean zonal wind profile. At the same time recalling that the statistics of the initial error field are isotropic, we may rewrite the equation above in the form

$$\left(\frac{\partial^2 R}{\partial t^2}\right)_o = \frac{1}{2}\,(2m^2 - \overline{M}^2)\;\overline{\mathbb{R} \cdot \mathbb{R}}\;U$$

where $\mathbb{R}$ is the vector error in the initial wind field. This equation, taken together with Eq. (18), leads to Eq. (3) of Section VI.

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Captions

Figure 1 (a) - A hypothetically "true" initial contour pattern.

Figure 1 (b) - The contour pattern reconstructed by interpolation between given "true" height values at the circled points.

Figure 1 (c) - A typical pattern of initial analysis error, formed by subtracting the height field of Fig. 1 (a) from that of Fig. 1 (b). Note the relation between the "station locations" and the positions and scale of the error centers.

Figure 2 - Percentage increase of error as a function of error scale and average vertical wind shear.

Figure 3 - The cost of maintaining a uniformly dense network of stations, plotted against the percentage increase of inherent error corresponding to that outlay. The circled point on the curve represents the existing situation.

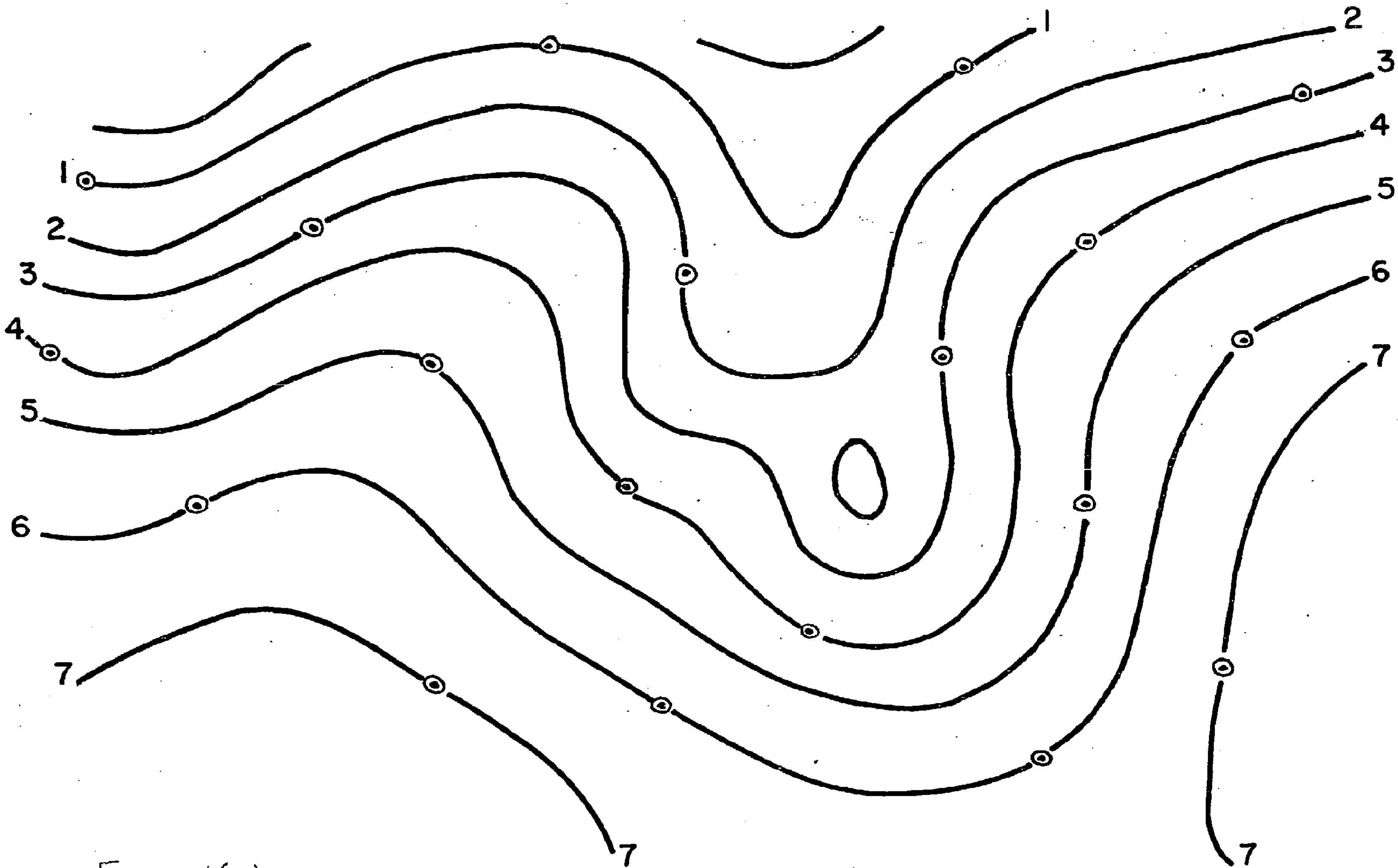


Figure 1(a)

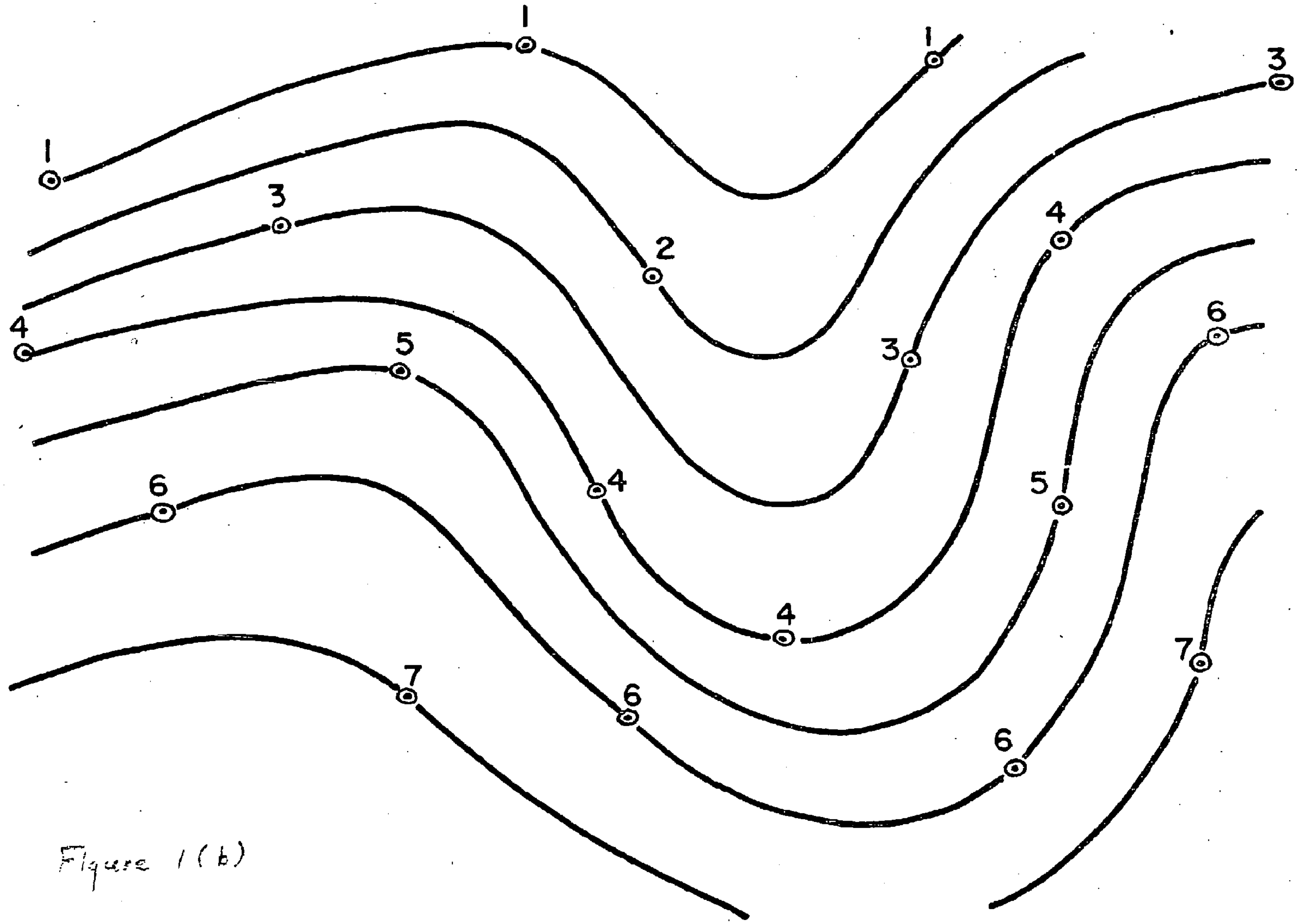


Figure 1(b)

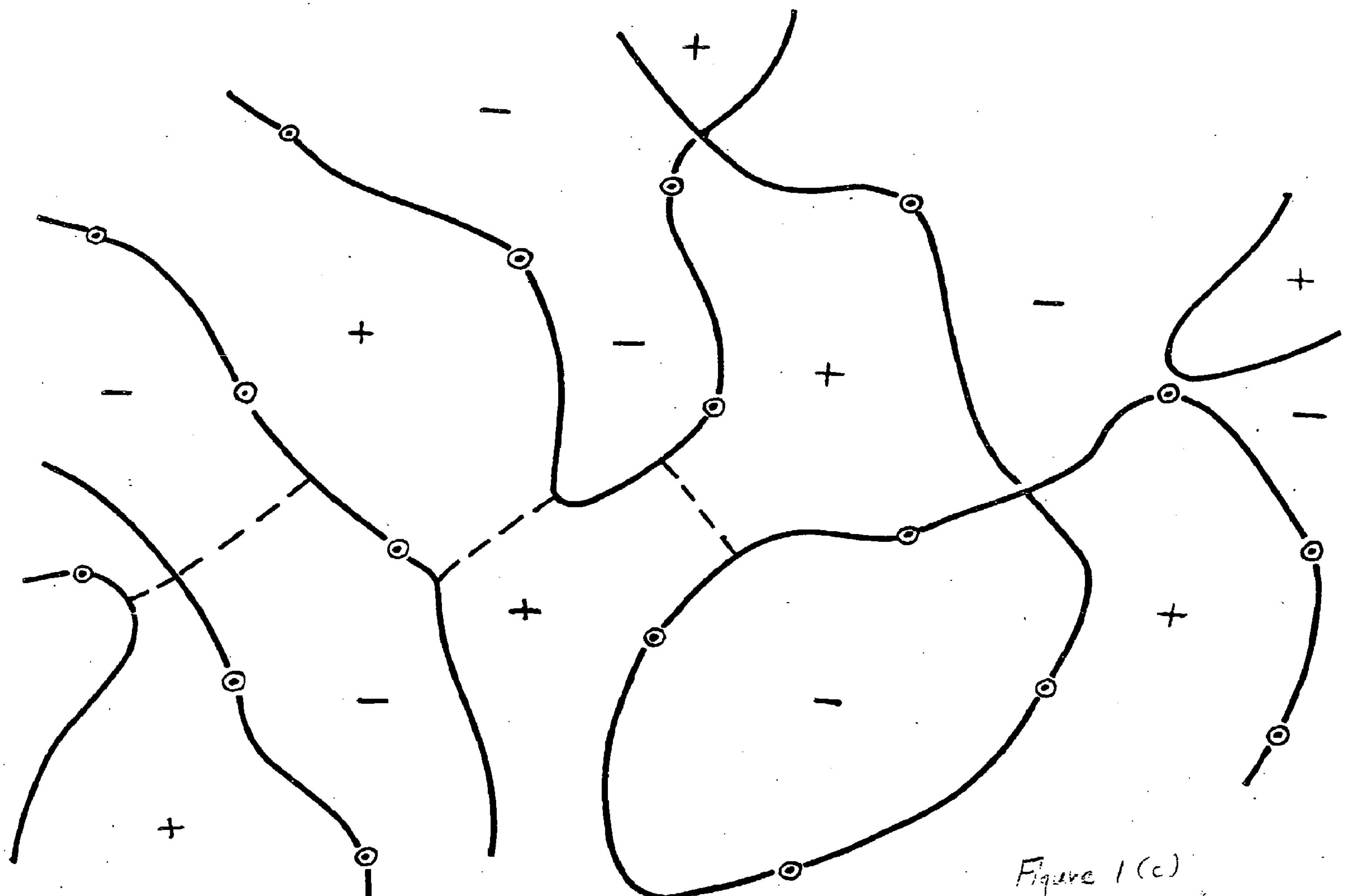


Figure 1 (c)

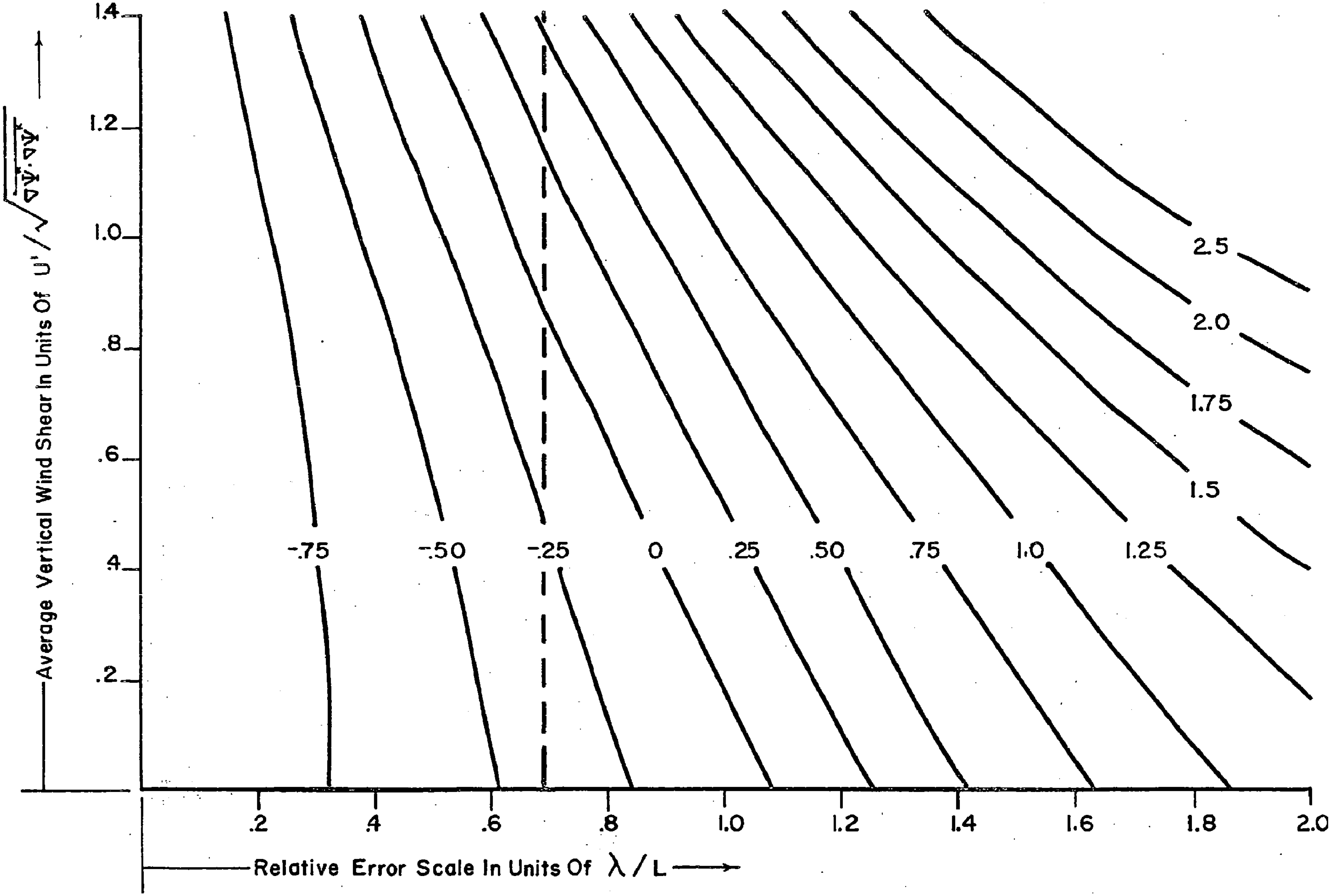
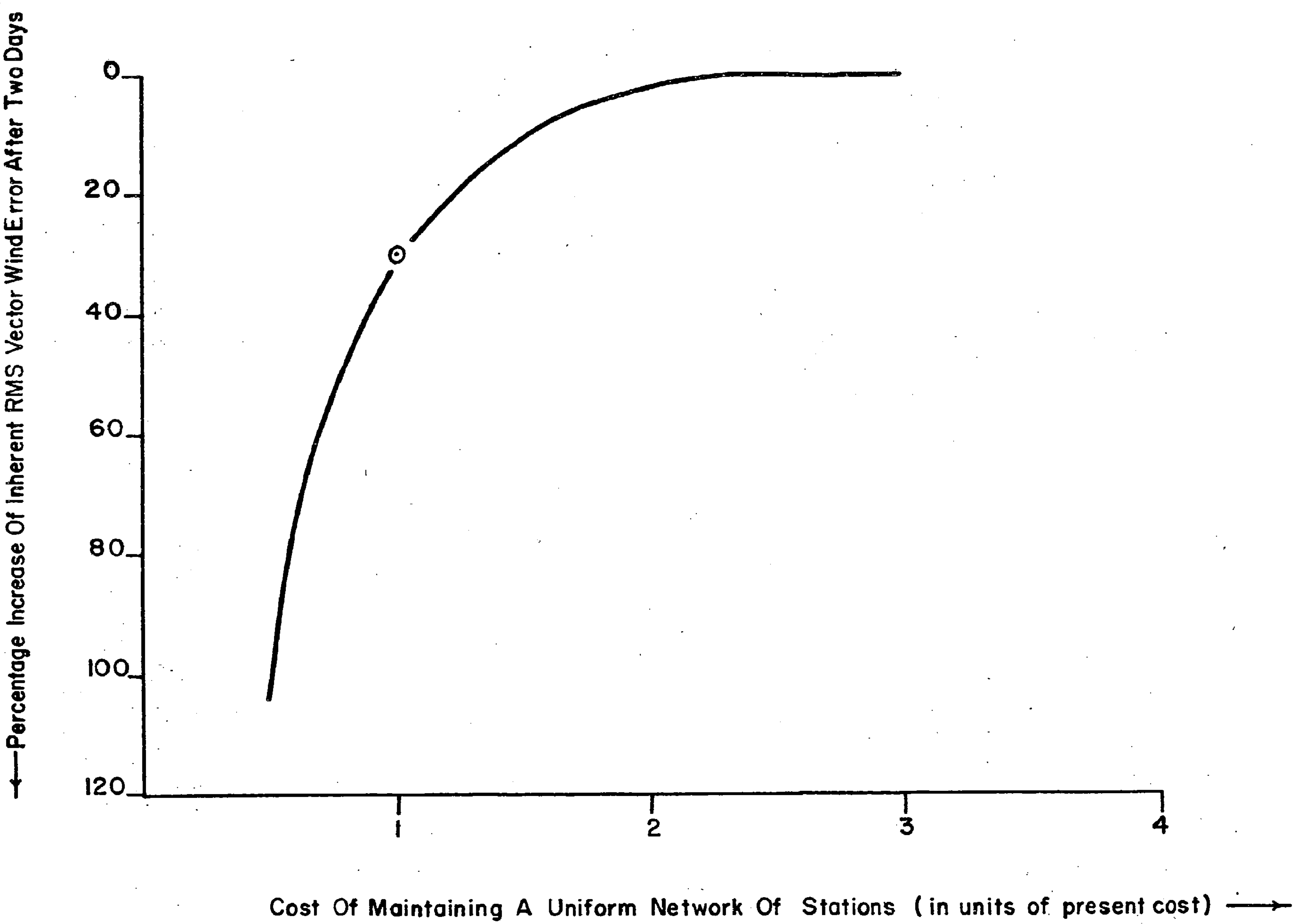

Average Vertical Wind Shear In Units Of $U' / \sqrt{\nabla\Psi \cdot \nabla\Psi}$
Relative Error Scale In Units Of λ / L
1.4
1.2
1.0
.8
.6
.4
.2
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
-.75
-.50
-.25
0
.25
.50
.75
1.0
1.25
1.5
1.75
2.0
2.5

Figure 2



Percentage Increase Of Inherent RMS Vector Wind Error After Two Days
0
20
40
60
80
100
120
1
2
3
4
Cost Of Maintaining A Uniform Network Of Stations (in units of present cost)

Figure 3.