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U.S. Joint Numerical Weather Prediction Unit

A Two-Parameter Nongeostrophic Model
Suitable for Routine Numerical Weather Forecasting

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1. Introduction

Over the past few months, the Joint Numerical Weather Prediction Unit has computed over sixty sets of 12, 24, and 36 hour numerical forecasts, based on the equations for a generalized form of the so-called "thermotropic" model. This two-parameter model is very similar to the one described by Thompson and Gates (1956), in that the terms representing the reorientation of vortex tubes and vertical advection of vorticity are omitted from the vorticity equation, but differs from it in the following important respects*:

(1) The absolute vorticity, where it enters as a factor of the "divergence" term in the vorticity equation, is not replaced by the Coriolis parameter.

(2) The total derivative of pressure at 1000 millibars is not set equal to zero, but is computed as the advection of standard pressure at the ground surface with the interpolated geostrophic wind at the ground surface.

(3) The finite-difference grid --- a rectangular array of 30 x 34 points, spaced about 300 kilometers apart along the grid axes --- covers an area much larger than that covered by earlier multi-parameter forecasts. Thus, since the restrictions still present are valid for disturbances of very large scale (where the Richardson number is large), the model in question approaches the most general two-parameter quasi-geostrophic model.

*A more detailed discussion of the "thermotropic" model is given by Thompson in Appendix I, Report of Progress from 15 February 1953 to 15 August 1953, Joint GRD-AWS Numerical Prediction Project, Geophysics Research Directorate, AF Cambridge Research Center.

A limited number of comparisons between forecasts that were identical in all other respects showed that the introduction of changes (1) and (2) above produced a slight, but significant improvement in the accuracy of both the 500 and 1000 millibar height forecasts. Similarly, the introduction of change (3) guaranteed that the forecasts in the central portion of the grid were almost completely unaffected by errors in the lateral boundary conditions for periods up to 36 hours. Further comparisons between otherwise identical models have revealed that forecasts computed from the initial streamfunction (Calculated from the balance equation for nondivergent flow, and treated as a "height") are slightly, but significantly better than forecasts computed from the initial height field itself.

In a more absolute sense, the 500 millibar height forecasts based on the current version of the thermotropic model are slightly, but significantly better than those based on the barotropic model, the JNWP Unit's earlier 3-level model --- and, for that matter, better than the 500 millibar height forecasts prepared subjectively by the National Weather Analysis Center. In a still more absolute sense, the 500 millibar thermotropic forecasts are essentially "correct", usually predicting the observed trend of the large-scale changes for periods up to 36 hours. A truly bad prediction of the large-scale circulation pattern at 500 millibars is almost a rarity.

The 1000 millibar "thermotropic" forecasts, however, still leave much to be desired. Taken on the whole, they are perhaps slightly better than the forecasts based on the JNWP Unit's earlier 3-level model, comparable with the subjective forecasts of the NWAC, and inferior to the 500 millibar forecasts computed by any of the accepted numerical methods. With change (1) above, spurious "baroclinic" anticyclogenesis (usually confined to the lowest layers) has been considerably reduced. Moreover, by treating the initial streamfunction for "balanced" flow as a height, spurious "barotropic" anticyclogenesis (extending through all layers, and most pronounced in subtropical highs) can be effectively controlled. Nevertheless, the 1000 millibar thermotropic forecasts frequently display highly unrealistic changes in the phase relations between the surface systems and the large-scale troughs and ridges aloft.

Considering that the JNWP Unit's thermotropic model does approach the general two-parameter quasi-geostrophic model, it is natural to ask whether the poor quality of the 1000 millibar forecasts is due to the lack of vertical resolution or to the quasi-geostrophic character of the model. With regard to the first alternative, it should be noted that the 1000 millibar forecasts based on the 3-level model (making due allowances for smaller grid area) were no better than the thermotropic forecasts. One is thus led to suspect that the main defect of the models currently under study is not merely that they do not detect all the minute variations in the vertical structure of the large-scale disturbances, but that they do not accurately reproduce the physical mechanism by which the flow in the lowest layers is coupled to that in the middle and upper troposphere. In view of the facts cited above, the system is evidently not governed by the equations for quasi-geostrophic or quasi-nondivergent flow, even beginning with initially nongeostrophic (balanced) initial conditions. It appears that the system is governed, rather, by the equations for an essentially nongeostrophic divergent flow.

The purpose of this memorandum is to derive the equations for a two-parameter nongeostrophic model, which is not much more complicated than the corresponding quasi-geostrophic or quasi-nondivergent models, and to outline a numerical method by which those equations can be solved.

2. The Vertical Distribution of Vertical Motion.

Since the manner in which the horizontal motions at different levels are coupled together depends crucially on the distribution of vertical motion, it is important that any model reproduce something like the correct vertical profile of vertical air speed or vertical mass transport. It is implicit in the thermotropic model and virtually all other two-parameter models that the profile of vertical mass transport either has a specified (usually parabolic) form, or that the vertical mass transport varies smoothly and remains of the same sign throughout the entire depth of the atmosphere. The validity of such crude representations of the distribution of vertical motion may be investigated with the aid of a diagnostic equation that relates ω, an approximate measure of vertical mass transport, to a given height distribution.

$$\frac{\sigma^2}{p^2} \nabla^2 \omega + f\eta \frac{\partial^2 \omega}{\partial p^2} = f \frac{\partial}{\partial p} \mathbb{V} \cdot \nabla \eta - \nabla^2 \mathbb{V} \cdot \nabla \frac{\partial \phi}{\partial p} = F(x,y,p) \qquad (1)$$

in which $\sigma^2 = (R^2T^2/g\,\theta)(\partial\theta/\partial z)$, a measure of the static stability; p is pressure; ∇^2 is the horizontal Laplacian operator; $\omega = \frac{dp}{dt}$; f is the Coriolis parameter; η is the absolute vorticity; $\mathbf{V}$ is the horizontal projection of the wind vector, ∇ is the horizontal vector gradient; and ϕ is the geopotential. Eq. (1) was obtained by differentiating the vorticity equation for quasi-geostrophic flow with respect to p, applying the Laplacian operator to the adiabatic equation, and eliminating the height tendency between the resulting pair of equations. It is most nearly exact at the levels where ω reaches its extreme values.

In the great majority of cases, when the flow is almost equivalent-barotropic, the character of the ω-profile is determined primarily by vertical variations in the advection of absolute vorticity --- i.e., by the first term on the right-hand side of Eq. (1). In general, although the sense of the temperature gradient reverses as one passes from troposphere to stratosphere, the direction of the isotherms remains about the same. Thus, the vorticity advection normally reaches its extreme values near the tropopause level, whence F changes sign at about that level. This, with Eq. (1), implies that the three-dimensional "curvature" of the ω-field also changes sign near the tropopause level. According, if the tropopause is low enough, ω itself may vanish at a height somewhat above tropopause level, and reverse its sense before reaching a value of zero at a pressure of zero millibars.

Since this qualitative description of the ω-profile does not agree with that provided by the thermotropic and other 2-parameter models, it is worth making a rough quantitative estimate of the effect of the tropopause in the normal case of equivalent-barotropic flow. This can be done most simply by applying Eq. (1) at the tropopause (here assumed to lie at 300 millibars), 650 millibars, and 150 millibars. Since ω oscillates around zero at any level, $\nabla^2\omega$ is highly correlated with ω, so that the resulting equations may be written in the approximate form:

$$\left(\frac{\partial^2\omega}{\partial p^2}\right)_3 - \frac{k_3^{\,2}}{p_3^{\,2}}\,\omega_3 = \frac{F_3}{f^2}$$

$$\left(\frac{\partial^2\omega}{\partial p^2}\right)_2 - \frac{k_2^{\,2}\omega_2}{p_2^{\,2}} = 0$$

$$\left(\frac{\partial^2\omega}{\partial p^2}\right)_1 - \frac{k_1{}^2\omega_1}{p_1{}^2} = -\frac{\alpha F_3}{f^2}$$

where the subscripts 1, 2, and 3 refer to conditions at 150, 300 and 650 millibars respectively; $k_1 = \frac{\nu\sigma_1}{f}$; ν is an inverse measure of scale, corresponding to the wavelength of maximum baroclinic instability; and α is a positive constant for equivalent-barotropic flow, reflecting the degree to which the temperature gradient reverses above the tropopause. When all vertical derivatives are replaced by corresponding ratios of finite differences, and the upper and boundary conditions are introduced, the equations above become a complete system of linear algebraic equations involving ω_1, ω_2, and ω_3. The solution of this system is illustrated by the solid curve on Fig. 1, which shows the typical ω-profile for $\alpha = 1$, $k_1{}^2 = 2$ (corresponding to isothermal lapse rate), $k_3{}^2 = 1$ (corresponding to $\partial\theta/\partial z = 3.5^\circ$ A/km.), and $k_2{}^2 = 1.5$. The most striking features of Fig. 1 are the reversal in the sense of ω at about 200 millibars, and the pronounced extreme of ω at a level roughly midway between 200 and 1000 millibars.

As far as the design of a 2-parameter model is concerned, the relevant question is how to choose the reference levels in such a way that $\partial^2\omega/\partial p^2$ is best approximated by finite differences. The most common practice is to approximate $\partial^2\omega/\partial p^2$ at 500 millibars as

$$\frac{\omega_o - 2\,\omega_{500} + \omega_{1000}}{(500\text{ mb})^2}$$

taking $\omega_o = 0$ and $\omega_{1000} = 0$ (in the case of flat terrain).

A better procedure, as may be seen from Fig. 1, is to approximate $\partial^2\omega/\partial p^2$ at <u>600</u> millibars as

$$\frac{\omega_{200} - 2\,\omega_{600} + \omega_{1000}}{(400\text{ mb})^2}$$

taking $\omega_{200} = 0$ and $\omega_{1000} = 0$. Stated in another way, the question is whether the actual ω-profile in the troposphere is best approximated by a parabola passing through ($\omega = 0$, $p = 0$), ($\omega = 0$, $p = 1000$) and ($\omega = |\omega|_{max}$, $p = 500$) or by a parabola passing through ($\omega = 0$, $p = 200$), ($\omega = 0$, $p = 1000$), and ($\omega = |\omega|_{max}$, $p = 600$). For reference, these parabolae are plotted on Fig. 1 as dashed and dotted curves, respectively. The latter is clearly the better approximation.

In summary, the overall effect of the existence of a tropopause in normal situations of almost equivalent-barotropic flow is to produce a reversal in the sense of ω at about 200 millibars if the tropopause lies at 300 millibars, and somewhat lower if the tropopause is also lower. The importance of this conclusion with regard to the construction of a 2-parameter model is simply this: The effective "top" of the atmosphere, the lowest level where ω vanishes, is generally around 200 millibars rather than at zero millibars*.

3. The Effect of the Quasi-geostrophic and Quasi-nondivergent Approximations on the Average Growth Rate.

The essential simplifying feature of the quasi-geostrophic and quasi-nondivergent approximations is that the absolute vorticity at each level is advected with a geostrophic or nondivergent wind, after the horizontal divergence has been replaced by $-\partial\omega/\partial p$ in the so-called "divergence" term of the vorticity equation. Experience with various quasi-geostrophic baroclinic models indicates that they have too great an average growth rate, and systematically tend to produce much stronger pressure gradients between surface highs and lows than are actually observed. Our present concern, therefore, is to see if these shortcomings are attributable to the quasi-geostrophic or quasi-nondivergent character of the models.

To examine this question, we shall set up a simple 2-parameter model with reference levels at 200, 400, 600, 800, and 1000 millibars, in accordance with the results of Section 2, letting ω vanish at 200 and 1000 millibars. Applying the vorticity equation at 400 and 800 millibars, and replacing vertical derivatives by ratios of finite-differences, we obtain

*Much the same conclusion has been stated by Sawyer and Bushby (1953), who have taken 200 millibars as a "lid" on their quasi-geostrophic model.

$$\frac{\partial \eta_1}{\partial t} + \mathbb{V}_1 \cdot \nabla \eta_1 + \frac{\omega_1}{P}(\eta_2 - \eta_1) - \frac{\eta_1 \omega}{P} = 0 \tag{2}$$

$$\frac{\partial \eta_2}{\partial t} + \mathbb{V}_2 \cdot \nabla \eta_2 + \frac{\omega_2}{P}(\eta_2 - \eta_1) + \frac{\eta_2 \omega}{P} = 0 \tag{3}$$

in which the subscripts 1 and 2 refer to conditions at 400 and 800 millibars, respectively; P = 400 millibars; and ω without subscript is its value at 600 millibars. It will be noted that the term representing the reorientation of the vortex tubes has been omitted, whereas the one representing the vertical advection of vorticity has not. The justification for this procedure is that the omission of the "twisting" effect merely changes the defined value of the potential vorticity --- i.e., the dynamical quantity that is conserved following the individual elements of fluid --- and not by more than a few percent of its exact value. Omission of the vertical advection of vorticity, on the other hand, results in a spurious change in the phase difference of potential vorticity patterns at different levels, an effect that may account for some of the past errors in predicted phase relationship.

As will be seen later, it is less convenient to deal with Eqs. (2) and (3) than with an equivalent pair of equations, obtained by forming the sum and difference of Eqs. (2) and (3). Letting $\omega_1 = \omega_2 = A\omega$ (ω-profile symmetrical around 600 millibars),

$$\frac{\partial \bar{\eta}}{\partial t} + \bar{\mathbb{V}} \cdot \nabla \bar{\eta} + \mathbb{V}' \cdot \nabla \eta' + \frac{(2A + 1)}{P} \omega \eta' = 0 \tag{4}$$

$$\frac{\partial \eta'}{\partial t} + \bar{\mathbb{V}} \cdot \nabla \eta' + \mathbb{V}' \cdot \nabla \bar{\eta} + \frac{\omega \bar{\eta}}{P} = 0 \tag{5}$$

where $\bar{\eta} = (\eta_2 + \eta_1)/2$, $\eta' = (\eta_2 - \eta_1)/2$, and similarly for all other "barred" or "primed" quantities. In order to calculate a measure of the growth rate, we now multiply Eq. (4) by $\bar{\eta}$, multiply Eq. (5) by η', and add them together, with the result that

$$\frac{\partial}{\partial t}\left(\frac{\overline{\eta}^2+\eta'^2}{2}\right)+\overline{\mathbb{V}}\cdot\nabla\left(\frac{\overline{\eta}^2+\eta'^2}{2}\right)+\mathbb{V}'\cdot\nabla(\overline{\eta}\,\eta')+(2A+2)\frac{\omega\,\overline{\eta}\,\eta'}{P}=0 \quad (6)$$

In an exactly analogous fashion, we apply the continuity equation at 400 and 800 millibars and replace all vertical derivatives by ratios of finite-differences.

$$\nabla\cdot\mathbb{V}_1+\frac{\omega}{P}=0$$

$$\nabla\cdot\mathbb{V}_2-\frac{\omega}{P}=0$$

Thus, forming the sum and difference of the equations above,

$$\nabla\cdot\overline{\mathbb{V}}=0 \quad (7)$$

$$\nabla\cdot\mathbb{V}'=\frac{\omega}{P} \quad (8)$$

We next combine Eqs. (6) and (7) in the form

$$\frac{\partial}{\partial t}\left(\frac{\overline{\eta}^2+\eta'^2}{2}\right)+\nabla\cdot\left(\frac{\overline{\eta}^2+\eta'^2}{2}\right)\overline{\mathbb{V}}+\nabla\cdot\overline{\eta}\;\;\eta'\,\mathbb{V}'-$$
$$-\overline{\eta}\;\eta'\,\nabla\cdot\mathbb{V}'+(2A+2)\frac{\omega\,\overline{\eta}\,\eta'}{P}=0$$

Integrating each term of the equation above over a closed surface (such as an isobaric surface) or over a closed region with cyclic boundary conditions, and noting that the area integral of the divergence of any continuous vector can be transformed by Gauss' theorem into a line integral around the boundary curve,

$$\frac{\partial}{\partial t}\int_S\left(\frac{\overline{\eta}^2+\eta'^2}{2}\right)dS-\int_S\overline{\eta}\;\eta'\,\nabla\cdot\mathbb{V}'\,dS+\left(\frac{2A+2}{P}\right)\int_S\omega\overline{\eta}\;\eta'\,dS=0 \quad (9)$$

where S is the region of integration and dS is an element of S. At this point, it should be noted that $\mathbf{V}'$ is regarded as a nondivergent vector in the quasi-nondivergent model, and is essentially nondivergent in the quasi-geostrophic model. Thus, in these models, the second term of Eq. (9) would be missing, so that

$$\frac{\partial}{\partial t}\int_S \left(\frac{\bar{\eta}^2 + \eta'^2}{2}\right) dS = -\left(\frac{2A + 2}{P}\right)\int_S \omega\bar{\eta}\eta' \, dS \qquad (10)$$

In actuality, however, $\nabla \cdot \mathbf{V}'$ does not vanish. According to Eq. (8), the exact equation corresponding to Eq. (10) is:

$$\frac{\partial}{\partial t}\int_S \left(\frac{\bar{\eta}^2 + \eta'^2}{2}\right) dS = -\left(\frac{2A + 1}{P}\right)\int_S \omega\bar{\eta}\eta' \, dS \qquad (11)$$

It remains to interpret the result stated in Eqs. (10) and (11). We first note that

$$\frac{\partial}{\partial t}\int_S \left(\frac{\bar{\eta}^2 + \eta'^2}{2}\right) dS = \frac{1}{4}\frac{\partial}{\partial t}\int_S (\zeta_2^{\,2} + \zeta_1^{\,2})\, dS + f\,\frac{\partial}{\partial t}\int_S \bar{\eta}\, dS$$

where ζ is the relative vorticity. Now, since the second term on the right-hand side of the equation above is generally negligible, the left-hand side of Eqs. (10) and (11) is essentially a measure of the rate of change of the total "rotational energy" of the flow. As has been shown by Thompson (1956), the right-hand sides of Eq. (10) and (11) are related to the rate at which potential and internal energy are converted, and are indirectly related to the rate at which the total kinetic energy of the growing disturbances is built up through the mechanism of baroclinic instability.

Comparing Eqs. (10) and (11) we see that the kind of mechanism by which the rotational energy of the quasi-geostrophic and quasi-nondivergent models increases is the same as for the general divergent nongeostrophic model. Other things being equal, however, this mechanism operates $(2A + 2)/(2A + 1)$ as rapidly in the former as it does in the latter. In short, the average growth rate in the quasi-nondivergent and quasi-geostrophic models is about 50% greater than that in the general divergent model. This result is in accord with the general observation mentioned earlier ---- namely, that forecasts based on quasi-geostrophic baroclinic models tend to exaggerate the pressure gradients between surface highs and lows.

The effect described above is clearly not serious in the great majority of cases, for in equivalent-barotropic flow the change in the total rotational energy is not very great. The discrepancy becomes greater, however, as the flow becomes more and more baroclinic. It is also evident that the effect on the quasi-nondivergent flow near 600 millibars will be considerably less than on the flow in the lowest levels ---- another distinction between the flows in the middle and lower troposphere that is borne out by experience.

In conclusion, we infer that the advection of vorticity with the divergent component of the wind is, on the average, quite as important as the direct "stretching effect" of divergence. Finally, whatever is markedly different in the mean must be even more markedly different in individual cases of baroclinic development.

4. The Equations for a Two-Parameter Nongeostrophic Model

In accordance with the results of the preceding sections, we shall consider a divergent nongeostrophic model which is governed partially by Eqs. (4), (5), (7) and (8). It should be noted that this system of four scalar equations is not complete, since it involves the scalar ω as well as the vectors $\overline{\mathbb{V}}$ and $\mathbb{V}'$. One of the equations necessary to complete the system is the thermodynamic energy equation for adiabatic flow, applied at 600 millibars. Replacing the vertical derivatives in this equation by ratios of finite-differences, we obtain

$$\frac{\partial \phi'}{\partial t} + \overline{\mathbb{V}} \cdot \nabla \phi' + \frac{\sigma^2 \omega}{2P} = 0 \qquad (12)$$

This equation, however, involves still another variable ---- namely, ϕ'. The remaining equation is the so-called "balance" equation, which, in its general form, is:

$$f\zeta' = \nabla^2 \phi' + 2J(\overline{v}, u') + 2J(v', \overline{u}) + \beta u' \qquad (13)$$

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where u and v are the eastward and northward components of $\mathbb{V}$, respectively. Eqs. (4), (5), (7), (8), (12), and (13) now comprise a complete system of six equations involving the six scalar variables $\overline{u}$, $\overline{v}$, u', v', ω, and ϕ'.

According to Helmholtz' theorem, it is always possible to express the horizontal velocity vector as the sum of an irrotational vector and a non-divergent vector. Together with Eqs. (7) and (8), this implies that

$$\overline{\mathbb{V}} = \mathbb{k} \times \nabla\overline{\psi} \qquad \overline{\eta} = \nabla^2\overline{\psi} + f$$

$$\mathbb{V}' = \nabla \chi + \mathbb{k} \times \nabla\psi' \qquad \zeta' = \eta' = \nabla^2\psi' \qquad \nabla\cdot\mathbb{V}' = \nabla^2\chi = \frac{\omega}{p}$$

where $\mathbb{k}$ is a unit vector directed vertically upward. Introducing these results into Eqs. (4), (5), (8), (12), and (13), we obtain

$$\nabla^2\left(\frac{\partial\overline{\psi}}{\partial t}\right) + J\left(\overline{\psi}, \nabla^2\overline{\psi} + f\right) + \nabla\chi\cdot\nabla\left(\nabla^2\psi'\right) + J\left(\psi', \nabla^2\psi'\right) + \frac{(2A+1)\ \nabla^2\psi'}{p}\,\omega = 0 \tag{14}$$

$$\nabla^2\left(\frac{\partial\psi'}{\partial t}\right) + J\left(\overline{\psi}, \nabla^2\psi'\right) + \nabla\chi\cdot\nabla\left(\nabla^2\overline{\psi} + f\right) + J\left(\psi', \nabla^2\overline{\psi} + f\right) + \frac{(\nabla^2\overline{\psi} + f)}{p}\,\omega = 0 \tag{15}$$

$$\nabla^2\chi = \frac{\omega}{p} \tag{16}$$

$$\frac{\partial\phi'}{\partial t} + J\left(\overline{\psi}, \phi'\right) + \frac{\sigma^2\,\omega}{2p} = 0 \tag{17}$$

$$f\nabla^2\psi' = \nabla^2\phi' + 2J\left(\frac{\partial\overline{\psi}}{\partial x}, \frac{\partial\chi}{\partial x} - \frac{\partial\psi'}{\partial y}\right) - 2J\left(\frac{\partial\chi}{\partial y} + \frac{\partial\psi'}{\partial x}, \frac{\partial\overline{\psi}}{\partial y}\right) + \beta\left(\frac{\partial\chi}{\partial x} - \frac{\partial\psi'}{\partial y}\right) \tag{18}$$

Counting up variables, we find that these five equations contain only the five quantities $\overline{\psi}$, χ, ψ', ω, and ϕ'. The equations summarized above are the equations for a general 2-parameter model.

It is a matter of general experience that the inversion of the balance equation (18) is an extremely time-consuming operation, and cannot be carried out economically at every time stage of a numerical integration. Thus, noting that ϕ' enters only into Eqs. (17) and (18), one is strongly tempted to apply the balance equation in approximate form for the purpose of computing ϕ' in terms of ψ', and for this purpose only. As will be seen later, this procedure modifies ω and χ slightly, but leaves the form of the prognostic equations intact. We shall assume, in fact; that

$$\frac{\partial \psi'}{\partial t} = \frac{1}{f}\frac{\partial \phi'}{\partial t} \qquad \text{and} \qquad \nabla\psi' = \frac{1}{f}\nabla\phi'$$

so that Eqs. (17) and (18) are replaced by the single equation

$$\frac{\partial \psi'}{\partial t} + J\,(\bar{\psi},\ \psi') + \frac{\sigma^2\,\omega}{2fP} = 0 \tag{19}$$

This assumption, it will be recognized, is very similar to the quasi-nondivergent assumption proposed by Kuo, but differs in that it is not introduced directly into the continuity equation or the vorticity equations. It is also noteworthy that the balance equation is no longer one of the equations that governs the evolution of the flow from one time stage to the next, but enters explicitly only in the determination of the initial fields of $\bar{\psi}$ and ψ' ---- an approximation that has already shown some improvement over the usual quasi-geostrophic approximation.

Applying the Laplacian operator to Eq. (19) and substituting for $\nabla^2\,(\partial\psi'/\partial t)$ in Eq. (15), we obtain a diagnostic equation similar to Eq. (1).

$$\frac{\sigma^2}{2fP}\,\nabla^2\omega - \frac{(\nabla^2\bar{\psi} + f)}{P}\,\omega = \nabla\chi\cdot\nabla\,(\nabla^2\bar{\psi} + f)$$
$$+ J\,(\bar{\psi},\ \nabla^2\,\psi') + J\,(\psi',\ \nabla^2\bar{\psi} + f) - \nabla^2\,J\,(\bar{\psi},\ \psi')$$

Finally, substituting from Eq. (16),

$$\frac{\sigma^2}{2f}\,\nabla^4\chi - (\nabla^2\bar{\psi} + f)\,\nabla^2\chi = \nabla\chi\cdot\nabla\,(\nabla^2\bar{\psi} + f)$$
$$+ J\,(\bar{\psi},\ \nabla^2\,\psi') + J\,(\psi',\ \nabla^2\bar{\psi} + f) - \nabla^2 J\,(\bar{\psi},\ \psi') \tag{20}$$

This equation is to be regarded as a means of computing χ from known fields of $\bar{\psi}$ and ψ'.

The method that is proposed for solving the system of Eqs. (14), (15), (16), and (20) is the following:

1) Beginning with the initial geopotential ϕ at 850 and 400 millibars, interpolate ϕ at 600 millibars and form $\bar{\phi}$ and ϕ'.
2) Compute the initial field of $\bar{\psi}$ by solving the balance equation for the nondivergent flow at 600 millibars.

$$f\nabla^2\bar{\psi} = \nabla^2\bar{\phi} - 2 J \left(\frac{\partial\bar{\psi}}{\partial x}, \frac{\partial\bar{\psi}}{\partial y}\right) - \beta \frac{\partial\bar{\psi}}{\partial y}$$

   This step is carried out only once, in order to calculate the initial $\bar{\psi}$-field.
3) Compute the initial fields of ψ' and χ by solving Eqs. (18) and (20) as a simultaneous system, regarding ψ' and χ as the unknowns and using the previously computed $\bar{\psi}$-field. This step is also carried out only once.
4) Compute the initial field of ω from Eq. (16), using the previously computed field of χ.
5) Compute $\partial\bar{\psi}/\partial t$ by inverting Eq. (14), using the previously computed fields of $\bar{\psi}$, ψ', χ, and ω. Extrapolate over the next time interval to obtain the predicted $\bar{\psi}$-field.
6) Compute $\frac{\partial\psi'}{\partial t}$ by inverting Eq. (15) using the previously computed fields of $\bar{\psi}$, ψ', χ and ω. Extrapolate over the next time interval to obtain the predicted ψ'-field.
7) Compute the predicted χ-field by solving Eq. (20), using the predicted fields of $\bar{\psi}$ and ψ'.
8) Compute the predicted field of ω from Eq. (16), using the previously computed field of χ
9) Return to Step 5 and iterate the cycle of Steps 5 through 9, until the forecast period is of the length desired.

It will be noted that the most time-consuming operation in the iterative scheme outlined above is Step 7. If the flow is not very divergent, the first term on the right-hand side of Eq. (20) is negligible in comparison with the remaining terms, in which case Eq. (20) may be solved for $\nabla^2\chi = \omega/P$. The χ field may then be computed by inverting $\nabla^2\chi$. This fact, incidentally indicates that the boundary conditions required to solve Eq. (20) for χ are the values of χ and $\nabla^2\chi$ around the boundary of a closed region, provided

$\nabla^2\psi + f > 0$. In practice, it is probably sufficient to take $\chi = 0$ and $\nabla^2\chi = 0$ around the edges of a large grid.

In conclusion, it should be pointed out that the model discussed here can easily be modified to include the large-scale effects of irregular terrain. The appropriate change is to replace $(\frac{\partial\omega}{\partial p})_2$ by $(\Omega - \omega)/P$ rather than by $- \omega/P$. The quantity Ω is the value of ω at 1000 millibars, and is given approximately by

$$\Omega = V_g \cdot \nabla p_g \qquad V_g = k \times \nabla\bar{\psi} + (\frac{2p_g}{P} - 3) \ (\nabla\chi + k \times \nabla\psi')$$

where p_g is the known standard pressure at the ground. Thus, Ω is expressible in terms of $\bar{\psi}$, ψ' and χ.

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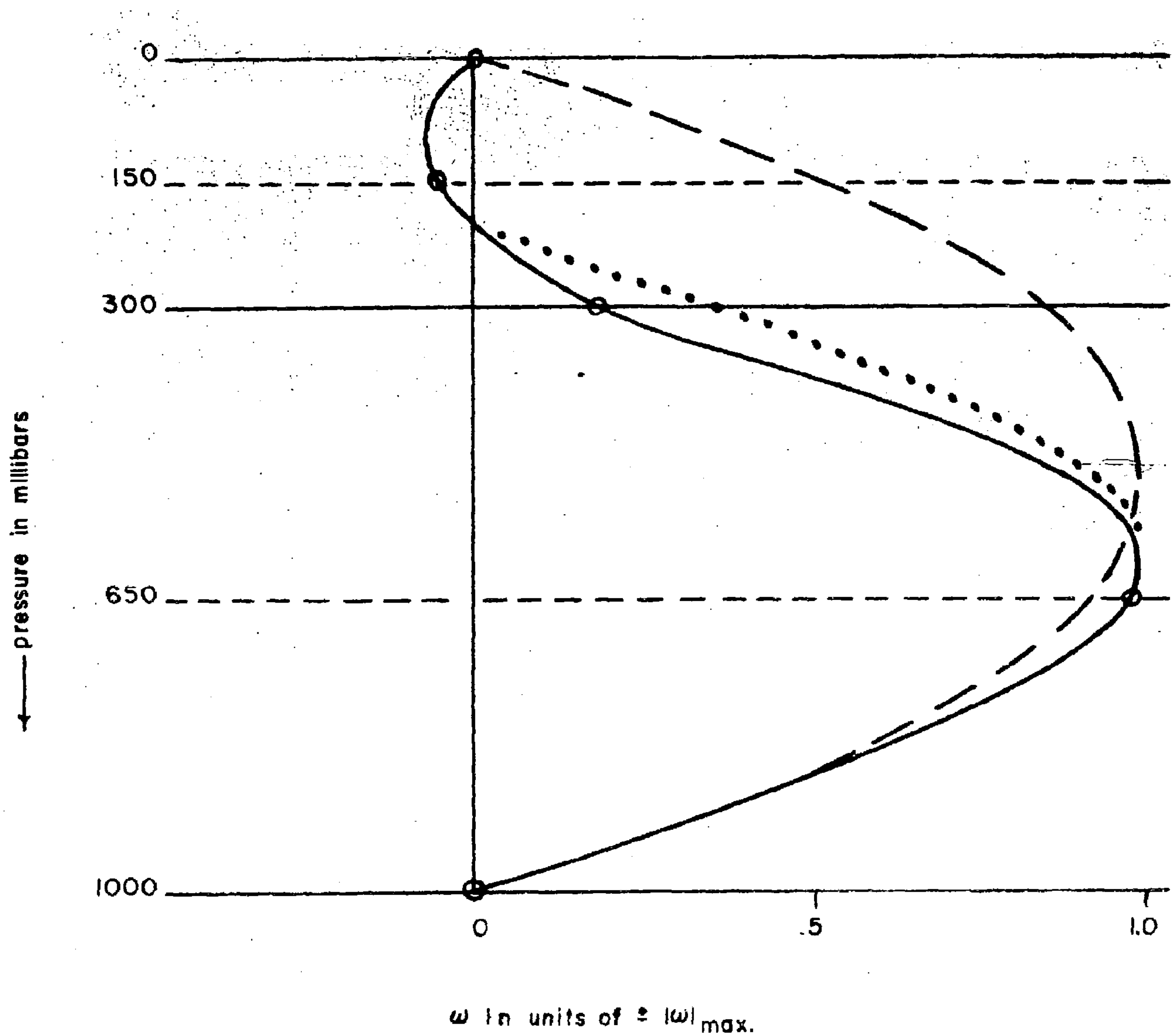
0
150
300
650
1000
pressure in millibars
0
.5
1.0
ω in units of $\pm |\omega|_{max}$.

Figure 1