Statistical Aspects of the Dynamics of Quasi-Nondivergent and Divergent Baroclinic Models

Philip Duncan Thompson
Lt. Colonel, U.S. Air Force

Joint Numerical Weather Prediction Unit
Washington, D. C.

25 May 1957

Commerce-Weather Bureau, Washington, D. C.

National Oceanic and Atmospheric Administration

U.S. Joint Numerical Weather Prediction Unit

ERRATA NOTICE

One or more conditions of the original document may affect the quality of the image, such as:

Discolored pages
Faded or light ink
Binding intrudes into the text

This has been a co-operative project between the NOAA Central Library, National Center for Environmental Prediction and the U.S. Air Force. This project includes the imaging of the full text of each document. To view the original documents, please contact the NOAA Central Library in Silver Spring, MD at (301) 713-2607 x124 or www.reference@nodc.noaa.gov.

LASON
Imaging Contractor
12200 Kiln Court
Beltsville, MD 20704-1387
April 13, 2004

Abstract

By combining the vorticity and continuity equations for the general two-level model, it has been found possible to derive relatively simple expressions for the average rate of growth (or increase of rotational energy) and the average rate of occlusion of large-scale baroclinic disturbances. The growth rate depends primarily on the area correlation between the vertical air-speed and the deviation of the temperature from its zonal average; the rate of occlusion depends on the area correlations between the vertical air speed and the deviations of both temperature and pressure from their zonal averages. Comparisons of the average rates of growth and occlusion for various types of approximations show that the "quasi-nondivergent" model cannot correctly reproduce all the important aspects of baroclinic development, and that markedly baroclinic flows are essentially nongeostrophic. It is also shown that the vertical advection of vorticity has a significant effect on the average growth rate, and cannot be neglected on the basis of simple order-of-magnitude estimates.

At least in cases when the horizontal velocity field is independent of the north-south coordinate, it is found that disturbances in which the temperature field precedes the pressure field tend to die out undetected. Disturbances in which the temperature field lags behind the pressure field, on the other hand, grow in amplitude as long as they maintain that phase relationship. Simultaneously and as a dynamical consequence of the growth process, the temperature field in such disturbances gradually catches up with the pressure field, and the flow approaches a state of quasi-barotropy. At the instant this state is reached, growth ceases and

there is no further change in the relative phase of the pressure and temperature fields. In the absence of heating and viscosity, the flow would tend to remain quasi-barotropic, verifying an hypothesis proposed by Rossby almost twenty years ago.

I. Introduction

Over the past two years, the Joint Numerical Weather Prediction Unit has built up a file of almost 1500 routine numerical forecasts and well over a hundred experimental forecasts. Of the routine forecasts, roughly 850 were based on either of two baroclinic models——one, a variant of the 3-level model proposed by Charney and Phillips (1953), and the other, a 2-level model generically the same as the "thermotropic" model described by Thompson and Gates (1956). The remaining 650 routine forecasts were based on the barotropic model. Since October 1956, both barotropic and baroclinic forecasts have been computed daily, permitting direct comparisons between the performances of physically different models. During a three month period from April through June 1956, experimental 2-level forecasts were computed daily and compared with the routine 3-level forecasts.

Although this entire mass of data has not been given the systematic and intensive study it deserves, certain conclusions can be drawn from statistical indices of the errors of numerical forecasts and from a case-by-case inspection of long series of individual forecasts. One inescapable and rather surprising conclusion is that the behavior of the large-scale flow patterns in midtroposphere is very similar to that of initially equivalent flow patterns in a barotropic fluid. On the whole, the barotropic 500 mb forecasts account for most of the day-to-day change in the large-scale flow patterns——at least in regions where the forecasts are relatively unaffected by conditions around the boundary of the forecast area and/or by large errors in the analyst's reconstruction of the initial flow pattern. It is perhaps even more significant that the root-mean-square wind error of the barotropic forecasts is no greater than that of the 500 mb forecasts based on either of the baroclinic models.

Superficially, at least, this result is not difficult to understand, since the atmosphere does maintain itself in a state of quasi-barotropy, in the sense that the isotherms of an isobaric surface in midtroposphere coincide remarkably well with its contours. It does, however, raise a fundamental question as to why the atmosphere tends toward a quasi-barotropic state, and of what dynamical properties this behavior is a consequence.

Prediction methods based on the barotropic model cannot, of course, forecast the creation of genuinely new circulation centers. It is our experience, however, that many of the developments which the weather forecaster might once have regarded as "new" are due to the reorganization of previously existing circulation centers——as, for example, the deepening and apparent retrogression of a long-wave trough as a vorticity maximum of smaller scale is carried through it by the large-scale northwesterly flow. Baroclinic development of really new circulation centers in midtroposphere appears to be relatively rare (especially in regions of good data coverage), and is generally confined to a few small areas at any given time.

The foregoing remarks do not, of course, apply to the flow patterns at the lowest levels, whose behavior is rather poorly reproduced by either of the baroclinic models. Errors in the forecast height gradient at 500 mb, although percentagewise small at that level, are transmitted undiminished to all lower levels and result in large percentage errors at 1000 mb ——owing merely to the fact that the 500 mb height gradient is very nearly compensated by the gradient of thickness between 1000 and 500 mb. Aside from errors of this simple type, moreover, we have observed a systematic tendency for both baroclinic models to overpredict increases in the intensity of surface circulations, and a consistent failure to preserve the correct phase relationship between the patterns of height and temperature. The latter is often manifested in a tendency for the predicted axes of the pressure systems to become vertical too rapidly, the temperature field rapidly overtaking the height field and sometimes, in fact, advancing a little beyond it. In general, it is our impression that the instability exhibited by the simple 2- and 3-level baroclinic models is of the correct type, but that the details of the predicted development are quantitatively incorrect.

Beyond the very general observations stated above, it is extremely difficult to come to any definite conclusions about the defects of the baroclinic models currently under study. In individual cases, the numerical forecasts are contaminated by many other types of error——truncation error, errors due to the imposition of arbitrary lateral boundary conditions, errors in the initial analysis, and errors in the verifying analysis——none of which is completely isolable or removable. Added to these difficulties is the fact that the equations for even the simplest of baroclinic models are mathematically quite complicated, making it all but impossible to interpret a numerical forecast in simple physical terms and to understand how it evolved from the given initial state.

In the face of such difficulties, it is natural to inquire whether or not one can derive equations that govern the statistical or "general" aspects of the behavior of various models, from the dynamical equations that describe their behavior in detail. If so, and if those equations are simple enough, they might serve as an aid in interpreting numerical forecasts, and might provide a clearer understanding of the general behavior of fairly complicated models. As we shall see later, dynamical equations that govern the overall state of motion may also be taken as a guide in designing improved baroclinic models; a model that is incorrect in the average is necessarily wrong in detail. In particular, it appears probable that any successful baroclinic model must contain the mechanism by which the atmosphere keeps itself near a state of quasi-barotropic motion, at least in a statistical sense.

Having discussed the various reasons for approaching the dynamics of baroclinic flow from a statistical standpoint, we can now state briefly the main purpose of this article. It is simply to derive a pair of equations that describe how, on the average, the phase of the temperature field (relative to the pressure field) changes with time, and how the average rotational or kinetic energy of the large-scale disturbances increases or decreases. These equations are first derived from the vorticity equations for a rather general nongeostrophic model, and are later specialized in the case of "quasi-nondivergent" flow——a procedure that enables us to isolate the effects of certain common types of approximations on the statistical behavior of the models.

Other things being equal, it turns out that the average rates of growth and occlusion are significantly greater in quasi-nondivergent and quasi-geostrophic models than in the general divergent model. Accordingly, one may infer that a baroclinic model cannot correctly reproduce the process of development and the attendant changes of vertical structure, unless the horizontal wind field (in which the vorticity is advected) is regarded as divergent. This, in turn, implies that any successful baroclinic model must be a fortiori nongeostrophic. It is also found that the vertical advection of vorticity has an important systematic effect on the average growth rate, and must be taken into account in order to predict correctly the essential features of baroclinic development —— a conclusion that illustrates the dangers of relying on order-of-magnitude estimates when dealing with inherently unstable systems.

By considering the evolution of a whole family of initial states, one can also deduce the existence of certain phenomena that are commonly observed in the true atmosphere. One finds, for example, that the atmosphere has a selective preference for disturbances in which the temperature pattern lags behind the pressure pattern; its dynamical properties are such that disturbances in which the temperature field precedes the pressure field die out and remain undetected, while disturbances in which the temperature field lags behind are amplified. Finally, through the very same mechanism by which "out-of-phase" disturbances grow, the temperature field is gradually brought into phase with the pressure field, and the atmosphere approaches a statistical state of quasi-barotropy. As the temperature and pressure fields come into phase, the growth of average rotational energy ceases, and there is no further change in the relative phases of the temperature and pressure fields. Left to itself, the atmosphere would remain in this quasi-barotropic state. This result may be taken as a rather belated theoretical justification for the intensive studies of the barotropic model proposed and initiated by Rossby (1939), and, in some measure, accounts for their remarkable and unexpected success.

The irreversibility of the sequence of events outlined above, together with the quasi-barotropy of the state toward which these events tend, make it clear that an entirely different process must be postulated to account for the removal of accumulated rotational or "eddy" kinetic energy. It is evident, moreover, that the mechanism required to do so is necessarily operative in quasi-barotropic flow, a result that is certainly in accord with the experimental and theoretical studies of Kuo (1951), Lorenz (1955) and Phillips (1956). Lastly, the fact that the process of development in an adiabatic nonviscous flow destroys the conditions that are initially favorable to growth strongly suggests that the ultimate causes of baroclinic development cannot lie within a closed system, but must arise from nonadiabatic heating — either by eddy conduction from the surface and/or by absorption or emission of radiant energy.

2. The equations for a nongeostrophic two-level model

We begin with the general vorticity equation for adiabatic nonviscous flow, written in the form

$$\frac{d\eta}{dt} + \eta \nabla_\theta \cdot \underset{\sim}{V} = 0 \tag{1}$$

in which $\eta = f + \zeta$; $\zeta = \underset{\sim}{k} \cdot \nabla_\theta \times \underset{\sim}{V}$; $\underset{\sim}{k}$ is a unit vector directed vertically upward; $\underset{\sim}{V}$ is the horizontal projection of the velocity vector; and ∇ is the horizontal vector derivative. The subscript θ indicates that the differentiation is to be carried out along surfaces of constant potential temperature θ . Unless otherwise stated, all notation is standard. Now, in general,

$$\nabla_\theta \cdot \mathbf{V} = \nabla \cdot \mathbf{V} + \frac{\partial \mathbf{V}}{\partial p} \cdot \nabla_\theta p$$

where the vector derivative without subscript denotes differentiation along isobaric surfaces, and differentiation with respect to the pressure p is carried out along the vertical. Since the flow is almost geostrophic, the vector $\partial \mathbf{V}/\partial p$ is very nearly perpendicular to the vector $\nabla_\theta p$, so that $\nabla_\theta \cdot \mathbf{V}$ is approximately equal to $\nabla \cdot \mathbf{V}$ Accordingly, the continuity equation may be written in the following form:

$$\nabla_\theta \cdot \mathbf{V} = \nabla \cdot \mathbf{V} = - \frac{\partial \omega}{\partial p} \tag{2}$$

where ω is the total derivative of pressure. Finally, combining Eqs. (1) and (2), we may rewrite the vorticity equation as

$$\frac{\partial \eta}{\partial t} + \mathbf{V} \cdot \nabla \eta + \omega \frac{\partial \eta}{\partial p} - \eta \frac{\partial \omega}{\partial p} = 0 \tag{3}$$

It is now understood that differentiations with respect to time are carried out with p held fixed. Eq. (3) is the general form of the vorticity equation on which most baroclinic models are based.

We next apply Eq. (3) at the 400 and 800 mb surfaces, and approximate the vertical derivatives $\partial \eta/\partial p$ and $\partial \omega/\partial p$ by finite-differences. For simplicity, we shall assume that ω vanishes at 200 and 1000 mb, and that its vertical profile is symmetrical around the 600 mb surface.* Thus,

$$\frac{\partial \eta_1}{\partial t} + \underset{\sim}{V}_1 \cdot \nabla \eta_1 + \frac{A\omega(\eta_2 - \eta_1)}{P} - \frac{\eta_1 \omega}{P} = 0$$

$$\frac{\partial \eta_2}{\partial t} + \underset{\sim}{V}_2 \cdot \nabla \eta_2 + \frac{A\omega(\eta_2 - \eta_1)}{P} + \frac{\eta_2 \omega}{P} = 0$$

in which the subscripts 1 and 2 denote conditions at the 400 and 800 mb surfaces, respectively, ω without subscript is its value at 600 mb, $\omega_1 = \omega_2 = A\omega$, and P = 400 mb. It turns out that it is more convenient for later purposes to consider an equivalent pair of equations, obtainable by forming the sum and difference of the equations above. After some rearrangement, we find that

$$\frac{\partial \eta^*}{\partial t} + \underset{\sim}{V}^* \cdot \nabla \eta^* + \underset{\sim}{V}' \cdot \nabla \eta' - \frac{(2A+1)\,\omega \eta'}{P} = 0 \tag{4}$$

$$\frac{\partial \eta'}{\partial t} + \underset{\sim}{V}^* \cdot \nabla \eta' + \underset{\sim}{V}' \cdot \nabla \eta^* - \frac{\omega \eta^*}{P} = 0 \tag{5}$$

where a "starred" quantity is half the sum of its values at the 400 and 800 mb surfaces, and a "primed" quantity is half the difference between its values (upper minus lower) at those surfaces.

In exactly the same way, we apply Eq. (2) at the 400 and 800 mb surfaces, with the result that

*It should be noted that ω is very nearly proportional to the vertical mass transport ρw. Thus, ω is much smaller in magnitude near a flat ground surface than it is in midtroposphere, where it reaches its extreme values. It has also been shown by Thompson (1956) that the vertical air speed must reverse sign somewhere above the tropopause, owing to the reversal of temperature gradient above that level.

$$\nabla \cdot \underset{\sim}{V}_1 + \frac{\omega}{P} = 0 \qquad \nabla \cdot \underset{\sim}{V}_2 - \frac{\omega}{P} = 0$$

from which it follows that

$$\nabla \cdot \underset{\sim}{V}^* = 0 \qquad \nabla \cdot \underset{\sim}{V}' = -\frac{\omega}{P} \tag{6}$$

Eqs.(4), (5) and (6) provide the physical basis for most of the remaining discussion.

3. An equation for the growth of total rotational energy

Our next concern is to derive an equation that relates the instantaneous increase of total rotational energy (or average amplitude) to the current fields of horizontal and vertical motion. To do so, we first multiply Eq. (4) by η^*, multiply Eq. (5) by η', and add those equations together. Factoring and making use of standard vector identities, we find that

$$\frac{1}{2}\frac{\partial}{\partial t}(\eta^{*2} + \eta'^2) + \frac{1}{2}\underset{\sim}{V}^* \cdot \nabla(\eta^{*2} + \eta'^2) + \underset{\sim}{V}' \cdot \nabla(\eta^*\eta') - \frac{(2A+2)\,\omega\eta^*\eta'}{P} = 0 \tag{7}$$

Now, from Eq. (6), it follows that

$$\underset{\sim}{V}^* \cdot \nabla(\eta^{*2} + \eta'^2) = \nabla \cdot (\eta^{*2} + \eta'^2)\underset{\sim}{V}^*$$

$$\underset{\sim}{V}' \cdot \nabla(\eta^*\eta') = \nabla \cdot \eta^*\eta'\underset{\sim}{V}' + \frac{D\omega\eta^*\eta'}{P}$$

where D is a constant which is actually unity. In the socalled "quasi-nondivergent" models, however, the vector $\underset{\sim}{V}'$ is regarded as nondivergent wherever it enters <u>explicitly</u> in Eqs. (4) and (5). Accordingly, the constant D takes on the value "zero" in the special case of "quasi-nondivergent" flow. Eq. (7), when combined with the expressions above, then becomes

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\eta^{*2}+\eta'^2\right) + \frac{1}{2}\nabla\cdot\left(\eta^{*2}+\eta'^2\right)\underset{\sim}{V}^* + \nabla\cdot\eta^*\eta'\underset{\sim}{V}' = \frac{(2A+2-D)\,\omega\eta^*\eta'}{P} \tag{8}$$

The next step is to form the average of each term of Eq. (8) taken over the entire area of the earth. Thus,

$$\frac{\partial}{\partial t}\overline{\left(\eta^{*2}+\eta'^2\right)} = \frac{2(2A+2-D)\,\overline{\omega\eta^*\eta'}}{P} \tag{9}$$

in which the "bar" above a quantity denotes its area-average. The area-averages of the second and third terms on the lefthand side of Eq. (8) vanish, since the area integral of the divergence of any continuous vector vanishes, when taken over a closed surface.

By making use of the statistical properties of the various fields, Eq. (9) may be simplified as follows: in the first place,

$$\frac{\partial}{\partial t}\overline{\left(\eta^{*2}+\eta'^2\right)} = \frac{\partial}{\partial t}\overline{\left(\zeta^{*2}+\zeta'^2\right)} + 2\,\overline{f\,\frac{\partial\zeta^*}{\partial t}}$$

Let us imagine that the integration implied by averaging is first carried out around latitude circles, along which f is constant. Now, along the whole of a latitude circle, the instantaneous tendency of ζ^* is as positive as it is negative, on the average, so that the second term on the righthand side of the equation above is negligible. By definition, moreover,

$$\zeta^{*2}+\zeta'^2 = \frac{1}{2}\left(\zeta_1^2+\zeta_2^2\right)$$

whence, combining the results above,

$$\frac{\partial}{\partial t}\overline{(\eta^{*2}+\eta'^2)} = \frac{\partial}{\partial t}\overline{(\zeta^{*2}+\zeta'^2)} = \frac{1}{2}\frac{\partial}{\partial t}\overline{(\zeta_1^2+\zeta_2^2)} \qquad (10)$$

The righthand side of Eq. (9) may be simplified by noting that

$$\overline{\omega\eta^*\eta'} = \overline{f\omega\zeta'} + \overline{\omega\zeta^*\zeta'}$$

The significant fact is that ω , ζ^*, and ζ' all oscillate around zero, and have the same characteristic scale. Under these conditions, it can be shown that three-factor correlations tend to be uniformly small, without regard to the relative phases of the individual factors.* Thus, substituting from Eq. (10) into Eq. (9), and introducing the last result, we find that

$$\frac{\partial}{\partial t}\overline{(\zeta^{*2}+\zeta'^2)} = \frac{1}{2}\frac{\partial}{\partial t}\overline{(\zeta_1^2+\zeta_2^2)} = \frac{2(2A+2-D)\overline{f\omega\zeta'}}{P} \qquad (11)$$

*This can be easily seen in the case of three sinusoidal functions whose wave-numbers are equal, but whose phases are arbitrary. The proof in the general case hinges on the identity $4\overline{\omega\zeta^*\zeta'} \equiv \overline{\omega(\zeta^*+\zeta')^2} - \overline{\omega(\zeta^*-\zeta')^2}$ and on the fact that ω, ζ^*, ζ', $(\zeta^*+\zeta')$, and $(\zeta^*-\zeta')$ all have the same characteristic scale. As will be shown later, both of the terms on the righthand side of the preceding equation tend to be uniformly small.

Since the quantity $\overline{(\zeta_1^2 + \zeta_2^2)}$ is a measure of the total "rotationality" of the flow, one may think of Eq. (11) as a relationship between the instantaneous increase (or decrease) of total rotational energy and the current fields of vertical and horizontal motion.

In order to interpret Eq. (11) in simple physical terms, let us estimate the correlation $\overline{f\omega\zeta'}$ in quasi-geostrophic flow. Replacing ζ'/P by $-\frac{1}{2}\,\partial\zeta/\partial p$, and making use of the hydrostatic equation, we find that

$$\overline{f\omega\zeta'} = \frac{R}{3}\,\overline{\omega\nabla^2 T}$$

where R is the universal gas constant, and T is the absolute temperature at 600 mb. Thus, assuming for the sake of argument that most of the power in the entire spectrum of disturbances is concentrated in a narrow band around wave-number α

$$\overline{f\omega\zeta'} = -\frac{R\alpha^2}{3}\,\overline{\omega(T - T_o)}$$

in which T_o is a zonally symmetric temperature field in which $\nabla^2 T_o = 0$, and whose average meridional gradient is that in the true atmosphere. Referring back to Eq. (11), we see that the total rotational energy increases when there is a negative correlation between ω and $(T - T_o)$ This corresponds to the case when, on the average around all latitude circles, cold air ($T < T_o$) is sinking ($\omega > 0$), and warm air is rising. This result, of course, is in accord with the classical view of the growth process.

To carry the interpretation a little further, we note that ω tends to oscillate around zero along a latitude circle, so that the preceding equation becomes approximately

$$\frac{\overline{f\omega\zeta'}}{P} = -\frac{R\alpha^2}{2p^{\kappa}}\overline{\omega T} = -\frac{\alpha^2}{2}\overline{\left(\frac{\omega}{\rho^*}\right)} \tag{12}$$

Thus, from the adiabatic and continuity equations

$$\frac{\overline{f\omega\zeta'}}{P} = -\frac{\alpha^2 C_p}{2}\overline{\frac{dT}{dt}} = -\frac{\alpha^2 C_p}{2}\left[\frac{\partial \overline{T}}{\partial t} + \overline{\nabla\cdot T\mathbb{V}} + \overline{\frac{\partial}{\partial p}(T\omega)}\right]$$

But the second and third terms in the square brackets of the equation above vanish: one, because $T\mathbb{V}$ is a continuous vector, and the other, because ω was assumed to vanish at 200 and 1000 mb. Finally, substituting the results above into Eq. (11),

$$\frac{1}{\alpha^2}\frac{\partial}{\partial t}\overline{(\zeta_1^2+\zeta_2^2)} + 2(2A+2-D)\frac{\partial}{\partial t}\overline{C_p T} = 0 \tag{13}$$

Now, it is a well-known fact that the potential energy of a fluid in hydrostatic equilibrium is proportional to its internal energy, and that the sum of these two forms of energy (per unit mass) is equal to $\overline{C_p T}$. Accordingly, Eq. (13) states that any transformation of internal and potential energy in the atmosphere must result in an increase of total rotational energy. Moreover, since the latter is essentially a measure of the degree to which the winds depart from horizontally uniform zonal flow, at least a part of the transformed potential energy must go directly into the kinetic energy of disturbances.

From more general considerations, it can be shown that the equation for conservation of total kinetic, internal and potential energy may be written as

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\overline{\mathbb{V}_1\cdot\mathbb{V}_1}+\overline{\mathbb{V}_2\cdot\mathbb{V}_2}\right)=-\frac{2R\,\overline{\omega T}}{p^*}$$

Now, combining Eq. (11) with Eq. (12), we find that

$$\frac{1}{2}\frac{\partial}{\partial t}\frac{(\overline{\zeta_1^2+\zeta_2^2})}{\alpha^2}=-\frac{(2A+2-D)\,R\,\overline{\omega T}}{p^*}$$

Thus, setting $D = 1$ and $A = \frac{1}{2}$, and comparing the two equations above, we see that the rate at which the total kinetic energy per unit mass $(\overline{\mathbb{V}_1\cdot\mathbb{V}_1}+\overline{\mathbb{V}_2\cdot\mathbb{V}_2})$ changes is just the rate at which $(\overline{\zeta_1^2+\zeta_2^2})/\alpha^2$ changes. Since the latter is very nearly a measure of the kinetic energy of the disturbances, we are led to conclude that all or a major fraction of the transformed potential energy goes directly into the disturbances, and that none or relatively little goes directly into the kinetic energy of the average westerly flow. This result, of course, is also in accord with the recent studies of Lorenz (1955) and Phillips (1956).

4. An equation for the average rate of occlusion.

It was pointed out in the preceding section that Eq. (11) may be regarded as a means of calculating the average increase in the amplitude of disturbances. Our next concern is to derive a similar equation that expresses the instantaneous change in the phase of the temperature field (relative to the pressure field) in terms of the current state of vertical and horizontal motion. To do so, we multiply Eq. (4) by η', multiply Eq. (5) by η^*, and add those two equations together. Factoring and making use of the rules for vector differentiation, we find that

$$\frac{\partial}{\partial t}(\eta^*\eta') + \underset{\sim}{\mathbb{V}}^* \cdot \nabla(\eta^*\eta') + \frac{1}{2}\underset{\sim}{\mathbb{V}}' \cdot \nabla(\eta^{*2} + \eta'^2) - \frac{\omega}{p}\left[\eta^{*2} + (2A+1)\eta'^2\right] = 0 \tag{14}$$

Now, from Eq. (6), it follows that

$$\underset{\sim}{\mathbb{V}}^* \cdot \nabla(\eta^*\eta') = \nabla \cdot \eta^*\eta'\underset{\sim}{\mathbb{V}}^*$$

$$\underset{\sim}{\mathbb{V}}' \cdot \nabla(\eta^{*2} + \eta'^2) = \nabla \cdot (\eta^{*2} + \eta'^2)\underset{\sim}{\mathbb{V}}' + \frac{D\omega(\eta^{*2} + \eta'^2)}{p}$$

Thus, when combined with the expressions above, Eq. (14) becomes

$$\frac{\partial}{\partial t}(\eta^*\eta') + \nabla \cdot \eta^*\eta'\underset{\sim}{\mathbb{V}}^* + \frac{1}{2}\nabla \cdot (\eta^{*2} + \eta'^2)\underset{\sim}{\mathbb{V}}' = \left(1 - \frac{D}{2}\right)\frac{\omega\eta^{*2}}{p} + \left(2A + 1 - \frac{D}{2}\right)\frac{\omega\eta'^2}{p} \tag{15}$$

As before, we next average each term of Eq. (15) over the entire area of the earth. Since the area integral of the divergence of any continuous vector vanishes when taken over a closed surface, the area averages of the second and third terms on the lefthand side of Eq. (15) vanish, with the result that

$$\frac{\partial}{\partial t}\overline{\eta^*\eta'} = \left(1 - \frac{D}{2}\right)\frac{\overline{\omega\eta^{*2}}}{p} + \left(2A + 1 - \frac{D}{2}\right)\frac{\overline{\omega\eta'^2}}{p} \tag{16}$$

Let us now investigate correlations of the type

$$\overline{\omega\eta'^2} = \overline{\omega\zeta'^2} = \bar{\omega}\,\overline{\zeta'^2} + \overline{\omega(\zeta'^2 - \overline{\zeta'^2})}$$

In the first place, Eq. (6) implies that $\bar{\omega} = 0$, so that the first term on the righthand side of the equation above vanishes. We next note that ζ' fluctuates fairly regularly around zero, so that ζ'^2 usually has a minimum (zero) between each minimum of ζ' and its adjacent maxima. On the average, therefore, ζ'^2 has twice as many minima as ζ' in any given direction, and the scale of fluctuations of ζ'^2 is half the scale of fluctuations of ζ'. Finally, since the characteristic scale of ω is the same as that of ζ' , and because $(\zeta'^2 - \overline{\zeta'^2})$ oscillates around zero with double the frequency of ω , the three-factor correlation $\overline{\omega\zeta'^2}$ tends to be uniformly small without regard to the relative phase of ω and ζ'.

The correlation $\overline{\omega\eta^{*2}}$ may be simplified in a similar way. By definition,

$$\overline{\omega\eta^{*2}} = \overline{\omega\zeta^{*2}} + 2\,\overline{f\omega\zeta^*} + \overline{\omega f^2}$$

Since ζ^* oscillates around zero and has the same characteristic scale as ω , the argument outlined in the preceding paragraph may be invoked to show that $\overline{\omega\zeta^{*2}}$ is negligible. Moreover, $\overline{\omega f^2}$ is generally negligible, because ω tends to fluctuate around zero along any latitude circle. Approximately, then, the equation above reduces to

$$\overline{\omega\eta^{*2}} = 2\,\overline{f\omega\zeta^*}$$

Finally, we note that

$$\frac{\partial}{\partial t}\,\overline{\eta^*\eta'} = \frac{\partial}{\partial t}\,\overline{\zeta^*\zeta'} + \overline{f\,\frac{\partial \zeta'}{\partial t}}$$

Now, along the whole of a latitude circle, $\partial\zeta'/\partial t$ is as positive as it is negative, on the average, so that the second term on the righthand side of the equation above may be omitted. Substituting all of the results above into Eq. (16), we find that

$$\frac{\partial}{\partial t}\,\overline{\zeta^*\zeta'} = \frac{(2-D)\,\overline{f\omega\,\zeta^*}}{P} \qquad (17)$$

Since $\overline{\zeta^*\zeta'}$ depends on the correlation between the fields of ζ^* and ζ', this equation obviously implies something about the rate at which the relative phase of ζ^* and ζ' changes with time.

The quantity $\overline{\zeta^*\zeta'}$ does not, however, depend solely on the phase of ζ^* relative to that of ζ', for it is conceivable that $\overline{\zeta^*\zeta'}$ might increase without change of relative phase —— merely as a result of an increase in the amplitudes of both ζ^* and ζ'. Accordingly, we shall define a kind of correlation index r, which is quasi-normalized with respect to the amplitudes of ζ^* and ζ'.

$$r = \frac{N}{K} \qquad N = \overline{\zeta^*\zeta'} \qquad K = \overline{\zeta^{*2} + \zeta'^2}$$

As can be seen by substituting a constant multiple of ζ' ($k\,\zeta'$, for example) for ζ', r is not independent of k and is not, therefore, entirely independent of amplitude. On the other hand, $\overline{\zeta^{*2}}$ and $\overline{\zeta'^2}$ tend to increase in about the same ratio, so that r is primarily a measure of the relative phase of the fields ζ^* and ζ'. Another important property of r, derivable by simple algebraic manipulation, is that it never exceeds $\frac{1}{2}$.

Differentiating r with respect to time, we obtain

$$\frac{\partial r}{\partial t} = \frac{1}{K}\left(\frac{\partial N}{\partial t} - r\frac{\partial K}{\partial t}\right)$$

or, substituting from Eqs. (11) and (17) for $\partial N/\partial t$ and $\partial K/\partial t$,

$$\frac{\partial r}{\partial t} = \frac{1}{K}\left[(2-D)\,\frac{\overline{f\omega\zeta^*}}{p} - 2r\,(2A+2-D)\,\frac{\overline{f\omega\zeta'}}{p}\right] \qquad (18)$$

This equation provides us with a means of calculating the average rate of occlusion — i.e., the rate at which the correlation (or relative phase) between the fields ζ^* and ζ' changes — in terms of the current fields of horizontal and vertical motion.

5. <u>The average rates of growth and occlusion in quasi-nondivergent and divergent baroclinic models</u>.

According to Eqs. (11) and (18), the average rates of growth and occlusion are linear combinations of the correlations $\overline{f\omega\zeta^*}$ and $\overline{f\omega\zeta'}$ It is important to note that the inclusion or omission of certain effects —as, for instance, that of vertical advection of vorticity, and the divergence of the wind with which the absolute vorticity is advected horizontally — has no effect on the form of the correlations, but is reflected only in the values of certain constant coefficients. This clearly indicates that the physical mechanisms of growth and occlusion are qualitatively the same in the quasi-nondivergent and divergent models, but differ in the rate at which they operate.

One of the factors in the average growth rate, for example, is the coefficient (2A + 2 - D), which is about 2 in the general divergent model. This value, as was shown earlier, leads to an increase of "rotational" kinetic energy that is just about equal to the transformed potential and internal energy. If, however, the effect of vertical advection of vorticity is included in a quasi-nondivergent model ($A \simeq \frac{1}{2}, D = 0$), the value of (2A + 2 - D) is about 3, implying that 50% more "rotational" kinetic energy is produced than there is potential energy transformed to supply it! Fortuitously enough, the quasi-nondivergent model on which the JNWP Unit's baroclinic forecasts are based does not include the effects of vertical vorticity advection ($A = 0$). Thus, the value of (2A + 2 - D) in this model is 2, which is close to the correct value. It must be pointed out, however, that one cannot expect a model which has the correct average growth rate to be correct in detail if the average rate of occlusion is wrong.

As can be seen from Eq. (18), the average rate of occlusion is very sensitive to the value of the coefficient (2 - D), whose correct value is unity. In the quasi-nondivergent models ($D = 0$), on the other hand, the value of (2 - D) is 2. This implies that the process of occlusion in such models operates 100% faster than it should in some cases, and may account for the observed tendency for too rapid occlusion in the JNWP Unit's baroclinic forecasts.

Because the rates of both growth and occlusion depend on A and D, it is clear that no model can accurately reproduce all of the essential aspects of baroclinic development unless it includes the effects of vertical vorticity advection and the divergent component of the advecting wind. From the latter, it follows immediately that any baroclinic model must be essentially nongeostrophic.

6. Dependence of the average rates of growth and occlusion on the relative phase of the pressure and temperature fields.

Since the average rates of growth and occlusion depend on the correlations $\overline{f\omega\zeta^*}$ and $\overline{f\omega\zeta'}$, they are crucially dependent on the phase of the vertical motion pattern relative to the horizontal velocity field. Our next objective is to find out how the phase of the ω-field is related to the horizontal velocity field and how, in turn, the rates of growth and occlusion depend on the relative phase of the pressure and temperature fields.

In order to retain the concept of "relative phase" in a clear and unambiguous way, we shall consider the growth and occlusion of disturbances in which the velocity field is independent of the north-south coordinate. If, in addition, the amplitudes of those disturbances are small, the general relationship between the ω-field and the horizontal velocity field is given by Eq. (8) of Thompson (1956). For sinusoidal perturbations whose orbital frequency is much greater than a pendulum day, that equation reduces to

$$\frac{f\omega}{p} = \sigma\left(\beta v' - 2\alpha^2 U' v^*\right) \tag{19}$$

where $\sigma^{-1} = 1 + \alpha^2 c^2/2f^2$; $c^2 = (R^2T^2/g\theta)\,\partial\theta/\partial z$; α is the wave-number; β is the northward derivative of the Coriolis parameter; v is the northward component of velocity; and U is the average zonal component of velocity. The "starred" and "primed" quantities carry their earlier meaning. In flows of the type under consideration,

$$\zeta^* = \frac{\partial v^*}{\partial x} \qquad\qquad \zeta' = \frac{\partial v'}{\partial x}$$

where x is the coordinate toward the east. Thus

$$\overline{v^*\zeta^*} = \overline{\frac{\partial}{\partial x}\left(\frac{v^{*2}}{2}\right)} = 0 \qquad\qquad \overline{v'\zeta'} = \overline{\frac{\partial}{\partial x}\left(\frac{v'^2}{2}\right)} = 0$$

Substituting from Eq. (19) into Eq. (18), and making use of the expressions above, we then find that

$$\frac{\partial h}{\partial t} = \frac{\sigma}{K}\left[(2-D)\,\beta\,\overline{v'\zeta^*} + 4\nu\alpha^2(2A+2-D)\,U'\,\overline{v^*\zeta'}\right] \tag{20}$$

We note, however, that

$$\overline{v^*\zeta'} = \overline{v^*\frac{\partial v'}{\partial x}} = \overline{\frac{\partial}{\partial x}(v^*v')} - \overline{v'\frac{\partial v^*}{\partial x}} = -\overline{v'\zeta^*}$$

whence Eq. (20) can be rewritten as

$$\frac{\partial h}{\partial t} = \frac{\sigma}{K}\left[(2-D)\,\beta - 4\nu\alpha^2(2A+2-D)\,U'\right]\overline{v'\zeta^*} \tag{21}$$

Similarly, substitution from Eq. (19) into Eq. (11) yields

$$\frac{\partial}{\partial t}\,\overline{\zeta_1^2 + \zeta_2^2} = 8\sigma\alpha^2(2A+2-D)\,U'\,\overline{v'\zeta^*} \tag{22}$$

The two equations above express the average rates of growth and occlusion in terms of the static stability, the scale of the disturbances, the average vertical wind shear, and the correlation between the fields of v' and ζ^*

Let us now consider a whole family of initial velocity fields in which the phase difference between v^* and v' takes on all possible values, and trace first the evolution of those disturbances in which the v'-field lags behind the v^*-field. A disturbance of this type is shown schematically in Figure 1, on which the solid lines are streamlines and the dashed lines are "streamlines" for the vertical wind shear. Approximately, of course, the solid lines are isobaric contours and the dashed lines are isotherms. In the case shown, v' is positive where ζ^* reaches its maximum values, and vice versa, so that $\overline{v'\zeta^*}$ is positive. We see, moreover, that $\overline{v'\zeta^*}$ is positive as long as the coldest air (along a latitude circle) lies in the shaded region between the pressure trough and the pressure ridge following it. Now, the average static stability is positive, and the average temperature gradient is directed from north to south, so that U' is also positive. Thus, according to Eq. (22), the average amplitude of the disturbances (or, more precisely, the total rotational energy) increases as long as the temperature field lags behind the pressure field. The rate of growth, of course, depends on the phase difference between the fields of pressure and temperature, being greatest when the temperature field is 90° behind the pressure field and becoming smaller and smaller as the fields are brought into phase.

Let us now suppose that the wave numbers of the disturbances are small —— small enough that, even if r approaches its maximum value of $\frac{1}{2}$, the expression in square brackets in Eq. (21) remains positive.*

*In actuality, r does not attain its theoretical upper limit, and the factor in question generally remains positive. It does, however, become very small when the temperature and pressure fields are in phase.

In the case discussed above, then, Eq. (21) states that the correlation between the pressure and temperature fields increases as long as the temperature field lags behind the pressure field; that is to say, the temperature field tends to catch up with the pressure field. It is important to note that the rates of growth and "occlusion" both depend on the same correlation $\overline{v'\zeta^*}$, and that the structure of the disturbances enters only into that correlation. This means that the physical mechanism of occlusion is the same as that by which the total rotational energy increases, and that occlusion takes place as an inevitable dynamical consequence of the process of baroclinic development.

Carrying the foregoing arguments to their conclusion, we see that a temperature field which initially lags behind the pressure field will continue to catch up until it is in phase with the pressure field. At that point, $\overline{v'\zeta^*}$ vanishes, the pressure and temperature fields undergo no further change in relative phase, and simultaneously the disturbances stop growing. Left to themselves, disturbances of this type would remain in this final state of quasi-barotropy.

We now turn to the remaining class of disturbances —— namely, those in which the temperature field precedes the pressure field. A disturbance of this type is shown in Figure 2. In this case, the correlation $\overline{v'\zeta^*}$ is negative as long as the coldest air (around any latitude circle) lies in the shaded region between the pressure trough and the pressure ridge preceding it. According to Eq. (21), therefore, the correlation between the pressure and temperature fields decreases, and the temperature field tends to become more and more out of phase with the pressure field. This process continues until the pressure and temperature fields are exactly 180° out of phase, when $\overline{\varpi'\zeta^*}$ vanishes and r can decrease no further. The important aspect of the behavior of this class of disturbances, however, is that their amplitudes decrease with time in accordance with Eq. (22). Thus, if disturbances of this type are very small to begin with, they will become even less perceptible as time goes on.

7. The general behavior of large-scale disturbances in baroclinic flow.

Let us next imagine that a horizontally uniform zonal flow is slightly and intermittently disturbed in such a way that the velocity field remains independent of the north-south coordinate, but such that the initial temperature field precedes the associated pressure field just as often as it lags behind it. According to the arguments outlined in the preceding section, however, one would not expect that equal amounts of energy would be found in the two ranges of relative phase at some later time. As we have seen, the disturbances in which the initial temperature field precedes the pressure field will die out undetected, but will remain in the same range of relative phase. The maximum energy will tend to accumulate in those disturbances in which the temperature and pressure fields are almost exactly in phase, but in which the temperature field originally lagged behind the pressure field. At any given time, on the average, one would also expect to find considerable energy associated with disturbances in which the temperature field lags behind the pressure field, the amount of energy varying inversely with the current phase difference between those fields. From one time to the next, disturbances of this latter type increase in amplitude; simultaneously, their temperature fields come more and more into phase with the pressure fields, and gradually approach a final state of quasi-barotropy.

This description of the general behavior of large-scale disturbances in baroclinic flow is, of course, necessarily lacking in detail —— being deduced as it was from very simple dynamical and statistical considerations. Nevertheless, it conforms remarkably well with the gross features of the observed behavior of disturbances in the true atmosphere. It is, in fact, observed that the atmosphere has a predilection for disturbances in which the temperature field lags behind the pressure field; the reverse phase relationship is rarely observed, and is never associated with large amplitudes. The sequence of events in the evolution of baroclinic disturbances, as deduced from the theory, is also in striking agreement with the history of the typical large-scale storm, even to the simultaneous operation of the processes of growth and occlusion. Finally, it appears that the observed tendency toward a state of quasi-barotropic motion is a consequence of the dynamical properties of relatively simple baroclinic models.

These results, coupled with the conclusions of section 5, indicate that the equations for nongeostrophic divergent flow will not only provide a basis for methods of predicting the general features of baroclinic development, but will eventually prove successful in accounting for the details of the processes of growth and occlusion.

8. Summary

A brief resumé of the main results is given in the last three paragraphs of the introduction.

References

Charney, J. G. and N. A. Phillips, 1953: Numerical integration of the quasi-geostrophic equations for barotropic and simple baroclinic flow. J. Meteor., 10, 71-99.

Kuo, H.-L., 1951: Dynamical aspects of the general circulation and the stability of zonal flow. Tellus, 3, 268-284.

Lorenz, E. N., 1955: Available potential energy and the maintenance of the general circulation. Tellus, 7, 157-167.

Phillips, N. A., 1956: The general circulation of the atmosphere: a numerical experiment. Quart. J. r. meteor. soc., 82, 123-164.

Rossby, C.-G., and collaborators, 1939: Relation between circulation of the atmosphere and the displacement of the semi-permanent centers of action. J. marine Res., 2, 38-55.

Thompson, P. D., 1956: A theory of large-scale disturbances in non-geostrophic flow. J. Meteor., 13, 251-261.

____________, 1956: A two-level nongeostrophic model suitable for routine numerical weather forecasting. JNWP Unit Tech. memo., 11.

____________ and W. L. Gates, 1956: A test of numerical prediction methods based on the barotropic and two-parameter baroclinic models. J. Meteor. 13, 127-141.

Captions

Fig. 1 - Streamlines (solid) and "streamlines" for the vertical wind shear (dashed) in a typical disturbance whose temperature field lags behind the pressure field.

Fig. 2 - Streamlines (solid) and "streamlines" for the vertical wind shear (dashed) in a typical disturbance whose temperature field precedes the pressure field.

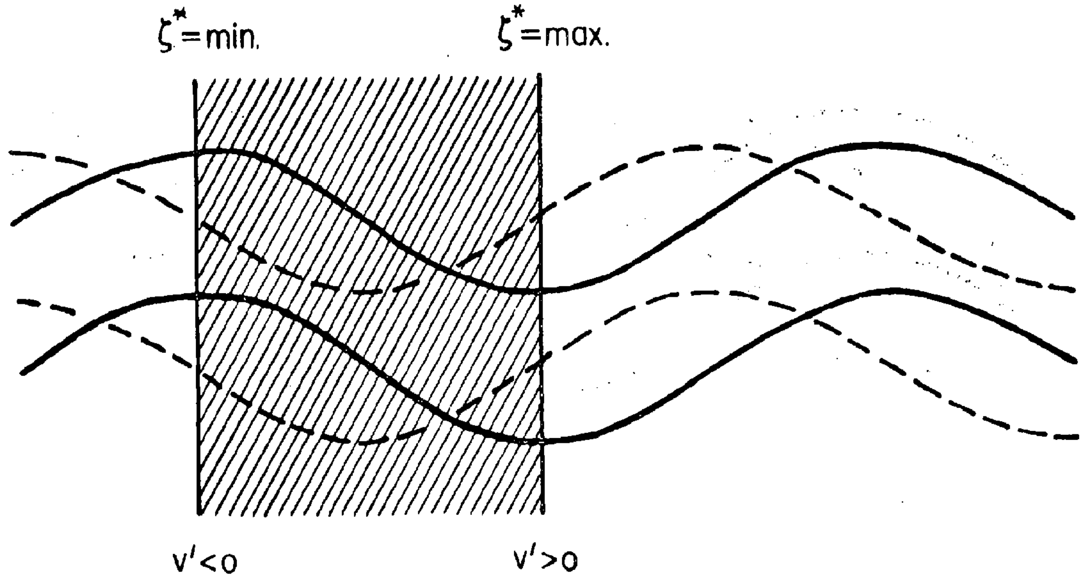

Figure 1

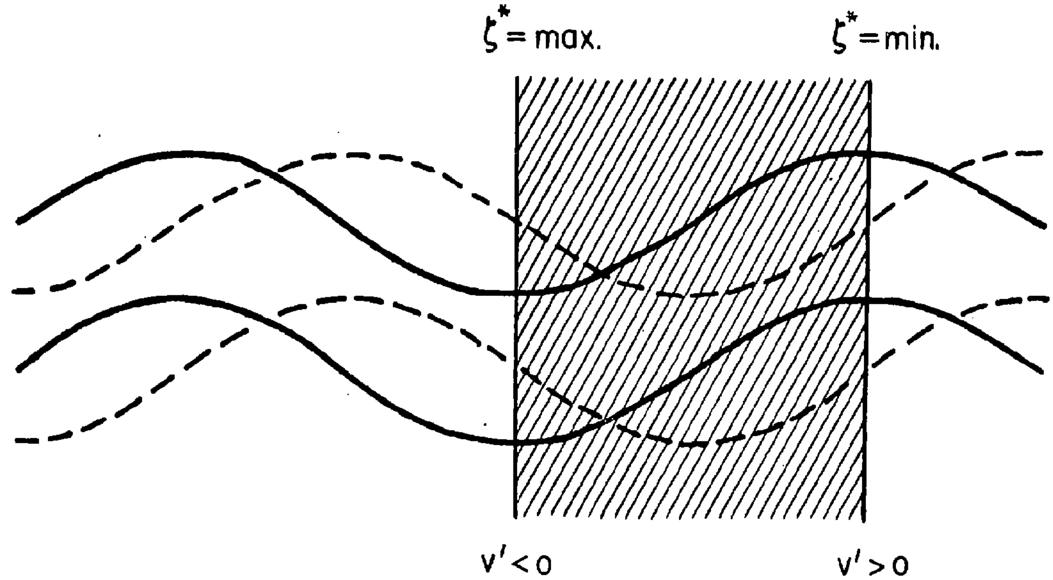

Figure 2