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A NOTE ON THE THERMAL STRUCTURE OF WAVES IN A SIMPLE BAROCLINIC MODEL

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ABSTRACT

The change in time in the relative phase between the temperature and pressure fields in a two-parameter model has been investigated by making one-dimensional forecasts for simple flcw pattern varying the initial phase-difference. The variation in amplitude of the pressure and temperature fields has also been investigated. It has been found that the two fields adjust rather rapidly to each other in the unstable wave in such a way that the temperature field after a while will lag behind the pressure field even if it initially were preceding the pressure field. It is further found that the amplitudes of the two fields will increase initially, if the temperature field lags behind the pressure field initially, but decrease initially if the opposite is the case. Due to the adjustment in the phase difference between the two fields the amplitudes will start to increase after a while, when the temperature field starts to lag behind the pressure field. The amplification during the first 24 hours depends therefore critically on the initial state and will seldom reach the estimated amplication rate obtained by a consideration of only the amplifying part of the solution.

The behavior of very long waves in the two-parameter model with no divergence in the mean flow has been investigated. It is found that the temperature field and the pressure field move almost independent of each other. The pressure field will retrograde with a speed comparable with the Rossby speed for non-divergent waves, while the temperature field will progress slowly. As

a result the latter field will precede the pressure field after a while verifying an earlier observation in these forecasts. Introducing a divergence in the pressure field it is found that not only is the retrogression greatly reduced in the pressure field, but a stronger coupling is found between the temperature and pressure fields.

1. Introduction

Recent investigations of the energy-conversions in the atmosphere, especially the conversion from potential to kinetic energy, by Wiin-Nielsen (1959 b) and Saltzman and Fleischer (1959) show a positive conversion on each day. The conversion of potential to kinetic energy is positive, if on the average the warm air is rising and the cold air is sinking. It is furthermore found that this arrangement is possible only if, on the average, the temperature waves are somewhat out of phase with the pressure waves in such a way that cold air is advected into the pressure troughs and warm air into the ridges. It is an important problem to find out how the atmosphere is able to keep the temperature and the pressure almost, but not quite, in phase and to keep the temperature wave lagging somewhat behind the pressure wave. From the energetical point of view it is obvious that the arrangement of the temperature field relative to the pressure has to be this way due to the fact that frictional dissipation continuously reduces the amount of kinetic energy and a conversion of energy from the source of potential energy has to take place in order to make up for the frictional loss. It is, however, of a great interest to describe, if possible, the major mechanisms which are responsible for keeping the atmosphere in a state where it is able to convert potential to kinetic energy.

The purpose of the following investigation was to demonstrate that the two-parameter model of the atmosphere will predict the desired lag of the temperature relative to the pressure field independent of the initial state of the atmosphere at least for sinusoidal initial flow pattern. We can demonstrate that this is the case simply by making a prediction using the two-parameter model for varying initial conditions. In order to avoid complicated numerical problems it was decided to make the problem one-dimensional and thereby reduce it to such an extent that a solution can be given in a closed form.

A similar investigation of the two-parameter model from a similar point of view, but using a statistical approach was made by Thompson (1959). However, his considerations can only give an initial tendency for "growth" and "occlusion" and can therefore not say anything about the state towards which the atmosphere develops. By obtaining the solution in a closed form we can easily investigate the development in time and also investigate the limiting state towards which the atmosphere develops. What is gained by obtaining the solution in a closed form is sacrified by the limited complexity of the flow pattern, which is possible.

In short, we are going to make one-dimensional forecasts with a two-parameter model for initial conditions having varying phase differences between the temperature and pressure.

2. A Solution of the Model Equations

The atmosphere will be represented by the simplest possible baroclinic model. We are going to use the 40 and 80 cb levels as information levels to assume $\omega = 0$ for p = 20 cb and p = 100 cb and to assume that the atmosphere is frictionless and adiabatic. The model equation for this model can be written:

$$\frac{\partial \zeta^*}{\partial t} + \underset{\sim}{V}^* \cdot \nabla(\zeta^* + f) + \underset{\sim}{V}' \cdot \nabla \zeta' = 0 \tag{2.1}$$

$$\frac{\partial \zeta'}{\partial t} - \lambda^2 \frac{\partial \psi'}{\partial t} + \underset{\sim}{V}^* \cdot \nabla \zeta' + \underset{\sim}{V}' \cdot \nabla(\zeta^* + f) - \lambda^2 \underset{\sim}{V}^* \cdot \nabla \psi' = 0 \tag{2.2}$$

In (2.1) and (2.2) $\underset{\sim}{V} = \underset{\sim}{k} \times \nabla \psi$ denotes a horizontal wind, $\zeta = \nabla^2 \psi$ the relative vorticity and f the Coriolis' parameter. Quantities with the superscript "star" refer to the arithmetic mean of the corresponding quantities at the two information levels, (40 and 80 cb), while a superscript "prime" refers to one half of the difference between the quantities at the same levels. The paramenter $\lambda^2 = 2 f_0^2 / \sigma P^2$ is assumed to be a constant. f_0 is a standard value of the Coriolis' parameter, $P = 40$ cb and $\sigma = -\alpha \partial \ln \theta / \partial p$ is a measure of static stability. In the following we are going to refer to the ψ'- field as the temperature field, while we will refer to the ψ^*- field as the pressure field.

We are going to find solutions to (2.1) and (2.2) in a linearized form. The perturbations will be assumed to have the form:

$$\left\{ \begin{aligned} \psi^*(x,t) &= \hat{\psi}^* e^{ik(x-ct)} \\ \psi'(x,t) &= \hat{\psi}' e^{ik(x-ct)} \end{aligned} \right. \tag{2.3}$$

$\hat{\psi}^*$ and $\hat{\psi}'$ denote the (complex) amplitudes, $k = 2\pi/L$ the wave-number and c the phase-speed. (2.3) will be the solutions to the linearized forms of (2.1) and (2.2) if the following two homogeneous linear equations are satisfied:

$$\text{(2.4)} \quad \left[c - \mathcal{U}^* + \beta/k^2\right]\hat{\psi}^* - \mathcal{U}'\hat{\psi}' = 0$$

$$\text{(2.5)} \quad \mathcal{U}'\left[\lambda^2/k^2 - 1\right]\hat{\psi}^* + \left[(1+\lambda^2/k^2)(c - \mathcal{U}^* + \beta/k^2) - \beta\lambda^2/k^4\right]\hat{\psi}' = 0$$

$\mathcal{U}^*$ and $\mathcal{U}'$ are the (constant) zonal wind speeds and $\beta = df/dy$, the Rossby-parameter. Non-trivial solutions to (2.4) and (2.5) are possible if the determinant is zero. This condition gives us an equation from which the wave-speed c can be determined. Denoting:

$$\text{(2.6)} \quad X = c - \mathcal{U}^* + \beta/k^2$$

we find

$$\text{(2.7)} \quad X = \frac{\beta\lambda^2/k^4 \pm \Delta^{\frac{1}{2}}}{2(1+\lambda^2/k^2)}$$

where

$$\text{(2.8)} \quad \Delta = (\beta\lambda^2/k^4)^2 + 4\mathcal{U}'^2(1 - \lambda^4/k^4)$$

The two solutions (2.7) will be denoted X_+ and X_- corresponding to the two possible signs in front of the square root. The corresponding values of c will be c_+ and c_-.

The complete solution to the model equations may now be written:

$$\psi^*(x,t) = \hat{\psi}_+^* e^{ik(x-c_+t)} + \hat{\psi}_-^* e^{ik(x-c_-t)} \tag{2.9}$$

$$\psi'(x,t) = \hat{\psi}_+' e^{ik(x-c_+t)} + \hat{\psi}_-' e^{ik(x-c_-t)} \tag{2.10}$$

because our equations are linear, and superposition of solutions is possible.

3. Determination of the Amplitudes

The four amplitudes, $\hat{\psi}_+^*, \hat{\psi}_-^*, \hat{\psi}_+'$ and $\hat{\psi}_-'$ may be determined from the following conditions. Firstly, the solutions (2.9) and (2.10) have to satisfy the initial conditions for the two fields ψ^* and ψ'. Let us assume that the initial fields are specified by the relations

$$\psi^*_{t=0} = A_0^* \cos kx \tag{3.1}$$

$$\psi'_{t=0} = A_0' \cos(kx + \alpha_0') \tag{3.2}$$

which means that we assume initially a phase-difference of α_0' between the temperature wave $\psi'_{t=0}$ and the pressure $\psi^*_{t=0}$. Note, that the temperature field is lagging behind the pressure field, if $\alpha_0' > 0$. In the complex notation we may write the initial conditions in the form

$$\psi^*_{t=0} = A_0^* e^{ikx} \tag{3.3}$$

$$\psi'_{t=0} = A_0' e^{i(kx+\alpha_0')} \tag{3.4}$$

where A_0^* and A_0' are real numbers.

Equating (2.9) and (2.10) for $t = o$ to (3.3) and (3.4) we obtain two of the necessary equations for a determination of the amplitudes, i.e.,

$$\hat{\psi}_+^* + \hat{\psi}_-^* = A_o^* \tag{3.5}$$

$$\hat{\psi}_+' + \hat{\psi}_-' = A_o' e^{i\alpha_o'} \tag{3.6}$$

The two remaining relations are obtained from either (2.4) or (2.5) which gives a relation between $\hat{\psi}_+^*$ and $\hat{\psi}_+'$, if $c = c_+$; and a relation between $\hat{\psi}_-^*$ and $\hat{\psi}_-'$, if $c = c_-$. The simplest forms are obtained from (2.4):

$$X_+ \hat{\psi}_+^* - \mathcal{U}' \hat{\psi}_+' = 0 \tag{3.7}$$

$$X_- \hat{\psi}_-^* - \mathcal{U}' \hat{\psi}_-' = 0 \tag{3.8}$$

The four equations (3.5), (3.6), (3.7), and (3.8) determine the amplitudes in terms of the initial conditions, the wave-number k and the parameters: $\mathcal{U}'$, χ^2 and β. The solution to this set of equations may be written:

$$\begin{aligned} \hat{\psi}_+^* &= \frac{\mathcal{U}' A_o' e^{i\alpha_o'} - X_- A_o^*}{X_+ - X_-}, \quad \hat{\psi}_+' = \hat{\psi}_+^* \frac{X_+}{\mathcal{U}'} \\ \hat{\psi}_-^* &= \frac{X_+ A_o^* - \mathcal{U}' A_o' e^{i\alpha_o'}}{X_+ - X_-}, \quad \hat{\psi}_-' = \hat{\psi}_-^* \frac{X_-}{\mathcal{U}'} \end{aligned} \tag{3.9}$$

The expressions (3.9) substituted in (2.9) and (2.10) determine the complete solution. In the following we are going to study some properties of the unstable, baroclinic wave. We note first of all that in this case we have:

$$c_+ = c_r + i c_i, \quad c_- = c_r - i c_i \tag{3.10}$$

where the subscripts r and c denote the real and imaginary parts of the numbers, and the expression for ψ^* may be written:

$$(3.11) \quad \psi^* = \hat{\psi}_+^* \cdot e^{kc_c t} \cdot e^{ik(x-c_r t)} + \hat{\psi}_-^* \cdot e^{-kc_c t} \cdot e^{ik(x-c_r t)}$$

In order to study some details of this solution we shall write:

$$(3.12) \quad \hat{\psi}_+^* = A_+^* \left(\cos\delta_+^* + i\sin\delta_+^*\right), \quad \hat{\psi}_-^* = A_-^* \left(\cos\delta_-^* + i\sin\delta_-^*\right)$$

The expressions (3.12) are to be considered as the definitions of A_+^*, A_-^*, δ_+^*, and δ_-^*.

Substituting (3.12) in (3.11) we can after some manipulations write ψ^* in the following form:

$$(3.13) \quad \psi^* = A^* \cos\left[k(x - c_r t) + \delta^*\right]$$

with

$$(3.14) \quad A^{*2} = A_+^{*2} e^{2kc_c t} + A_-^{*2} e^{-2kc_c t} + 2A_+^* A_-^* \cos(\delta_+^* - \delta_-^*)$$

and

$$(3.15) \quad \tan\delta^* = \frac{A_+^* e^{kc_c t} \sin\delta_+^* + A_-^* e^{-kc_c t} \sin\delta_-^*}{A_+^* e^{kc_c t} \cos\delta_+^* + A_-^* e^{-kc_c t} \cos\delta_-^*}$$

In an analogous way, we define

$$(3.16) \quad \hat{\psi}_+' = A_+'\left(\cos\delta_+' + i\sin\delta_+'\right), \quad \hat{\psi}_-' = A_-'\left(\cos\delta_-' + i\sin\delta_-'\right)$$

and we can then write the solution for ψ' in the form:

$$\psi' = A' \cos\left[k(x - c_r t) + \delta'\right] \tag{3.17}$$

with

$$A'^2 = A_+'^2 e^{2kc_i t} + A_-'^2 e^{-2kc_i t} + 2A_+' A_-' \cos(\delta_+' - \delta_-') \tag{3.18}$$

and

$$\tan \delta' = \frac{A_+' e^{kc_i t} \sin \delta_+' + A_-' e^{-kc_i t} \sin \delta_-'}{A_+' e^{kc_i t} \cos \delta_+' + A_-' e^{-kc_i t} \cos \delta_-'} \tag{3.19}$$

The expressions (3.13) and (3.17) give the variation of ψ^* and ψ' as simple sinusoidal waves, however, with an amplitude and a phase, which are functions of time.

It is the problem to investigate the solutions given above in some detail in the next section

4. Some Characteristic Features of the Unstable Baroclinic Wave

Even the very simple initial flow pattern which is considered here shows a great deal of complexity as it develops in time. We shall first investigate the phase difference between the thermal wave and the pressure wave, i.e., $(\delta' - \delta^*)$. From (3.15) and (3.19) it is seen that the limiting phase angle as time approaches infinity will be $\delta^* = \delta_+^*$ and $\delta' = \delta_+'$. The phase difference will therefore in the limit approach the phase difference in the amplifying wave.

From (3.12) and (3.16) and (3.9), we find in the amplifying case:

$$\tan \delta_+^* = \frac{-U' A_0' \cos \alpha_0' + A_0^* \beta \lambda^2/k^4 \left[2(1+\lambda^2/k^2)\right]^{-1}}{U' A_0' \sin \alpha_0' + A_0^* (-\Delta)^{\frac{1}{2}} \left[2\,(1+\lambda^2/k^2)\right]^{-1}} \tag{4.1}$$

and

$$\tan \delta_+' = \frac{A_0' \sin \alpha_0' (-\Delta)^{\frac{1}{2}} - A_0' \cos \alpha_0' \beta \lambda^2/k^4 - 2 A_0^* U' (1-\lambda^2/k^2)}{A_0' \sin \alpha_0' \beta \lambda^2/k^2 + A_0' \cos \alpha_0' (-\Delta)^{\frac{1}{2}}} \tag{4.2}$$

From (4.1) and (4.2) we can derive the interesting result that

$$\tan(\delta_+' - \delta_+^*) = \frac{(-\Delta)^{\frac{1}{2}}}{\beta \lambda^2/k^4} \tag{4.3}$$

(4.3) can be rearranged using (2.8) to the form:

$$\lim_{t \to \infty} \left[\tan(\delta' - \delta^*)\right] = \left[4U'^2 k^4 \beta^{-2} (1 - k^4/\lambda^4) - 1\right]^{\frac{1}{2}} \tag{4.4}$$

(4.4) shows that the baroclinic, unstable wave will develop towards a state, where the temperature field has a certain lag behind the pressure field. The limiting phase difference depends on the vertical shear, the β -effect, the stability parameter λ^2 and the wave number, k, but is independent of the initial state.

From (2.8) it is seen that the criterion for instability is:

$$U'^2 \geq (\beta \lambda^2/k^4)^2 \cdot \left[4\lambda^4/k^4 - 1\right]^{-1} \tag{4.5}$$

It is interesting to note that (4.5) also expresses the condition that the square root in (4.4) is real. We may therefore conclude that any unstable wave will ultimately have a positive phase lag. If the equality sign in (4.5) holds, we have waves in neutral stability. (4.4) shows that these waves are equivalent barotropic in the sense that isotherms and contours are parallel. The more unstable the wave is the larger is the limiting phase-lag. Figure 1 shows $\mathcal{U}'$ as a function of wave length in the baroclinic unstable waves with a certain phase-lag. The curve $(\delta' - \delta^*) = 0$ is the same as the curve for neutral stability.

Equation (4.4) gives the phase-lag in the baroclinic wave for time approaching infinity. It is interesting to note that if we can apply the results of this analysis to the real atmosphere, we find that the atmosphere constantly is tending toward a state with a positive lag between the temperature and pressure field. This result is somewhat different from the statement made by Thompson (1959) who arrives at the result that the atmosphere constantly tends towards a state of quasi-barotropy. These results are obtained from essentially the same model, but are based upon a tendency computation for "growth" $(\partial(\overline{\zeta^{*2}+\zeta'^2})/\partial t$ and "occlusion" $(\partial(\overline{\zeta^*\zeta'})/\partial t)$. He finds that waves, where the temperature field initially lags behind the pressure field will grow and occlude, while waves, where the temperature field initially precedes the pressure field will decrease in amplitude and have a tendency to an increasing phase difference between the temperature and pressure field in such a way that the temperature field at a later time will precede the pressure field even more. Although these results are true in most cases as tendency computations, the present results show that the later development of the flow always will result in an amplifying wave, which will tend towards a state of a positive phase lag. As a further correction to Thompson's results we find that the phase difference between the temperature wave and the pressure wave do not decrease in a monotonic way if the temperature field initially lags behind the pressure field, but the results suggest that the phase difference will decrease if the initial phase difference is greater than the limiting one, and increase, if it is initially smaller than the limiting phase difference.

It is naturally somewhat doubtful whether one can apply the results of a linearized treatment of the dynamic equations very far in time. It becomes therefore interesting to compute a few examples which will show the development in time of amplitude and phase for the baroclinic wave and for different initial states. We have here chosen a wave with a wave length of 5000 km, maximum meridional winds in the "star" field of 20 msec^{-1} initially and in the "prime" field of 10 msec^{-1} initially. The zonal winds have been chosen to be $U^* = 20$ msec^{-1}, $U' = 10$ msec^{-1}.

The initial phase-lag has been chosen to be $\alpha_0' = \pi/2, \pi/4, 0, -\pi/4, -\pi/2$. With a wave length of 5000 km these values correspond to about 16, 8, 0, -8, and -16 degrees of longitude. The formula (3.14), (3.15), (3.17), (3.18), and (3.9) were used to compute the phase difference $(\delta' - \delta^*)$ and the amplitudes A^* and A' as a function of time. Multiplying A^* and A' by the wave number k, we obtain the amplitude in the meridional wind component.

Figure 2 shows the phase difference $(\delta' - \delta^*)$ as a function of time up to t = 36 hours. For our particular choice of parameters the limiting hase difference turns out to be 57° or roughly 1/6 of the wave length. It is seen on fig. 2 that the initial states characterized by a phase difference between $-\pi/4$ and $\pi/2$ approaches the limiting phase difference uniformly. The initial phase difference of $-\pi/2$ is an example of a case, where the phase difference becomes numerically greater until it finally approaches the limiting phase difference. The curves on figure 2 indicate that the adjustment between the temperature field and pressure field takes place rather rapidly. Already after 36 hours, the temperature field is lagging behind the pressure field in all the waves which have been investigated.

Figure 3 gives $V^* = kA^*$ as function of time computed for the five different initial states from (3.14). It is seen that the amplitude increases monotonically with time in the case where the temperature field initially lags behind the pressure field and even in the case where the two fields are initially in phase. In the two cases where the temperature field initially precedes the pressure field the amplitudes decrease at first, but have, at least in the examples computed here, started to increase after 36 hours.

The amplification rate computed from the amplifying wave alone is $\exp.(kc_c t)$ This factor turns out to be 1.8 for the wave length of 5000 km chosen here and t = 24 hours. From fig. 3 it is seen that the amplification rate in the complete solution, where we have the amplifying, as well as the dampening wave present, is smaller than the value above in all cases. The factor 1.8 would correspond to an amplitude of 36 msec^{-1} for (kA^*) at $t = 24$ hours. Only in the case of an initial phase of $\pi/2$ do we get a similar amplification rate. The meridional velocity, $v' = kA'$, varies as a function of time in a way very similar to $v^* = kA^*$ and the corresponding curves have not been reproduced here. It is, however, interesting to note that the thermal field amplifies in much the same way as the pressure field.

We find as the main conclusion from this section that the phase difference between the temperature and pressure fields rapidly become positive if it were negative in the initial state. This explains in a qualitative way why we always find a positive conversion between the potential and kinetic energy.

With respect to the amplification rate we find that this quantity in general depends to a large extent on the initial arrangement of the temperature field relative to the pressure field. Fig. 3 shows that the amplification rate is much smaller than the one estimated from the amplifying wave alone in cases where the two fields are nearly in phase initially. As we, according to the present theory, most often will find the two fields only slightly out of phase it is to be expected that the amplification will be small in most short-range predictions (24-36 hour forecasts).

5. On the Behavior of Very Long Waves in the Two-Parameter Model

In this section we are going to investigate the behavior of very long waves in the two-parameter model. From earlier investigations, see for instance Wiin-Nielsen (1959a), it is known that very long waves are always stable for values of the vertical shear appearing in the atmosphere. This means that the two roots, c_+ and c_-, and therefore also the quantities x_+ and x_-, will be real numbers. In order to keep the investigations relatively simple we shall investigate what happens, if $k \to 0$, although it is realized that there is an upper limit for the wave length in the atmosphere.

We return to the expressions (3.9) for the amplitudes. Denoting

$$c_R = \mathcal{U}^* - \beta / k^2 \tag{5.1}$$

we have

$$x_+ = c_+ - c_R \ , \quad x_- = c_- - c_R \tag{5.2}$$

From the expression (5.1), (2.6) and (2.7) it can be shown that

$$c_R \to -\infty \ , \quad x_+ \to +\infty \ , \quad x_- \to 0 \quad \text{for } k \to 0 \tag{5.3}$$

From the results (5.3) it follows using (3.9) that

$$\left.\begin{aligned} \hat{\psi}_+^* / A_0^* &\to 0 \\ \hat{\psi}_-^* / A_0^* &\to 1 \\ \hat{\psi}_+' / A_0' e^{i\alpha_0'} &\to 1 \\ \hat{\psi}_-' / A_0' e^{i\alpha_0'} &\to 0 \end{aligned}\right\} \text{for } k \to 0 \tag{5.4}$$

The results obtained in (5.4) are interesting because they show that the stream function ψ^* for very large values of the wave length predominantly will move with the speed c_-, which is close to the Rossby speed and therefore numerically large and negative for very long waves, while the amplitude $\hat{\psi}_+^*$

in the part of the solution moving with the speed c_+ will be very small. However, the last two expressions in (5.4) shows that the thermal wave ψ' behaves opposite. Only a very small part of the initial amplitude will appear in the wave moving with the speed c_-, while the major part will move with the speed c_+, which for small values of k is positive, but small. The pressure wave ψ^* and the thermal wave, ψ', move therefore in opposite directions and with speeds which are very different. We may as an illustration take the case where the two waves, ψ^*, and ψ', are initially in phase. According to our results in (5.4) we will find the ψ^*-wave will retrograde with a large speed, while the ψ'-wave will move slowly toward the east. Due to this result we will find the temperature wave preceding the pressure wave in this forecast. It has been noted earlier that this actually takes place in forecasts with this model (Thompson, 1959), although it was not stated whether it tended to appear for very long waves. Although our results are obtained for infinitely long waves it is likely that similar but slightly modified results apply also for the planetary waves on the hemispheric forecasts. The waves in which it is found that the temperature wave gradually will precede the pressure wave are therefore not the unstable waves, but rather sufficiently long stable waves.

In an earlier effort to control the behavior of very long waves in a two-parameter model it has been suggested (Wiin-Nielsen, 1959a) to modify the prognostic equations (2.1) to the following equation:

$$\frac{\partial \zeta^*}{\partial t} + \underset{\sim}{V}^* \cdot \nabla(\zeta^* + f) + \underset{\sim}{V}' \cdot \nabla \zeta' = r^2 \frac{\partial \psi^*}{\partial t} \tag{5.5}$$

but to leave the other prognostic equation (2.2) unchanged. The modification of (2.1) to (5.5) means that we have introduced a certain amount of divergence, i.e.,

(5.6) $$\nabla \cdot \underset{\sim}{V}^* = -\frac{r^2}{f_0}\frac{\partial \psi^*}{\partial t}$$

which will have a characteristic distribution relative to the waves in the ψ^*- field. Assuming for a moment that

(5.7) $$\psi^* = A \sin k(x-ct)$$

we find

(5.8) $$\nabla \cdot \underset{\sim}{V}^* = \frac{r^2}{f_0} c\, kA \cos k(x-ct) = \frac{r^2}{f_0} c\, v^*$$

For very long waves, where $c<0$, it means that we have introduced a convergence, where $v^*>0$ i.e., between the trough and the ridge, and divergence between the ridge and the trough. Although the introduction of these divergence patterns was based on an equivalent barotropic structure of the very long waves, which not always is found, it is interesting to note that a divergence pattern of this type will appear in a three-parameter model, if the vertical wind profile has a jet-type structure, i.e., $d^2U/dp^2>0$. The distribution of vertical velocity and divergence in a three-parameter model will be described in a paper to appear later. It is, however, pertinent to mention here that the term, $r^2\partial\psi^*/\partial t$, may be justified also from the point of view that we have introduced a pattern of convergence which cannot be represented directly by two parameters, because it depends on the deviation from a linear vertical wind profile, but which is present in the real atmosphere due to the vertical structure, especially the property that d^2u/dp^2 is positive in the midtroposphere at least in a jet-stream region. With this further justification for the presence of this term it becomes interesting to investigate whether the modified two-parameter model will give a coupling between the waves in the ψ^*-field and the ψ'-field, especially for very long waves. From the analysis in the earlier paper (Wiin-Nielsen, 1959a), it is evident that the two speeds, c_+ and c_-, are numerically smaller. In the limit we found:

$$c_+ = u^* - \beta/\lambda^2, \quad c_- = -\beta/r^2 \quad \text{for } k \to 0 \tag{5.9}$$

In mid-latitudes (φ = 45) and with a value of $\lambda^2 = 4 \times 10^{-12}\, m^{-2}$, we find $\beta/\lambda^2 = 4\, m\, sec^{-1}$. This means that c_+ in general will be a positive quantity, $c_+ = 16\, m\, sec^{-1}$ for $u^* = 20\, m\, sec^{-1}$. A value of $r^2 = 2 \times 10^{-12}\, m^{-2}$ was estimated for this quantity in mid-latitudes. With this value we obtain $c_- = -8\, m\, sec^{-1}$. Even if the results obtained for the model with no mean divergence should hold here, we find, due to the smaller speeds, that the tendency to let the temperature field precede the pressure field would be greatly reduced. The main point is, however, that it is possible to show that we, indeed, will obtain a coupling between the pressure and temperature fields in this case. In order to show this, it is necessary to repeat the analysis for the modified model. Due to the more complicated equations the analysis becomes somewhat more laborious and only the major steps will be outlined here.

Introducing the same type of perturbations as before (see 2.3), we arrive this time at the following set of linear, homogeneous equations:

$$(5.10)\quad \left[(1+r^2/k^2)c-\mathcal{U}^*+\beta/k^2\right]\hat{\psi}^* - \mathcal{U}'\hat{\psi}' = 0$$

$$(5.11)\quad -\mathcal{U}'(1-\lambda^2/k^2)\hat{\psi}^* + \left[(1+\lambda^2/k^2)(c-\mathcal{U}^*+\beta/k^2)-\beta\lambda^2/k^4\right]\hat{\psi}'$$

The condition that the determinant has to be zero in order to obtain non-trival solution gives us an equation from which we can determine the phase speed c, which becomes:

$$(5.12)\quad c = \frac{2+r^2/k^2}{2(1+r^2/k^2)}\mathcal{U}^* - \frac{(2+r^2/k^2+\lambda^2/k^2)\beta/k^2}{2(1+r^2/k^2)(1+\lambda^2/k^2)} \pm \frac{D^{\frac{1}{2}}}{2(1+r^2/k^2)(1+\lambda^2/k^2)}$$

where

$$(5.13)\quad D = \left[(1+\lambda^2/k^2)r^2\mathcal{U}^*/k^2 + (\lambda^2/k^2-r^2/k^2)\beta/k^2\right]^2 + 4(1+r^2/k^2)(1-\lambda^2/k^2)\mathcal{U}'^2$$

If we now define

$$(5.14)\quad \begin{cases} X_+ = (1+r^2/k^2)\,c_+ - c_R \\ X_- = (1+r^2/k^2)\,c_- - c_R \end{cases}$$

we obtain the same relations (3.9) to determine the four amplitudes in terms of initial conditions with the interpretation (5.14) of X_+ and X_-.

It is immediately obvious that as $c_R \to -\infty$, while $c_+ \to \mathcal{U}^* - \beta/\lambda^2 > 0$ we have $\chi_+ \to \infty$ for $k \to 0$. Due to the compensating effect between the two terms in c_- in (5.14), it is somewhat more laborious to find $\lim_{k\to 0} c_-$. It is, however, possible to find

$$\lim_{k\to 0} \chi_- = \frac{\mathcal{U}'^2}{\mathcal{U}^* + (\beta/r^2 - \beta/\lambda^2)} \tag{5.15}$$

through the use of l'Hospital's rule twice. The computations leading to (5.15) will not be reproduced here.

With the result (5.15) and the fact that $\lim_{k\to 0} \chi_+ = +\infty$, we can easily derive the following results:

$$\begin{aligned}
&\hat{\psi}^*_+ / A^*_0 \to 0 \\
&\hat{\psi}^*_- / A^*_0 \to 1 \\
&\hat{\psi}'_+ / A'_0 e^{i\alpha'_0} \to 1 - \frac{\mathcal{U}'}{\mathcal{U}^* + (\beta/r^2 - \beta/\lambda^2)} \cdot \frac{A^*_0}{A'_0 e^{i\alpha'_0}} \\
&\hat{\psi}'_- / A'_0 e^{i\alpha'_0} \to \frac{\mathcal{U}'}{\mathcal{U}^* + (\beta/r^2 - \beta/\lambda^2)} \cdot \frac{A^*_0}{A'_0 e^{i\alpha'_0}}
\end{aligned} \tag{5.16}$$

for $k \to 0$.

We find that the behavior of the pressure wave, ψ^*, is unchanged in the limit, i.e., the ψ^*-wave will retrograde with the speed c_-. However, the behavior of the ψ'-wave is different. In order to obtain an idea about the amplitude in the ψ'-wave traveling with the speed c_-, we may estimate:

$$\left| \frac{A^*_0}{A'_0 e^{i\alpha'_0}} \right| \sim 2 \tag{5.17}$$

assuming a linear increase with height of the amplitude. Further we find that

$$(5.18) \qquad \frac{u'}{U^{*} + (\beta/r^2 - \beta/\lambda^2)} \sim 0.4$$

and consequently:

$$(5.19) \qquad \hat{\psi}'_- / | A'_0 e^{i\alpha'_0} | \sim 0.8$$

We find therefore that the major portion of the thermal wave will retrograde in this case, and we have a rather strong coupling between the thermal wave and the pressure wave.

The main conclusion from this section is that we obtain a stronger coupling between the $\overset{*}{\psi}$- and the ψ'-wave if we have a certain pattern of divergence in the $\overset{*}{\psi}$-field, namely such patterns as are described by (5.6). This divergence has to be introduced as a special requirement in a two-parameter model, but will appear as part of the inherent solution in a model, when the deviations from linearity in the vertical wind profile can be considered. The simplest model having this property is a three-parameter model.

6. General Conclusions

From the complete solutions of the two-parameter prognostic equations for certain simple flow-pattern, it has been found that the temperature field and the pressure field in the unstable wave tend to adjust to each other in such a way that the temperature field after a while will be lagging behind the pressure field. The limiting phase difference does not depend on the initial arrangement of the two fields, but the initial state is important for the development within the first 24 to 36 hours.

It is further found that the amplitude in the temperature field and the pressure field will decrease initially if the temperature field precedes the pressure field initially, but due to the adjustment in the phase difference they will start to increase when the phase difference becomes positive. As shown in the first part, this state will appear sooner or later.

The amplification rate within the first 24 to 36 hours is very sensitive to the initial state. Only in the case where the temperature field initially lags behind the pressure field by about 1/4 of a wave-length will we find amplification rates comparable to the one found if only the amplifying wave is considered.

In a two-parameter model with no mean divergence we find that there is practically no coupling between the temperature and pressure fields for very long waves. The pressure field will retrograde with a speed comparable to the Rossby speed for non-divergent waves, while the temperature wave will progress with a small positive speed leading to a situation, where the temperature field will precede the pressure field as it sometimes is found in forecasts using the model. If we introduce a divergence in the pressure field we find that the pressure field will retrograde much slower and further that a stronger coupling now will exist between the pressure and temperature fields.

The results obtained in this study are not necessarily directly applicable to the development of the flow in a two-parameter model or in the atmosphere, based as they are on extremely simple flow patterns. Especially the lack of shear in the zonal winds and the non-existence of non-linear interaction between different wave components are severe restrictions in the solutions considered here. The results should be looked upon as describing the behavior of simple baroclinic waves and could serve as background information for empirical studies of baroclinic development in cases where an analytical solution is impossible due to the great complexity of the equations.

The investigation described in this note is of course closely related to earlier investigations of baroclinic instability, which are too many to be listed in the references. The attention has here been focused on the nature of the complete solution, the evolution of the phase difference and the amplitudes as functions of time in the first 36 hours and their dependence on the initial state.

POSTSCRIPT

After the completion of this paper it was pointed out to the author that the results in sections 2-4 are similar to those obtained by Ogura (Journal of Meteorology, 1957, vol. 14, pp. 60-64). The dicussion of several initial situations showing the rapid adjustment of the temperature field to the pressure field in the unstable wave, the comparison with Thompson's results, the discussion of the dependence of the growth rate on the initial fields, and the discussion of the very long waves are probably still of interest as an extension of Ogura's investigation.

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LEGENDS

Figure 1: The vertical windshear, $u'(m\,sec^{-1})$, as a function of wave length, $L \times 10^{-3}$ (km) in unstable baroclinic waves having varying limiting phase difference. The curve, $(\delta' - \delta^*) = 0$, coincides with the curve for neutral stability.

Figure 2: The phase difference between the temperature field and the pressure field as a function of time in unstable baroclinic waves from various initial conditions.

Figure 3: The meridional velocity, $v^* = kA^*(m\,sec^{-1})$, as a function of time in unstable baroclinic waves. Each curve corresponds to a different initial phase difference.

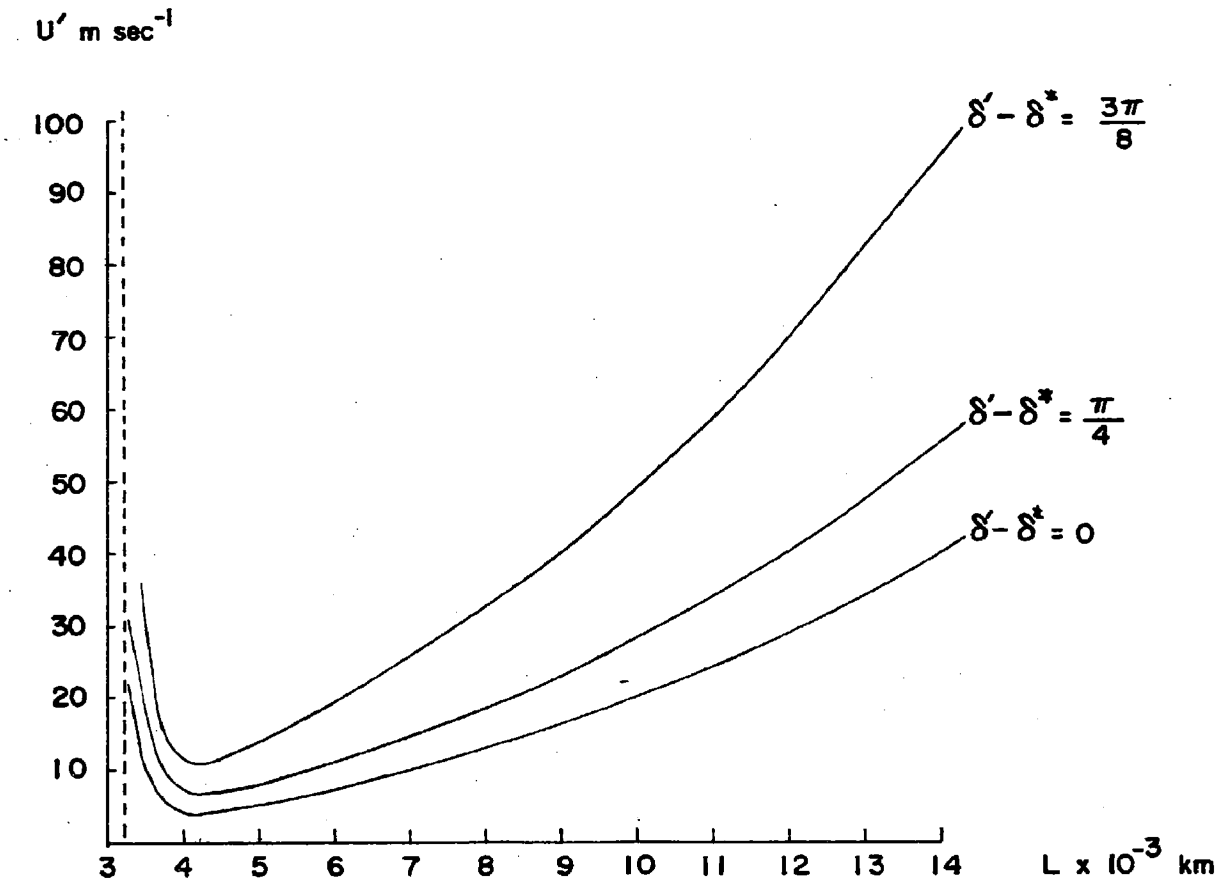
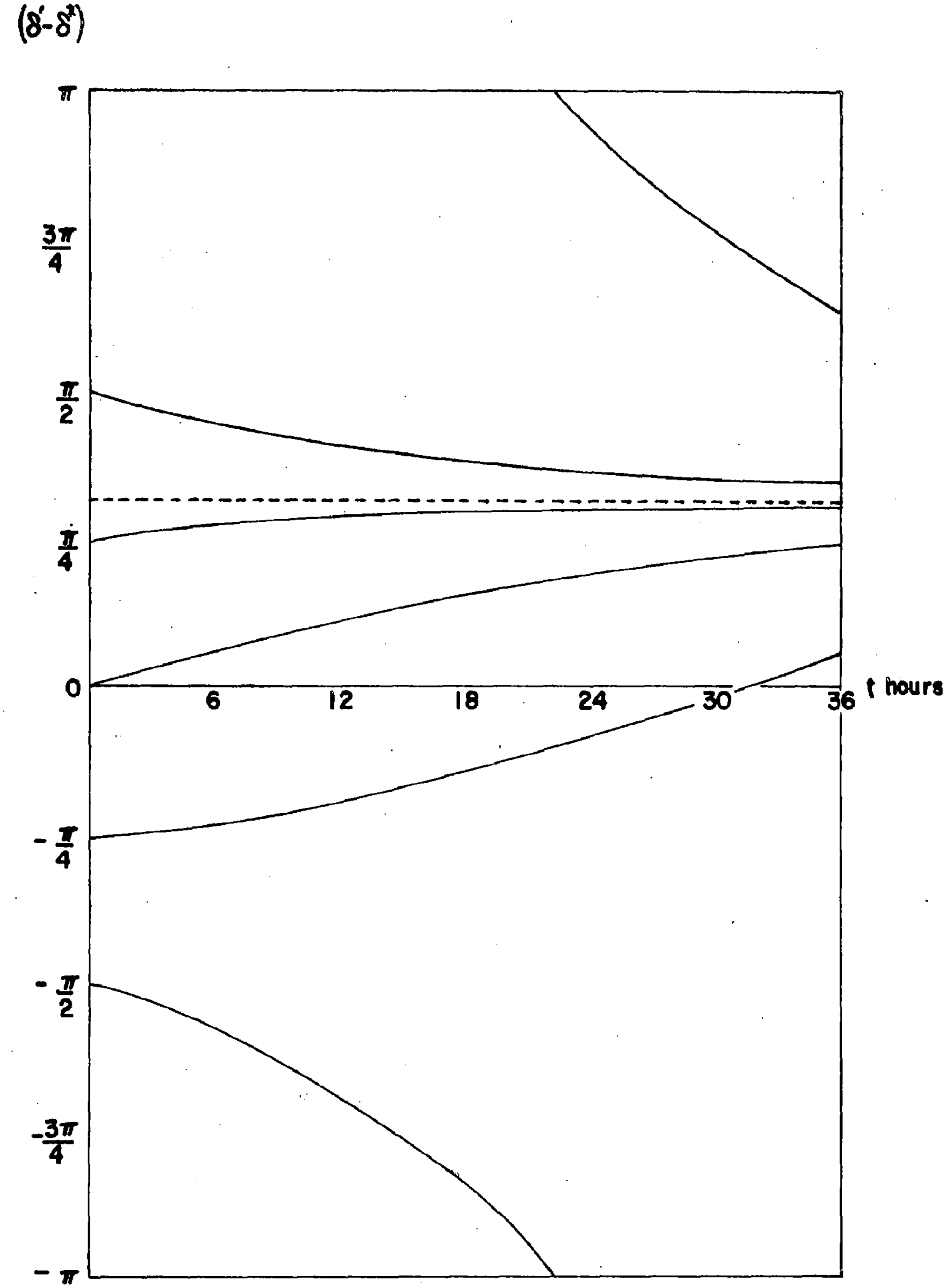
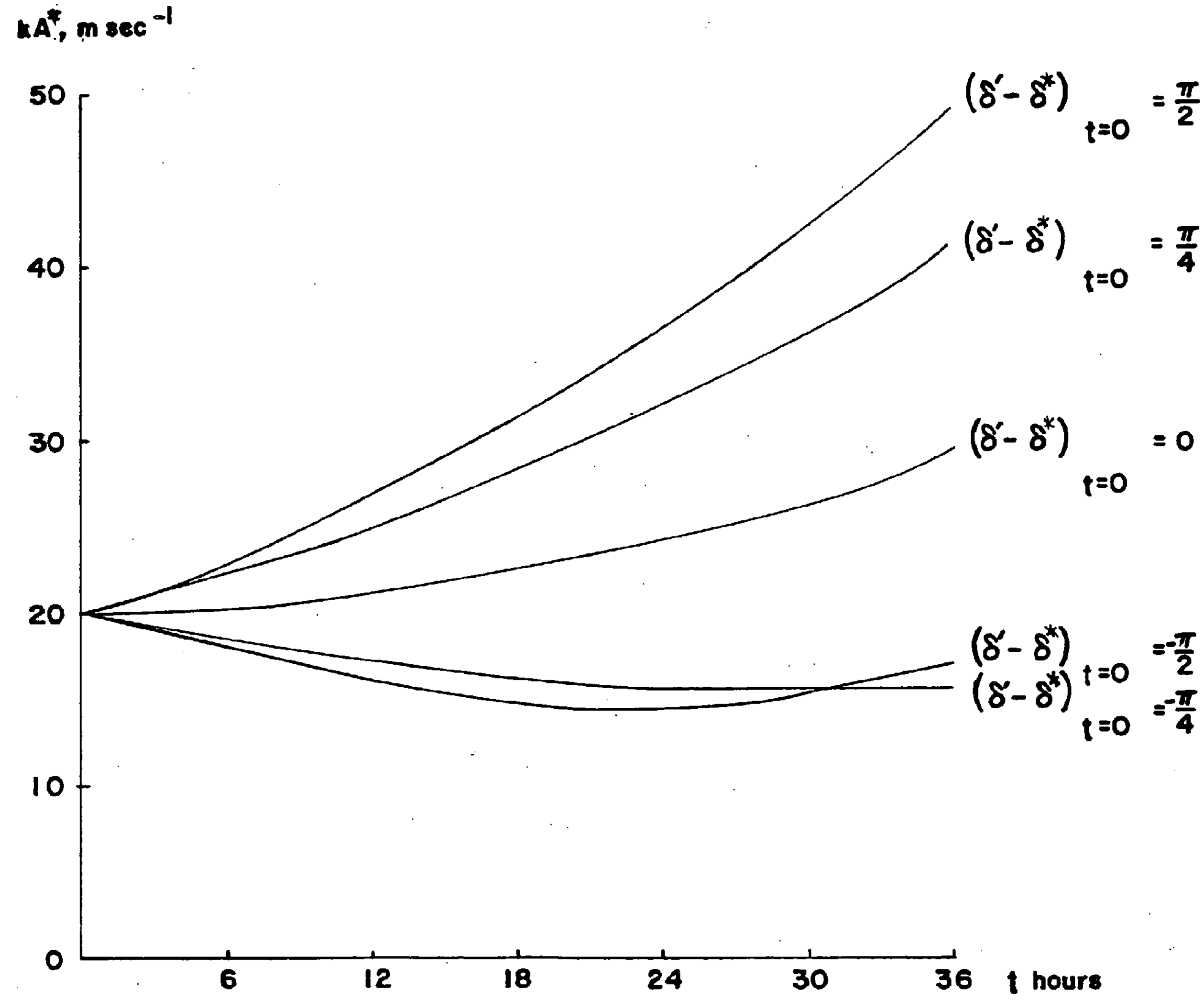

U′ m sec^{-1}
100
90
80
70
60
50
40
30
20
10
3
4
5
6
7
8
9
10
11
12
13
14
L x 10^{-3} km
$\delta' - \delta^* = \frac{3\pi}{8}$
$\delta' - \delta^* = \frac{\pi}{4}$
$\delta' - \delta^* = 0$

Figure 1



$(\delta' - \delta'')$
π
$\frac{3\pi}{4}$
$\frac{\pi}{2}$
$\frac{\pi}{4}$
0
$-\frac{\pi}{4}$
$-\frac{\pi}{2}$
$-\frac{3\pi}{4}$
$-\pi$
6
12
18
24
30
36
t hours

Figure 2



kA^*, m sec^{-1}
50
40
30
20
10
0
6
12
18
24
30
36
t hours
$(\delta' - \delta^*)_{t=0} = \frac{\pi}{2}$
$(\delta' - \delta^*)_{t=0} = \frac{\pi}{4}$
$(\delta' - \delta^*)_{t=0} = 0$
$(\delta' - \delta^*)_{t=0} = -\frac{\pi}{2}$
$(\delta' - \delta^*)_{t=0} = -\frac{\pi}{4}$

Figure 3