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STUDIES OF A MINIMUM ORBITAL UNMANNED SATELLITE OF THE EARTH (MOUSE).

PART II. ORBITS AND LIFETIMES OF MINIMUM SATELLITES

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STUDIES OF A MINIMUM ORBITAL UNMANNED SATELLITE OF THE EARTH (MOUSE).

PART II. ORBITS AND LIFETIMES OF MINIMUM SATELLITES*

By

S. F. Singer

ABSTRACT

The purpose of this paper is two-fold:

(1) To describe the (elliptical) orbit of an artificial satellite in terms of the launching conditions (i.e. at "burnout"); in particular, to give perigee (minimum) and apogee (maximum) altitudes of the ellipse as a function of launching altitudes and of errors in launching velocity and angle. Criteria are then developed for controlling the launching conditions.

(2) To investigate the subsequent behavior of the orbit under the influence of the drag of the upper atmosphere. The "lifetime" of the satellite depends on its area and mass, and is related to the upper atmosphere densities; hence these densities can be determined from measured changes of the orbit by methods discussed in detail. Both circular and elliptic orbits are investigated using appropriate approximation methods. The appendices take up (a) some features of orbit theory and (b) upper atmosphere densities.

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*I apologize for the delay in the publication of this paper. The reason is two-fold: (1) I was planning to present a more sophisticated treatment of orbital theory, including a detailed discussion of astronomical perturbations; I have decided to postpone this to Part III. (2) I was not satisfied with the atmospheric density data obtained from earlier rocket experiments since they may be subject to a large systematic error; recent measurements of solar X-ray absorption have confirmed this belief. However, I have decided not to wait any longer for improved data. Over the past year I have distributed the lifetime graphs widely, and have been asked about their basis and have been urged repeatedly (especially by the ARS office) to set down the methods of calculation and basic assumptions and data. In the meantime I have had a chance to compare my results with those of Krafft Ehricke and Norman Petersen, who presented papers on similar topics. They use similar atmospheric models, and our results agree quite closely.

INTRODUCTION

Before discussing its orbit, it is clearly important to specify first the utility of an artificial satellite, to discuss the applications to which it is to be put, and to solve some of the technical problems of its instrumentation, including power supply and means of orientation[1]. For a satellite like the MOUSE, whose main application is geophysical and astrophysical research, many features of the orbit are of minor importance, e.g. the exact altitude and shape. It is only desirable to keep it above the ionosphere for a sufficiently long period of time so that it can carry out its observations.

But some important questions must be asked. How accurately do we have to place the satellite in its orbit? What is the effect of launching errors on the lifetime? How important is the choice of launching altitude, and the satellite's size and weight? These and other questions must be answered to decide, e.g. on the degree of precision, and therefore on the weight of the guidance system in the rocket stages, and thus allow the designer to achieve an optimum balance of the propulsion and control requirements. For this purpose we will present the results of our calculations in the form of graphs wherever possible.

(1) We will first derive the perigee and apogee altitudes as a function of launching errors.

(2) We will then calculate the lifetime as a function of perigee and apogee altitudes.

SATELLITE ORBIT

At the instant of burn-out of the last propulsion stage the satellite is "launched" in its orbit. In the inverse-square law, central force field of the earth's gravitation the orbit is given by the well-known equation for an ellipse[2]:

$$r = a\,(1 - \epsilon^2)(1 + \epsilon \cos \phi)^{-1} \tag{1}$$

For our purposes it is more convenient to introduce the altitudes of apogee and perigee of the ellipse, h_A and h_P:

$$h_A = R_A - R_E \qquad\qquad h_P = R_P - R_E \tag{2}$$

In Equation (1), both ϵ and a can be expressed in terms of R_A and R_P:

$$\epsilon = \frac{R_A - R_P}{R_A + R_P} \qquad\qquad a = \frac{R_A + R_P}{2} \tag{3}$$

(These relationships can be seen most clearly by inspecting Figure 1). We easily derive

$$R_A = a\,(1 + \epsilon), \qquad\qquad R_P = a(1 - \epsilon), \tag{4}$$

and

$$r = \frac{2\,R_A\,R_P}{(R_A + R_P) + (R_A - R_P)\cos \phi} \tag{5}$$

It is sometimes useful to give the expression for geocentric angle

$$\cos \phi = \frac{2\,R_A\,R_P/r - R_A - R_P}{R_A - R_P}$$

It is useful also to take the Keplerian expression for the period T of an elliptical orbit and express it in terms of R_A and R_P:

$$T = \frac{2\pi\,a^{3/2}}{(G\,M_E)^{1/2}} = 2\,\pi\,(G\,M_E)^{-1/2}\left(\frac{R_A + R_P}{2}\right)^{3/2} \tag{6}$$

T is shown for various values of h_P and h_A in Figure 2.

EFFECT OF LAUNCHING ERRORS

The simplest orbit is the circular one. At any altitude h the gravitational force (weight) is mg_h, and this is balanced by the centrifugal force $\frac{m\,v^2}{r}$, so that

$$\frac{G\,M_E\,m}{r^2} = mg_h = \frac{m\,v_c^2}{r}\;; \tag{7}$$

Now g_h varies with altitude: $g_h = g_o\,(\frac{R_E}{R_E + h})^2$ (see Figure 3) according

to the inverse square law:

Therefore, the circular velocity is given by $v_C = \left[g_o \frac{R_E^2}{R_E + h}\right]^{1/2}$ (8)

(See Figure 4)

Now in general it is quite impossible to achieve a mathematically perfect circular orbit; we always have a launching velocity v_L which is slightly different from v_C by an amount Δv; $v_L = v_C + \Delta V$; and we have an angular error (in the plane of the orbit) $\Delta\theta$. (See Figure 1).

PROBLEM

The problem we want to solve is the following:

Given the launching altitude h_L, and the launching errors Δv and $\Delta\theta$, we want to find the perigee and apogee altitudes of the resulting orbit.

SOLUTION

We simply use only the principles of (a) conservation of angular momentum, and (b) conservation of energy (see Appendix I).

(a) $(v_L \cos \Delta\theta)\, R_L = v_A\, R_A = v_P\, R_P$ (9)

(b) $G\, M_E \left(\frac{1}{R_L} - \frac{1}{R_P}\right) = \frac{1}{2}\left(v_L^2 - v_P^2\right)$,

$$\therefore\ v_P^2 = 2\, G\, M_E \left(\frac{1}{R_P} - \frac{1}{R_L}\right) + v_L^2 \qquad (10)$$

It is most convenient to solve (9) for $\cos \Delta\theta$ and use (10):

$$\cos \Delta\theta = \frac{R_P}{R_L\, v_L}\, v_P = \frac{R_P}{R_L\, v_L}\left[v_L^2 + 2\, G\, M_E \left(\frac{1}{R_P} - \frac{1}{R_L}\right)\right]^{1/2}$$

$$= \frac{R_P}{R_L}\left[1 + 2\,\frac{G\, M_E}{v_L^2}\left(\frac{1}{R_P} - \frac{1}{R_L}\right)\right]^{1/2} \qquad (11)$$

We now define $y_L \equiv \frac{v_L}{v_C}$; because of (7): $v_C^2 = \frac{G\, M_E}{r}$, $\therefore y_L^2 = \frac{v_L^2\, R_L}{G\, M_E}$, and substituting we arrive at: $\cos \Delta\theta = \frac{R_P}{R_L}\left[1 + 2\,\frac{1}{y_L^2}\left(\frac{R_L - R_P}{R_P}\right)\right]^{1/2}$ (12)

For a nearly circular orbit, $(R_L - R_P)$ is small, and we can write

$$\cos \Delta \theta \sim \frac{R_P}{R_L}\left[1 + \frac{1}{y_L^2}\left(\frac{R_L - R_P}{R_P}\right)\right] \tag{13}$$

Note that a circular orbit, i.e. $R_L = R_P$, implies $\Delta\theta = 0$.

PRESENTATION OF RESULTS

After a number of not altogether successful attempts I arrived at the presentation shown in Figures 5 and 6. The problem was to make the results of the preceeding section useful to a design engineer who wants to see at one glance what effect combinations of launching errors have on the perigee and apogee altitude. I decided to use the launching altitude R_L as a fixed parameter*; Figure 5 is for h_L=300 km (187 mi) and Figure 6 is for h_L=600 km (375 mi). The launching (burn-out) velocity is expressed as y_L, i.e. in terms of the circular velocity corresponding to R_L.

EXAMPLES

(a) If one fixes h_L at 300 km, and takes the launching velocity as 1% higher than circular, and $\Delta\theta = 1^\circ$ (either up or down), then from Figure 5: $y_L = 1.01$ intersects the two $\Delta\theta = 1^\circ$ lines at abscissa values of 260 km and 625 km. These are the resulting perigee and apogee altitudes.

(b) If one wishes to keep h_P above 200 km, then $\Delta\theta$ can be $\sim 1^\circ$. If $\Delta\theta = 2^\circ$, then we must raise y_L to 1.02, but this results in an apogee altitude of 960 km which may be undesirable.

CONCLUSION

The advantages of a higher launch altitude are clearly seen in Figure 6; much larger errors are then allowed. This satellite orbit, however, requires more propulsion, the ascent control problem may be much more difficult, and because of the increased altitude the tracking and radiotelemetering problems for the satellite become more difficult. The results presented here are a

*J. Jensen at the ARS Meeting, Chicago, November 1955 has used a presentation applicable to a range of R_L, if $R_L \sim R_E$.

convenient aid in arriving at a compromise solution.

It is apparent that it is not necessary or even desirable to try to attain a circular orbit velocity by fuel cutoff; complete burn-out allows larger angular errors in the velocity vector.

LIFETIME IN CIRCULAR ORBIT

Introduction

Various authors[3] have considered the effects of air drag at high altitudes in great detail, but no discussion of the lifetime of a satellite vehicle seems to exist in the literature. This may be because designers generally considered quite elaborate satellites or space stations, and placed them high enough to be above any atmospheric effects. When we are faced, however, with the limited means of rocket propulsion presently at our disposal, we must reorient our thinking towards a minimum, unmanned satellite. If properly instrumented this object can be extremely useful for various geophysical and astrophysical investigations.[1] It would be valuable even if the observations extended over a few hours or days. We will define here "lifetime" as due to atmospheric effects only. In reality, and particularly for longer periods, the effective or useful life may be limited by meteor hits, cosmic ray bombardment and deterioration of components of the satellite instrumentation.

The drag of the atmosphere will change the characteristics of the orbit gradually. We will be interested in deriving the variation of perigee and apogee altitude as a function of time. In this calculation we will neglect all astronomical perturbations and consider the earth a perfect sphere. Our justification is the following: For short lifetimes one can generally neglect astronomical perturbations anyway. For very long lifetimes we will consider them (Part III of this series) but neglect the drag. For intermediate cases it would be best to work out particular examples on high-speed

computers. But by treating separately drag and astronomical perturbations we preserve a physical insight into the resulting orbit changes, and are able to give better qualitative arguments if, e.g. our picture of upper atmosphere densities changes.

Drag Loss

We will consider a (nearly) spherical satellite of (average) projected area A. A squat cylinder would do equally well. We will consider the drag coefficient C_D appropriate to the molecular flow region as ~ 2. Any errors in A or C_D are completely absorbed by the uncertainty in the atmospheric density ρ. $\rho(h)$ is plotted vs h in Figure 7; its derivation is discussed critically in Appendix II.

The drag force $F_D = \frac{1}{2} C_D A v^2 \rho(h)$ (14)

To appreciate the effect of this drag force on the satellite's orbit, I would like to point out first a fallacy by means of a numerical example: For the case of the MOUSE at e.g. 210 km:

$A = 0.1\ m^2$, $V = 7{,}780$ m/sec, $\rho = 10^{-10}$ kg/m^3, $m = 20$ kg

$\therefore F_D = 6 \times 10^{-4}$ newtons

The drag deceleration $= F_D/m = 3 \times 10^{-5}$ m/sec^2

Multiplying the deceleration by the orbital period at 210 km (5310 sec) one would gain the impression that the satellite loses 1.6×10^{-1} m/sec in velocity. But this is not correct; in fact the satellite speeds up. What happens is that it loses altitude; this recovery of potential energy overcomes the drag loss and even increases the kinetic energy of the satellite, so that its velocity becomes almost equal (but not quite) to the circular velocity corresponding to the lower altitude. Figure 4 shows how v_C increases with lower altitude.

Perturbation Calculation

In order to avoid errors we use straightaway a method of energy balance

to describe the shrinking orbit.

(a) We first express the total energy (potential with respect to sealevel + kinetic) of the satellite per unit mass of satellite: the total "specific" energy E_{TC} (in a circular orbit), measured in joules/kg:

Potential Energy V: $-m\left(\frac{G\,M_E}{R_E + h} - \frac{G\,M_E}{R_E}\right) = \frac{m\,G\,M_E}{R_E}\left(\frac{h}{R_E + h}\right)$

Kinetic Energy T: $\frac{1}{2}\,m\,v_C^{\,2} = \frac{1}{2}\,m\,\frac{G\,M_E}{R_E + h}$ from (7)

$$\therefore \quad E_{TC} = \frac{G\,M_E}{R_E}\left(\frac{h}{R_E + h} + \frac{1}{2}\,\frac{R_E}{R_E + h}\right) \tag{15}$$

We can also express (15) as

$$E_T = \frac{1}{2}\;g_o\,R_E\,\left(1 + \frac{h}{R_E + h}\right) \quad \text{(see Fig. 8)} \tag{16}$$

(b) Next we express the energy loss per orbit, and using (8)

$$(-\Delta E) = F_D \cdot 2\,\pi\,r = \left[\frac{1}{2}\,C_D\,A\,v_C^{\,2}\,\rho(h)\right] \cdot 2\,\pi\,(R_E + h)$$

$$= \pi\,C_D\,A\,(g_o\,R_E^{\,2})\;\rho(h) \tag{17}$$

We see therefore that the energy loss per unit projected area depends on h through $\rho(h)$ only; namely $(-\Delta E/A) = 2.5 \times 10^{15}\;\rho$ joule/m^2 (18)

Note that here is the only place where C_D is used, and that it appears in combination with ρ. Any slight error in C_D is thus absorbed by the much larger uncertainty in ρ.

(c) The determination of the change in orbit, and of lifetime is now a very simple process. We use Figure 7, or an analytical expression for $\rho(h)$, evaluate $(-\Delta E)$ from (17) and divide by m to reduce it to the specific energy loss per orbit revolution. We then apply this loss $(-\Delta E_T)$ on the plot of Figure 8 to determine the new altitude after one orbit. (In practice it may be more convenient to take a larger number of orbit revolutions, and use average values of ρ).

(d) Note that in our perturbation method we assume that both the energy

loss, and therefore altitude loss, are so small that we can use a mean value for h and completely neglect this altitude loss <u>during</u> one orbit revolution. Our method breaks down in the very dense atmosphere near sealevel where the energy loss is large, and the altitude decreases rapidly. There we can no longer average over one orbit, but we must apply a step-by-step numerical calculation.

(e) We can express our procedure (c) analytically: By differentiating (16) we have:

$$E_T = \frac{1}{2} g_o \left(\frac{R_E}{R_E + h}\right)^2 \Delta h;$$

$$\therefore \Delta h/\text{per orbit} = \Delta E_T \left[\frac{1}{2} g_o \left(\frac{R_E}{R_E + h}\right)^2\right]^{-1}$$

Expressing $\Delta E_T = \Delta E/m$, and using (17), we end up with

$$\Delta h/\text{orbit} = 2\pi C_D \cdot \left(\frac{A}{m}\right) \cdot (R_E + h)^2 \cdot \rho(h) \tag{19}$$

In order to study h as a function of time, we remember the dependence of the orbital period on altitude given by (6); then

$$\begin{aligned} \frac{\Delta h}{\Delta t} &= C_D (G M_E)^{\frac{1}{2}} \cdot \left(\frac{A}{m}\right) \cdot (R_E + h)^{\frac{1}{2}} \rho(h) \\ &= 4 \times 10^7 \left(\frac{A}{m}\right) \cdot (R_E + h)^{\frac{1}{2}} \rho(h) \end{aligned} \tag{20}$$

To obtain the time required to descend from altitude h_1 to h_2, we integrate

$$\Delta t_{12} = 2.5 \times 10^{-8} \left(\frac{m}{A}\right) \int_{h_2}^{h_1} \frac{dh}{(R_E + h)^{\frac{1}{2}} \rho(h)} \tag{21}$$

<u>Lifetime</u>

We define "lifetime" t_L as the time required to descend from an initial altitude to sealevel. Figure 9 shows plots of t_L for different satellites*; (a) a densely packed one corresponding to a MOUSE or similar instrumented satellite, (b) a large, light-weight "balloon" satellite designed for visibility rather than for carrying instrumentation.

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*Here we have used upper air densities which are rather similar to Model (4) Appendix II. Figure 16 gives universal lifetime curves for various density models and any m/A.

MEASUREMENT OF AIR DENSITY

Equation (19) or (20) and Figure 9 show clearly how observations of changes of the orbit can be used to deduce the density of the upper atmosphere. We see also how the sensitivity of the determination can be improved by suitably designing A/m.

In actual practice it may be easier to determine the change in orbital period, and deduce Δh from Equation (6), rather than to measure Δh directly.

It would appear that the method described here is the only one known at present for measuring densities in the very high atmosphere.

LIFETIME IN ELLIPTIC ORBIT

Introduction

As pointed out earlier, the satellite orbit will in general be an ellipse, whose eccentricity (or "elongation") depends on the magnitude of the launching errors. Clearly then, since the altitude now varies considerably over the orbit, we can no longer use the perturbation method applied above to circular orbits. Instead we now introduce a new method based on an impulse approximation. We assume (and this holds quite well for very elongated ellipses) (a) that all of the drag loss can be concentrated at the perigee point, and (b) that the perigee remains fixed. We imagine therefore all of the drag is compressed into an "impulse" which lasts for such a short time that it changes only the velocity, but not the position of the satellite.

Reduction of v_P, the velocity at perigee, without change in h_P leads to a decrease in the apogee altitude. This is shown graphically in Figure 10, and can be calculated most conveniently from Equations (9) to (13), remembering that $\Delta\theta = 0$ at the perigee and apogee. The results of this calculation are presented in Figure 11.

Impulse Approximation Calculation

We proceed as follows:

1. Find energy loss due to atmospheric drag per orbit.

2. Calculate the new energy, apply impulse approximation discussed above, and find new value for h_A.

3. Repeat this procedure until $h_A \sim h_P$. The impulse approximation then breaks down and we return to the circular case.

(a) We first construct the satellite orbit trajectory in the vicinity of the perigee, in the manner shown in Figure 12. (For the sake of definiteness I have chosen h_P values of 100, 150 and 200 km.) These "dip curves" show how quickly the satellite dips into and leaves the dense atmosphere around the perigee. (We recall that the atmospheric falls off exponentially with increasing altitude).

(b) We now calculate the energy loss for all the dip curves by integrating (numerically) the drag force (14) over the trajectory. The results are shown in Figure 13. Note that ΔE does not depend too strongly on h_A (for large values of h_A), but breaks away sharply when h_A becomes $\sim 2\ h_P$. This consideration establishes roughly the region of application of our approximation method.

(c) As in the circular orbit case, we multiply by $(\frac{A}{m})$, and apply the resultant ΔE_T to Figure 14, which shows the total specific energy as a function of h_A and h_P. Here we remember to keep h_P fixed, and slide down, e.g. the h_P = 100 km line to find our new value of h_A. $E_{T(ell)}$ is easily calculated using the same method which led to Equation (15), but using $\frac{1}{2}\ m\ v_P^2$ rather than $\frac{1}{2}\ m\ v_C^2$ for the kinetic energy.

Defining $$y_P^2 = \frac{v_P^2}{v_C^2} = \frac{2\ R_A}{R_A + R_P} = \frac{2\ (R_E + h_A)}{2\ R_E + h_A + h_P}\ , \tag{22}$$

$$E_{T(ell)} = \frac{G\ M_E}{R_E}\left[\frac{h_P}{R_E + h_P} + \frac{1}{2}\ \frac{R_E}{R_E + h_P}\ y_P^2\right] = g_o\ \frac{R_E}{R_E + h_P}\left[h_P + \frac{1}{2}\ R_E\ y_P^2\right] \tag{23}$$

(d) We have calculated (and plotted in Figure 15) the variation of h_A with number of orbits (and with time) for the three values of h_P: 100, 150 and

200 km, and for a particular $\frac{A}{m}$. In Figure 15 we note again how sensitive the lifetime is to the perigee altitude; t_L increases by over a factor 1000 in going from h_P = 100 km to 200 km.

Analytical Method

From the general energy relation in an ellipse (Equation (33) of Appendix I)

$$\frac{1}{2} v^2 = G M_E \left(\frac{1}{r} - \frac{1}{2a}\right)$$

We derive by differentiation, and since $T/m = \frac{1}{2} v^2$

$$\Delta a = \frac{2 a^2}{G M_E} \frac{\Delta T}{m} \tag{24}$$

Assuming no change in h_P

$$\Delta h_A/\text{orbit} = 2(\Delta a) = \frac{(R_A + R_P)^2}{2 G M_E m} C_D A v_P^2 \cdot \int \rho \, ds$$

$$\sim \frac{1}{4 \times 10^{14}} \left[(R_A + R_P) v_P\right]^2 \cdot \left(\frac{A}{m}\right) \cdot \int \rho \, ds \tag{25}$$

It is apparent that the slope is proportional again to A/m

Again we can use Equation (6) to derive

$$\frac{\Delta h_A}{\Delta t} = \frac{C_D}{\pi} (G M_E)^{\frac{1}{2}} \frac{R_A}{\left(\frac{R_A + R_P}{2}\right)^{\frac{1}{2}} R_P} \cdot \left(\frac{A}{m}\right) \cdot \int \rho \, ds \tag{26}$$

$$\sim 5 \times 10^3 \cdot \frac{A}{m} \cdot \int \rho \, ds$$

(Here we assume R_A and R_P not much greater than R_E).

Discussion

Figure 15 shows the apogee altitude decreasing rather uniformly until a near-circular orbit is attained. This feature is, of course, a result of our approximation. What actually happens is that the perigee altitude decreases also, although much less rapidly than h_A. As pointed out earlier the effect becomes important when $h_A \sim 2 h_P$. But even small decreases of h_P affect the lifetime considerably (see Figure 15). We conclude, therefore, that our method holds well when $h_A \gg h_P$ since then the greatest portion of

the satellite lifetime is spent in h_A decreasing down to 2 h_P, and the subsequent shortening of the remaining life can be sometimes neglected, e.g. in Figure 15, for h_P = 200 km, it takes 280 days for h_A to decrease from 1000 km to 400 km, with the remaining life only about 50 days, but probably somewhat less according to the reasoning given above. We must, therefore, regard our approximate methods as convenient but rough guides when appropriately used, but we must resort to detailed numerical orbit integrations when real accuracy is mandatory.

APPENDIX I

BASIC RELATIONS ABOUT ORBITS

We start with the equations of motion in polar coordinates

$$m\,(\ddot{r} - r\,\dot{\phi}^2) = f\,(r) \tag{27}$$

$$m\,(r\ddot{\phi} + 2\,r\dot{\phi}) = f\,(\phi) \tag{28}$$

We will neglect the atmospheric drag but introduce it later as a perturbation. We, therefore, set $f\,(\phi) = 0$, and arrive at a pure central force law; this is the Newtonian gravitational law:

$$f\,(r) = G\,M_E\,m\,r^{-2} \tag{29}$$

Equations (27) and (28) are of second order. However, (28) can be written as $\frac{m}{r}\frac{d}{dt}\,(r^2\dot{\phi}) = 0$ and integrated to $m\,r^2\,\dot{\phi} = J =$ Constant, i.e. the angular momentum is constant. (30)

Solution of the equations of motion leads to the familiar Keplerian ellipse provided the initial conditions are such as to give a bound orbit. These conditions are satisfied if the kinetic energy T of the satellite is less than its potential energy.

The energy integral can be set up as

$$T + V = W = \text{ constant}$$

Where $T = \frac{1}{2}\,m\,v^2$

$$V = -\int_{\infty}^{r} f\,(r)\,dr = -\frac{G\,M_E\,m}{r}$$

In order for the orbit to be bound, i.e. elliptic or circular:

$|T|$ must be less than $|V|$

This can be seen from the expression for the eccentricity ϵ of the conic section (which is the orbit)

$$\epsilon = \left(\frac{2\,W\,J^2}{G\,M_E\,m} + 1\right)^{\frac{1}{2}} \tag{31}$$

$$\text{or } W = \frac{G\,M_E\,m}{2\,J^2}\,(\epsilon^2 - 1)$$

For a circular orbit: $\epsilon = 0$, $\therefore W_C = -\frac{G\,M_E\,m}{2\,J^2}$

For an elliptic orbit: $0 < \epsilon < 1$, $\therefore\ W_C < W_{ell} < 0$

We want to state in detail some important properties of orbits. A body having zero velocity at infinity and falling towards the earth acquires kinetic energy at the expense of potential energy

$$\frac{1}{2} m v^2 = G M_E m \left(\frac{1}{r} - \frac{1}{\infty}\right) \tag{32}$$

where $v = \left(\frac{2 G M_E}{r}\right)^{\frac{1}{2}}$ is now the "escape velocity" since the body follows a parabolic orbit: $W = 0$, as can be seen from (32).

For an elliptic orbit the kinetic energy is less, and corresponds to a fall from an initial circle of radius 2a,[2] where a is the semi-major axis, so that

$$\frac{1}{2} m v^2 = G M_E m \left(\frac{1}{r} - \frac{1}{2a}\right) \tag{33}$$

Since the body's potential energy is $\frac{G M_E m}{r}$, its total energy

$$W_{ell} = - \frac{G M_E}{2a} m \tag{34}$$

and is seen to depend only on a. It has been shown that the orbital period depends also only on a. (see Equation (6)). Conversely, when the speed at any point r of the orbit is given, W is determined, and therefore, a. The launching conditions R_L and v_L therefore determine $R_A + R_P$, since

$$\frac{1}{2} v_L^2 = G M_E \left(\frac{1}{R_L} - \frac{1}{R_A + R_P}\right)$$

$$\therefore R_A + R_P = \left(\frac{1}{R_L} - \frac{v_L^2}{2 G M_E}\right)^{-1} = R_L \left(1 - \frac{y_L^2}{2}\right)^{-1}, \tag{35}$$

using $y_L^2 \equiv \frac{v_L^2}{v_C^2} = \frac{v_L^2 R_L}{G M_E}$

Note that the $\Delta\theta$ does not enter; only v_L^2. It is immediately obvious that the eccentricity does depend on $\Delta\theta$, being a minimum for $\Delta\theta = 0^\circ$ and a maximum (i.e. $\epsilon = 1$) for $\Delta\theta = 90^\circ$. Using (31) and by suitable substitutions we derive ϵ in terms of the launching conditions y_L & $\Delta\theta$:

$$\epsilon = \left[1 - 2 y_L^2 \left(1 - \frac{y_L^2}{2}\right) \cos^2 \Delta\theta\right]^{\frac{1}{2}} \tag{36}$$

$$\text{Since } \epsilon = \frac{R_A - R_P}{R_A + R_P}, \tag{37}$$

we have now derived our ellipse in terms of the familiar R_A and R_P as functions of y_L, $\cos \Delta\theta$, and R_L.

We can finally express the orbit equation

$$r = a\,(1 - \epsilon^2)\,(1 + \epsilon \cos \phi)^{-1}$$

as $$(R_E + h) = R_L\, y_L^2 \cos^2 \Delta\theta\,(1 + \epsilon \cos \phi)^{-1} \qquad (38)$$

APPENDIX II

UPPER AIR DENSITIES

Values for upper air densities up to 220 km have been given based on rocket measurements.* Beyond this altitude all density values are deduced indirectly or arrived at by calculating particular atmospheric models. To get good density values one needs to know the atmospheric constitution and the temperature. It is generally agreed now that O_2 dissociates around the 100 km level and that at 150 km it is present only in its atomic form. Somewhere near this altitude diffusive equilibrium sets in between O and N_2 so that one can neglect the contribution of N_2 to atmospheric density beyond 400 km.

The temperature in the upper layers is arrived at from a great variety of considerations: optical observation of night glow and aurorae, radio measurements of scale heights, and computations of the escape of helium from the exosphere. It is thought that the temperature at the base of the exosphere is 500°K, or even 1000°K, and may rise to even higher values during solar flares. Above 300 - 400 km the temperature probably does not increase much further and may gradually approach to that of interplanetary space.

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*There is some uncertainty about the rocket data beyond even 100 km. Recent measurements by Friedman (Phys. Rev. 1956; see also Ann. de Geophys. II, 174, 1955) based on the absorption of solar X-rays give quite reliable density values in the 110-130 km region which are only 1/3 of the Rocket Panel densities.

There is evidently a great deal of uncertainty in these extrapolations, and it may be hoped that observation of the satellite's orbit will settle not only values of density but, at the same time, many other interrelated problems about the structure of the upper atmosphere. No final conclusion is possible at the present time; only a compilation of values is given, together with references.

Using Equation (21) we have plotted universal lifetime curves for density models (1 and 2), (4), (1 and 5). (Figure 16). t_L is shown for $\frac{m}{A} = 1\ kg/m^2$; to use these curves for any other satellite mass and area, one should simply multiply the value of t_L in Figure 16 by the new $\frac{m}{A}$.

LIST OF SYMBOLS

r = distance of satellite from center of the earth

a = semimajor axis of orbit ellipse

ϕ = polar angle of orbit ellipse (measured from perigee) (geocentric angle)

ϵ = eccentricity of orbit ellipse

R_A = apogee distance

R_P = perigee distance

R_L = distance of launching point from center of the earth

R_E = (mean earth's radius) 6.37×10^6 meter

h = altitude above sealevel

h_A = apogee altitude

h_P = perigee altitude

h_L = altitude of launching point from center of the earth

G = (Newton's gravitational constant) 6.67×10^{-11} nt $-m^2$ $-kg^{-2}$

M_E = (mass of the earth) 5.98×10^{24} kg

m = mass of the satellite

T = kinetic energy of satellite

V = potential energy of satellite

E_T = total specific energy, E_{TC} for circular orbit, $E_{T(ell)}$ for elliptic orbit, in joules/kg

g_o = gravitational acceleration at sealevel 9.8 m/sec^2

g_h = gravitational acceleration at altitude h

v_L = launching velocity

v_A = velocity at apogee

v_P = velocity at perigee

Δv = magnitude error in the launching velocity

$\Delta \theta$ = angular error (in the vertical plane) in the launching velocity

A = projected area of satellite

C_D = drag coefficient

ρ = atmospheric density kg/m^3

F_D = drag force

$\gamma_L = \frac{v_L}{v_C}$

$\gamma_P = \frac{v_P}{v_C}$

Δt_{12} = time to descend from altitude h_1 to h_2

t_L = lifetime, time to descend to sealevel

J = orbital angular momentum of satellite

SOME CONVERSION FACTORS

Density	$1\ Kg/m^3 = 1.94 \times 10^{-3}\ slug/ft^3$
Force	1 newton = 0.225 lb
Length	1 Km = .6215 miles = 3280 ft
Area	$1\ m^2 = 10.76\ ft^2 = 1550\ in^2$
Energy	1 joule = 0.738 ft -lb

COMPILATION OF UPPER ATMOSPHERE DENSITIES

	Altitude (Km)						
Author	100	150	200	250	300	400	500
(1)	8×10^{-7}	3.5×10^{-9}	1.7×10^{-10}				
(2)			1.7×10^{-10}	2.6×10^{-11}	8×10^{-12}	1×10^{-12}	1.7×10^{-13}
(3)			1.7×10^{-10}	2.6×10^{-11}	7.5×10^{-12}	1.3×10^{-12}	4.6×10^{-13}
(4)		9.6×10^{-9}	6.4×10^{-10}	8×10^{-11}	2×10^{-11}	2.8×10^{-12}	6.7×10^{-13}
(5)			1.5×10^{-10}		7.6×10^{-12}	1.2×10^{-12}	4.2×10^{-13}
(6)	7×10^{-7}	10^{-9}	9×10^{-11}	2.6×10^{-11}	8×10^{-12}	1.3×10^{-12}	4.6×10^{-13}

All Values in Kg/m^3

(1) Rocket Panel, Phys. Rev. 88, 1027, 1952.

(2) Bates, (isothermal above 250 km), in "Rocket Exploration of the Upper Atmosphere", p. 350, London, 1954.

(3) Bates, (isothermal above 400 km), in "Rocket Exploration of the Upper Atmosphere," p. 350, London, 1954.

(4) Mitra, "The Upper Atmosphere", p. 582, Calcutta, 1952.

(5) Nicolet, in "The Earth as a Planet", p. 654, Chicago, 1954.

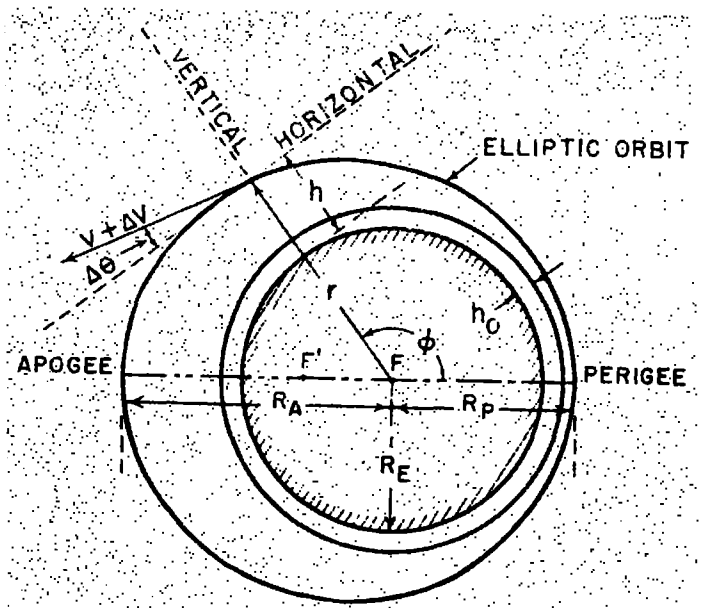
(6) Present analysis: Based on and extrapolated from Friedman's data, isothermal above 400 km.

REFERENCES

1. Singer, S. F. "Studies of a Minimum Orbital Satellite, Part I. Geophysical and Astrophysical Applications." Astronautica Acts, Vol. I, No. 4, 1955.

2. Any textbook on mechanics or theoretical physics, e.g. "Introduction to Celestial Mechanics", by F. R. Moulton, New York 1914.

3. Saenger, E. and Bredt, I., "A Rocket Drive for Long Range Bombers", German Aviation Research UM #3538, August 1944.
Ehricke, K. A., "On the Descent of Winged Orbital Vehicles," Astronautic Acta, Vol. I, No. 3, 1955.



R_E = MEAN RADIUS OF EARTH
R_P = PERIGEE DISTANCE
R_A = APOGEE DISTANCE
h = ALTITUDE
h_0 = CRITICAL DRAG ALTITUDE
F, F' = FOCI OF ELLIPTIC ORBIT

EQUATION OF ORBIT:

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\phi}$$

$$\epsilon = \left(\frac{R_A - R_P}{R_A + R_P}\right)$$

$$a = \frac{R_A + R_P}{2}$$

Fig.1

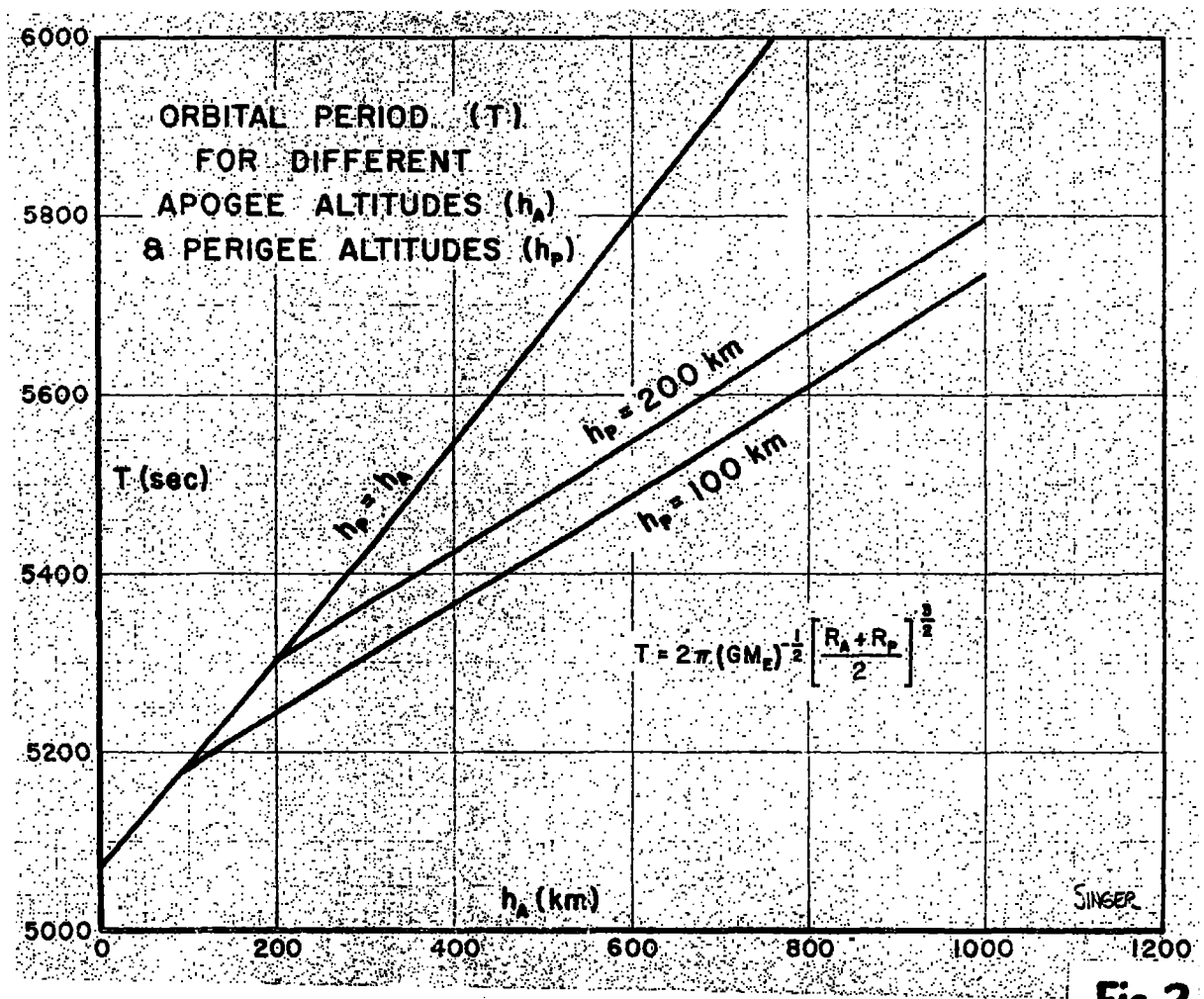


Fig.2

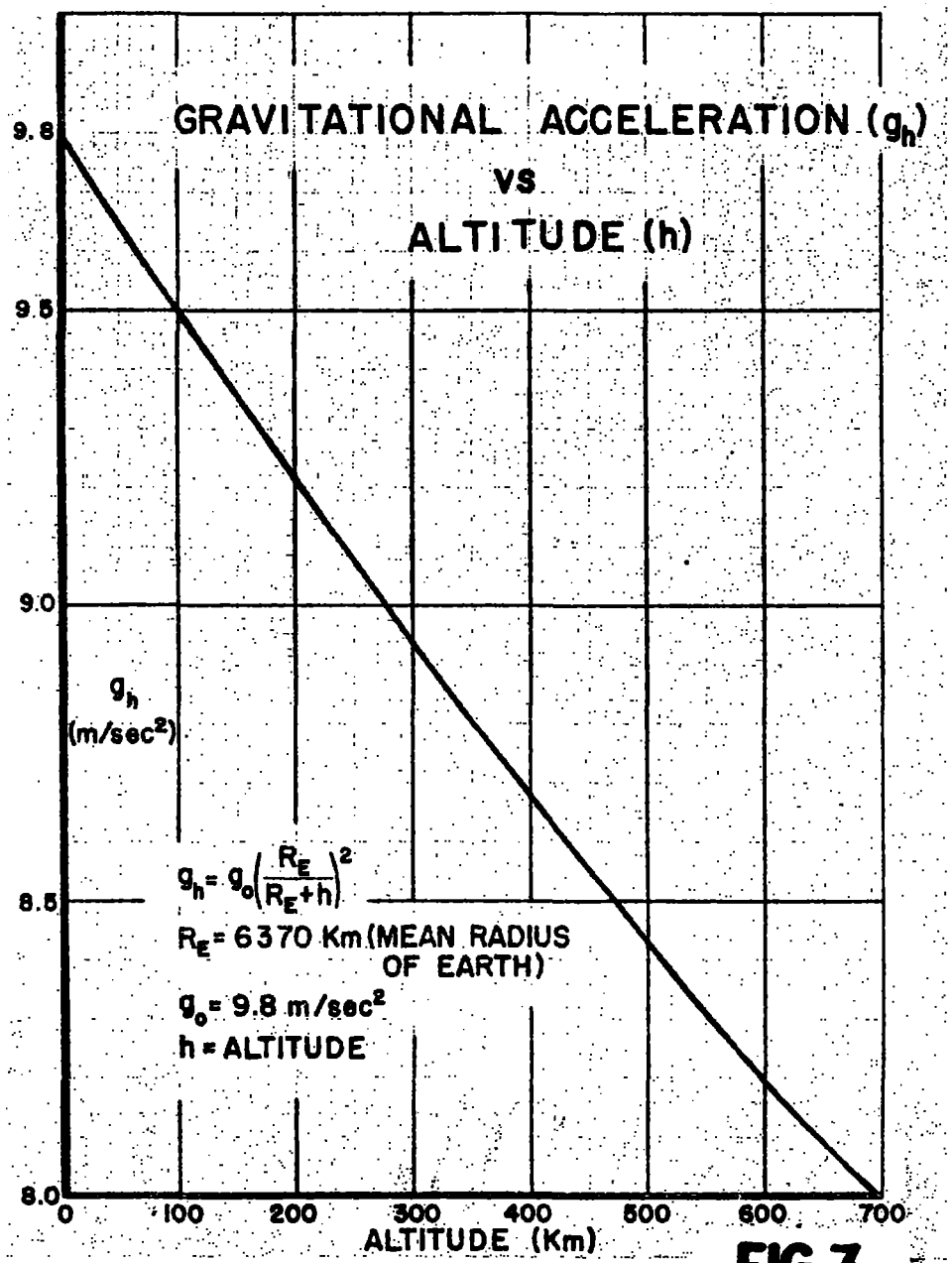

GRAVITATIONAL ACCELERATION (g_h)
VS
ALTITUDE (h)
9.8
9.5
9.0
8.5
8.0
g_h
(m/sec²)
$g_h = g_o\left(\frac{R_E}{R_E+h}\right)^2$
R_E = 6370 Km (MEAN RADIUS OF EARTH)
g_o = 9.8 m/sec²
h = ALTITUDE
0
100
200
300
400
500
600
700
ALTITUDE (Km)

FIG.3

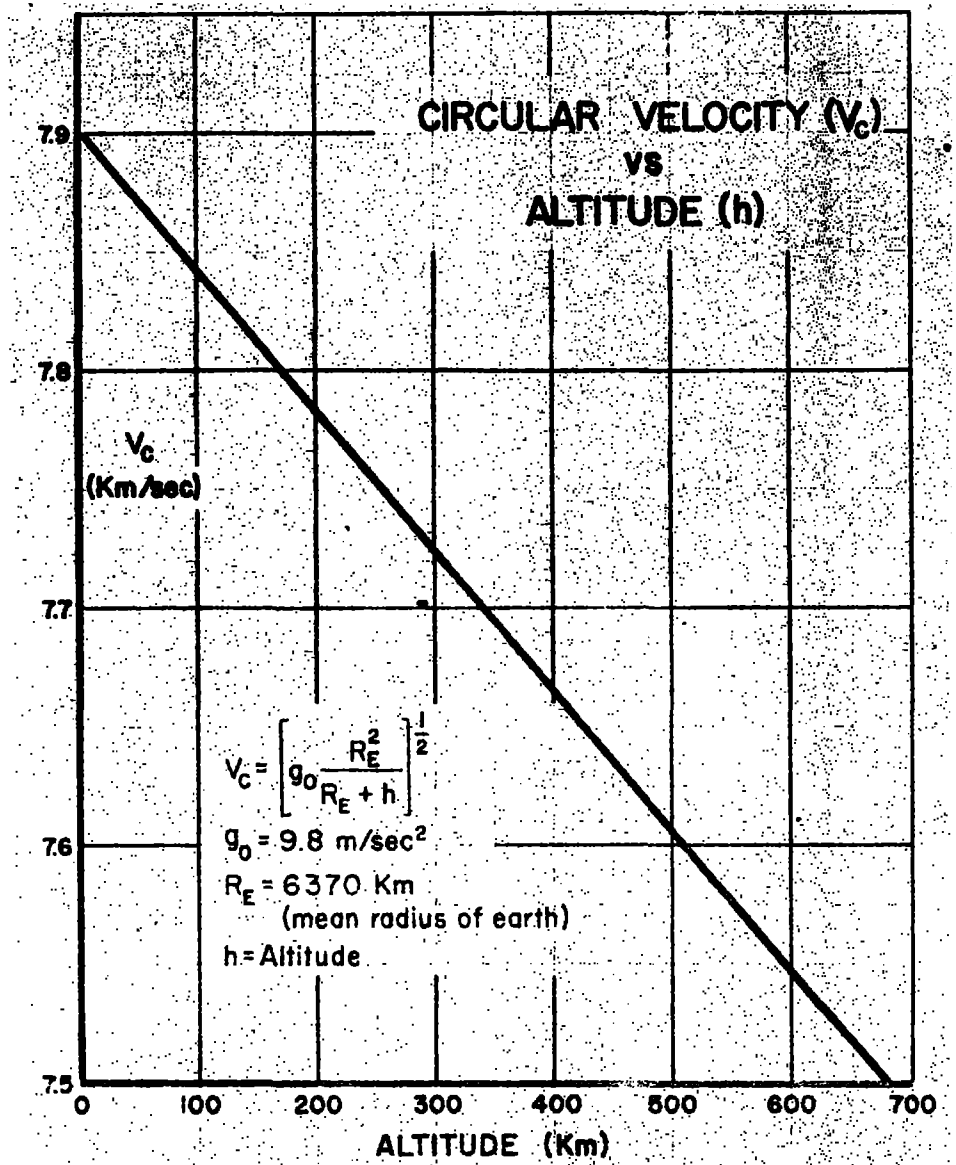

CIRCULAR VELOCITY (V_c)
VS
ALTITUDE (h)
7.9
7.8
7.7
7.6
7.5
V_c
(Km/sec)
$V_c = \left[g_o \frac{R_E^2}{R_E + h}\right]^{\frac{1}{2}}$
g_o = 9.8 m/sec²
R_E = 6370 Km
(mean radius of earth)
h = Altitude
0
100
200
300
400
500
600
700
ALTITUDE (Km)

FIG.4

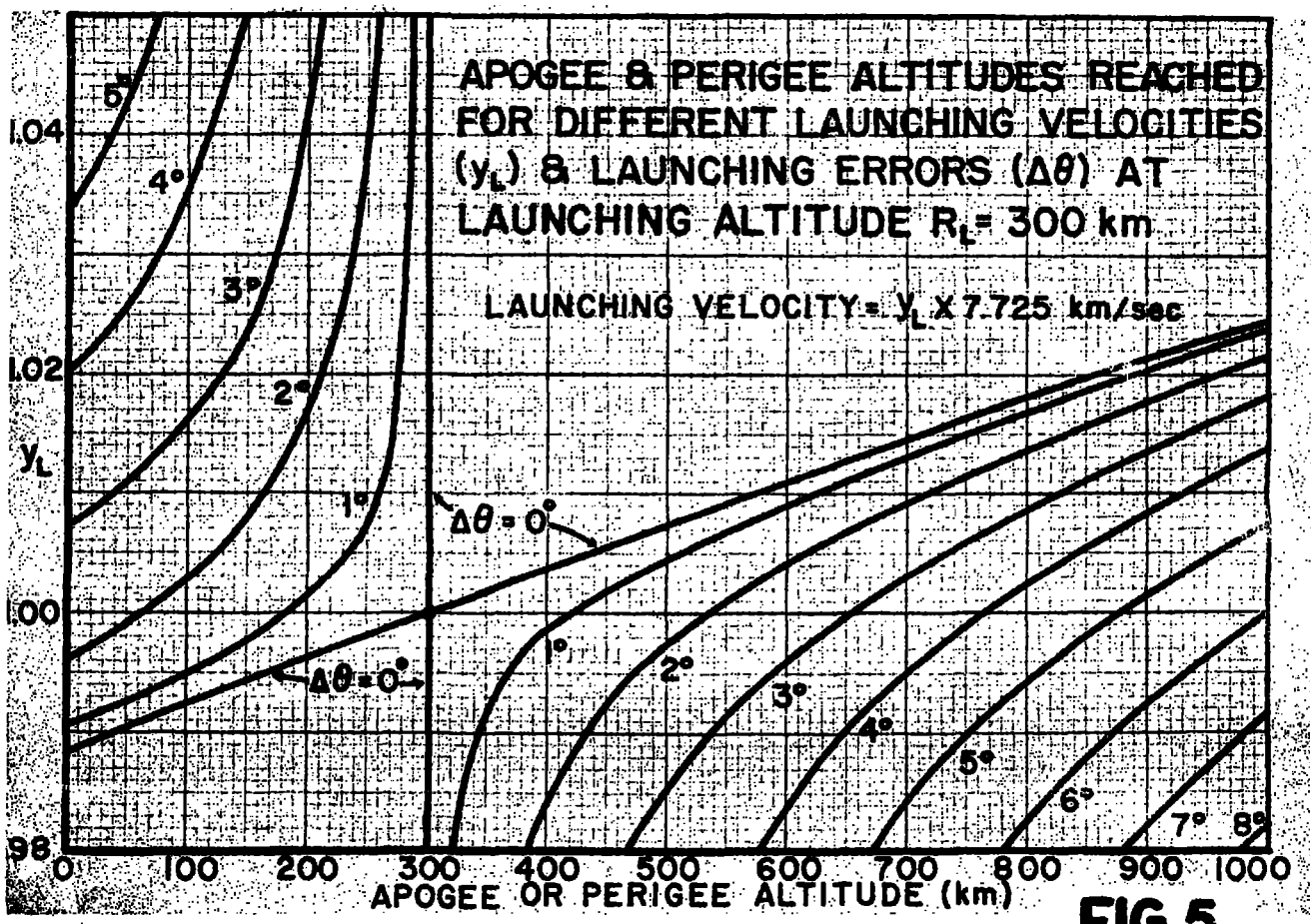

APOGEE & PERIGEE ALTITUDES REACHED FOR DIFFERENT LAUNCHING VELOCITIES (y_L) & LAUNCHING ERRORS ($\Delta\theta$) AT LAUNCHING ALTITUDE R_L = 300 km
LAUNCHING VELOCITY = y_L X 7.725 km/sec
y_L
1.04
1.02
1.00
.98
0
100
200
300
400
500
600
700
800
900
1000
APOGEE OR PERIGEE ALTITUDE (km)
$\Delta\theta = 0°$
$\Delta\theta = 0°$
1°
2°
3°
4°
5°
6°
7°
8°

FIG.5

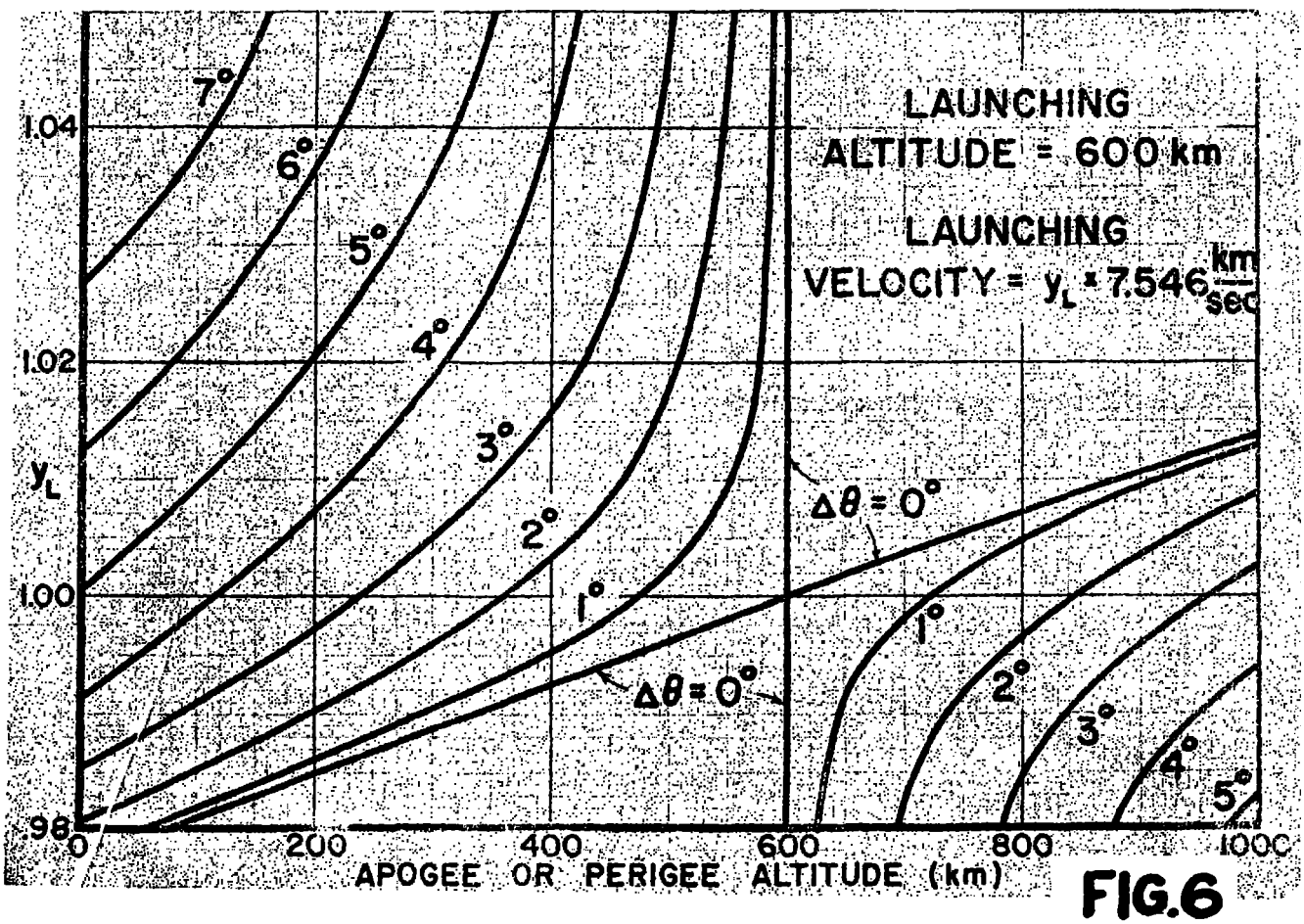

LAUNCHING ALTITUDE = 600 km
LAUNCHING VELOCITY = y_L * 7.546 $\frac{km}{sec}$
y_L
1.04
1.02
1.00
.98
0
200
400
600
800
1000
APOGEE OR PERIGEE ALTITUDE (km)
$\Delta\theta = 0°$
$\Delta\theta = 0°$
1°
2°
3°
4°
5°
6°
7°

FIG.6

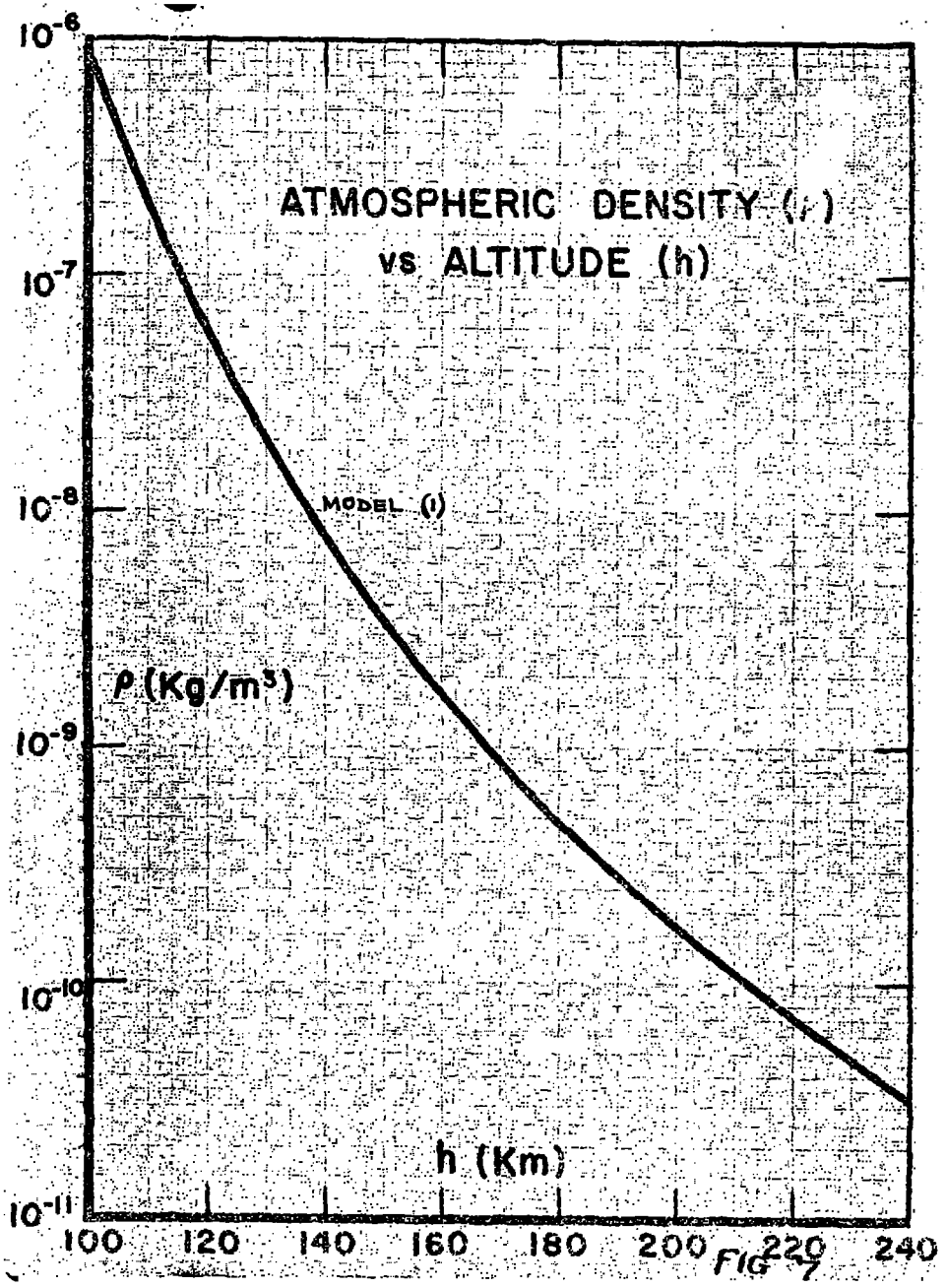

ATMOSPHERIC DENSITY (ρ)
vs ALTITUDE (h)
MODEL (I)
ρ (Kg/m³)
h (Km)
10⁻⁶
10⁻⁷
10⁻⁸
10⁻⁹
10⁻¹⁰
10⁻¹¹
100
120
140
160
180
200
220
240

FIG 7

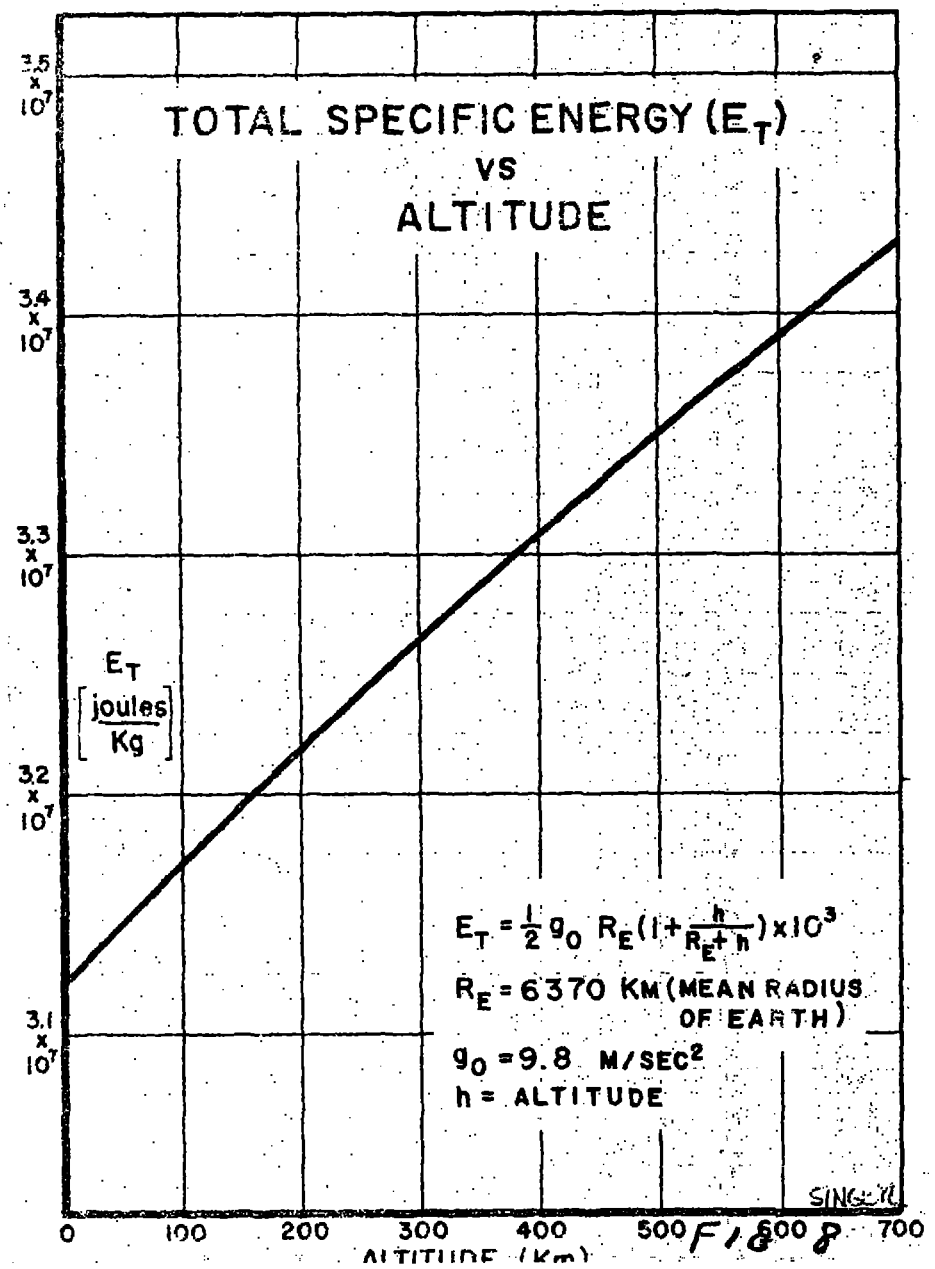

TOTAL SPECIFIC ENERGY (E_T)
vs
ALTITUDE
E_T [joules/Kg]
3.5 x 10⁷
3.4 x 10⁷
3.3 x 10⁷
3.2 x 10⁷
3.1 x 10⁷
E_T = ½ g_0 R_E (1 + h/(R_E + h)) x 10³
R_E = 6370 KM (MEAN RADIUS OF EARTH)
g_0 = 9.8 M/SEC²
h = ALTITUDE
0
100
200
300
400
500
600
700
ALTITUDE (Km)

FIG 8

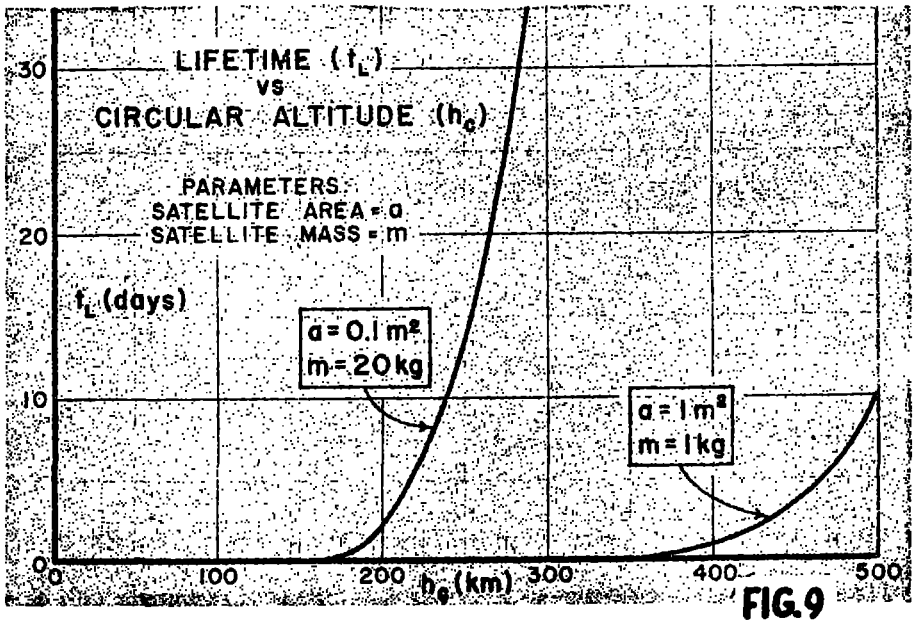


FIG. 9

DECREASE OF ECCENTRICITY OF ELLIPTICAL ORBIT DUE TO DRAG AT PERIGEE

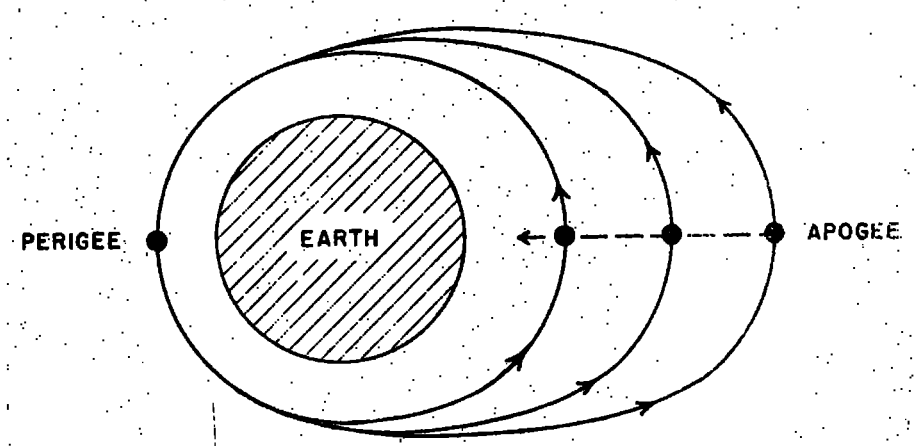


FIG. 10

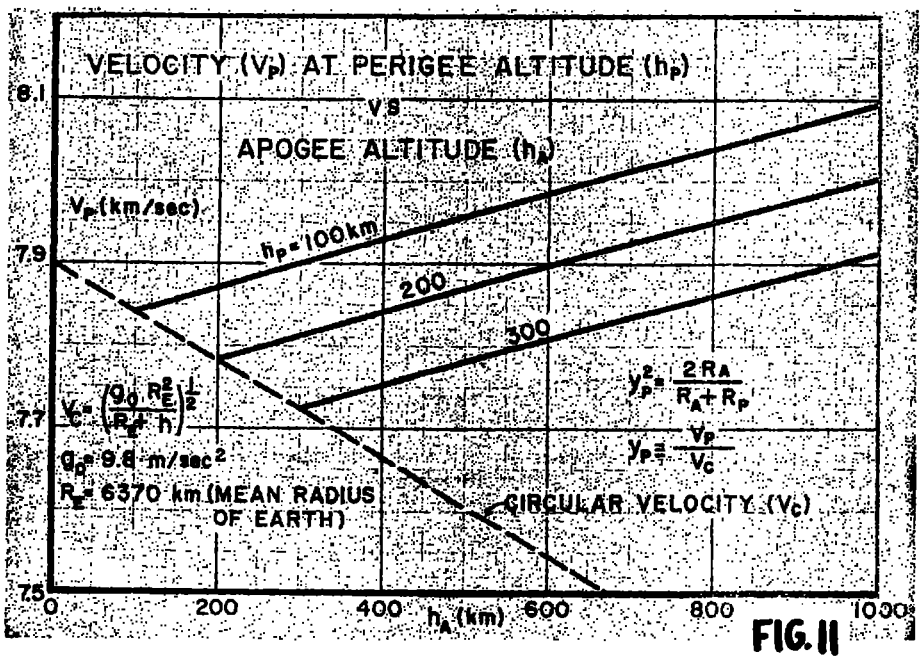


FIG. 11

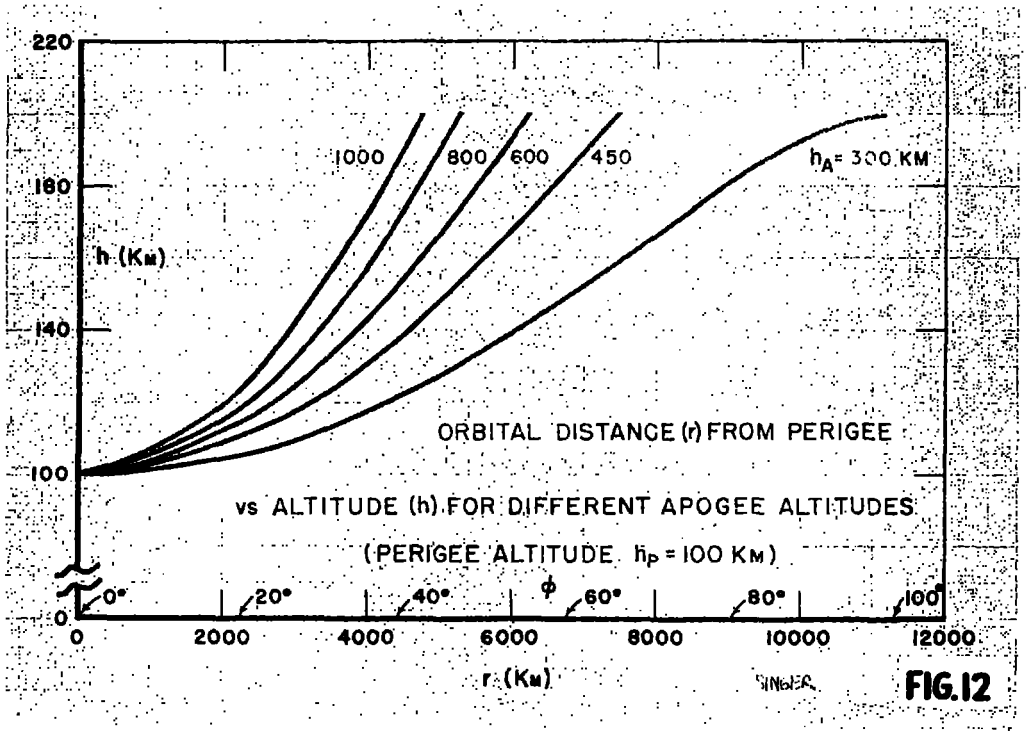


FIG. 12

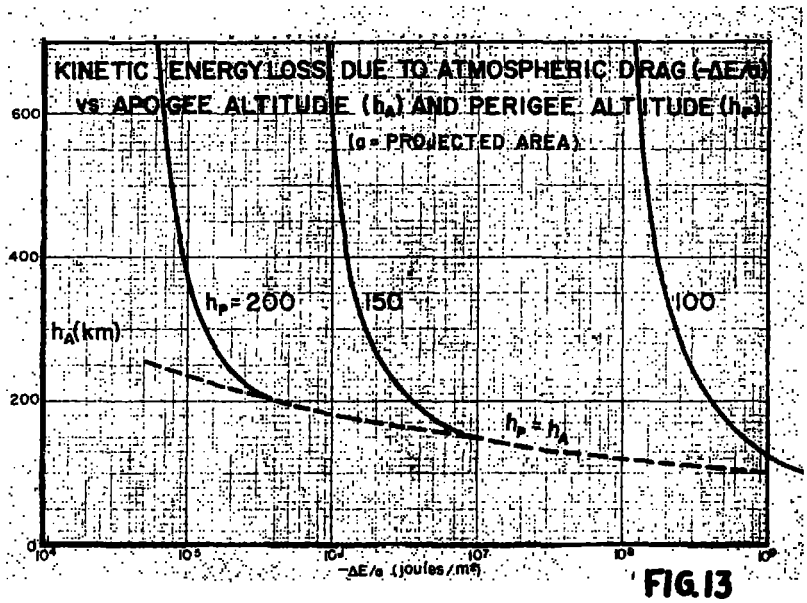


FIG. 13

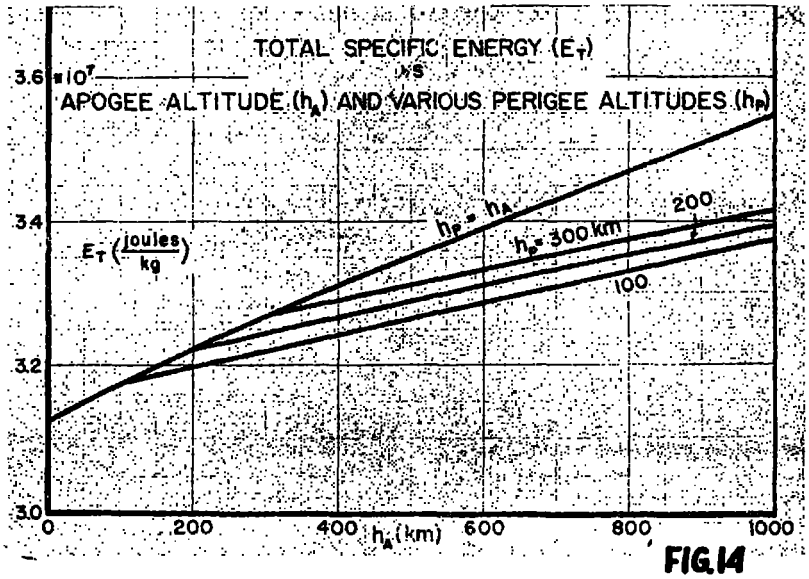


FIG. 14

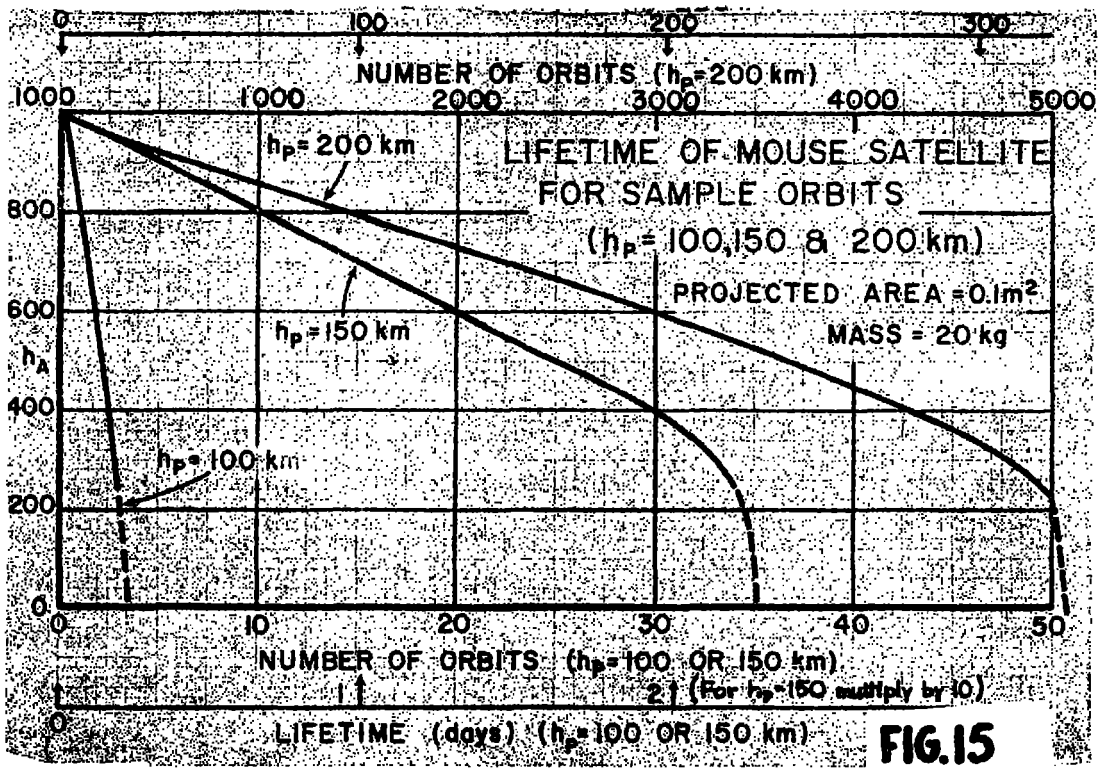

NUMBER OF ORBITS (h_p=200 km)
LIFETIME OF MOUSE SATELLITE
FOR SAMPLE ORBITS
(h_p= 100,150 & 200 km)
PROJECTED AREA = 0.1m²
MASS = 20 kg
h_p= 200 km
h_p= 150 km
h_p= 100 km
h_A
NUMBER OF ORBITS (h_p=100 OR 150 km)
(For h_p=150 multiply by 10)
LIFETIME (days) (h_p=100 OR 150 km)

FIG. 15

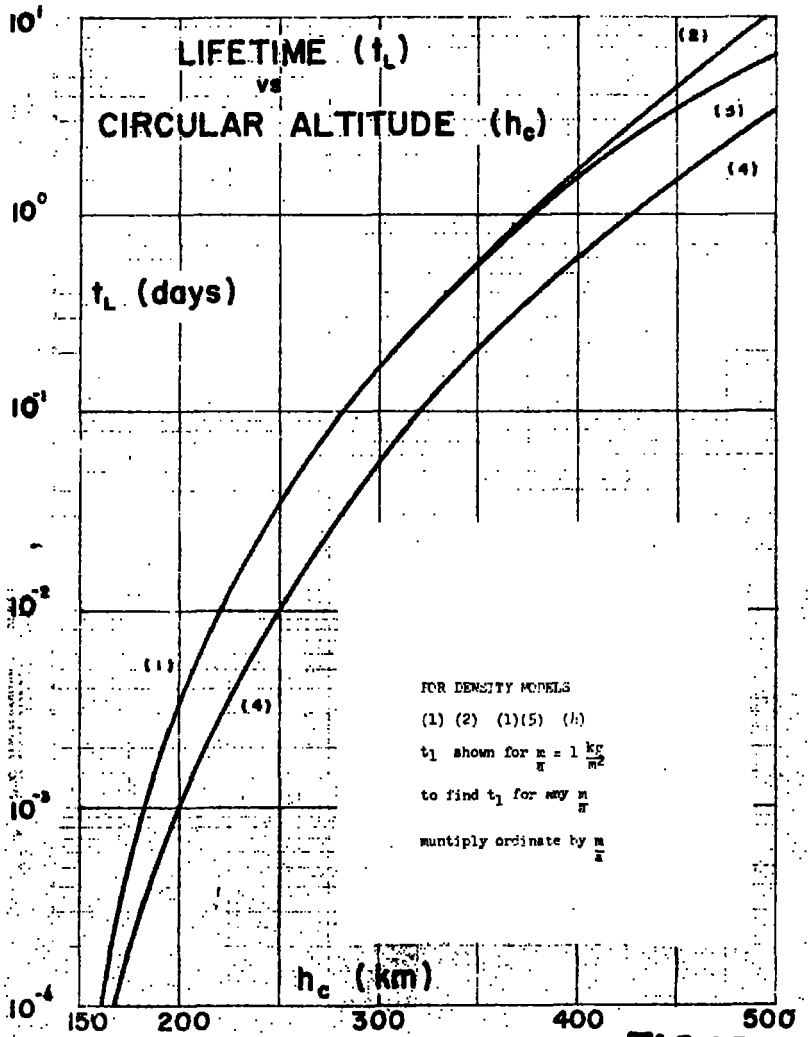

LIFETIME (t_L)
vs
CIRCULAR ALTITUDE (h_c)
t_L (days)
h_c (km)
(1)
(2)
(3)
(4)
FOR DENSITY MODELS
(1) (2) (1)(5) (4)
t_1 shown for $\frac{m}{a}$ = 1 $\frac{kg}{m^2}$
to find t_1 for any $\frac{m}{a}$
multiply ordinate by $\frac{m}{a}$

FIG. 16

APPENDIX III

ANALYTICAL METHOD FOR DERIVING CHANGE IN PERIGEE ALTITUDE

In calculating lifetime in an elliptical orbit we have assumed no change in h_p. For a near-circular orbit we can derive Δh_p by perturbation methods.

From Ref (2) p 405 we obtain the effect of a tangential perturbation acceleration T on the orbit parameters a and ϵ :

$$\frac{da}{dt} = \frac{2\sqrt{1+\epsilon^2+2\epsilon\cos v}}{n\sqrt{1-\epsilon^2}}\,T \tag{39}$$

$$\frac{d\epsilon}{dt} = \frac{2\sqrt{1-\epsilon^2}\,(\cos v+\epsilon)}{n a\sqrt{1+\epsilon^2+2\epsilon\cos v}}\,T \tag{40}$$

Since

$$\left.\begin{aligned} \Delta a &= \tfrac{1}{2}\left(\Delta R_A + \Delta R_p\right) \\ \Delta(\epsilon a) &= \epsilon\Delta a + a\Delta\epsilon = \tfrac{1}{2}\left(\Delta R_A - \Delta R_p\right) \end{aligned}\right\} \text{from (3)}$$

we get

$$\Delta R_p = \Delta a - \Delta(\epsilon a) = (1-\epsilon)\Delta a - a\Delta\epsilon \tag{41}$$

If $\epsilon \sim 0$, we neglect terms in ϵ^2; using (39) and (40)

$$\Delta R_p \sim \frac{2T}{n}\Delta t\left[(1-\epsilon)(1+\epsilon\cos v) - \frac{\cos v+\epsilon}{1+\epsilon\cos v}\right]$$

$$= \frac{2T}{n}\Delta t\left(\frac{1+2\epsilon\cos v-2\epsilon-\cos v}{1+\epsilon\cos v}\right) \tag{42}$$

Where n is the mean motion (angular velocity) in radians/sec and v the geocentric angle measured from apogee.

At perigee, $v = 180°$, and with $\epsilon \sim 0$

$$\Delta R_p \sim \frac{4T}{n} \Delta t \tag{43}$$

We can give some numerical values to illustrate the order of magnitude of ΔR_p.

We take $n = \frac{2\pi}{5500}$; $T = \frac{F_D}{m} = \left(\frac{A}{m}\right) v^2 \rho$; $\Delta t = \frac{\Delta s}{v}$;

then $\Delta R_p = \Delta h_p \sim 4 \frac{2\pi}{5500} \left(\frac{A}{m}\right) v \quad (\rho \cdot \Delta s)$.

Taking $\frac{A}{m} = \frac{0.1}{20}$, $v \sim 8 \, km/sec$

$\Delta h_p/orbit \sim 500 \, m$ for $h_p = 150 \, km$, $h_A = 400 \, km$

$\sim 50 \, m$ for $h_p = 200 \, km$, $h_A = 450 \, km$

The effect of a change in h_p as against our assumption of fixed h_p becomes important. Its effect on lifetime can be judged by examination of Fig. 15.