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DISTRIBUTION COEFFICIENTS OF MAGNETS

By

GEORGE HARTNELL
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DISTRIBUTION COEFFICIENTS OF MAGNETS

By George Hartnell

ABSTRACT

The theory of the distribution coefficients of magnets has been treated by Börgen,[1] Schmidt[2] and others. The treatment in this paper follows along a different line. It endeavors to simplify the mathematical analysis and to render the formulas available for practical use; at the same time it emphasizes the physical and practical aspects of the subject. It is pointed out that the distribution coefficients P and Q are not independent, and that the usual methods of deriving P and Q, which assume their independence, lead to erroneous and inconsistent results. In this paper chief attention is given to the theory. In a future paper will be given the application of the formulas to variometers and magnetometers.

1. **Introduction.**—Neither theory nor experiment have furnished an accurate account of the distribution of magnetism in a magnet. We know that the potential of an attracting or repelling body may be expressed

$$V=\frac{m}{r}+\frac{K_1+K_2+K_3-3I}{2r^2}$$

Here the origin of coordinates is at the centroid of the distribution, r = the distance of the point P, K_1, K_2, K_3, I = the moments of inertia around coordinate axes and line OP, respectively, and m = mass. For a homogeneous spherical shell, for a homogeneous sphere, or for a sphere composed of homogeneous layers the potential is represented by the first term, indicating that the body acts as if concentrated at the center or pole. Such a body is called centrobaric. The earth, on account of its spheroidal shape, is an example of a body that is not centrobaric; for the direction of gravity as determined by the plumb line and the level does not pass exactly through the center, nor do the resultant attractions of the other celestial bodies, like the sun or the moon, supposed centrobaric, pass through the earth's center, thus producing the phenomena of precession and nutation. Again, in the case of a thin spheroidal distribution in which one of the axes is very small, terms beyond the first are negligibly small, so that the distribution is very nearly centrobaric. In any case the number of terms used depends on the accuracy required.

[1] Ableitung des Ausdrucks für die Ablenkung eines Magnetnadel durch einen Magnet. Doctor Börgen, Hamburg, 1891.

[2] Ueber die gegenseitige Einwirkung Zweier Magnete in beliebiger Lage. Adolf Schmidt, Terr. Mag. and A. E., Dec., 1912, and Juee, 1913.

We therefore adopt as our concrete model of a magnet two equal and opposite distributions of magnetism acting as if concentrated at their centers or poles and separated a finite distance along the axis of the magnet.

This model offers several advantages—it avoids the use of definite integrals and simplifies the mathematical analysis; it affords a physical interpretation of the distribution coefficients; any desired accuracy may be attained by carrying the mathematical expressions far enough.

2. **Statement of the problem.**—Our problem is to determine the turning effect of one or more deflecting magnets upon a suspended magnet free to turn in a magnetic field. The turning effect, or

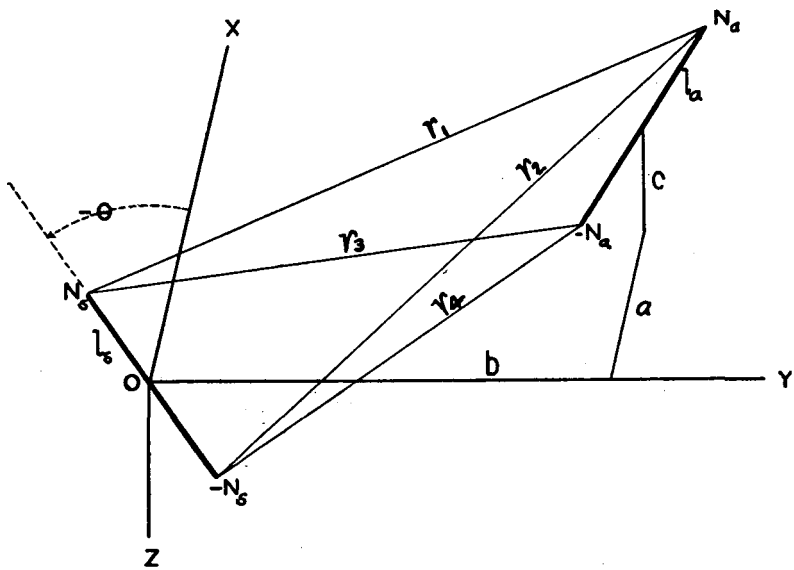

FIGURE 1.—Axes of reference and symbols

torque, will be expressed as a series of inverse powers of the distances between the centers of the magnets. In this paper the series is carried to the inverse seventh power, which has been found to be amply accurate for all practical purposes.

3. **Division of the subject.**—The present publication will be mathematical and will develop the equations. A later publication will deal with the application of the equations to magnetometers and variometers.

4. **Axes of reference and symbols** (fig. 1).—We take a right-handed system of rectangular coordinates, with the origin at the center of the suspended magnet. For concreteness the x axis is directed toward magnetic north, the y axis toward magnetic east, and the z axis is directed vertically downward. In the practical

applications this set of axes is regarded as fixed or movable, according to circumstances.

N_a = pole strength of deflecting magnet.
l_a = pole distance of deflecting magnet from center to pole.
$M_a = 2N_a l_a$ = magnetic moment of deflecting magnet.
N_s = pole strength of suspended magnet.
l_s = pole distance of suspended magnet.
$M_s = 2N_s l_s$ = magnetic moment of suspended magn
P = first distribution coefficient.
Q = second distribution coefficient.
$A = 1 + \frac{P}{d^2} + \frac{Q}{d^4}$ = distribution factor, A position.
B = distribution factor, B position.
C = distribution factor, C position.
D = distribution factor, D position.
d = distance between centers of magnets.
r_1, r_2, r_3, r_4 = distances between poles.
x_a, y_a, z_a = coordinates of poles of deflecting magnet.
a, b, c = coordinates of center of deflecting magnet.
λ_a, μ_a, ν_a = direction cosines of deflecting magnet.
θ, ψ = position angles of suspended magnet.
θ is measured from x to y and ψ from x to z.
u = deflection angle = angle between axis of suspended magnet and earth's horizontal field.
$\lambda_s = \cos\psi \cos\theta$ = direction cosine of suspended magnet, x axis.
$\mu_s = \cos\psi \sin\theta$ = direction cosine of suspended magnet, y axis.
$\nu_s = \sin\psi$ = direction cosine of suspended magnet, z axis.
x_s, y_s, z_s = coordinates of poles of suspended magnet.
V = mutual potential energy.

Other symbols will be explained in the text.

5. **Mutual potential energy of two magnets.**—The distances between any pair of poles is

$$r^2 = (x_a - x_s)^2 + (y_a - y_s)^2 + (z_a - z_s)^2$$

$$\text{Now } x_a = a \pm l_a\lambda_a,\ y_a = b_a \pm l_a\mu_a, \text{ and } z_a = c \pm l_a\nu_a$$

$$\text{Also } x_s = \pm l_s\lambda_s,\ y_s = \pm l_s\mu_s, \text{ and } z_s = \pm l_s\nu_s$$

By substitution, the distances between the poles are

$$r_1^2 = (a + l_a\lambda_a - l_s\lambda_s)^2 + (b + l_a\mu_a - l_s\mu_s)^2 + (c + l_a\nu_a - l_s\nu_s)^2$$
$$r_2^2 = (a + l_a\lambda_a + l_s\lambda_s)^2 + (b + l_a\mu_a + l_s\mu_s)^2 + (c + l_a\nu_a + l_s\nu_s)^2$$
$$r_3^2 = (a - l_a\lambda_a - l_s\lambda_s)^2 + (b - l_a\mu_a - l_s\mu_s)^2 + (c - l_a\nu_a - l_s\nu_s)^2$$
$$r_4^2 = (a - l_a\lambda_a + l_s\lambda_s)^2 + (b - l_a\mu_a + l_s\mu_s)^2 + (c - l_a\nu_a + l_s\nu_s)^2$$

By expansion

$$r_1^2 = d^2 + l_a^2 + l_s^2 + 2dl_a\left(\frac{a}{d}\lambda_a + \frac{b}{d}\mu_a + \frac{c}{d}\nu_a\right) - 2dl_s\left(\frac{a}{d}\lambda_s + \frac{b}{d}\mu_s + \frac{c}{d}\nu_s\right)$$
$$- 2l_a l_s(\lambda_a\lambda_s + \mu_a\mu_s + \nu_a\nu_s)$$

$\text{Cos } \delta = \frac{a}{d}\lambda_a + \frac{b}{d}\mu_a + \frac{c}{d}\nu_a$, in which δ is the angle between deflecting magnet and the line d,

$\cos \xi = \frac{a}{d}\lambda_s + \frac{b}{d}\mu_s + \frac{c}{d}\nu_s$, in which ξ is the angle between the suspended magnet and the line d,

$\cos \eta = \lambda_a\lambda_s + \mu_a\mu_s + \nu_a\nu_s$ in which η is the angle between the axes of the magnets.

Then

$$r_1^2 = d^2 + l_a^2 + l_s^2 + 2dl_a \cos \delta - 2dl_s \cos \xi - 2l_al_s \cos \eta$$
$$r_2^2 = d^2 + l_a^2 + l_s^2 + 2dl_a \cos \delta + 2dl_s \cos \xi + 2l_al_s \cos \eta$$
$$r_3^2 = d^2 + l_a^2 + l_s^2 - 2dl_a \cos \delta - 2dl_s \cos \xi + 2l_al_s \cos \eta$$
$$r_4^2 = d^2 + l_a^2 + l_s^2 - 2dl_a \cos \delta + 2dl_s \cos \xi - 2l_al_s \cos \eta$$

These may be written

$$r_1^2 = (d + l_a \cos \delta - l_s \cos \xi)^2 + l_a^2 \sin^2 \delta + l_s^2 \sin^2 \xi - 2l_al_s (\cos \eta - \cos \delta \cos \xi)$$
$$r_2^2 = (d + l_a \cos \delta + l_s \cos \xi)^2 + l_a^2 \sin^2 \delta + l_s^2 \sin^2 \xi + 2l_al_s (\cos \eta - \cos \delta^2 \cos \xi)$$
$$r_3^2 = (d - l_a \cos \delta - l_s \cos \xi) + l_a^2 \sin^2 \delta + l_s^2 \sin^2 \xi + 2l_al_s (\cos \eta - \cos \delta^2 \cos \xi)$$
$$r_4^2 = (d - l_a \cos \delta + l_s \cos \xi) + l_a^2 \sin^2 \delta + l_s^2 \sin^2 \xi - 2l_al_s (\cos \eta - \cos \delta \cos \xi)$$

Let $\overset{+}{U} = l_a \cos \delta + l_s \cos \xi$ and $\bar{U} = l_a \cos \delta - l_s \cos \xi$

$$\overset{+}{W} = l_a^2 \sin^2 \delta + l_s^2 \sin^2 \xi + 2\, l_al_s(\cos \eta - \cos \delta \cos \xi)$$

$$\bar{W} = l_a^2 \sin^2 \delta + l_s^2 \sin^2 \xi - 2\, l_al_s(\cos \eta - \cos \delta . \cos \xi)$$

We then have

$$r_1^2 = (d + \bar{U})^2 + \bar{W} \qquad r_2^2 = (d + \overset{+}{U})^2 + \overset{+}{W}$$
$$r_3^2 = (d - \overset{+}{U})^2 + \overset{+}{W} \qquad r_4^2 = (d - \bar{U})^2 + \bar{W}$$

For the mutual potential energy of the two magnets we have

$$V = N_aN_s\left(\frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3} + \frac{1}{r_4}\right)$$

By substitution

$$V = N_aN_s\left(\frac{1}{[(d + \bar{U})^2 + \bar{W}]^{\frac{1}{2}}} - \frac{1}{[(d + \overset{+}{U})^2 + \overset{+}{W}]^{\frac{1}{2}}} - \frac{1}{[(d - \overset{+}{U})^2 + \overset{+}{W}]^{\frac{1}{2}}} + \frac{1}{[(d - \bar{U})^2 + \bar{W}]^{\frac{1}{2}}}\right)$$

6. **Development of expression for mutual potential energy of two magnets.**—Expanding by the binomial theorem each term of the expression of the mutual potential energy and arranging according to the inverse powers of d we obtain

$$V = \frac{M_aM_s}{l_al_sd^3}\left[\frac{1}{2}(\bar{U}^2 - \overset{+}{U}^2) + \frac{1}{4}(\overset{+}{W} - \bar{W})\right] + \frac{M_aM_s}{l_al_sd^5}\left[\frac{1}{2}(\bar{U}^4 - \overset{+}{U}^4) + \frac{3}{2}(\overset{+}{U}^2\overset{+}{W} - \bar{U}^2\bar{W}) + \frac{3}{16}(\bar{W}^2 - \overset{+}{W}^2)\right] + \frac{M_aM_s}{l_al_sd^7}\left[\frac{1}{2}(\bar{U}^6 - \overset{+}{U}^6) + \frac{15}{4}(\overset{+}{U}^4\overset{+}{W} - \bar{U}^4\bar{W}) + \frac{5}{32}(\overset{+}{W}^3 - \bar{W}^3) + \frac{45}{32}(\bar{U}^2\bar{W}^2 - \overset{+}{U}^2\overset{+}{W}^2)\right] \quad (1)$$

7. Turning force on suspended magnet.—To obtain the torque acting on the suspended magnet, we differentiate the mutual potential energy with respect to the desired coordinate. The direction cosines of the suspended magnet are

$$\lambda_s = \cos\psi\cos\theta \qquad \mu_s = \cos\psi\sin\theta \qquad \nu_s = \sin\psi$$

in which θ is the angle between the suspended magnet and the x axis measured in the xy plane and ψ is the inclination of the suspended magnet to the xy plane. The most frequent case in practice is the turning of the suspended magnet around the vertical or z axis or, in other words, the effect of the deflecting magnet in increasing the angle θ.

The torque acting on the suspended magnet is $-\frac{\delta V}{\delta\theta}$. Performing the differentiations and entering as equations (3) to (9), inclusive, the equations of paragraph 5 with substitutions in the case of those for $\cos\xi$ and $\cos\eta$, as indicated in this paragraph, we have

$$\begin{aligned}
-\frac{\delta V}{\delta\theta} = {} & \frac{M_a M_s}{l_a l_s d^3}\left(\overset{+}{U}\frac{\delta\overset{+}{U}}{\delta\theta} - \overset{-}{U}\frac{\delta\overset{-}{U}}{\delta\theta} - \frac{1}{4}\frac{\delta\overset{+}{W}}{\delta\theta} + \frac{1}{4}\frac{\delta\overset{-}{W}}{\delta\theta}\right) + \frac{M_a M_s}{l_a l_s d^5}\left(2\overset{+}{U}{}^3\frac{\delta\overset{+}{U}}{\delta\theta} - 2\overset{-}{U}{}^3\frac{\delta\overset{-}{U}}{\delta\theta}\right. \\
& \left. - 3\overset{+}{W}\overset{+}{U}\frac{\delta\overset{+}{U}}{\delta\theta} + 3\overset{-}{W}\overset{-}{U}\frac{\delta\overset{-}{U}}{\delta\theta} - \frac{3}{2}\overset{+}{U}{}^2\frac{\delta\overset{+}{W}}{\delta\theta} + \frac{3}{2}\overset{-}{U}{}^2\frac{\delta\overset{-}{W}}{\delta\theta} + \frac{3}{8}\overset{+}{W}\frac{\delta\overset{+}{W}}{\delta\theta} - \frac{3}{8}\overset{-}{W}\frac{\delta\overset{-}{W}}{\delta\theta}\right) \\
& + \frac{M_a M_s}{l_a l_s d^7}\left(3\overset{+}{U}{}^5\frac{\delta\overset{+}{U}}{\delta\theta} - 3\overset{-}{U}{}^5\frac{\delta\overset{-}{U}}{\delta\theta} - 15\overset{+}{W}\overset{+}{U}{}^3\frac{\delta\overset{+}{U}}{\delta\theta} + 15\overset{-}{W}\overset{-}{U}{}^3\frac{\delta\overset{-}{U}}{\delta\theta} - \frac{15}{4}\overset{+}{U}{}^4\frac{\delta\overset{+}{W}}{\delta\theta}\right. \\
& + \frac{15}{4}\overset{-}{U}{}^4\frac{\delta\overset{-}{W}}{\delta\theta} - \frac{15}{32}\overset{+}{W}{}^2\frac{\delta\overset{+}{W}}{\delta\theta} + \frac{15}{32}\overset{-}{W}{}^2\frac{\delta\overset{-}{W}}{\delta\theta} + \frac{45}{8}\overset{+}{W}{}^2\overset{+}{U}\frac{\delta\overset{+}{U}}{\delta\theta} - \frac{45}{8}\overset{-}{W}{}^2\overset{-}{U}\frac{\delta\overset{-}{U}}{\delta\theta} \\
& \left. + \frac{45}{8}\overset{+}{W}\overset{+}{U}{}^2\frac{\delta\overset{+}{W}}{\delta\theta} - \frac{45}{8}\overset{-}{W}\overset{-}{U}{}^2\frac{\delta\overset{-}{W}}{\delta\theta}\right) \qquad (2)
\end{aligned}$$

$$\cos\delta = \frac{a}{d}\lambda_a + \frac{b}{d}\mu_a + \frac{c}{d}\nu_a \qquad (3)$$

$$\cos\xi = \frac{a}{d}\cos\psi\cos\theta + \frac{b}{d}\cos\psi\sin\theta + \frac{c}{d}\sin\psi \qquad (4)$$

$$\cos\eta = \lambda_a\cos\psi\cos\theta + \mu_a\cos\psi\sin\theta + \nu_a\sin\psi \qquad (5)$$

$$\overset{+}{U} = l_a\cos\delta + l_s\cos\xi \qquad (6)$$

$$\overset{-}{U} = l_a\cos\delta - l_s\cos\xi \qquad (7)$$

$$\overset{+}{W} = l_a^2\sin^2\delta + l_s^2\sin^2\xi + 2\,l_a l_s(\cos\eta - \cos\delta\cos\xi) \qquad (8)$$

$$\overset{-}{W} = l_a^2\sin^2\delta + l_s^2\sin^2\xi - 2\,l_a l_s(\cos\eta - \cos\delta\cos\xi) \qquad (9)$$

$$\frac{\delta\overset{+}{U}}{\delta\theta} = l_s\cos\psi\left(-\frac{a}{d}\sin\theta + \frac{b}{d}\cos\theta\right) \qquad (10)$$

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$$\frac{\delta \bar{U}}{\delta\theta} = -l_s \cos\psi\left(-\frac{a}{d}\sin\theta + \frac{b}{d}\cos\theta\right) \tag{11}$$

$$\frac{\delta \overset{+}{W}}{\delta\theta} = -2\, l_s^2 \cos\psi \cos\xi\left(-\frac{a}{d}\sin\theta + \frac{b}{d}\cos\theta\right) + 2\, l_a l_s \cos\psi\, [-\lambda_a \sin\theta + \mu_a \cos\theta - \cos\delta\left(-\frac{a}{d}\sin\theta + \frac{b}{d}\cos\theta\right)] \tag{12}$$

$$\frac{\delta \bar{W}}{\delta\theta} = 2\, l_s^2 \cos\psi \cos\xi\left(-\frac{a}{d}\sin\theta + \frac{b}{d}\cos\theta\right) - 2\, l_a l_s \cos\psi\, [-\lambda_a \sin\theta + \mu_a \cos\theta - \cos\delta\left(-\frac{a}{d}\sin\theta + \frac{b}{d}\cos\theta\right)] \tag{13}$$

Equations (1) to (13) constitute the basis of our study of the interaction and distribution coefficients of two magnets. The effects of any desired number of deflecting magnets on the suspended magnet may be determined by computing the effect of each magnet separately and adding the results.

8. **Significance of the term in $\frac{1}{d^3}$.**—Substituting equations (6), (7), (8), and (9) in the first term of (1) and reducing we obtain

$$V = \frac{M_a M_s}{d^3}(-3\cos\delta\cos\xi + \cos\eta)$$

This is the general expression for the mutual potential energy of two very small or elementary magnets. The torque exerted on the suspended magnet is

$$-\frac{\delta V}{\delta\theta} = \frac{M_a M_s}{d^3}\left(3\cos\delta\frac{\delta\cos\xi}{\delta\theta} - \frac{\delta\cos\eta}{\delta\theta}\right) \tag{14}$$

As an example, we shall apply this equation to the case of two magnets in the same horizontal plane; that is, xy. In this case $\psi = c = 0$. Let $\frac{a}{d} = \cos\alpha$, $\frac{b}{d} = \sin\alpha$, the angle α and the distance d defining the position of the center of the deflecting magnet. Also let $\lambda_a = \cos\phi$ and $\mu_a = \sin\phi$, the angle ϕ defining the orientation of the deflecting magnet. Then

$$\begin{aligned}
\cos\delta &= \cos\alpha\cos\phi + \sin\alpha\sin\phi = \cos(\phi - \alpha)\\
\cos\xi &= \cos\alpha\cos\theta + \sin\alpha\sin\theta = \cos(\alpha - \theta)\\
\cos\eta &= \cos\phi\cos\theta + \sin\phi\sin\theta = \cos(\phi - \theta)
\end{aligned} \tag{15}$$

$$\frac{\delta\cos\xi}{\delta\theta} = \sin(\alpha - \theta)$$

$$\frac{\delta\cos\eta}{\delta\theta} = \sin(\phi - \theta)$$

By substitution

$$-\frac{\delta V}{\delta\theta}=\frac{M_a M_s}{d^3}[3\cos(\phi-\alpha)\sin(\alpha-\theta)-\sin(\phi-\theta)] \qquad (16)$$

If we place the deflecting magnet along the y axis, $\phi=\alpha=\frac{\pi}{2}$. The torque is then

$$-\frac{\delta V}{\delta\theta}=\frac{2M_a M_s}{d^3}\cos\theta$$

Equating this to the couple exerted by the earth's horizontal field, we get

$$H\tan u=\frac{2M_a}{d^3} \qquad (17)$$

in which u is the deflection angle, that is, the angle between the axis of the suspended magnet and the direction of the earth's horizontal field. Also, if the magnets are kept perpendicular to each other, and we turn the system of axes around the vertical, so that $\theta=0$, and

$$H\sin u=\frac{2M_a}{d^3} \qquad (18)$$

In equation (16) the pole distances do not appear. In other words, as far as the inverse cube of the distance is concerned the magnets act like magnetic doublets. This is true of all positions involving the inverse cube distances only.

9. **Terms in $\frac{1}{d^3}$, deflecting magnet in the xy plane.**—Interesting cases of deflection occur when the center of the deflecting magnet lies in the coordinate planes, and at the same time the axis of the deflecting magnet is always parallel to the y axis.

Let α = the angle between the center of the deflecting magnet and the x axis. Then

$$\frac{a}{d}=\cos\alpha \qquad \frac{\bar{b}}{d}=\sin\alpha \qquad \lambda_a=\psi=c=\nu=0$$

Substituting in equations (3), (4), and (5) we have

$$\cos\delta=\sin\alpha \qquad \cos\xi=\cos(\alpha-\theta) \qquad \cos\eta=\sin\theta$$

$$\frac{\delta\cos\xi}{\delta\theta}=\sin(\alpha-\theta) \qquad \frac{\delta\cos\eta}{\delta\theta}=\cos\theta$$

These substituted in equation (14) give for the torque

$$-\frac{\delta V}{\delta\theta}=\frac{M_a M_s}{d^3}(3\sin\alpha\sin(\alpha-\theta)-\cos\theta)$$

Equating this to the couple exerted by the earth's horizontal field and keeping the axes of the magnets perpendicular to each other—that is, $\theta=0$—as we do in the magnetometer

$$H \sin u=\frac{M_a}{d^3}(3 \sin^2 \alpha-1) \tag{19}$$

If the magnets are not perpendicular to each other, the full expression is

$$H \tan u=\frac{M_a}{d^3}(3 \sin^2 \alpha-3 \sin \alpha \cos \alpha \tan u-1)$$

The middle term on the right vanishes when $\alpha=0$ or $\frac{\pi}{2}$. It is always small when the deflection angle u is small, as in the case of deflections of a variometer. The last equation may, without sensible error, be written

$$H \tan u=\frac{M_a}{d^3}(3 \sin^2 \alpha-1) \tag{20}$$

There will be no deflections when $3 \sin^2 \alpha=1$; that is, when $\alpha=35° 16'$.

10. **Terms in $\frac{1}{d^3}$, deflecting magnet in yz plane.**—Let $\beta=$ the angle of elevation of the center of the magnet above the horizontal plane. Then

$$\psi=a=\lambda_a=\nu_a=0 \qquad \sin \beta=\frac{c}{d} \qquad \cos \beta=\frac{b}{d} \qquad \mu_a=1$$

From equations (3), (4), and (5)

$$\cos \delta=\cos \beta \qquad \cos \xi=\cos \alpha \sin \theta \qquad \cos \eta=\sin \theta$$

$$\frac{\delta \cos \xi}{\delta \theta}=\cos \beta \cos \theta \qquad \frac{\delta \cos \eta}{\delta \theta}=\cos \theta$$

The equation for the torque (14) becomes

$$-\frac{\delta V}{\delta \theta}=\frac{M_a M_s}{d^3}(3 \cos^2 \beta-1) \cos \theta$$

For the variometer deflections

$$H \tan u=\frac{M_a}{d^3}(3 \cos^2 \beta-1) \tag{21}$$

and for magnetometer deflections

$$H \sin u=\frac{M_a}{d^3}(3 \cos^2 \beta-1) \tag{22}$$

There will be no deflections when $3 \cos^2 \beta-1=0$; that is, when $\beta=54° 44'$.

11. **Terms in $\frac{1}{d^3}$, deflecting magnet in xz plane.**—In this case let γ = angle of elevation from horizontal plane. Then

$$\sin\gamma=\frac{c}{d} \qquad \cos\gamma=\frac{a}{d} \qquad \psi=\lambda_a=\nu_a=b=0 \qquad \mu_a=1$$

$$\cos\delta=0 \qquad \cos\xi=\cos\beta\cos\theta \qquad \cos\eta=\sin\theta \qquad \frac{\delta\cos\eta}{\delta\theta}=\cos\theta$$

Equation (14) becomes

$$-\frac{\delta V}{\delta\theta}=-\frac{M_a M_s}{d^3}\cos\theta$$

The deflection equations are

$$H\sin u=\frac{M_a}{d^3} \text{ for a magnetometer} \tag{23}$$

$$H\tan u=\frac{M_a}{d^3} \text{ for a variometer} \tag{24}$$

In both these cases the deflection angle is always the same for all angles of elevation and is never zero.

12. **Terms in $\frac{1}{d^3}$, deflecting magnet vertical in yz plane.**—In determining the induction coefficient the long magnet of a magnetometer is vertical in the yz plane, above and below the center of the suspended short magnet. In this case let ϵ = the angle of elevation. Then

$$a=\theta=\lambda_a=\mu_a=\psi=0 \qquad \nu_a=1 \qquad \frac{c}{d}=\sin\epsilon \qquad \frac{b}{d}=\cos\epsilon$$

By substitution is equations (3), (4), and (5) as before

$$\cos\delta=\sin\epsilon \qquad \cos\xi=0 \qquad \cos\eta=0 \qquad \frac{\delta\cos\xi}{\delta\theta}=\cos\epsilon \qquad \frac{\delta\cos\eta}{\delta\theta}=0$$

Equation (14) becomes

$$-\frac{\delta V}{\delta\theta}=\frac{3M_a M_s}{d^3}\sin\epsilon\cos\epsilon$$

The deflection equations are

$$H\sin u=\frac{3M_a M_s}{d^3}\sin\epsilon\cos\epsilon \text{ for a magnetometer} \tag{25}$$

$$\text{and } H\tan u=\frac{3M_a M_s}{d^3}\sin\epsilon\cos\epsilon \text{ for a variometer} \tag{26}$$

For a given distance d the deflection will be a maximum when $\epsilon=45°$ The deflection will be zero when $\epsilon=0°$ or $90°$.

Usually we wish to know for a given horizontal distance on the bar of the magnetometer the height of the deflecting magnet for a maximum deflection. For this purpose let D = the horizontal distance and h the height of the deflecting magnet.
Then

$$\sin \epsilon = \frac{h}{d} \cdot \quad \cos \epsilon = \frac{D}{d} \qquad d^2 = D^2 + h^2$$

Equation (25) becomes

$$H \sin u = \frac{3\,hD}{(D^2+h^2)^{\frac{5}{2}}} \tag{27}$$

By equating the derivative of the right-hand member with respect to h to zero and solving for h we obtain

$$h = \frac{D}{2} \tag{28}$$

Therefore, when determining the induction coefficient, the maximum deflection is obtained when the height of the deflecting magnet is one-half the horizontal distance on the bar.

13. **Standard positions.**—In practice, the deflecting magnet is usually placed in a particular position with reference to the suspended magnet. Four positions are in common use, which we shall call the A, B, C, D positions. Each of these positions has two forms, according as the sine or tangent of the deflection angle u appears in the formulas.

In A position the deflecting magnet lies on and along the y axis; in B position, on the x axis parallel to the y axis; in C position, on the z axis parallel to the y axis; and in D position, vertically in yz plane as in section 12.

In these definitions the suspended magnet points toward the north and is deflected in the horizontal plane, in accordance with the selected orientation of the coordinate axes. The relative positions are, however, the essential features.

14. **Terms in $\frac{1}{d^5}$, A position.**—In this case

$$\psi = c = a = \lambda_a = \nu_a = 0 \qquad b = d \qquad \mu_a = 1$$

From equations (3) to (13) we get

$$\cos \delta = 1 \qquad \cos \xi = \sin \theta \qquad \cos \eta = \sin \theta$$

$$\overset{+}{U} = l_a + l_s \sin \theta \qquad \bar{U} = l_a - l_s \sin \theta$$

$$\frac{\delta \overset{+}{U}}{\delta \theta} = l_s \cos \theta \qquad \frac{\delta \bar{U}}{\delta \theta} = -l_s \cos \theta$$

$$\overset{+}{W} = l_s^2 \cos^2 \theta \qquad \bar{W} = l_s^2 \cos^2 \theta$$

$$\frac{\delta \overset{+}{W}}{\delta \theta} = -2\, l_s^2 \sin \theta \cos \theta \qquad \frac{\delta \bar{W}}{\delta \theta} = -2\, l_s^2 \sin \theta \cos \theta$$

By substitution in equation (2).

$$\left(\text{Terms in } \frac{1}{d^5}\right) = \frac{2 M_a M_s}{d^5} \cos\theta \, [2 l_a^2 - 3 l_s^2 (1 - 5 \sin^2\theta)] \tag{29}$$

$$= \frac{2 M_a M_s}{d^3} \cos\theta \left(\frac{P}{d^2}\right)$$

in which

$$P = 2 l_a^2 - 3 l_s^2 (1 - 5 \sin^2\theta) \tag{30}$$

P is one of the so-called distribution coefficients. In general, it is a function of the deflection angle θ as well as of the pole distances.

When the magnets are kept perpendicular to each other, for example, by turning the magnetometer around a vertical axis, $\theta = 0$.

$$\left(\text{Terms in } \frac{1}{d^5}\right) = \frac{2 M_a M_s}{d^3} \left(\frac{P}{d^2}\right) \tag{31}$$

in which

$$P = 2 l_a^2 - 3 l_s^2$$

15. **Terms in $\frac{1}{d^7}$, A position.**—By substitution in equation (2)

$$\left(\text{Terms in } \frac{1}{d^7}\right) = \frac{2 M_a M_s}{d^7} \cos\theta \, [3 l_a^4 - 15 l_a^2 l_s^2 (1 - 5 \sin^2\theta)$$

$$+ \frac{45}{8} l_s^4 (1 - 14 \sin^2\theta + 21 \sin^4\theta)] \tag{32}$$

$$= \frac{2 M_a M_s}{d^3} \cos\theta \left(\frac{Q}{d^4}\right)$$

in which Q is the second distribution coefficient.
Q, like P, is a function of the pole distances of the magnets and also of the deflection angle θ.

When $\theta = 0$.

$$\left(\text{Terms in } \frac{1}{d^7}\right) = \frac{2 M_a M_s}{d^7} \left(3 l_a^4 - 15 l_a^2 l_s^2 + \frac{45}{8} l_s^4\right) \tag{33}$$

$$= \frac{2 M_a M_s}{d^3} \left(\frac{Q}{d^4}\right)$$

When the deflection angle θ is very small—as, for example, when determining the scale values of a magnetograph by means of an auxiliary deflecting magnet—the sine terms in the distribution coefficients may, without sensible error, be dropped.

Equating the sums of equations (18), (31), and (33) to the couple produced by the earth's horizontal field

$$M_s H \sin\theta = \frac{2 M_a M_s}{d^3} \cos\theta \left(1 + \frac{P}{d^2} + \frac{Q}{d^4}\right)$$

or

$$H \tan\theta = \frac{2 M_a}{d^3} \left(1 + \frac{P}{d^2} + \frac{Q}{d^4}\right) \quad \text{A position, tangent form} \tag{34}$$

$$H \sin\theta = \frac{2 M_a}{d^3} \left(1 + \frac{P}{d^2} + \frac{Q}{d^4}\right) \quad \text{A position, sine form} \tag{35}$$

in which $$P=2\,l_a^2-3\,l_s^4 \tag{36}$$

$$Q=3\,l_a^4-15\,l_a^2 l_s^2+\frac{45}{8}\,l_s^4 \tag{37}$$

The expressions for P and Q in equations (36) and (37) may be very easily derived from equations (2) to (13) by taking $\theta=0$ at the outset. We then have

$$a=c=\lambda_a=\nu_a=\psi=\theta=0 \qquad \mu_a=\frac{b}{d}=1$$

$$\cos\delta=1 \qquad \cos\xi=0 \qquad \cos\eta=0$$

$$\overset{+}{U}=l_a \qquad \bar{U}=l_a \qquad \frac{\delta\overset{+}{U}}{\delta\theta}=l_s \qquad \frac{\delta\bar{U}}{\delta\theta}=-l_s$$

$$\overset{+}{W}=l_s^2 \qquad \bar{W}_s=l^2_s \qquad \frac{\delta\overset{+}{W}}{\delta\theta}=0 \qquad \frac{\delta\bar{W}}{\delta\theta}=0$$

By substitution in equation (2) we obtain the formulas for P and Q as above.

16. **Distribution coefficients, B position.**—In this position

$$\nu_a=\lambda_a \qquad b=\psi=c=\theta=0 \qquad \mu_a=1 \qquad a=d$$

$$\cos\delta=\cos\eta=0 \qquad \cos\xi=1$$

$$\overset{+}{U}=l_s \qquad \bar{U}=-l_s \qquad \overset{+}{W}=\bar{W}=l_a^2$$

$$\frac{\delta\overset{+}{U}}{\delta\theta}=\frac{\delta\bar{U}}{\delta\theta}=0 \qquad \frac{\delta\overset{+}{W}}{\delta\theta}=2\,l_a l_s \qquad \frac{\delta\bar{W}}{\delta\theta}=-2\,l_a l_s$$

With these values and equation (2), bearing in mind that the couple is minus and that, therefore, for comparison with the A position, in which the torque is positive, all the terms in the B position must be reversed, we derive

$$P=6\,l_s^2-\frac{3}{2}\,l_a^2 \qquad \text{B position} \tag{38}$$

$$Q=\frac{15}{8}\,l_a^4-\frac{45}{2}\,l_a^2 l_s^2+15\,l^4 \qquad \text{B position} \tag{39}$$

17. **Distribution coefficients, C position.**—In this case

$$a=b=\lambda_a=\psi=\theta=\nu_a=0 \qquad \mu_a=1 \qquad c=d$$

$$\overset{+}{U}=\bar{U}=\frac{\delta\overset{+}{U}}{\delta\theta}=\frac{\delta\bar{U}}{\delta\theta}=0 \qquad \overset{+}{W}=\bar{W}=l_a^2+l_s^2$$

$$\frac{\delta\overset{+}{W}}{\delta\theta}=2\,l_a l_s \qquad \frac{\delta\bar{W}}{\delta\theta}=-2\,l_a l_s$$

By substitution, as before, in equation (2), reversing signs of all terms for positive torque

$$P=-\frac{3}{2}(l_a^2+l_s^2) \quad \text{C position} \tag{40}$$

$$Q=\frac{15}{8}(l_a^2+l_s^2) \quad \text{C position} \tag{41}$$

18. **Deflection equations, sine form.**—

$$H \sin u=\frac{2M_a}{d^3}\left(1+\frac{P_a}{d^2}+\frac{Q_a}{d^4}\right) \quad \text{A position} \tag{42}$$

$$H \sin u=\frac{M_a}{d^3}\left(1+\frac{P_b}{d^2}+\frac{Q_b}{d^4}\right) \quad \text{B position} \tag{43}$$

$$H \sin u=\frac{M_a}{d^3}\left(1+\frac{P_c}{d^2}+\frac{Q_c}{d^4}\right) \quad \text{C position} \tag{44}$$

$$H \sin u=\frac{3M_a}{d^3}\sin\epsilon\cos\epsilon\left(1+\frac{P_d}{d^2}+\frac{Q_d}{d^4}\right) \quad \text{D position} \tag{45}$$

19. **Another method of deriving P and Q.**—In view of the importance of these equations, we give another method of deriving P and Q for the standard positions of A, B, and C. The method is based on the consideration of the field of a magnet parallel to its axis.

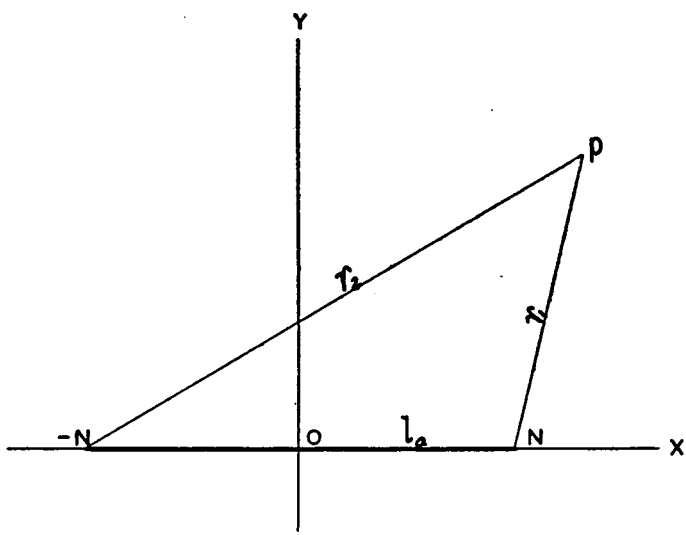

FIGURE 2.—Intensity of magnetic field at any point due to a magnet

In Figure 2 the center of the magnet is the origin, and l_a is the pole distance. We require the field intensity parallel to the axis of the magnet at a point p whose coordinates are x and y.

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The potential at point p is

$$V = N\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

in which

$$r_1^2 = (x - l_a)^2 + y^2 \qquad r_2^2 = (x + l_a)^2 + y^2$$

The field parallel to the direction of the x axis is

$$X = -\frac{\delta V}{\delta x} = N\left(\frac{1}{r_1^2}\frac{\delta r_1}{\delta x} - \frac{1}{r_2^2}\frac{\delta r_2}{\delta x}\right)$$

Inserting the derivatives we obtain

$$X = N\left(\frac{x - l_a}{[(x - l_a)^2 + y^2]^{\frac{3}{2}}} - \frac{x + l_a}{[(x + l_a)^2 + y^2]^{\frac{3}{2}}}\right) \tag{46}$$

Expanding by the binomial theorem and arranging according to inverse powers of x

$$X = \frac{2M_a}{x^3}\left(1 + \frac{2l_a^2 - 3y^2}{x^2} + \frac{3l_a^4 - 15l_a^2y^2 + \frac{45}{8}y^4}{x^4}\right) \tag{47}$$

We have here the inverse cube relation $\frac{2M_a}{x^3}$, together with additional terms which involve the pole distances of the magnet and the coordinate y.

20. **Properties of the X intensity.**—From equation (47) we see that the X intensity is greater for points on the x axis, where we have

$$X = \frac{2M_a}{x^3}\left(1 + \frac{2l_a^2}{x^2} + \frac{3l_a^4}{x^4}\right) \tag{48}$$

At points near the x axis the X intensity is greater than the inverse cube relation. Proceeding from a point on the x axis in the direction of the y axis we reach a point where the third term in the parentheses is zero. Proceeding further in the same direction we reach a point where the second term becomes zero.

For greater values of y, the X intensity is less than the inverse cube relation.

21. **X intensity on the y axis.**—Letting $x = 0$ in equation (46)

$$X = -\frac{M_a}{(y^2 + l_a^2)^{\frac{3}{2}}} \tag{49}$$

and expanding

$$X = -\frac{M_a}{y^3}\left(1 - \frac{3}{2}\frac{l_a^2}{y^2} + \frac{15}{8}\frac{l_a^4}{y^4}\right) \tag{50}$$

The X intensity is minus and is parallel to the negative direction of the x axis and, furthermore, is less than that due to the inverse cube relation.

22. **Torque on magnet, A position.**—In the A position the suspended magnet lies on the x axis and is perpendicular to this axis. The torque on the suspended magnet is

$$\text{Torque} = 2XN_s l_s = XM_s \tag{51}$$

Replacing in equation (47) the y coordinate by the pole distance l_s, and the x coordinate by d, and multiplying by M_s, we obtain

$$\text{Torque} = \frac{2M_aM_s}{d^3}\left(1 + \frac{2l_a^2 - 3l_s^2}{d^2} + \frac{3l_a^4 - 15l_a^2l_s^2 + \frac{45}{8}l_s^4}{d^4}\right) \tag{52}$$

$$= \frac{2M_aM_s}{d^3}\left(1 + \frac{P_a}{d^2} + \frac{Q_a}{d^4}\right) = \frac{2M_aM_s}{d^3}A \quad \text{A position}$$

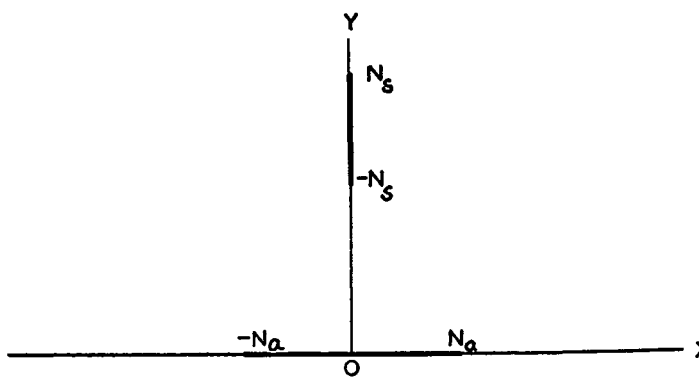

FIGURE 3.—B position

23. **Torque on magnet, B position** (fig. 3).—The y coordinate of $+N_s$ is $d+l_s$ and the coordinate of $-N_s$ is $d-l_s$. The turning force on $+N_s$ is, from equation (51),

$$XN_sl_s = -\frac{M_aN_sl_s}{(d+l_s)^3}\left(1 - \frac{3}{2}\frac{l_a^2}{(d+l_s)^2} + \frac{15}{8}\frac{l_a^4}{(d+l_s)^4}\right)$$

The turning force on $-N_s$ is

$$XN_sl_s = -\frac{M_aN_sl_s}{(d-l_s)^3}\left(1 - \frac{3}{2}\frac{l_a^2}{(d-l_s)^2} + \frac{15}{8}\frac{l_a^4}{(d+l_s)^4}\right)$$

The total turning force is

$$\text{Torque} = -\frac{M_aM_s}{2}\left[\frac{1}{(d+l_s)^3} + \frac{1}{(d-l_s)^3} - \frac{3}{2}l_a^2\left(\frac{1}{(d+l_s)^2} + \frac{1}{(d-l_s)^2}\right)\right.$$
$$\left. + \frac{15}{8}l_a^4\left(\frac{1}{(d+l_s)^4} + \frac{1}{(d-l_s)^4}\right)\right]$$

Expanding and collecting terms

$$\text{Torque} = -\frac{M_a M_s}{d^3}\left(1 - \frac{\frac{3}{2} l_a^2 - 6 l_s^2}{d^2} + \frac{\frac{15}{8} l_a^4 - \frac{45}{2} l_s^2 + 15 l_a^2 l_s^4}{d^4}\right) \tag{53}$$

$$= -\frac{M_a M_s}{d^3}\left(1 - \frac{P_b}{d^2} + \frac{Q_b}{d^4}\right)$$

$$= -\frac{M_a M_s}{d^3} B$$

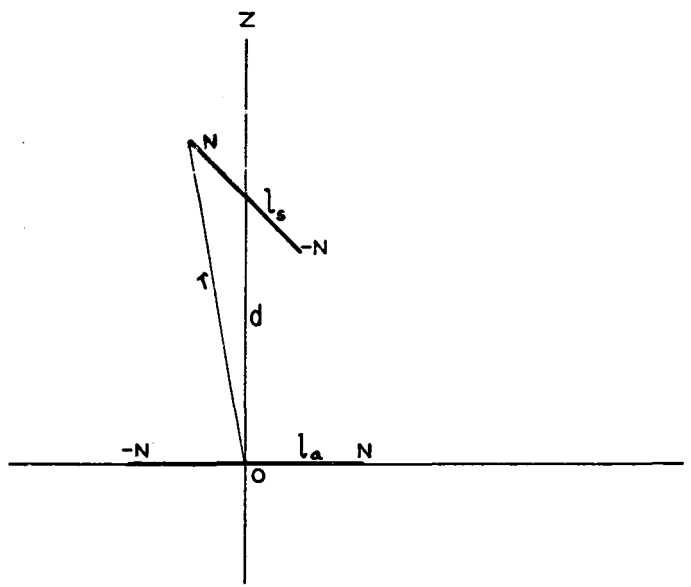

FIGURE 4.—C position

24. **Torque on magnet, C position** (fig. 4).—In this case the distance of the poles $-N_s$ and $+N_s$ from the center of the deflecting magnet is $d^2+l_s^2$, which takes the place of the y coordinate in equation (50).

The turning force on the suspended magnet is

Torque $= 2XN_s l_s = XM_s$

From equation (50)

$$\text{Torque} = -\frac{M_a M_s}{(d^2+l_s)^3}\left(1 - \frac{3}{2}\frac{l_s^2}{(d^2+l_s)^2}\right) + \frac{15}{8}\frac{l_s^4}{(d^2+l_s)^4}$$

Expanding

$$\text{Torque} = -\frac{M_a M_s}{d^3}\left[1 - \frac{3}{2}\frac{l_a^2+l_s^2}{d^2} + \frac{15}{8}\frac{(l_a^2+l_s^2)^2}{d^4}\right] \tag{54}$$

$$= -\frac{M_a M_s}{d^3}\left(1 - \frac{P_c}{d^2} + \frac{Q_c}{d^4}\right) = -\frac{M_a M_s}{d^3} C \quad \text{C position}$$

25. **Definition of distribution coefficients in coordinate planes.**—In the coordinate planes P is defined as the coefficient of $\frac{M_a}{d^5}$ in the deflection equations (65), (66), and (67), or as the coefficient of $\frac{3M_a}{d^5}$ $\sin \epsilon \cos \epsilon$ in the deflection equation (68) for position D. The plane referred to is indicated by the subscripts. Again, P may be defined as the coefficient of $\frac{M_a M_s}{d^5 l_a l_s}$ in the torque equation (2) or as the coefficient of $\frac{3 M_a M_s}{l_a l_s}$ $\sin \epsilon \cos \epsilon$ in the torque equation (2) after the substitutions have been made.

Q is defined as the coefficient of $\frac{M_a}{d^7}$ in the deflection equations (65), (66), and (67), or as the coefficient of $\frac{3 M_a}{d^7}$ $\sin \epsilon \cos \epsilon$ in equation (68) for the D position. Or, again, Q is defined as the coefficient of $\frac{M_a M_s}{d^7 l_a l_s}$, or as the coefficient of $\frac{3 M_a M_s}{d^7 l_a l_s}$ in the torque equation (2) after the substitutions have been made. For the D position the distribution coefficients are written P_{DYZ} and Q_{DYZ}.

26. **Distribution coefficients in coordinate xy planes.**—In sections (9), (10), (11), and (12) we have considered terms in $\frac{1}{d^3}$ when the deflecting magnet lies in the coordinate planes with its axis always parallel to the y axis and also when it lies in the yz plane with axis parallel to the z axis. We proceed to derive the formulas for P and Q for the same positions of the deflecting magnet. The formulas are developed specifically for a magnetometer, so that $\theta = 0$, and the resulting deflection equations have the sine form. In the case of variometers the deflection angle is very small, so that the formulas are still applicable, since the sine and tangent of a small angle are sensibly equal.

In the xy plane

$$\psi = \lambda_a = \theta = \nu_a = 0 \qquad \frac{a}{d} = \cos \alpha \qquad \frac{b}{d} = \sin \alpha$$

From equations (3), (4), and (5)

$$\cos \delta = \sin \alpha \qquad \cos \xi = \cos \alpha \qquad \cos \eta = 0$$

The work will be much simplified by using symbols of abbreviation. For the xy plane, for example, we may write

$$s \equiv l_a \sin \alpha \qquad t \equiv l_s \cos \alpha \qquad u \equiv l_a \cos \alpha \qquad v \equiv l_s \sin \alpha$$

Equations (6) to (13) become

$$\overset{+}{U}=l_a \sin\alpha+l_s\cos\alpha=s+t$$

$$\frac{\delta\overset{+}{U}}{\delta\theta}=l_s\sin\alpha=v$$

$$\bar{U}=l_a\sin\alpha-l_s\cos\alpha=s-t$$

$$\frac{\delta\bar{U}}{\delta\theta}=-l_s\sin\alpha=-v$$

$$\overset{+}{W}=l_a^2\cos^2\alpha+l_s^2\sin^2\alpha-2\,l_al_s\sin\alpha\cos\alpha=(u-\iota)^2$$

$$\frac{\delta\overset{+}{W}}{\delta\theta}=-2\,l_s^2\sin\alpha\cos\alpha+2\,l_al_s\cos^2\alpha=2\,t(u-v)$$

$$\bar{W}=l_a^2\cos^2\alpha+l_s^2\sin^2\alpha+2\,l_al_s\sin\alpha\cos\alpha=(u+v)^2$$

$$\frac{\delta\bar{W}}{\delta\theta}=-2\,l_s^2\sin\alpha\cos\alpha-2\,l_al_s\cos^2\alpha=-2\,t(u+v)$$

Substituting in equation (2) we obtain

$$P_{XY}=4\,l_a^2\left(\frac{3}{8}-\frac{15}{4}\sin^2\alpha+\frac{35}{8}\sin^4\alpha\right)-6\,l_s^2\left(1-\frac{35}{4}\sin^2\alpha+\frac{35}{4}\sin^4\alpha\right) \quad (55)$$

$$\begin{aligned}Q_{XY}=l_a^4&\left(-\frac{15}{8}+\frac{315}{8}\sin^2\alpha-\frac{945}{8}\sin^4\alpha+\frac{693}{8}\sin^6\alpha\right)\\&+l_a^2l_s^2\left(\frac{45}{2}-\frac{1785}{4}\sin^2\alpha+1260\sin^4\alpha-\frac{3465}{4}\sin^6\alpha\right)\\&+l_s^4\left(-15+\frac{525}{2}\sin^2\alpha-\frac{5355}{8}\sin^4\alpha+\frac{3465}{8}\sin^6\alpha\right)\end{aligned} \quad (56)$$

27. **Distribution coefficients in coordinate *yz* planes.**—In this case

$$a=\lambda_a=\nu_a=\theta=\psi=0 \qquad \mu_a=1 \qquad \frac{b}{d}=\cos\beta \qquad \frac{c}{d}=\sin\beta$$

$$\overset{+}{U}=l_a\cos\beta \qquad \frac{\delta\overset{+}{U}}{\delta\theta}=l_s\cos\beta \qquad \bar{U}=l_a\cos\beta \qquad \frac{\delta\bar{U}}{\delta\theta}=-l_s\cos\beta$$

$$\overset{+}{W}=l_a^2\sin^2\beta+l_s^2 \qquad \frac{\delta\overset{+}{W}}{\delta\theta}=2\,l_al_s\sin^2\beta$$

$$W=l_a^2\sin^2\beta+l_s^2 \qquad \frac{\delta\bar{W}}{\delta\theta}=-2\,l_al_s\sin^2\beta$$

Substitution in equation (2) gives

$$P_{YZ}=4\,l_a^2\left(1-5\sin^2\beta+\frac{35}{8}\sin^4\beta\right)-6\,l_s^2\left(1-\frac{5}{4}\sin^2\beta\right) \quad (57)$$

$$Q_{YZ}=l_a^4\left(6-63\sin^2\beta+\frac{567}{4}\sin^4\beta-\frac{693}{8}\sin^6\beta\right)$$
$$+l_a^2l_s^2\left(-30+105\sin^2\beta-\frac{315}{4}\sin^4\beta\right)$$
$$+l_s^4\left(\frac{45}{4}-\frac{105}{8}\sin^2\beta\right) \quad (58)$$

28. **Distribution coefficients in coordinate *xz* planes.**—In this case

$$b=\lambda_a=\psi=\nu_a=\theta=0 \quad \mu_a=1 \quad \frac{a}{d}=\cos\gamma \quad \cos\delta=0 \quad \cos\xi=\cos\gamma \quad \cos\eta=0$$

$$\overset{+}{U}=l_s\cos\gamma \qquad \frac{\delta\overset{+}{U}}{\delta\theta}=0 \qquad \bar{U}=l_s\cos\gamma \qquad \frac{\delta\bar{U}}{\delta\theta}=0$$

$$\overset{+}{W}=l_s^2\sin^2\gamma+l_a^2 \qquad \frac{\delta\overset{+}{W}}{\delta\theta}=2\,l_al_s$$

$$W=l_s^2\sin^2\gamma+l_a^2 \qquad \frac{\delta W}{\delta\theta}=-2\,l_al_s$$

By substitution we obtain

$$P_{XZ}=\frac{3}{2}l_a^2-6\,l_s^2\left(1-\frac{5}{4}\sin^2\gamma\right) \quad (59)$$

$$Q_{XZ}=-\frac{15}{8}l_a^4+l_a^2l_s^2\left(\frac{45}{2}-\frac{105}{4}\sin^2\gamma\right)$$
$$+l_s^4\left(-15+\frac{105}{2}\sin^2\gamma-\frac{315}{4}\sin^4\gamma\right) \quad (60)$$

29. **Distribution coefficients in coordinate *yz* planes, magnet vertical.**—This is the D position defined in section 8. Let ϵ = the angle of elevation of the center of the deflecting magnet from the y axis toward the z axis. Then

$$\lambda_a=\mu_a=a=\psi=\theta=0 \quad \nu_a=1 \quad \cos\delta=\frac{c}{d}=\sin\epsilon \quad \cos\xi=0 \quad \cos\eta=0$$

$$\overset{+}{U}=l_a\sin\epsilon \quad \bar{U}=l_a\sin\epsilon \quad \overset{+}{W}=l_a^2\cos^2\epsilon+l_s^2 \quad \bar{W}=l_a^2\cos^2\epsilon+l_s^2$$

$$\frac{\delta\overset{+}{U}}{\delta\theta}=l_s\cos\epsilon \quad \frac{\delta\bar{U}}{\delta\theta}=-l_s\cos\epsilon$$

$$\frac{\delta\overset{+}{W}}{\delta\theta}=-2\,l_al_s\sin\epsilon\cos\epsilon \quad \frac{\delta\bar{W}}{\delta\theta}=2\,l_al_s\sin\epsilon\cos\epsilon.$$

By substitution in equation (2) we obtain

$$\text{Coefficient of } \frac{1}{d^5} = 3\ M_a M_s \sin\epsilon \cos\epsilon \left[l_a^2\left(\frac{35}{6}\sin^2\epsilon - \frac{5}{2}\right) - \frac{5}{2} l_s^2 \right] \tag{61}$$

$$= 3\ M_a M_s \sin\epsilon \cos\epsilon\ P_{DYZ}$$

$$\text{Coefficient of } \frac{1}{d^7} = 3\ M_a M_s \sin\epsilon \cos\epsilon \left[l_a^4\left(\frac{35}{8} - \frac{105}{4}\sin^2\epsilon + \frac{231}{8}\sin^4\epsilon\right) + l_a^2 l_s^2\left(\frac{35}{4} - \frac{105}{4}\sin^2\epsilon\right) + \frac{35}{8} l_s^4 \right] \tag{62}$$

$$= 3\ M_a M_s \sin\epsilon \cos\epsilon\ Q_{DYZ}$$

Equating the sum of terms in $\frac{1}{d^3}$, $\frac{1}{d^5}$, and $\frac{1}{d^7}$ to the torque exerted by the earth's horizontal field we derive the deflection equation

$$H \sin u = \frac{3 M_a}{d^3} \sin\epsilon \cos\epsilon \left(1 + \frac{P_{DYZ}}{d^2} + \frac{Q_{DYZ}}{d^4}\right) \tag{63}$$

For some purposes the distribution coefficients are more conveniently expressed in terms of pole distance of the deflecting magnet. For example, in the A position

$$P = 2\ l_a^2 - 3\ l_a^2$$

$$\text{Now let } \kappa = \frac{l_s}{l_a} \quad \text{then } P = l_a^2\,(2 - 3\kappa^2) \tag{64}$$

In the practical applications of the equations to deflections in the coordinate planes we shall consider the effects of P only, the effect of Q being very small. We therefore summarize the equations in the following tables.

30. **Summary of formulas in coordinate planes.**—

TABLE 1

Plane	Term in $\frac{1}{d^3}$	P
xy Magnet ‖ to y axis	$M_a M_s(3\sin^2\alpha - 1)$	$4\,l_a^2\left[\frac{3}{8} - \frac{15}{4}\sin^2\alpha + \frac{35}{8}\sin^4\alpha - \frac{3}{2}\kappa^2\left(1 - \frac{35}{4}\sin^2\alpha + \frac{35}{8}\sin^4\alpha\right)\right]$
yz Magnet ‖ to y axis	$M_a M_s(3\cos^2\beta - 1)$	$4\,l_a^2\left[1 - 5\sin^2\beta + \frac{35}{8}\sin^4\beta - \frac{3}{2}\kappa^2\left(1 - \frac{5}{4}\sin^2\beta\right)\right]$
xz Magnet ‖ to y axis	$M_a M_s$	$l_a^2\left[\frac{3}{2} - 6\kappa^2\left(1 - \frac{5}{4}\sin^2\gamma\right)\right]$
yz Magnet vertical ‖ to z axis	$3 M_a M_s \sin\epsilon \cos\epsilon$	$l_a^2\left(\frac{35}{6}\sin^2\epsilon - \frac{5}{2} - \frac{5}{2}\kappa^2\right)$

TABLE 1—Continued

Plane	Deflection equations	
xy	$H \tan u = \frac{M_a}{d^3}\left(3 \sin^2 \alpha - 1 + \frac{P_{XY}}{d^2} + \frac{Q_{XY}}{d^4}\right)$	(65)
yz	$H \tan u = \frac{M_a}{d^3}\left(3 \cos^2 \beta - 1 + \frac{P_{YZ}}{d^2} + \frac{Q_{YZ}}{d^4}\right)$	(66)
xz	$H \tan u = \frac{M_a}{d^3}\left(1 + \frac{P_{XZ}}{d^2} + \frac{Q_{XZ}}{d^4}\right)$	(67)
yz Magnet vertical	$H \sin u = \frac{3M_a}{d^3} \sin \epsilon \cos \epsilon \left(1 + \frac{P_{DYZ}}{d^2} + \frac{Q_{DYZ}}{d^4}\right)$	(68)

31. **Conditions for $P=0$ in coordinate planes.**—We have already shown that in some cases the term in $\frac{1}{d^3}$ vanishes. Interest attaches to the first three values of P given in Table 1, in that P also may vanish. Equating the P's to zero and solving for the angles we obtain the results shown in

TABLE 2

Plane	$P=0$ when—	
xy Magnet ‖ to y axis	$\sin^2 \alpha = \frac{3}{7}\left(\frac{1-\frac{7}{2}\kappa^2}{1-3\kappa^2}\right) \pm \left[\frac{9}{49}\left(\frac{1-\frac{7}{2}\kappa^2}{1-3\kappa^2}\right)^2 + \frac{3}{35}\left(\frac{4\kappa^2-1}{1-3\kappa^2}\right)\right]^{\frac{1}{2}}$	(69)
yz Magnet ‖ to y axis	$\sin^2 \beta = \frac{4}{7}\left(1-\frac{3}{8}\kappa^2\right) \pm \left[\frac{16}{49}\left(1-\frac{3}{8}\kappa^2\right)^2 + \frac{4}{35}(3\kappa^2-2)\right]^{\frac{1}{2}}$	(70)
xz Magnet ‖ to y axis	$\sin^2 \gamma = \frac{1}{5}\left(4-\frac{1}{\kappa^2}\right)$	(71)

These equations are in convenient form for practical application. When the suspended magnet is several times the length of the deflecting magnet, P becomes so large that its effects are readily shown by deflections on variometers.

We now turn our attention to the standard A, B, C, and D positions. For these positions the equations are given in the following tables.

32. Summary of formulas, A, B, C, and D positions.—

TABLE 3

Position	P	Q	Factor
A	$2l_a^2-3l_s^2$	$3l^4_a-15l_a^2l_s^2+\frac{45}{8}l_s^4$	$A=1+\frac{P}{d^2}+\frac{Q}{d^4}$
B	$\frac{3}{2}l_a^2-6l_s^2$	$\frac{15}{8}l_a^4-\frac{45}{2}l_a^2l_s^2+15l_s^4$	$B=1-\frac{P}{d^2}+\frac{Q}{d^4}$
C	$\frac{3}{2}(l_a^2+l_s^2)$	$\frac{15}{8}(l_a^2+l_s^2)^2$	$C=1-\frac{P}{d^2}+\frac{Q}{d^4}$
D	$l_a^2\left(\frac{35}{6}-\frac{5}{2}\sin^2\epsilon\right)-\frac{5}{2}l_s^2$	$l_a^4\left(\frac{35}{8}-\frac{105}{4}\sin^2\epsilon+\frac{231}{8}\sin^4\epsilon\right)+l_a^2l_s^2\left(\frac{35}{4}-\frac{105}{4}\sin^2\epsilon\right)+\frac{35}{8}l_s^4$	$D=1+\frac{P}{d^2}+\frac{Q}{d^4}$

Distribution coefficients expressed in terms of l_a and κ

Position	P	Q
A	$l_a^2(2-3\kappa^2)$	$l_a^4\left(3-15\kappa^2+\frac{45}{8}\kappa^4\right)$
B	$l_a^2\left(\frac{3}{2}-6\kappa^2\right)$	$l_a^4\left(\frac{15}{8}-\frac{45}{2}\kappa^2+15\kappa^4\right)$
C	$\frac{3}{2}l_a^2(1+\kappa^2)$	$\frac{15}{8}l_a^4(1+\kappa^2)^2$
D	$l_a^2\left(\frac{35}{6}-\frac{5}{2}\sin^2\epsilon-\frac{5}{2}\kappa^2\right)$	$l_a^4\left[\frac{35}{8}-\frac{105}{4}\sin^2\epsilon+\frac{231}{8}\sin^4\epsilon+\kappa^2\left(\frac{35}{4}-\frac{105}{4}\sin^2\epsilon\right)+\frac{35}{8}\kappa^4\right]$

DEFLECTION EQUATIONS

$$H\sin u=\frac{2M_a}{d^3}A \quad \text{A position, sine form.} \tag{72}$$

$$H\tan u=\frac{2M_a}{d^3}A \quad \text{A position, tangent form.} \tag{73}$$

$$H\tan u=\frac{M_a}{d^3}B \quad \text{B position, tangent form.} \tag{74}$$

$$H\tan u=\frac{M_a}{d^3}C \quad \text{C position, tangent form.} \tag{75}$$

$$H\sin u=\frac{3M_a}{d^3}\sin\epsilon\cos\epsilon\, D \quad \text{D position, sine form.} \tag{76}$$

33. **Scale value equations for variometers.**—To determine the scale values of H and Z variometers by the magnetic method, an auxiliary magnet is used to deflect the D variometer and also the H and Z variometers in the same relative positions and distances as regards the magnets. Two distances are used in order to assess the effects of distribution. Usually the variometers are provided with bars to hold the deflecting magnet in the proper position. Any of the A, B, and C positions may be used, but generally the A position is preferred, since the largest deflections are obtained and the errors of reading or scaling are proportionately less.

The field intensity F of the auxiliary magnet deflecting the D variometer in the A position is given by the equation

$$2\frac{M_a}{d^3}A_D = F = H \tan u \tag{77}$$

in which A_D involves the distribution coefficients P and Q for the deflecting magnet and the D magnet. The field intensity of the auxiliary magnet deflecting the H variometer in the A position is

$$2\frac{M_a}{d^3}A_H = F = su' \times 10^{-5} \tag{78}$$

in which s is the scale value in gammas per unit distance and u' is the deflection. A_H involves the distribution coefficients for the auxiliary deflecting magnet and the recording magnet of the H variometer. Now, if the same auxiliary magnet and the same distances are used, M_a may be eliminated from these equations, resulting in the following scale value equation

$$s = H\gamma \frac{\tan u}{u'} \frac{A_H}{A_D} \tag{79}$$

Thus, in general, the distribution coefficients must be taken into account in determining the scale values of magnetographs. If, however, the magnets of the variometers are of the same shape and size A_H and A_H may be assumed to be equal and will therefore cancel. If the magnets are small and the distances used are large compared with the lengths of the magnets, or if the deflecting magnet is strong enough so that relatively long distances can be used, the A's tend toward unity and will again practically cancel. Under these conditions the scale value equation will be

$$s = H\gamma \frac{\tan u}{u'} \tag{80}$$

34. **Magnetometer equation.**—Equation (42) is used in the deflection observations when determining the absolute values of H with a magnetometer. It is usually written in the form

$$\frac{H}{M_a} = \frac{2}{d^3 \sin u}\left(1 + \frac{P}{d^2} + \frac{Q}{d^4}\right)\left(1 - \frac{2\mu}{d^3}\right) \tag{81}$$

in which the coefficient μ allows for the effects of induction.

35. **Graphs of distribution coefficients.**—Figures 5, 6, and 7 show the graphs of the P' and Q' values. In these graphs the pole distance l_a of the deflecting magnet is taken as unity. Accordingly the abscissas represent the ratios of the pole distance of the suspended magnet to that of the deflecting magnet. We call the curves the P' and Q' curves. The relation between P and P' and between Q and Q' is given by the equations

$$P=l_a^2P' \quad Q=l_a^4Q' \tag{82}$$

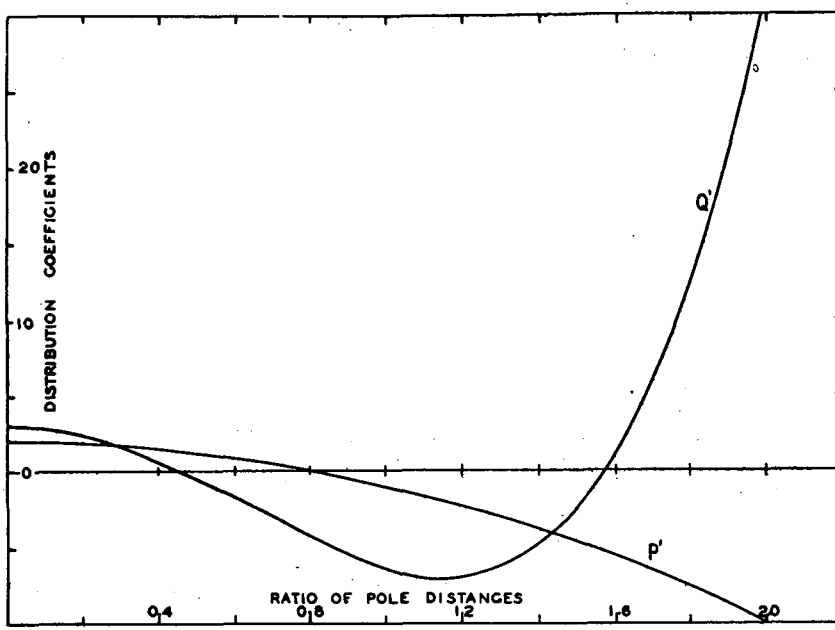

FIGURE 5.—A position, variation of P′ and Q′ with ratio of pole distances

The P' curves are parabolas and have their vertices on the y axis. In the A and B positions they cross the x axis and may therefore be positive, zero, or negative. In the C position P' is always positive.

The Q' curves are equations of the fourth degree. In the A and B positions they cross the x axis and are zero at two points between which they have a minimum and a point of inflection. In position C, Q' is always positive. Both curves are symmetrical to the y axis but we are here concerned only with the positive part of the x axis.

36. **Discussion of P' and Q' curves, A position** (fig. 5).—Since P and Q are functions of the pole distances of the magnets, l_a and l_s may be expressed in terms of P and Q. Solving the equations

$$2\,l_a^2-3\,l_s^2=P$$

$$3\,l_a^4-15\,l_a^2\,l_s^2+\frac{45}{8}l_s^4=Q$$

we obtain

$$l_a^2 = \frac{5}{18}P \pm \left(\frac{35}{162}P^2 - \frac{2}{9}Q\right)^{\frac{1}{2}} \tag{83}$$

$$l_s^2 = -\frac{4}{27}P \pm \left(\frac{70}{81}P^2 - \frac{4}{9}Q\right)^{\frac{1}{2}} \tag{84}$$

Returning to the discussion of the P' and Q' curves, we have, by eliminating l_s from the equations

$$2 - 3\,l_s^2 = P'$$

$$3 - 15\,l_s^2 + \frac{45}{8}l_s^4 = Q'$$

the following equations which express the relation between P' and Q'

$$Q' = \frac{5}{8}P'^2 + \frac{5}{2}P' - \frac{9}{2} \tag{85}$$

$$P' = -2 \pm \left(\frac{56}{5} + \frac{8}{5}Q'\right)^{\frac{1}{2}} \tag{86}$$

Referring to Figure 5, when the suspended magnet is very short l_s may be regarded as zero and P' has the maximum value $=2$. P' becomes zero at a point determined by the equation

$$2 - 3\,l_s^2 = 0$$

This gives $l_s = \sqrt{\frac{2}{3}} = 0.8165$ (87)

Therefore, assuming that the ratio of the geometrical lengths of the magnets is the same as the ratio of the pole distances, we can make P zero by having the suspended magnet 0.8165 times the length of the deflecting magnet. For ratios greater than 0.82, P is negative.

When the suspended magnet is very short $Q' = 3$. It is zero at a point determined by the equation

$$3 - 15\,l_s^2 + \frac{45}{8}l_s^4 = 0. \quad \text{Thus } l_s = 0.4667 \text{ and } 1.5649 \tag{88}$$

Therefore, to have Q zero, the suspended magnet must be either 0.47 or 1.56 times the length of the deflecting magnet.

For the minimum of Q'

$$\frac{\delta Q'}{\delta l_s} = 0 \text{ or } l_s = \sqrt{\frac{4}{3}} = 1.15 \tag{89}$$

The minimum value of Q' is -7.0, while at the same time $P' = -2.0$. For the point of inflection

$$\frac{\delta^2 Q'}{\delta l_s^2} = 0 \text{ from which } l_s = \frac{2}{3}$$

Also we have $Q' = -4.5$ when $P' = 0$.

37. **Discussion of P' and Q' curves, B position** (fig. 6).—Solving the equations

$$6\,l_s^2-\frac{3}{2}l_a^2=-P$$

$$\frac{15}{8}l_a^4-\frac{45}{2}l_a^2\,l_s^2+15\,l_s^4=Q$$

We obtain for the pole distances

$$l_a^2=-\frac{4}{9}P\pm\left(\frac{16}{45}Q+\frac{4}{81}P^2\right)^{\frac{1}{2}} \tag{90}$$

$$l_s^2=\frac{1}{18}P\pm\frac{1}{4}\left(\frac{16}{45}Q+\frac{4}{81}P^2\right)^{\frac{1}{2}} \tag{91}$$

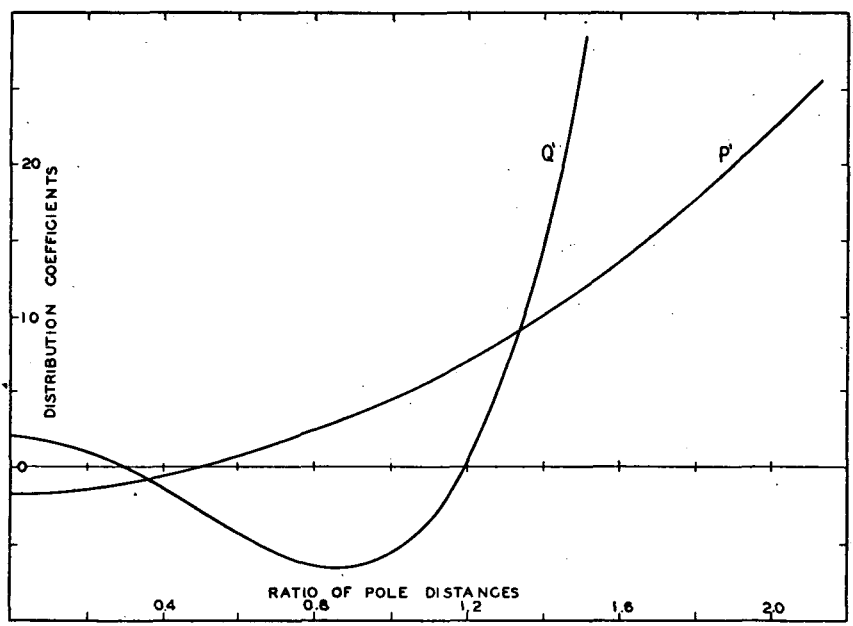

FIGURE 6.—B position, variation of P' and Q' with ratio of pole distances

For the P' and Q' curves

$$Q'=\frac{5}{12}P'^2-\frac{5}{2}P'-\frac{45}{16} \tag{92}$$

$$P'=3\pm\left(\frac{12}{5}Q'+\frac{63}{4}\right)^{\frac{1}{2}} \tag{93}$$

Referring to figure 6, when l_s is very short, P' has the minimum value of -1.67. P' becomes 0 at a point determined by the equation

$$6\,l_s^2-\frac{3}{2}=0 \text{ or } l_s=\frac{1}{2} \tag{94}$$

Assuming that the pole distances have the same ratio as the lengths of the magnets, P may be made 0 by making the suspended magnet one-half the length of the deflecting magnet.

When the suspended magnet is very short, $Q'=1.67$. It is zero at a point determined by the equation

$$\frac{15}{8}-\frac{45}{2}l_s^2+15\,l_s^4=0 \tag{95}$$

Thus $l_s=1.18804$ and 0.29759.

Therefore, to have Q zero in the B position, the suspended magnet must be either 0.3 times or 1.2 times the length of the deflecting magnet.

For the minimum of Q'

$$\frac{\delta^2 Q'}{\delta l_s^2}=0 \text{ or } l_s=\frac{1}{2}\sqrt{3}=0.866 \tag{96}$$

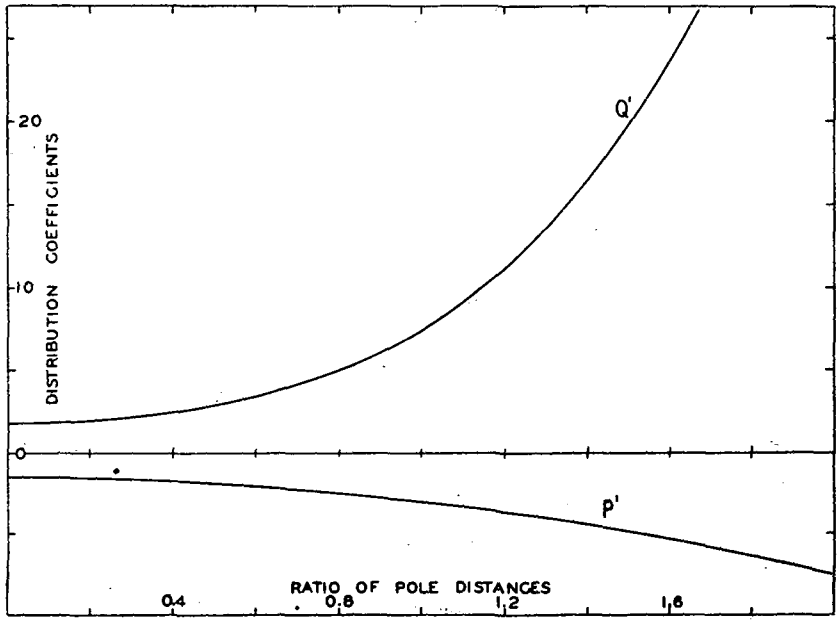

FIGURE 7.—C position, variation of P' and Q' with ratio of pole distances

For the point of inflection

$$\frac{\delta^2 Q'}{\delta l_s^2}=0 \text{ from which } l_s=0.5 \tag{97}$$

38. **Discussion of P' and Q' curves, C position** (fig. 7).—In this case P' and Q' are both parabolic in form. They are both positive and never become zero.

39. **Discussion of P and Q curves, D position** (fig. 8).—Figure 8 gives a concrete example of the values of P and Q in D position when

$l_a=3.8$ and $\kappa=0.467$, for different values of $\sin \epsilon$. For a short suspended magnet, P has large negative values, and becomes zero when

$$\sin^2 \epsilon=\frac{3}{7}(1+\kappa^2) \tag{98}$$

In this special case P is zero when $\sin \epsilon=0.7225$ ($\epsilon=46° 15'.6$). For a pair of magnets whose ratio is $\kappa=0.8165$, P is zero when $\sin \epsilon=0.845$ ($\epsilon=57° 41'.3$). In determining the induction coefficient of the long magnet of a magnetometer, the maximum deflection occurs when $\sin \epsilon=0.447$ ($\epsilon=26° 34'$). In practice the angle is a little less. So in the induction observations P always has a large negative value, and therefore the effect of Q being much smaller than that of P, the deflections are always smaller than those given by the equation

$$H \sin u=\frac{3 M_a}{d^3}\sin \epsilon \cos \epsilon \tag{99}$$

The curve P is, of course, a parabola.

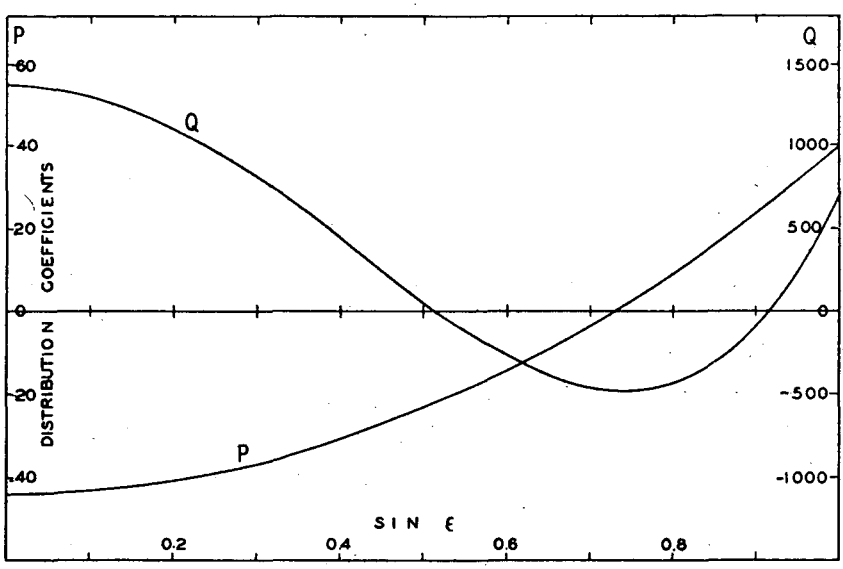

FIGURE 8.—D position, variation of P and Q with sine of angle of elevation of deflector

The curve Q, D position (fig. 8), is drawn for $l_a=3.8$ and $\kappa=0.467$ for different values of $\sin \epsilon$. It is a curve of the fourth degree and resembles the Q' curve, position A. It has a stationary value determined by the equation

$$\sin^2 \epsilon=\frac{5}{11}(1+\kappa^2) \tag{100}$$

which is $\sin \epsilon=0.744$ ($\epsilon=48° 04'.8$) when $\kappa=0.467$. It has a point of inflection when

$$\sin^2 \epsilon=\frac{5}{33}(1+\kappa^2) \tag{101}$$

which is $\sin \epsilon = 0.43$ ($\epsilon = 25° \, 26'.5$) when $\kappa = 0.467$. It crosses the $\sin \epsilon$ axis at two points.

40. **Effect on $\frac{H}{M}$ of magnets of different lengths, A position.**—For this purpose the magnetometer equation is written

$$\frac{H}{M} = \frac{2}{d^3 \sin u} A \tag{42}$$

With the aid of Figure 5 we can trace the effect of magnets of different lengths on the values of $\frac{H}{M}$ at different distances, two distances being sufficient for our purpose. Starting with a very short suspended magnet, A will be greater than unity and $\frac{H}{M}$ derived from the equation

$$\frac{H}{M} = \frac{2}{d^3 \sin u}$$

will be smaller at the short distance.

The reason for this is that the poles of the suspended magnet lie close to the line through the axis of the deflecting magnet, where the field intensity is greatest, and therefore the deflection greatest. Increasing the length of the suspended magnet we shall find a length such that $Q = 0$. The factor A is still greater than unity and, as before, $\frac{H}{M}$ will still be smaller at the short distance. From this point as the magnet lengthens Q becomes negative and $\frac{H}{M}$ for the two distances becomes more nearly equal. As we lengthen the suspended magnet still more, P will become zero, and just before it reaches this value A will be less than unity, and $\frac{H}{M}$ will be larger for the short distance. This indicates that the poles of the suspended magnet are now in a field weaker than that due to the inverse cube law, so that the deflection is relatively smaller. As the magnet still lengthens, A will be less than unity and Q will again become zero. For greater lengths $\frac{H}{M}$ will continue to be larger at the short distance.

41. **Effect of distribution on deflections, A position** (fig. 9).—Figure 9 shows a concrete example of the effects of distribution on the deflections of magnets in the A position. The line u_0, which disregards the effects of distribution, is computed from the formula

$$\sin u_0 = \frac{2M}{r^3 H} \tag{102}$$

and gives the deflection $39° \, 47'.5$. The curve u shows the effect of distribution. For a short suspended magnet—that is, small values

of κ—the deflection is larger than that given by the inverse cube relation (102) and becomes equal to it just before $\kappa = 0.8$. It is thus possible to make the magnets of such lengths that the effects of distribution will vanish when the deflecting magnet is at some particular

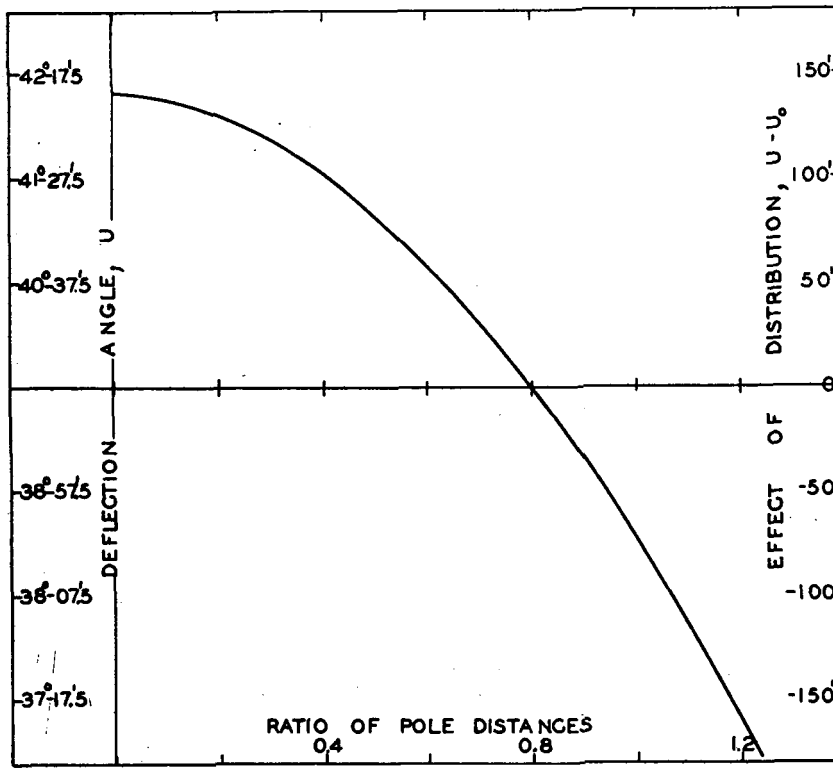

FIGURE 9.—A position, effect of distribution on deflections, with change in ratio of pole distances

position on the bar of the magnetometer. For larger values of κ the deflection is smaller than the inverse relation (102). The general trend of the curve u is approximately a parabola in form, since $\frac{P}{d^2}$ is a parabola and is always greater than $\frac{Q}{d^4}$.

O