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CONFORMAL PROJECTIONS IN GEODESY AND CARTOGRAPHY

By

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FOREWORD

The conformal projections used most in mapping and in geodetic computational work of the U. S. Coast and Geodetic Survey are the Lambert conic, the stereographic, the Mercator, the transverse Mercator, and oblique Mercator.

The mathematical development of the Lambert conformal conic projection is given by O. S. Adams in Special Publication No. 53. The development of the Mercator projection with tables is given in Special Publication No. 68, by C. H. Deetz and O. S. Adams. The development of the transverse Mercator and the stereographic conformal projections may be found in various Coast Survey publications, for instance in the "Manual of Plane-Coordinate Computation," Special Publication No. 193, by O. S. Adams and C. N. Claire, and in "General Theory of Polyconic Projections," Special Publication No. 57, by O. S. Adams.

The purpose of this publication is to bring together in one volume and to give in detail the mathematical development of the formulas (or source references) for these projections in their various forms for the convenience of the geodetic computers and cartographers of the Coast and Geodetic Survey. It will supersede Special Publication No. 53, since it will incorporate the essential material contained therein.

The format, differing somewhat from that of previous Coast Survey publications on projections, has been designed for the convenience of the computer or engineer. All the formulas for the projections are listed first. The mathematical developments or references to their source have been placed last for the convenience of those who would like to check the derivations of the formulas.

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PREFACE

In many of the published treatises on map projections, the autogonal or conformal projections are conceived as conical or cylindrical, the idea being that the ellipsoid is conformally developed on a cone or cylinder which is then in turn developed in the plane, i. e., cut along an element and "rolled out" in the plane. The stereographic and the Mercator are then conceived as being the two limiting positions of the Lambert conformal conic projection. That is, beginning with a tangent cone whose vertex is on the minor axis of the ellipsoid, the vertex is moved away from the spheroid along the axis to an infinite distance which generates in the limiting position a cylinder tangent to the Equator and under the conformal property results in the Mercator projection. If the vertex is moved toward the ellipsoid along the minor axis until it lies on the surface, the tangent plane at the pole is the limiting position and under the conformal property the stereographic projection results. The difficulty with this type of presentation is that these projections are not all perspective, hence the actual point-to-point correspondence is not exhibited and often an erroneous idea is conveyed. Then in some of the transverse or oblique positions of these projections with respect to the ellipsoid, the transition is not easily conceived or obtained.

In most conformal projections the point-to-point correspondence between points on the ellipsoid and points on the plane is not perspective. In fact it would be difficult if not impossible to describe geometrically the method of projection in each case. It is true that by using limiting processes or transformations on the mapping coordinates themselves one can make the transition from Lambert conformal conic to stereographic or Mercator.

It seems better, since the properties of an analytic function of a complex variable lend themselves so admirably to derivation of conformal mapping equations, to discuss autogonal projections from this standpoint, classifying the projection according to the conditions which the map must satisfy as to form and scale of map elements. In this manner, through the medium of the analytic function of a complex variable, we obtain a one-to-one correspondence between points on the ellipsoid and points on the plane without regard to the intermediate or developable surfaces implicitly involved. This method, which is not new, will be followed here in deriving the mapping equations for the autogonal projections.

It should be noted that most of the conformal projections in use today were in existence before complex variable theory had been developed. Deriving the mapping coordinates by this theory is not necessary, but is more elegant from a collectivization standpoint, since we can write down a general analytic function of a complex variable from which all conformal maps of the ellipsoid may be obtained.

The concept, often introduced, of the conformal sphere, that is, the mapping of the ellipsoid upon the conformal sphere and this sphere in turn mapped conformally upon the plane, leading to the development of the conformal latitude, is a useful one and will be demonstrated in some cases in the subsequent developments. It is possible, in some cases, to develop only the sphere conformally upon the plane, the ellipsoid then being taken into account by using the conformal latitudes which have been extensively tabulated by the War Department, Corps of Engineers, U. S. Lake Survey, Military Grid Unit, for several spheroids. (See the bibliography.)

An attempt has been made to keep the mathematical procedure in derivation of formulas as simple as practicable. It was thought necessary to give some account of the elementary parts of complex variable theory since few cartographic engineers are familiar with it and since it is the basis here for the development of the conformal projections. Some of the more essential theorems of the differential geometry of curves and surfaces have been included.

In this publication I have made free use of material in other publications. Particular references in most cases are avoided for sake of continuity but the publications consulted are listed in the bibliography.

Short historical accounts are given with each projection and other sources are indicated in the text or included in the bibliography.

I wish to gratefully acknowledge the valuable assistance given by Erwin Schmid who checked the mathematics of the manuscript, C. N. Claire who edited the manuscript and illustrations, Marjorie L. Moffett who typed the manuscript.

PAUL D. THOMAS.

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JOHANN HEINRICH LAMBERT

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"It is no more than just, therefore, to date the beginning of a new epoch in the science of map making from the appearance of Lambert's work".

O. S. Adams

FRONTISPIECE

CONFORMAL PROJECTIONS IN GEODESY AND CARTOGRAPHY

MAPPING FORMULAS

MERCATOR PROJECTION

The spheroid. Mapping equations,

$$x = a\lambda$$

$$y = \frac{a}{M} \log \left[\tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right) \left(\frac{1 - \epsilon \sin \phi}{1 + \epsilon \sin \phi} \right)^{\epsilon/2} \right]$$

$$= \frac{a}{M} \log \tan \left(\frac{\pi}{4} + \frac{\chi}{2} \right) = \frac{a}{M} \log \cot \frac{z}{2};$$

Magnification,

$$k = \frac{a}{N} \sec \phi;$$

where ϕ, χ, and z are respectively the geodetic latitude, the conformal latitude, and the conformal colatitude; ϵ is the eccentricity of the meridian ellipse; a is usually expressed in units of minutes on the Equator, $a = 3{,}437'.7467708$; M is the modulus of common logarithms, $M = 0.4342944819$; hence $a/M = 7{,}915'.704468$. With λ expressed in radians we have $x = \frac{10{,}800}{\pi} \lambda$ (radians). If λ is expressed in minutes, $x = \lambda'$. N is the radius of curvature normal to the meridian in latitude ϕ, $N = a/\sqrt{1 - \epsilon^2 \sin^2 \phi}$. Series approximation for y in terms of ϕ and ϵ:

$$y = 7{,}915'.704468 \log \tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right)$$

$$-3{,}437'.747 \left[\left(\epsilon^2 + \frac{\epsilon^4}{4} + \frac{\epsilon^6}{8} + \frac{5\epsilon^8}{64} + \ldots \right) \sin \phi \right.$$

$$- \left(\frac{\epsilon^4}{12} + \frac{\epsilon^6}{16} + \frac{3\epsilon^8}{64} + \ldots \right) \sin 3\phi$$

$$\left. + \left(\frac{\epsilon^6}{80} + \frac{\epsilon^8}{64} + \ldots \right) \sin 5\phi - \left(\frac{\epsilon^8}{448} + \ldots \right) \sin 7\phi \right].$$

Series approximation for y in terms of ϕ and f:

$$y = 7{,}915'.704468 \log \tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right)$$

$$-6{,}875'.494 f \left[\sin \phi - \frac{1}{3} \frac{f}{(2-f)} \sin 3\phi + \frac{1}{5} \left(\frac{f}{2-f} \right)^2 \sin 5\phi - \frac{1}{7} \left(\frac{f}{2-f} \right)^3 \sin 7\phi \right]$$

where $f = 1 - b/a = 1 - \sqrt{1 - \epsilon^2}$ = flattening or the compression.

The sphere. Mapping equations,

$$x=a\lambda,\ y=\frac{a}{M}\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)=\frac{a}{M}\log\cot\frac{p}{2};$$

Magnification,

$$k=\sec\phi;$$

where p is the colatitude; the constants a and M are the same as for the spheroid.

TRANSVERSE MERCATOR PROJECTION

Spheroid. Band size, 10 to 12 degrees in longitude—61 degrees latitude north. Conversion of geographic coordinates to rectangular coordinates.

$$\frac{x}{N}=\frac{\Delta\lambda}{\rho}\cos\phi+\frac{\Delta\lambda^3\cos^3\phi}{6\rho^3}(1-t^2+\eta^2)+$$

$$\frac{\Delta\lambda^5\cos^5\phi}{120\rho^5}(5-18t^2+t^4+14\eta^2-58t^2\eta^2+13\eta^4+4\eta^6-64\eta^4t^2-24\eta^6t^2)+$$

$$\frac{\Delta\lambda^7\cos^7\phi}{5{,}040\rho^7}(61-479t^2+179t^4-t^6)$$

$$\frac{y}{N}=\frac{S_\phi}{N}+\frac{\Delta\lambda^2}{2\rho^2}\sin\phi\cos\phi+\frac{\Delta\lambda^4}{24\rho^4}\sin\phi\cos^3\phi(5-t^2+9\eta^2+4\eta^4)+$$

$$\frac{\Delta\lambda^6}{720\rho^6}\sin\phi\cos^5\phi\begin{pmatrix}61-58t^2+t^4+270\eta^2-330t^2\eta^2+445\eta^4+324\eta^6\\-680\eta^4t^2+88\eta^8-600\eta^6t^2-192\eta^8t^2\end{pmatrix}+$$

$$\frac{\Delta\lambda^8}{40{,}320\rho^8}\sin\phi\cos^7\phi(1{,}385-3{,}111t^2+543t^4-t^6),$$

where $\rho=\operatorname{cosec}1''$, $t=\tan\phi$, $\eta^2=\delta\cos^2\phi=\frac{\epsilon^2}{1-\epsilon^2}\cos^2\phi$, S_ϕ=meridian arc from the Equator to latitude ϕ, $\Delta\lambda=\lambda-\lambda_0$=longitude difference from the central meridian λ_0, $N=a/\sqrt{1-\epsilon^2\sin^2\phi}$=the radius of curvature normal to the meridian.

Latitude and longitude from rectangular coordinates.

$$\frac{\Delta\phi}{\rho t_1}=\frac{\phi-\phi_1}{t_1}=-\frac{x^2}{2R_1N_1}+\frac{x^4}{24R_1N_1^3}(5+3t_1^2+\eta_1^2-4\eta_1^4-9\eta_1^2t_1^2)$$

$$-\frac{x^6}{720R_1N_1^5}\begin{pmatrix}61+90t_1^2+46\eta_1^2+45t_1^4-252t_1^2\eta_1^2-3\eta_1^4+100\eta_1^6-66t_1^2\eta_1^4\\-90t_1^4\eta_1^2+88\eta_1^8+225t_1^4\eta_1^4+84t_1^2\eta_1^6-192t_1^2\eta_1^8\end{pmatrix}$$

$$+\frac{x^8}{40{,}320R_1N_1^7}(1{,}385+3{,}633t_1^2+4{,}095t_1^4+1{,}574t_1^6),$$

$$\frac{\Delta\lambda}{\rho\sec\phi_1}=\frac{\lambda-\lambda_0}{\sec\phi_1}=\frac{x}{N_1}-\frac{1}{6}\left(\frac{x}{N_1}\right)^3(1+2t_1^2+\eta_1^2)+\frac{1}{120}\left(\frac{x}{N_1}\right)^5\begin{pmatrix}5+6\eta_1^2+28t_1^2-3\eta_1^4+8t_1^2\eta_1^2+\\24t_1^4-4\eta_1^6+4t_1^2\eta_1^4+24t_1^2\eta_1^6\end{pmatrix}$$

$$-\frac{1}{5{,}040}\left(\frac{x}{N_1}\right)^7(61+662t_1^2+1{,}320t_1^4+720t_1^6),$$

where ϕ_1=footpoint latitude (see fig. 27, p. 100), R_1=radius of curvature of the meridian corresponding to ϕ_1, N_1=radius of curvature normal to the meridian in latitude ϕ_1, $t_1=\tan\phi_1$, $\eta_1^2=\delta\cos^2\phi_1=\frac{\epsilon^2}{1-\epsilon^2}\cos^2\phi_1$, $\rho=\operatorname{cosec}1''$.

For formulas with higher-order terms in the coefficients see equations (288), (289), (314), and (324).

Meridian convergence from geographic coordinates.

$$\frac{\gamma}{\Delta\lambda \sin \phi}=1+\frac{\Delta\lambda^2 \cos^2 \phi}{3\rho^2}(1+3\eta^2+2\eta^4)+$$

$$\frac{\Delta\lambda^4 \cos^4 \phi}{15\rho^4}(2-t^2+15\eta^2+35\eta^4-15\eta^2t^2+33\eta^6-50\eta^4t^2+11\eta^8-60t^2\eta^6-24t^2\eta^8)+$$

$$\frac{\Delta\lambda^6 \cos^6 \phi}{315\rho^6}(17-26t^2+2t^4).$$

Meridian convergence from rectangular coordinates.

$$\frac{\gamma}{\rho t_1}=\frac{x}{N_1}-\frac{1}{3}\left(\frac{x}{N_1}\right)^3(1+t_1^2-\eta_1^2-2\eta_1^4)+$$

$$\frac{1}{15}\left(\frac{x}{N_1}\right)^5(2+5t_1^2+2\eta_1^2+3t_1^4+t_1^2\eta_1^2+9\eta_1^4+20\eta_1^6-7t_1^2\eta_1^4-27t_1^2\eta_1^6+11\eta_1^8-24t_1^2\eta_1^8)-$$

$$\frac{1}{315}\left(\frac{x}{N_1}\right)^7(17+77t_1^2+105t_1^4+45t_1^6).$$

The scale from geographic coordinates.

$$k=1+\frac{\Delta\lambda^2 \cos^2 \phi}{2}(1+\eta^2)+$$

$$\frac{\Delta\lambda^4 \cos^4 \phi}{24}(5-4t^2+14\eta^2+13\eta^4-28t^2\eta^2+4\eta^6-48t^2\eta^4-24t^2\eta^6)+$$

$$\frac{\Delta\lambda^6 \cos^6 \phi}{720}(61-148t^2+16t^4).$$

The scale from rectangular coordinates.

$$k=1+\frac{1}{2}\left(\frac{x}{N_1}\right)^2(1+\eta_1^2)+$$

$$\frac{1}{24}\left(\frac{x}{N_1}\right)^4(1+6\eta_1^2+9\eta_1^4+4\eta_1^6-24t_1^2\eta_1^4-24t_1^2\eta_1^6)+\frac{1}{720}\left(\frac{x}{N_1}\right)^6.$$

Reciprocal of the scale from rectangular coordinates.

$$\frac{1}{k}=1-\frac{1}{2}\left(\frac{x}{N_1}\right)^2(1+\eta_1^2)+\frac{1}{24}\left(\frac{x}{N_1}\right)^4(5+6\eta_1^2-3\eta_1^4-4\eta_1^6+24t_1^2\eta_1^4+24t_1^2\eta_1^6)-\frac{61}{720}\left(\frac{x}{N_1}\right)^6.$$

The $(t-T)$ corrections. (See fig. 25, p. 75.)

$$\frac{(t_1-T_1)}{\rho}=-\frac{1}{6R_m^2}(y_2-y_1)(x_2+2x_1)+\frac{\eta_1^2 t_1}{3R_m^3}(y_2-y_1)^2(x_1+x_2)$$

$$-\frac{\eta_1^2 t_1}{6R_m^3}(x_2-x_1)(x_2^2+2x_2x_1+3x_1^2)+\ . \ . \ . \ ,$$

$$\frac{(t_2-T_2)}{\rho}=-\frac{1}{6R_m^2}(y_1-y_2)(x_1+2x_2)-\frac{\eta_1^2 t_1}{3R_m^3}(y_1-y_2)^2(x_1+3x_2)$$

$$-\frac{\eta_1^2 t_1}{6R_m^3}(x_1-x_2)(x_1^2+2x_1x_2+3x_2^2)+\ . \ . \ . \ ,$$

where $R_m=\sqrt{R_1N_1}$, mean radius at the point P_1; ρ=cosec 1″.

For examples of the tabulation of coefficients in the above equations and applications of the formulas see the following publications: Army Map Service Technical Manual No. 19, Universal Transverse Mercator Grid, Corps of Engineers, Department of the Army, Washington, D. C.; Ordnance Survey, Constants, Formulae and Methods Used in Transverse Mercator Projection; Projection Tables for the Transverse Mercator Projection of Great Britain, London, 1950.

The following more simple formulas are for smaller bands, about 2 degrees each side of the central meridian, as used for computing triangulation and for State plane coordinate systems.

Transverse Mercator coordinates, scale, and convergence from geographic coordinates.

$$\frac{x}{N}=\frac{\Delta\lambda}{\rho}\cos\phi+\frac{\Delta\lambda^3\cos^3\phi}{6\rho^3}(1-t^2+\eta^2)+\frac{\Delta\lambda^5\cos^5\phi}{120\rho^5}(5-18t^2+t^4),$$

$$\frac{y}{N}=\frac{S_\phi}{N}+\frac{\Delta\lambda^2}{2\rho^2}\sin\phi\cos\phi+\frac{\Delta\lambda^4}{24\rho^4}(\sin\phi\cos^3\phi)\ (5-t^2),$$

$$k=1+\frac{\Delta\lambda^2\cos^2\phi}{2}(1+\eta^2)+\frac{\Delta\lambda^4\cos^4\phi}{24}(5-4t^2),$$

$$\gamma=\Delta\lambda\sin\phi\left[1+\frac{\Delta\lambda^2\cos^2\phi}{3\rho^2}(1+3\eta^2)+\frac{\Delta\lambda^4\cos^4\phi}{15\rho^4}(2-t^2)\right].$$

Geographic coordinates, scale, and convergence from rectangular coordinates.

$$\Delta\phi=\phi-\phi_1=\rho t_1\left[-\frac{x^2}{2R_1N_1}+\frac{x^4}{24R_1N_1^3}(5+3t_1^2)\right],$$

$$\Delta\lambda=\lambda-\lambda_0=\rho\sec\phi_1\left[\frac{x}{N_1}-\frac{1}{6}\left(\frac{x}{N_1}\right)^3(1+2t_1^2+\eta_1^2)+\frac{1}{120}\left(\frac{x}{N_1}\right)^5(5+28t_1^2+24t_1^4)\right],$$

$$k=1+\frac{1}{2}\left(\frac{x}{N_1}\right)^2(1+\eta_1^2)+\frac{1}{24}\left(\frac{x}{N_1}\right)^4(1+6\eta_1^2),$$

$$\gamma=\rho t_1\left[\frac{x}{N_1}-\frac{1}{3}\left(\frac{x}{N_1}\right)^3(1+t_1^2-\eta_1^2)+\frac{1}{15}\left(\frac{x}{N_1}\right)^5(2+5t_1^2+3t_1^4)\right].$$

The $(t-T)$ corrections.

$$t_1-T_1=-\frac{\rho}{6R_m^2}(y_2-y_1)(x_2+2x_1),$$

$$t_2-T_2=-\frac{\rho}{6R_m^2}(y_1-y_2)(x_1+2x_2).$$

Coordinates from bearing and distance.

From figure 25 (p. 75) we have $x_2=x_1+d \sin t_1$, $y_2=y_1+d \cos t_1$. For a first approximation assume the projected geodesic and its rectilinear chord to be coincident. We have then $x_2=x_1+S \sin \alpha$, $y_2=y_1+S \cos \alpha$ where S is the spheroidal geodesic distance. With approximate values of x_2 and y_2 computed from these formulas we then compute s and β from the formulas

$$s=S+\frac{S}{2R_m^2}\left[\left(\frac{x_1+x_2}{2}\right)^2+\frac{1}{3}\left(\frac{x_2-x_1}{2}\right)^2\right],$$

$$\beta=\alpha-\frac{\rho(y_2-y_1)(x_2+2x_1)}{6R_m^2}.$$

Then

$$x_2=x_1+s \sin \beta, \qquad y_2=y_1+s \cos \beta$$

and

$$\alpha'=\beta\pm 180°+\frac{\rho(y_1-y_2)(x_1+2x_2)}{6R_m^2}.$$

Bearing and distance from coordinates.

$$\beta=\tan^{-1}\frac{x_2-x_1}{y_2-y_1}, \qquad s=(x_2-x_1)\csc \beta=(y_2-y_1)\sec \beta,$$

$$S=s-\frac{s}{2R_m^2}\left[\left(\frac{x_1+x_2}{2}\right)^2+\frac{1}{3}\left(\frac{x_2-x_1}{2}\right)^2\right],$$

$$\alpha=\beta+\frac{\rho(y_2-y_1)(x_2+2x_1)}{6R_m^2}$$

$$\alpha'=\beta\pm 180°+\frac{\rho(y_1-y_2)(x_1+2x_2)}{6R_m^2}$$

The bearings in the above formulas are taken from true north.

Examples of the above formulas are found in Plane and Geodetic Surveying, D. Clark, Fourth Edition, 1951, Chapter V. The application of the transverse Mercator projection to computation of State plane coordinate systems is found in the Manual of Plane-Coordinate Computation and the Manual of Traverse Computation on the Transverse Mercator Grid by O. S. Adams and C. N. Claire, U. S. Coast and Geodetic Survey Special Publications Nos. 193 and 195.

Formulas for the sphere.

When $\epsilon=0$, $\eta^2=\frac{\epsilon^2}{1-\epsilon^2}\cot^2\phi=0$, $N=R=a$. Hence the spherical formulas are easily obtained from the spheroidal formulas above by placing $\eta^2=0$, $N=R=a$. Therefore only the spherical formulas in closed form will be listed here.

Mapping equations and scale in closed form.

$$x=\frac{a}{2}\ln\left(\frac{1+\cos\phi\sin\lambda}{1-\cos\phi\sin\lambda}\right)=a\tanh^{-1}(\cos\phi\sin\lambda),$$

$$y=a\tan^{-1}(\tan\phi\sec\lambda).$$

$$k=1/\sqrt{1-\cos^2\phi\sin^2\lambda}.$$

Equations of meridians and parallels.

$$\sin^2\lambda\coth^2\frac{x}{a}-\cos^2\lambda\tan^2\frac{y}{a}=1,\quad \text{(meridians)}$$

$$\sec^2\phi\tanh^2\frac{x}{a}+\tan^2\phi\cot^2\frac{y}{a}=1,\quad \text{(parallels)}.$$

OBLIQUE MERCATOR PROJECTION

The sphere. Formulas in closed form.

Mapping equations, and scale.

$$x=\frac{a}{2}\ln\frac{1+\sin\phi_0\sin\phi+\cos\phi_0\cos\phi\sin\Delta\lambda}{1-\sin\phi_0\sin\phi-\cos\phi_0\cos\phi\sin\Delta\lambda}$$

$$=a\tanh^{-1}(\sin\phi_0\sin\phi+\cos\phi_0\cos\phi\sin\Delta\lambda),$$

$$y=a\tan^{-1}\frac{\sin\phi_0\cos\phi\sin\Delta\lambda-\cos\phi_0\sin\phi}{\cos\phi\cos\Delta\lambda},$$

$$k=1/\sqrt{1-(\sin\phi_0\sin\phi+\cos\phi_0\cos\phi\sin\Delta\lambda)^2},$$

where $\Delta\lambda=\lambda_0-\lambda$, λ_0 being the central map meridian.

Coordinates of the pole of the great circle of true scale, (ϕ_0, λ_0).

Origin of coordinates $\phi=0$, $\Delta\lambda=\lambda_0-\lambda=0$.

Equations of meridians and parallels.

$$\left(\sin\Delta\lambda-\sin\phi_0\cos\Delta\lambda\tan\frac{y}{a}\right)^2\coth^2\frac{x}{a}-\left(\sin\phi_0\sin\Delta\lambda-\cos\Delta\lambda\tan\frac{y}{a}\right)^2=\cos^2\phi_0,$$
(meridians)

$$\sec^2\phi_0\left(\tanh\frac{x}{a}-\sin\phi_0\sin\phi\right)^2+\cot^2\frac{y}{a}\left[\tan\phi_0\left(\tanh\frac{x}{a}-\sin\phi_0\sin\phi\right)-\cos\phi_0\sin\phi\right]^2=\cos^2\phi.$$
(parallels)

If $\lambda_0=0$, then $\Delta\lambda=-\lambda$ and we must therefore replace in the above formulas $\sin\Delta\lambda$ by $-\sin\lambda$ and $\cos\Delta\lambda$ by $\cos\lambda$. Longitude is then measured from the point where the true scale great circle crosses the Equator, i. e., the point O' in figure 28 (p. 110). The required changes in the above formulas are easily made so no relisting of the formulas is necessary.

Great circle through two given points.

When the great circle to be held true to scale is that joining two given points $Q_1(\phi_1, \lambda_1)$ and $Q_2(\phi_2, \lambda_2)$ as shown in figure 29 (p. 113), we must compute the coordinates λ_0 and ϕ_0 of the vertex or the point where the great circle is orthogonal to a meridian, i. e. the point $O(\phi_0, \lambda_0)$ in figure 29 (p. 113) or the point $Q(\phi_0, \lambda_0)$ in figure 30 (p. 114).

Equations for λ_0 and ϕ_0.

$$\tan \lambda_0 = \frac{\tan \phi_2 \cos \lambda_1 - \tan \phi_1 \cos \lambda_2}{\tan \phi_1 \sin \lambda_2 - \tan \phi_2 \sin \lambda_1},$$

$$\cot \phi_0 = \cot \phi_1 \cos (\lambda_1 - \lambda_0) = \cot \phi_2 \cos (\lambda_2 - \lambda_0).$$

Equation of the great circle $Q_1 Q_2$.

$$\cos (\lambda - \lambda_0) = \cot \phi_0 \tan \phi.$$

Mapping equations referred to the vertex $Q(\phi_0, \lambda_0)$ of the great circle as shown in figure 30 (p. 114).

$$x = a \tan^{-1} \frac{\cos \phi \sin \Delta\lambda}{\sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \cos \Delta\lambda},$$

$$y = \frac{a}{2} \ln \frac{1 + \sin \phi \cos \phi_0 - \cos \phi \sin \phi_0 \cos \Delta\lambda}{1 - \sin \phi \cos \phi_0 + \cos \phi \sin \phi_0 \cos \Delta\lambda}$$

$$= a \tanh^{-1} (\sin \phi \cos \phi_0 - \cos \phi \sin \phi_0 \cos \Delta\lambda),$$

where x and y axes have their orientation as shown in figure 30 (p. 114).

The spheroid. The following formulas are essentially those developed by Brigadier M. Hotine and published in the Empire Survey Review, Vol. IX, No. 64. This approximation to the oblique Mercator projection of the spheroid is called the rectified skew orthomorphic projection.

Computation of constants.

Two widely spaced points are selected on a line running centrally with respect to the skewed area to be mapped. The geographic coordinates of these two points are then determined. The corresponding isometric latitudes are computed or obtained from tables. Suppose the coordinates of the two points are (τ_1, λ_1), (τ_2, λ_2) where τ is isometric latitude, i. e. $\tau = \ln \left[\tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right) \left(\frac{1 - \epsilon \sin \phi}{1 + \epsilon \sin \phi} \right)^{\frac{\epsilon}{2}} \right]$, and λ is longitude (positive westward from Greenwich). The constants γ_0 and λ_0 are then computed from

$$-\tan \gamma_0 = \frac{\sin [B(\lambda_1 - \lambda_0)]}{\sinh (B\tau_1 + C)} = \frac{\sin [B(\lambda_2 - \lambda_0)]}{\sinh (B\tau_2 + C)},$$

$$\tan \left\{ \frac{1}{2} B(\lambda_1 + \lambda_2) - B\lambda_0 \right\} = \frac{\tan \left[\frac{1}{2} B(\lambda_1 - \lambda_2) \right] \tanh \left\{ \frac{1}{2} B(\tau_1 + \tau_2) + C \right\}}{\tanh \left[\frac{1}{2} B(\tau_1 - \tau_2) \right]},$$

where $B = (1 + \epsilon_1^2 \cos^4 \phi_0)^{\frac{1}{2}}$, $A = B(R_0 N_0)^{\frac{1}{2}}$, $C = \cosh^{-1}(A/r_0) - B\tau_0$, $\epsilon_1^2 = \epsilon^2/(1 - \epsilon^2)$, ϵ is the eccentricity of the meridian ellipse, and ϕ_0, N_0, R_0, $r_0 = N_0 \cos \phi_0$, τ_0 are evaluated for the particular latitude where minimum distortion is required.

The projection formulas.

With the constants A, B, C, γ_0, λ_0 and the scale k the projection formulas are:

$$\frac{kr}{A} \cos \gamma = \cos \gamma_0 \cos \frac{By}{A} \cosh \frac{Bx}{A},$$

$$\frac{kr}{A} \sin \gamma = \sin \gamma_0 - \cos \gamma_0 \sin \frac{By}{A} \sinh \frac{Bx}{A},$$

$$\frac{A}{kr}\cos\gamma=\cos\gamma_0\cos[B(\lambda-\lambda_0)]\cosh(B\tau+C),$$

$$\frac{A}{kr}\sin\gamma=\sin\gamma_0-\cos\gamma_0\sin[B(\lambda-\lambda_0)]\sinh(B\tau+C),$$

$$\tan[B(\lambda-\lambda_0)]=\left(\cos\gamma_0\sinh\frac{Bx}{A}-\sin\gamma_0\sin\frac{By}{A}\right)\Big/\cos\frac{By}{A},$$

$$\tanh(B\tau+C)=\left(\cos\gamma_0\sin\frac{By}{A}+\sin\gamma_0\sinh\frac{Bx}{A}\right)\Big/\cosh\frac{Bx}{A},$$

$$\tanh\frac{Bx}{A}=\{\cos\gamma_0\sin[B(\lambda-\lambda_0)]+\sin\gamma_0\sinh(B\tau+C)\}/\cosh(B\tau+C),$$

$$\tan\frac{By}{A}=\{\cos\gamma_0\sinh(B\tau+C)-\sin\gamma_0\sin[B(\lambda-\lambda_0)]\}/\cos[B(\lambda-\lambda_0)].$$

Tabulated functions.

To effectively employ the above projection formulas the following 16 functions were tabulated at suitable intervals for the Malaya and Borneo projections:

For argument ϕ	For argument λ
I $\cosh(B\tau+C)$	V $\cos[B(\lambda-\lambda_0)]$
II $\tanh(B\tau+C)$	VI $\tan[B(\lambda-\lambda_0)]$
III $\cos\gamma_0\sinh(B\tau+C)$	VII $\cos\gamma_0\sin[B(\lambda-\lambda_0)]$
IV $\sin\gamma_0\sinh(B\tau+C)$	VIII $\sin\gamma_0\sin[B(\lambda-\lambda_0)]$
Functions of y	**Functions of x**
IX $\cos(By/A)$	XIII $\cosh(Bx/A)$
X $\tan(By/A)$	XIV $\tanh(Bx/A)$
XI $\cos\gamma_0\sin(By/A)$	XV $\cos\gamma_0\sinh(Bx/A)$
XII $\sin\gamma_0\sin(By/A)$	XVI $\sin\gamma_0\sinh(Bx/A)$

In the following formulas the roman numerals refer to the above tabulated functions.
Rectangular coordinates from geographic coordinates.
x is found by interpolation from XIV=(VII+IV)/I,
y is found by interpolation from X=(III−VIII)/V.
Geographic coordinates from rectangular coordinates.
ϕ is found by interpolation from II=(XI+XVI)/XIII,
λ is found by interpolation from VI=(XV−XII)/IX.

Skew convergence of meridians.

From geographic coordinates $\tan\gamma=[\tan\gamma_0-\sec^2\gamma_0\,(\text{III})(\text{VII})]/(\text{I})(\text{V})$. From rectangular coordinates $\tan\gamma=[\tan\gamma_0-\sec^2\gamma_0\,(\text{XI})(\text{XV})]/(\text{IX})(\text{XIII})$.

The scale factor.

At any point, $$k=\frac{A}{N\cos\phi}\cdot\frac{\cosh(Bx/A)}{\cosh(B\tau+C)}=\frac{A}{N\cos\phi}\cdot\frac{\text{XIII}}{\text{I}}=\frac{A}{N\cos\phi}\cdot\frac{\cos(By/A)}{\cos B(\lambda-\lambda_0)}.$$

The scale factor for a line may be computed by $k=(1/6)(k_1+4k_m+k_2)$ where k_1 and k_2 are the scale factors at the ends of the line and k_m the scale factor at the midpoint.

An over-all scale factor to reduce the extreme scale error may be incorporated in the value of the constant A. For any line we may take the scale factor as

$$\frac{A}{N_m \cos \phi_m \cosh (B\tau_m+C)}\left[1+\frac{1}{6}\frac{B^2}{A^2}(x_1^2+x_1 x_2+x_2^2)\right],$$

where the subscript m signifies the values of the functions at the midpoint of the line.
The $(t-T)$ correction.

$$t_1-T_1=\frac{1}{2}(y_2-y_1)\frac{B}{A \sin 1''} \tanh \left\{\frac{B}{A}\frac{2x_1+x_2}{3}\right\}+\epsilon_1^2\frac{R_0}{N_0}\sin \phi_0\cdot$$

$$\left\{\sin \frac{1}{3}(2\phi_1+\phi_2)-\sin \phi_0\right\}^2 (\lambda_1-\lambda_2),$$

where $\lambda_1-\lambda_2$ is in seconds. For lines not over 70 miles in length the maximum value of the second term is 0.″007. It can therefore usually be neglected and placing

$\tanh \left\{\frac{B}{A}\frac{2x_1+x_2}{3}\right\}=\frac{B}{A}\cdot\frac{2x_1+x_2}{3}$ we obtain for subsidiary work, the formula

$$t_1-T_1=\frac{B^2}{6\,A^2 \sin 1''}(y_2-y_1)(2x_1+x_2).$$

Rectified coordinates.

If N is the Northing map coordinate and E the Easting map coordinate we have by the ordinary rotation formulas for a plane rectangular coordinate system that

$$\begin{aligned} N&=y \cos \gamma_0+x \sin \gamma_0 & \qquad x&=-E \cos \gamma_0+N \sin \gamma_0\\ E&=y \sin \gamma_0-x \cos \gamma_0, \text{ or} & y&=\ \ E \sin \gamma_0+N \cos \gamma_0 \end{aligned}$$

where γ_0 is the skew convergence of the meridian through the origin or the angle which the center line of the skew projection makes with the meridian at the origin. The formulas for the computation of γ_0 are given above under computation of constants. Note that false Northings or Eastings may be added to the above rectified coordinates as is usually done to avoid negative plane coordinates.

Convergence of map meridians.

The convergence of map meridians is defined as the angle, measured positively clockwise, from true North to rectified grid North and is denoted by γ_R.

$\gamma_R=\gamma-\gamma_0$, where γ is computed by the formulas given above under skew convergence of meridians.

For examples of the application of this projection to actual skew areas see the publications entitled Projection Tables for British Commonwealth Territories in Borneo (Malaya), prepared by Directorate of Colonial Surveys, Teddington, Middlesex, England.

For tabulating the expressions above involving hyperbolic functions, there are available the following useful tables: Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments, National Bureau of Standards, U. S. Government Printing Office, 1949; Tables of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments, Columbia University Press, New York, 1943.

LAMBERT CONFORMAL CONIC PROJECTION

Spheroid. One standard parallel.

Mapping Equations—origin at the pole.

$x=r\cos l\lambda,\ y=r\sin l\lambda.$

Coordinates plotted from the intersection of the parallel in latitude ϕ with the central meridian.

$$x=r\sin l\lambda,\ y=r(1-\cos l\lambda)=x\tan\frac{l\lambda}{2}.$$

$$r=Ke^{-l\tau}=K\tan^{l}\frac{Z}{2},\ K=e^{l\tau_0}N_0\cot\phi_0=\cot^{l}\frac{Z_0}{2}N_0\cot\phi_0,\ l=\sin\phi_0.$$

Magnification

$k=rl/N\cos\phi,$

where ϕ_0=standard parallel

λ=longitude from the central meridian

Z=the conformal colatitude, i. e.

$$e^{-\tau}=\cot\left(\frac{\pi}{4}+\frac{\chi}{2}\right)=\cot\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1+\epsilon\sin\phi}{1-\epsilon\sin\phi}\right)^{\frac{\epsilon}{2}}=\tan\frac{Z}{2}.$$

Mapping Equations—origin at the intersection of the fixed parallel ϕ_0 with the central meridian.

$$x=r\sin l\lambda,\ y=r_0-r\cos l\lambda,$$

where for $\phi\neq\phi_0$ we have

$$r=r_0\mp\Delta r,$$

$$\Delta r=S+\frac{S^3}{6R_0N_0}\pm\frac{S^4(5R_0-4N_0)\tan\phi_0}{24R_0^2N_0^2}+\frac{S^5(5+3\tan^2\phi_0)}{120R_0N_0^3}\pm\frac{S^6(7+4\tan^2\phi_0)\tan\phi_0}{240R_0N_0^4},$$

$$r_0=N_0\cot\phi_0,$$

R_0, N_0=principal radii in latitude ϕ_0,

S=meridional arc of the spheroid measured from the parallel ϕ_0—positive with decreasing latitude; obtained from tables.

λ=longitude from the central meridian.

NOTE: For the rigorous series for Δr see equation (402) on page 120.

Spheroid. Two standard parallels.

Mapping equations—origin at the pole.

$$x=r\cos l\lambda,\ y=r\sin l\lambda.$$

Coordinates plotted from the intersection of the parallel in latitude ϕ with the central meridian.

$$x=r\sin l\lambda,\ y=r(1-\cos l\lambda)=x\tan\frac{l\lambda}{2}.$$

$$r=K\tan^{l}\frac{Z}{2},\ K=\frac{N_1\cos\phi_1}{l\tan^{l}\frac{Z_1}{2}}=\frac{N_2\cos\phi_2}{l\tan^{l}\frac{Z_2}{2}},$$

$$l=\frac{\log\cos\phi_1-\log\cos\phi_2+\log N_1-\log N_2}{\log\tan\frac{Z_1}{2}-\log\tan\frac{Z_2}{2}}.$$

Magnification

$$k=lr/N\cos\phi.$$

Mapping Equations—origin at the intersection of the central meridian and the parallel ϕ_0 of the corresponding one-standard-parallel projection.

$$\phi_0=\text{arc}\sin l=\text{arc}\sin\frac{\log\cos\phi_1-\log\cos\phi_2+\log N_1-\log N_2}{\log\tan\frac{Z_1}{2}-\log\tan\frac{Z_2}{2}}$$

$$x=r\sin l\lambda,\ y=r_0-r\cos l\lambda,$$

$r_0=N_0\cot\phi_0$; for $\phi\neq\phi_0$, $r=r_0\mp k_s\,\Delta r$, where Δr is the same as given above for the one-standard-parallel projection, and k_s is the scale reduction for Δr at ϕ_0 given by

$$k_s=1+\frac{S^2}{2R_0N_0}\pm\frac{S^3(5R_0-4N_0)\tan\phi_0}{6R_0^2N_0^2}+\frac{S^4(5+3\tan^2\phi_0)}{24R_0N_0^3}\pm\frac{S^5(7+4\tan^2\phi_0)\tan\phi_0}{40R_0N_0^4}.$$

NOTE: For the rigorous formula for k_s see equation (424) on page 123.

Conversion of geographic to Lambert rectangular coordinates.

$$r_0=r(\phi_0)=N_0\cot\phi_0,\ \Delta r=S+\frac{S^3}{6R_0N_0}\pm\frac{S^4(5R_0-4N_0)\tan\phi_0}{24R_0^2N_0^2}$$

$$+\frac{S^5(5+3\tan^2\phi_0)}{120R_0N_0^3}\pm\frac{S^6(7+4\tan^2\phi_0)\tan\phi_0}{240R_0N_0^4}$$

$$\Delta\lambda=\lambda-\lambda_0,\ \gamma=\Delta\lambda\sin\phi_0,\ x=(r_0\mp\Delta r)\sin\gamma,\ y=\Delta r+x\tan\frac{1}{2}\gamma.$$

$\Delta\lambda$ is positive east of the central meridian, negative west of the central meridian. S is the meridian distance between latitudes ϕ_0 and ϕ. Δr is tabulated for suitable intervals of ϕ.

Conversion of Lambert rectangular coordinates to geographic coordinates. $\tan\gamma=\frac{x}{r_0-y}$, $\Delta r=y-x\tan\frac{1}{2}\gamma$, $\Delta\lambda=\gamma\operatorname{cosec}\phi_0$.

Knowing Δr, the corresponding latitude can be obtained by interpolation from the table giving Δr for different values of ϕ. Alternatively, S can be obtained by successive approximations from the formula above for Δr and ϕ then found from a table of meridional distances.

Computation of Lambert rectangular coordinates from bearing and distance.

$$y_2-y_1=S\cos\alpha-\frac{y_1S^2\sin^2\alpha}{2R_m^2}-\frac{S\cos\alpha\cdot S^2\sin^2\alpha}{6R_m^2}+\frac{y_2^3-y_1^3}{6R_m^2}+\frac{Sy_1^3\cos\alpha\sec^3\delta\cos3\delta\tan\phi_0}{6R_m^2}$$

$$+\frac{S^4\cos4\alpha\tan\phi_0}{24R_m^3}+\frac{S^2y_1^2\sec^2\delta\cos2(\alpha+\delta)\tan\phi_0}{4R_m^3}+\frac{S^3y_1\sec\delta\cos(3\alpha+\delta)\tan\phi_0}{6R_m^3},$$

$$x_2-x_1=S\sin\alpha+\frac{y_2^2S\sin\alpha}{2R_m^2}-\frac{S^2\cos^2\alpha\cdot S\sin\alpha}{6R_m^2}+\frac{Sy_1^3\sin\alpha\sec^3\delta\cos3\delta\tan\phi_0}{6R_m^3}$$

$$+\frac{S^2y_1^2 \sec^2\delta \sin 2(\alpha+\delta) \tan\phi_0}{4R_m^3}+\frac{S^3y_1 \sec\delta \sin(3\alpha+\delta)\tan\phi_0}{6R_m^3}+\frac{S^4 \sin 4\alpha \tan\phi_0}{24R_m^3},$$

$$\alpha'=\alpha\pm 180°+\frac{S \sin\alpha(y_1+y_2)}{2R_m^2 \sin 1''},$$

where $\tan\delta=\frac{x_1}{y_1}$, ϕ_0=latitude of the origin, $R_m^3=R_0N_0^2$.

When a line is only a few miles in length and not more than about 150 miles from the origin, coordinates may be computed from:

$$s=S\left[1+\frac{m^2}{2R_0N_0}+\frac{m^3\tan\phi_0}{6R_0N_0^2}+\frac{m^4(5+3\tan^2\phi_0)}{24R_0N_0^3}\right],$$

$$\beta=\alpha+\frac{(x_2-x_1)(2y_1+y_2)}{6R_m^2\sin 1''}, \qquad x_2=x_1+s\sin\beta,$$

$$\alpha'=\beta\pm 180°+\frac{(x_2-x_1)(2y_2+y_1)}{6R_m^2\sin 1''}, \qquad y_2=y_1+s\cos\beta,$$

where m is the true meridian distance of the midpoint of the line and β is the grid bearing.

When the lines are long or the x's and y's are large we compute s from the above formula, using the scale factor for the midpoint of the line and the angle θ computed

from $\theta=\alpha+\dfrac{S\sin\alpha\left(y_1+\frac{1}{2}S\cos\alpha\right)}{2R_m^2\sin 1''}$.

Then $x_2=x_1+s\sin\theta$, $y_2=y_1+s\cos\theta$

$$\alpha'=\alpha\pm 180°+\frac{S\sin\alpha(y_1+y_2)}{2R_m^2\sin 1''}.$$

For lines about 30 miles in length the latter formulas will give accuracy to about 1/100,000. For much longer lines where highest degree of accuracy is required no really satisfactory formulas have been derived for point-to-point working directly in terms of Lambert conformal coordinates. In such cases one may compute geographic coordinates by Puissant's or Clarke's formulas and then transform these geographic coordinates into Lambert conformal coordinates.

Distances and bearings from Lambert rectangular coordinates.

One may use successive approximations in the formulas above for computing Lambert conformal coordinates from bearing and distance.

A first approximation from these gives

$$\tan\alpha=\frac{x_2-x_1}{y_2-y_1},\ S=(y_2-y_1)\sec\alpha=(x_2-x_1)\operatorname{cosec}\alpha.$$

These values are used in the second and succeeding terms to get new values for $S\sin\alpha$ and $S\cos\alpha$.

For short lines not too far from the origin calculate β and s from $\tan \beta = \dfrac{x_2 - x_1}{y_2 - y_1}$, $s = (y_2 - y_1) \sec \beta = (x_2 - x_1) \operatorname{cosec} \beta$.
Calculate S and α from

$$S = s\left(1 - \frac{m^2}{2R_0N_0} - \frac{m^3 \tan \phi_0}{6R_0N_0^2}\right), \quad \alpha = \beta - \frac{(x_2 - x_1)(2y_1 + y_2)}{6R_0N_0 \sin 1''}.$$

Scale and scale error.

For long lines $m_2 = m_1 + S \cos A$, where m_2 is the approximate meridional distance of the end point of the line, m_1 that of the beginning, and

$$s = S\left[1 + \frac{1}{2R_0N_0}\left\{\left(\frac{m_1 + m_2}{2}\right)^2 + \frac{1}{3}\left(\frac{m_2 - m_1}{2}\right)^2\right\} + \frac{m_1 m_2 (m_1 + m_2) \tan \phi_0}{12R_0N_0^2} + S^2 \frac{(m_1 + m_2)}{24R_0N_0^2} \tan \phi_0\right].$$

ϕ_0 is the latitude of the origin, ϕ that of the initial point of the line and A the azimuth at that point. For appropriate length lines, any of the above formulas connecting s and S may be used.

Reduction of scale error.

If a negative scale error is introduced along the central parallel, there will be two parallels, one north, the other south of the central parallel along which the scale error is zero. Between the two standard parallels thus introduced the scale error is negative. Outside them it is positive. If the scale error is to be reduced in this way by the factor $\dfrac{1}{F}$, all measured distances, meridian distances and geodetic functions must be reduced by multiplying them by $1 - 1/F$.

Formulas for the Lambert conformal conic projection of the sphere.

In all the above formulas for the spheroid we have but to place $\epsilon = 0$, $N = R = a$, Z (conformal colatitude) $= p$ (geodetic colatitude), to produce the corresponding formulas for the sphere. Hence it is not necessary to relist them here.

Most of the above formulas are found in Clark, Plane and Geodetic Surveying, Volume II, Fourth edition, London, 1951, pages 370–376, where numerical examples of applications are given.

The application of Lambert conformal coordinates to State plane coordinate systems is found in the Manual of Plane-Coordinate Computation, U. S. C. and G. S. Special Publication No. 193 by O. S. Adams and C. N. Claire.

The computation of traverse in Lambert conformal coordinates is explained in the Manual of Traverse Computation on the Lambert Grid, U. S. C. and G. S. Special Publication No. 194 by O. S. Adams and C. N. Claire.

STEREOGRAPHIC PROJECTION

POLAR STEREOGRAPHIC PROJECTION OF THE SPHEROID

Mapping equations.

$$x=r\cos\lambda,\ y=r\sin\lambda,$$

$$r=k_0\frac{a^2}{b}\left(\frac{1-\epsilon}{1+\epsilon}\right)^{\frac{\epsilon}{2}}\tan\frac{z}{2}.$$

Scale factor.

$$k=\frac{2a^2}{bN\cos\phi}\left(\frac{1-\epsilon}{1+\epsilon}\right)^{\frac{\epsilon}{2}}\tan\frac{z}{2}.$$

Series expansion for r.

$$r=\frac{k_0a}{(1-\epsilon^2)^{\frac{1}{2}}}\left[p+\frac{1-7\epsilon^2}{12(1-\epsilon^2)}p^3+\frac{1-2\epsilon^2+46\epsilon^4}{120(1-\epsilon^2)^2}p^5+\right.$$

$$\frac{17-93\epsilon^2-1{,}335\epsilon^4-4{,}889\epsilon^6}{20{,}160(1-\epsilon^2)^3}p^7+\frac{31-184\epsilon^2+3{,}831\epsilon^4+41{,}906\epsilon^6+53{,}641\epsilon^8}{362{,}880(1-\epsilon^2)^4}p^9+$$

$$\left.\frac{691-4{,}841\epsilon^2-44{,}966\epsilon^4-2{,}420{,}926\epsilon^6-10{,}194{,}436\epsilon^8-6{,}982{,}072\epsilon^{10}}{79{,}833{,}600(1-\epsilon^2)^5}p^{11}\right],$$

where z=conformal colatitude,

a, b, ϵ=semimajor axis, semiminor axis, eccentricity of the meridian ellipse

k_0=scale factor at the pole; an arbitrary reduction applied to all geodetic lengths to reduce the maximum scale distortion of the projection,

ϕ, λ=geodetic latitude and longitude,

p=geodetic colatitude,

N=principal radius of curvature orthogonal to the meridian in latitude ϕ (the great normal).

NOTE: See equations (438) and (439) on page 129 for formulas with the coefficients in the expansion of r above evaluated for the international spheroid of reference.

Geographic coordinates from rectangular coordinates.

$$\tan\lambda=\frac{y}{x},\ \text{or}\ \lambda=\tan^{-1}\frac{y}{x}, r=x\sec\lambda=y\csc\lambda,$$

where ϕ for the corresponding value of r is interpolated from computed tables of r with ϕ or p as argument.

Polar stereographic projection of the sphere.

In the above formulas place $\epsilon=0$, $N=a=b$, $Z=p$ to obtain the spherical forms which will not be listed separately here.

STEREOGRAPHIC MERIDIAN PROJECTION

The sphere. Mapping equations.

$$x=\frac{a\cos\phi\sin\lambda}{1+\cos\phi\cos\lambda},\qquad y=\frac{a\sin\phi}{1+\cos\phi\cos\lambda}.$$

Scale factor.

$$k=1/(1+\cos\phi\cos\lambda).$$

Equation of meridians.

Circles

$$(x+a\cot\lambda)^2+y^2=a^2\csc^2\lambda,$$

with centers $x=-a\cot\lambda$, $y=0$; radii $r_\lambda=a\csc\lambda$.

Equation of parallels.

Circles

$$x^2+(y-a \csc \phi)^2=a^2 \cot^2 \phi,$$

with centers $x=0$, $y=a \csc \phi$; radii $r_\phi=a \cot \phi$.

For the graphical construction of the stereographic meridian projection of the sphere see U. S. C. and G. S. Special Publication No. 57, page 34.

The spheroid. Replace ϕ by the conformal latitude χ.

Mapping equations.

$$x=\frac{a \cos \chi \sin \lambda}{1+\cos \chi \cos \lambda},\ y=\frac{a \sin \chi}{1+\cos \chi \cos \lambda}.$$

Scale factor.

$$k=a \cos \chi/[N \cos \phi\ (1+\cos \chi \cos \lambda)].$$

STEREOGRAPHIC HORIZON PROJECTION

Sphere. Mapping equations.

$$x=\frac{a \sin \lambda \cos \phi}{1+ \sin \phi \sin \phi_0+\cos \phi \cos \phi_0 \cos \lambda},$$

$$y=\frac{a(\sin \phi \cos \phi_0-\sin \phi_0 \cos \phi \cos \lambda)}{1+\sin \phi \sin \phi_0+\cos \phi \cos \phi_0 \cos \lambda}.$$

Scale factor.

$$k=1/(1+\sin \phi \sin \phi_0+\cos \phi \cos \phi_0 \cos \lambda),$$

where ϕ_0=latitude of the origin.

Equation of meridians.

Circles $(x+a \sec \phi_0 \cot \lambda)^2+(y+a \tan \phi_0)^2=a^2 \sec^2 \phi_0 \csc^2 \lambda$, with centers $x=-a \sec \phi_0 \cot \lambda$, $y=-a \tan \phi_0$; radii $r_\lambda=a \sec \phi_0 \csc \lambda$.

Equation of parallels.

$$\text{Circles } x^2+\left(y-\frac{a \cos \phi_0}{\sin \phi_0+\sin \phi}\right)^2=\frac{a^2 \cos^2 \phi}{(\sin \phi_0+\sin \phi)^2},$$

$$\text{with centers } x=0,\ y=\frac{a \cos \phi_0}{\sin \phi_0+\sin \phi};\ \text{radii } r_\phi=\frac{a \cos \phi}{\sin \phi_0+\sin \phi}.$$

POLAR COORDINATES FOR THE STEREOGRAPHIC PROJECTIONS OF THE SPHERE

Horizon. $\theta=\alpha$, $\rho=a \tan \frac{1}{2} D$,

$$\tan \alpha=\frac{\cos \phi_0 \tan \phi - \sin \phi_0 \cos \lambda}{\sin \lambda},\ \sin D=\frac{\cos \phi \sin \lambda}{\cos \alpha}.$$

For the meridian and polar stereographic projections we have but to place $\phi_0=0, \frac{\pi}{2}$ in these equations.

The graphical construction of the stereographic horizon projection is described in U. S. C. and G. S. Special Publication No. 57, page 48.

DERIVATION OF MAPPING FORMULAS

ELEMENTS OF COMPLEX VARIABLE THEORY

Before proceeding to the derivation of the formulas for the conformal projections we will give a short account of complex numbers, some of the properties of a complex variable and of analytic functions of a complex variable. No attempt is made to give rigorous proofs, the idea being to enable the reader to grasp a working knowledge by demonstrating the properties of an analytic function of a complex variable which allow the development of all conformal projections from such a function. Those who may be interested in further investigation of the theory will find the presentations whose titles are included in the bibliography most helpful.

Usually the first time one encounters the quantity $i=\sqrt{-1}$ is in obtaining the solutions of quadratic equations. The solution of the quadratic $ax^2+2bx+c=0$ is given by the well-known formula $x=\frac{-b\pm\sqrt{b^2-ac}}{a}$. Now if the quantity under the radical sign (the discriminant) is negative, i. e.,

$$b^2-ac<0, \text{ the solution is } x=-\frac{b}{a}\pm\frac{\sqrt{-k}}{a} \text{ where } k=|b^2-ac|.$$

We may then write

$$x=-\frac{b}{a}\pm\sqrt{-1}\frac{\sqrt{k}}{a}=-\frac{b}{a}\pm i\,\frac{\sqrt{k}}{a}, \tag{1}$$

and we say the equation has complex roots and that i is the imaginary unit. Consequently the roots are also said to be imaginary.

First let us demonstrate an important property of the imaginary unit $i=\sqrt{-1}$. If we multiply this unit by itself we have $i^2=\sqrt{-1}\sqrt{-1}=-1$. Multiplying both sides of this last by i we have $i^3=-i$. Continuing $i^4=-i^2=+1$, $i^5=i$, $i^6=i^2=-1$, etc., so that regardless of how many times we repeat the operation of multiplication of i by itself only four values are obtained and always in the same order each time, that is,

$$i^{4n-4}=+1,\, i^{4n-3}=+i,\, i^{4n-2}=-1,\, i^{4n-1}=-i, \tag{2}$$

for all integral values of n.

Now think of an ordinary pair of orthogonal axes, x and y as shown in figure 1. Let distances along the y-axis be multiples of the imaginary unit i, and those along the x-axis be the real numbers. The y-axis is then the axis of imaginaries, the x-axis is the axis of reals. The plane determined by these axes is called the complex plane. By examining the relations (2) as one places $n=1,2,3$ . . . . . it is seen that the effect of repeated multiplication of i by itself may be interpreted geometrically as a rotation of the unit vector $R=OT$ through increments of 90°. We see also from this operation that multiplication by -1 corresponds to a rotation through 180° leaving lengths unchanged. Hence multiplication by $-k$ would correspond to a rotation through 180° and a magnification in length in the ratio k to 1.

But what about some other point P on this circle? From figure 1 it is seen that the regular cartesian coordinates of P are x,y. But we say the point P in the complex plane corresponds to the complex number $x+iy$. That is, we have a double coordinate system but referred to the same orthogonal axes. One might think of it as a complex plane superimposed upon a real plane. x and y are both real numbers. When $x=0$, then $0+iy=iy$ is said to be a pure imaginary. When $y=0$, $x+0 \cdot i=x$ is a real number.

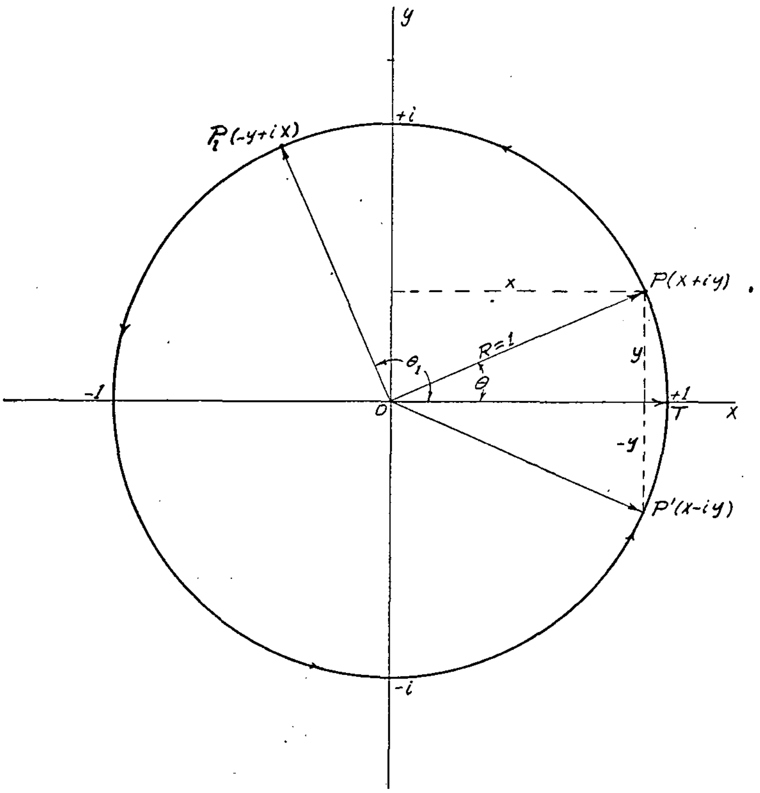

FIGURE 1.—Continued multiplication of $i=\sqrt{-1}$ by itself interpreted as rotation of a unit vector in the complex plane.

Hence it is seen that the formulation of the complex number system is a generalization of the real number system since it includes it. Clearly if the complex number $x+iy=0$, then $x=y=0$.

Again from figure 1 we have $OP=\sqrt{x^2+y^2}=R=1$ and $\tan\,\theta=y/x$. Now it is seen that we do not need to limit P to the unit circle or to any circle. We may say that $R=\sqrt{x^2+y^2}=|x+iy|$ is the numerical value of the complex number $z=x+iy$ as represented geometrically by the length of the line OP. In the formal terminology R is called the modulus. R always has a unique direction specified by $\theta=\tan^{-1}y/x$. θ is called the amplitude or the argument of z. If we impose the condition $R=\sqrt{x^2+y^2}=|x+iy|\leqq k$ the point P $(x+iy)$ is confined to the interior and the bounding circle of radius k. If we write $R=\sqrt{x^2+y^2}=|x+iy|<k$ then P is confined to the interior of the circle of radius k but excluded from the points of the circle itself. This idea of limiting a complex variable to a circle, to an area enclosed by a circle, or to the area contained between two circles is fundamental in the study of analytic functions of a complex variable, particularly in discussing the convergence of their power series expansions.[1] Series expansions will be used later in connection with the derivation of some of the autogonal projections.

[1] R. V. Churchill, Introduction to Complex Variables and Applications, p. 98.

From the above discussion it is seen that we have at once the polar or vector representation of complex numbers, namely

$$R=\sqrt{x^2+y^2}=|x+iy|=|z|,\ \theta=\tan^{-1} y/x,$$

$$x=R\cos\theta,\ y=R\sin\theta. \tag{3}$$

The vector nature of complex numbers may be demonstrated in performing the elementary arithmetical operations on complex numbers. For instance consider the sum of the complex numbers $3+i$, and $4+6i$. We have $3+i+4+6i=7+7i$. Now in figure 2 it is seen that $OP=7+7i$ is the vector sum of $OQ=3+i$ and $OS=4+6i$, since OP is the diagonal of the parallelogram $OSPQ$. The other arithmetic processes

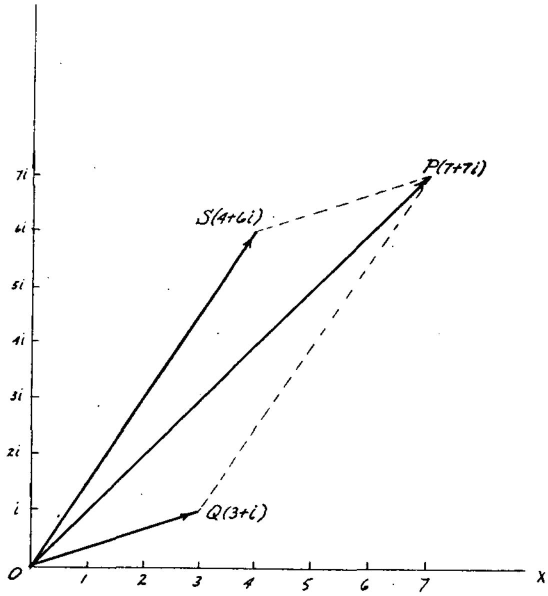

FIGURE 2.—The vector nature of complex numbers.

are as validly performed with complex numbers as with real numbers making use of relations (2) to reduce powers of i, and these processes always lead to another complex number. As an example of multiplication of complex numbers we have

$$(2+3i)(3+4i)=6+17i+12i^2=6+17i-12=-6+17i.$$

Division is performed by rationalizing the denominator (since i is a radical), i. e.,

$$\frac{6+7i}{3-5i}=\frac{(6+7i)(3+5i)}{(3-5i)(3+5i)}=\frac{18+51i+35i^2}{9-25i^2}=\frac{18+51i-35}{9+25}=-\frac{1}{2}+\frac{51}{34}\,i.$$

It is easy to show that the distributive and commutative laws hold in arithmetical operations on complex numbers.

We saw that multiplying i successively by itself corresponds to the four intercepts of the unit circle on the axes of the complex plane, giving in effect the cyclic rotation from one intercept to the other—that is, through 90° increments. Suppose that we multiply the complex number $z=x+iy$ by i to get $z_1=-y+ix$. If θ and θ_1 are the amplitudes of z and z_1 respectively we have $\tan\theta=\frac{y}{x}$, $\tan\theta_1=-\frac{x}{y}$ and since $\tan\theta_1$ is the negative reciprocal of tan θ, the line OP has been rotated through 90° as shown in figure 1.

In figure 1, the point P' is the symmetric of P with respect to the axis of reals or the reflection of P in the axis of reals. We have then the two complex numbers $z=x+iy$, $\overline{z}=x-iy$ which are called conjugate complex numbers. They have, among many other interesting and useful properties, the property that their product is always a real number, that is,

$$z\cdot\overline{z}=(x+iy)(x-iy)=x^2-i^2y^2=x^2+y^2=R^2=|z|^2. \tag{4}$$

We note that, in division of complex numbers, the rationalizing factor is the conjugate of the denominator. See the illustration above.

Returning now to equations (1), it is seen that the solutions of our quadratic equation represent a pair of conjugate complex numbers in the complex plane.

Now let $w=x+iy$, and $z=\lambda+i\tau$ and suppose that the complex variable w is a function of the complex variable z, that is,

$$w=x+iy=f(z)=f(\lambda+i\tau). \tag{5}$$

Then x and y are real functions of the real variables λ and τ, that is, $x=x(\lambda,\tau)$, $y=y(\lambda,\tau)$ and they are obtained by equating the real and imaginary parts of (5). For example if $f(\lambda+i\tau)=\lambda^2-\tau^2+2i\lambda\tau$ then $x+iy=\lambda^2-\tau^2+2i\lambda\tau$ and consequently $x=\lambda^2-\tau^2$, $y=2\lambda\tau$.

Let us write the differential form of (5). Since w is a function of two variables x and y we must use the differential form $dw=\frac{\partial w}{\partial x}\,dx+\frac{\partial w}{\partial y}\,dy$. We have then

$$dw=\frac{\partial w}{\partial x}\,dx+\frac{\partial w}{\partial y}\,dy=f'(z)dz=f'(\lambda+i\tau)\left(\frac{\partial z}{\partial\lambda}\,d\lambda+\frac{\partial z}{\partial\tau}\,d\tau\right). \tag{6}$$

But $\frac{\partial w}{\partial x}=\frac{\partial z}{\partial\lambda}=1$, $\frac{\partial w}{\partial y}=\frac{\partial z}{\partial\tau}=i$ so that we have from (6)

$$dw=dx+idy=f'(z)dz=f'(\lambda+i\tau)(d\lambda+id\tau). \tag{7}$$

Similarly for the conjugate complex function $x-iy=f(\lambda-i\tau)$, we have

$$d\overline{w}=dx-idy=f'(\overline{z})d\overline{z}=f'(\lambda-i\tau)(d\lambda-id\tau). \tag{8}$$

Now multiply respective members of (7) and (8) together to obtain

$$dx^2+dy^2=f'(\lambda-i\tau)f'(\lambda+i\tau)(d\lambda^2+d\tau^2). \tag{9}$$

If the derivative $f'(z)$ exists at the point z, it may be proved [2] that

$$\begin{aligned} f'(z)&=f'(\lambda+i\tau)=\frac{\partial x}{\partial\lambda}+i\,\frac{\partial y}{\partial\lambda}=\frac{\partial y}{\partial\tau}-i\,\frac{\partial x}{\partial\tau},\\ f'(\overline{z})&=f'(\lambda-i\tau)=\frac{\partial x}{\partial\lambda}-i\,\frac{\partial y}{\partial\lambda}=\frac{\partial y}{\partial\tau}+i\,\frac{\partial x}{\partial\tau}. \end{aligned} \tag{10}$$

[2] R. V. Churchill, Introduction to Complex Variables and Applications, p. 29.

The following demonstration will make equations (10) meaningful. In figure 3, the point A in the z-plane is the point $z=\lambda+i\tau$ and the corresponding point in the w-plane for $w=x+iy=f(z)=f(\lambda+i\tau)$ is the point C. Suppose that z is given the

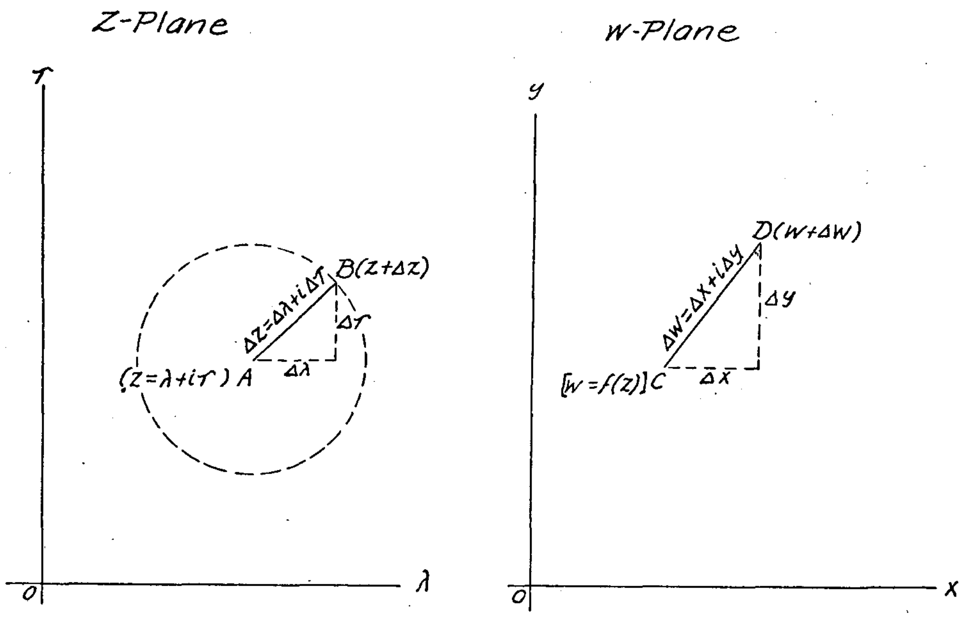

FIGURE 3.—Derivation of the derivative of an analytic function.

increment $AB=\Delta z=\Delta\lambda+i\Delta\tau$ as shown in the z-plane. Then $w=f(z)$ will get the increment $CD=\Delta w=\Delta x+i\Delta y$ as shown in the w-plane. Now $f'(z)=\lim\limits_{\Delta z\to 0}\dfrac{\Delta w}{\Delta z}$ in the notation of the differential calculus and if the limit exists the value should be unique.

Since $w=x+iy$ and $x=x(\lambda,\tau), y=y(\lambda,\tau)$, then

$$\Delta w=\frac{\partial x}{\partial\lambda}\Delta\lambda+\frac{\partial x}{\partial\tau}\Delta\tau+i\left(\frac{\partial y}{\partial\lambda}\Delta\lambda+\frac{\partial y}{\partial\tau}\Delta\tau\right), \tag{11}$$

where we have ignored additional infinitesimal terms which vanish when $\Delta\tau$ and $\Delta\lambda$ tend to zero.

With $\Delta z=\Delta\lambda+i\Delta\tau$ we may write (11) as

$$\frac{\Delta w}{\Delta z}=\frac{\dfrac{\partial x}{\partial\lambda}+i\dfrac{\partial y}{\partial\lambda}+\left(\dfrac{\partial x}{\partial\tau}+i\dfrac{\partial y}{\partial\tau}\right)\dfrac{\Delta\tau}{\Delta\lambda}}{1+i\dfrac{\Delta\tau}{\Delta\lambda}}, \tag{12}$$

whence

$$f'(z)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\substack{\Delta z\to 0\\ \left(\substack{\Delta\tau\to 0\\ \Delta\lambda\to 0}\right)}}\frac{\dfrac{\partial x}{\partial\lambda}+i\dfrac{\partial y}{\partial\lambda}+\left(\dfrac{\partial x}{\partial\tau}+i\dfrac{\partial y}{\partial\tau}\right)\dfrac{\Delta\tau}{\Delta\lambda}}{1+i\dfrac{\Delta\tau}{\Delta\lambda}}$$

or

$$f'(z)=\frac{\dfrac{\partial x}{\partial\lambda}+i\dfrac{\partial y}{\partial\lambda}+\left(\dfrac{\partial x}{\partial\tau}+i\dfrac{\partial y}{\partial\tau}\right)\dfrac{d\tau}{d\lambda}}{1+i\dfrac{d\tau}{d\lambda}}. \tag{13}$$

But clearly the right member of (13) depends on the value of $\frac{d\tau}{d\lambda}$ which in figure 3 is seen to be the slope of the line AB in the limit, i. e., $\lim_{\Delta z\to 0}\frac{\Delta\tau}{\Delta\lambda}=\frac{d\tau}{d\lambda}$. Hence the derivative $f'(z)$ as defined by (13) would not be unique at the point $A(z=\lambda+i\tau)$, since the point B may be any point on the circle about A of radius Δz, each point determining a different value of $\frac{d\tau}{d\lambda}$. We must, therefore, impose some conditions on x and y which will cause equation (13) to be free of $\frac{d\tau}{d\lambda}$. This will also cause the representation to be conformal. We note that if we place $\frac{d\tau}{d\lambda}=0$, $\frac{d\tau}{d\lambda}=\infty$ (equivalently take the directions of AB parallel to the λ and τ axes) we will obtain from (13)

$$f'(z)=\frac{\partial x}{\partial\lambda}+i\,\frac{\partial y}{\partial\lambda},\qquad f'(z)=\frac{1}{i}\left(\frac{\partial x}{\partial\tau}+i\,\frac{\partial y}{\partial\tau}\right),\tag{14}$$

which are free of $\frac{d\tau}{d\lambda}$.

But if $f'(z)$ is to be unique at the point z, then the two values of $f'(z)$ as given by (14) must be equal, and equating them we obtain the first of equations (10), namely

$$f'(z)=\frac{\partial x}{\partial\lambda}+i\,\frac{\partial y}{\partial\lambda}=\frac{\partial y}{\partial\tau}-i\,\frac{\partial x}{\partial\tau}.$$

Analogously we may give a demonstration of the second of equations (10).

If we equate real and imaginary parts in equations (10), we obtain

$$\frac{\partial x}{\partial\lambda}-\frac{\partial y}{\partial\tau}=0,\ \frac{\partial x}{\partial\tau}+\frac{\partial y}{\partial\lambda}=0,\tag{15}$$

which are known as the Cauchy-Riemann equations. They are the conditions which must be satisfied by the real functions $x(\lambda,\tau)$, $y(\lambda,\tau)$ if $f'(z)$ or $f'(\bar{z})$ exists at a point z, the existence of the derivative through (15) imposing, at the same time, the conformal mapping of one plane upon the other. This will be subsequently demonstrated.

Multiplying the right members of (10) together in all possible ways, making use of equations (15), we have

$$f'(z)\cdot f'(\bar{z})=f'(\lambda+i\tau)\cdot f'(\lambda-i\tau)=\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2=\left(\frac{\partial x}{\partial\tau}\right)^2+\left(\frac{\partial y}{\partial\tau}\right)^2=J\left(\frac{x,y}{\lambda,\tau}\right),\tag{16}$$

where $J\left(\frac{x,y}{\lambda,\tau}\right)=\begin{vmatrix}\frac{\partial x}{\partial\lambda} & \frac{\partial x}{\partial\tau}\\ \frac{\partial y}{\partial\lambda} & \frac{\partial y}{\partial\tau}\end{vmatrix}$ is the Jacobian functional determinant. Hence the product of the derivative of a function of a complex variable and the derivative of its conjugate is a real function since x and y are real functions of λ and τ.

We may wish to express the complex variables $\lambda+i\tau$, $\lambda-i\tau$ and hence the conjugate complex functions $f(\lambda+i\tau)$, $f(\lambda-i\tau)$ in polar form. From equations (3) we have $\lambda=R\cos\theta$, $\tau=R\sin\theta$ and (5) may then be written

$$\begin{aligned}w&=x+iy=f(z)=f(\lambda+i\tau)=f[R(\cos\theta+i\sin\theta)]=f(Re^{i\theta}),\\ \bar{w}&=x-iy=f(\bar{z})=f(\lambda-i\tau)=f[R(\cos\theta-i\sin\theta)]=f(Re^{-i\theta}).\end{aligned}\tag{17}$$

In (17) we have used the identities $e^{i\theta}=\cos\theta+i\sin\theta$, $e^{-i\theta}=\cos\theta-i\sin\theta$ which may be easily demonstrated by means of the Maclaurin expansions for the functions e^{θ}, $\cos\theta$, and $\sin\theta$ as follows:

$$e^{\theta}=1+\theta+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+\frac{\theta^5}{5!}+\cdots,$$

$$\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots,$$

$$\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots,$$

$$\begin{aligned}e^{i\theta}&=1+i\theta-\frac{\theta^2}{2!}-i\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+i\frac{\theta^5}{5!}-\cdots\\&=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)\\&=\cos\theta+i\sin\theta.\end{aligned}$$

$$\begin{aligned}e^{-i\theta}&=1-i\theta-\frac{\theta^2}{2!}+i\frac{\theta^3}{3!}+\frac{\theta^4}{4!}-i\frac{\theta^5}{5!}-\cdots\\&=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots\right)-i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)\\&=\cos\theta-i\sin\theta.\end{aligned}$$

The Cauchy-Riemann equations (15) become for the polar form

$$\frac{\partial x}{\partial R}-\frac{1}{R}\frac{\partial y}{\partial\theta}=0,\ \frac{\partial y}{\partial R}+\frac{1}{R}\frac{\partial x}{\partial\theta}=0. \tag{18}$$

Let us derive equations (18) as follows:

From (5) $w=x+iy=f(\lambda,\tau)$, where $x=x(\lambda,\tau)$, $y=y(\lambda,\tau)$ and we are changing to polar form by means of the transformation $\lambda=R\cos\theta$, $\tau=R\sin\theta$.

Then

$$\begin{aligned}\frac{\partial w}{\partial R}&=\frac{\partial x}{\partial R}+i\frac{\partial y}{\partial R}=\frac{\partial f}{\partial\lambda}\frac{\partial\lambda}{\partial R}+\frac{\partial f}{\partial\tau}\frac{\partial\tau}{\partial R}=\frac{\partial f}{\partial\lambda}\cos\theta+\frac{\partial f}{\partial\tau}\sin\theta,\\\frac{\partial w}{\partial\theta}&=\frac{\partial x}{\partial\theta}+i\frac{\partial y}{\partial\theta}=\frac{\partial f}{\partial\lambda}\frac{\partial\lambda}{\partial\theta}+\frac{\partial f}{\partial\tau}\frac{\partial\tau}{\partial\theta}=\frac{\partial f}{\partial\lambda}(-R\sin\theta)+\frac{\partial f}{\partial\tau}(R\cos\theta).\end{aligned} \tag{19}$$

But $\frac{\partial f}{\partial\lambda}=\frac{\partial w}{\partial\lambda}=\frac{\partial x}{\partial\lambda}+i\frac{\partial y}{\partial\lambda}$, $\frac{\partial f}{\partial\tau}=\frac{\partial w}{\partial\tau}=\frac{\partial x}{\partial\tau}+i\frac{\partial y}{\partial\tau}$ and these values placed in the right members of (19) give

$$\begin{aligned}\frac{\partial x}{\partial R}+i\frac{\partial y}{\partial R}&=\left(\frac{\partial x}{\partial\lambda}+i\frac{\partial y}{\partial\lambda}\right)\cos\theta+\left(\frac{\partial x}{\partial\tau}+i\frac{\partial y}{\partial\tau}\right)\sin\theta,\\\frac{\partial x}{\partial\theta}+i\frac{\partial y}{\partial\theta}&=\left(\frac{\partial x}{\partial\lambda}+i\frac{\partial y}{\partial\lambda}\right)(-R\sin\theta)+\left(\frac{\partial x}{\partial\tau}+i\frac{\partial y}{\partial\tau}\right)(R\cos\theta).\end{aligned} \tag{20}$$

Now multiply the second of equations (20) by i/R and then add respective members of the equations to obtain

$$\frac{\partial x}{\partial R}-\frac{1}{R}\frac{\partial y}{\partial \theta}+i\left(\frac{\partial y}{\partial R}+\frac{1}{R}\frac{\partial x}{\partial \theta}\right)=\left[\left(\frac{\partial x}{\partial \lambda}-\frac{\partial y}{\partial \tau}\right)\cos\theta+\left(\frac{\partial x}{\partial \tau}+\frac{\partial y}{\partial \lambda}\right)\sin\theta\right]+$$
$$i\left[\left(\frac{\partial x}{\partial \tau}+\frac{\partial y}{\partial \lambda}\right)\cos\theta-\left(\frac{\partial x}{\partial \lambda}-\frac{\partial y}{\partial \tau}\right)\sin\theta\right]. \tag{21}$$

Equating real and imaginary parts in (21) we have

$$\begin{aligned}\frac{\partial x}{\partial R}-\frac{1}{R}\frac{\partial y}{\partial \theta}&=\left(\frac{\partial x}{\partial \lambda}-\frac{\partial y}{\partial \tau}\right)\cos\theta+\left(\frac{\partial x}{\partial \tau}+\frac{\partial y}{\partial \lambda}\right)\sin\theta,\\ \frac{\partial y}{\partial R}+\frac{1}{R}\frac{\partial x}{\partial \theta}&=\left(\frac{\partial x}{\partial \tau}+\frac{\partial y}{\partial \lambda}\right)\cos\theta-\left(\frac{\partial x}{\partial \lambda}-\frac{\partial y}{\partial \tau}\right)\sin\theta.\end{aligned} \tag{22}$$

The right members of (22) are identically zero if $\frac{\partial x}{\partial \lambda}-\frac{\partial y}{\partial \tau}=0$, $\frac{\partial x}{\partial \tau}+\frac{\partial y}{\partial \lambda}=0$. But these last are the Cauchy-Riemann equations (15), hence from the left members of (22) we have equations (18).

A function of a complex variable is said to be analytic in a region if its derivative exists at every point of the region. Let us now state the conditions under which (5) is an analytic function. If $x=x(\lambda,\tau)$, $y=y(\lambda,\tau)$, together with their partial derivatives of first order, are continuous, single-valued and satisfy the Cauchy-Riemann equations (15) throughout some open two-dimensional region, then the function (5) is analytic at all points of the region.[3]

The properties of an analytic function of a complex variable which make it a natural medium for the development of the formulas for conformal maps are: (1) at each point where a function $f(z)$, as given by equation (5), is analytic and $f'(z)\neq 0$, the mapping $w=f(z)$ is conformal and; (2) the curves $x(\lambda,\tau)=c_1$, $y(\lambda,\tau)=c_2$ that intersect at that point under the above conditions are mapped into the lines $x=c_1$, $y=c_2$ in the w-plane. Since these lines in the w-plane are orthogonal, the curves $x(\lambda,\tau)=c_1$, $y(\lambda,\tau)=c_2$ are orthogonal in the z-plane, and conversely.

Since the mathematical figure of the earth, considered a sphere or spheroid, is referred to its orthogonal system of meridians and parallels, we recognize the importance of these properties. This will be discussed in more detail when we show that the spheroid can be mapped conformally upon a plane.

We will now demonstrate these properties by an example.

Suppose that equation (5) is given by $w=x+iy=f(z)=e^{+i(\lambda-i\tau)}=f(\lambda-i\tau)$. We have then that $x+iy=e^{\tau}e^{+i\lambda}$, and from (17), $e^{+i\lambda}=\cos\lambda+i\sin\lambda$ so that $x+iy=e^{\tau}(\cos\lambda+i\sin\lambda)$. Equating real and imaginary parts we have

$$x=e^{\tau}\cos\lambda,\ y=+e^{\tau}\sin\lambda. \tag{23}$$

From (23) we have $\frac{\partial x}{\partial \tau}=e^{\tau}\cos\lambda$, $\frac{\partial x}{\partial \lambda}=-e^{\tau}\sin\lambda$, $\frac{\partial y}{\partial \tau}=+e^{\tau}\sin\lambda$, $\frac{\partial y}{\partial \lambda}=+e^{\tau}\cos\lambda$, whence $\frac{\partial x}{\partial \lambda}=-\frac{\partial y}{\partial \tau}=-e^{\tau}\sin\lambda$, $\frac{\partial x}{\partial \tau}=+\frac{\partial y}{\partial \lambda}=e^{\tau}\cos\lambda$ and the Cauchy-Riemann equations (15) are thus satisfied, the sign reversals being due to the use of the conjugate function $f(\lambda-i\tau)$ instead of $f(\lambda+i\tau)$.

Squaring and adding, then dividing respective members of (23) we obtain the equations

$$x^2+y^2=e^{2\tau},\ y=+x\tan\lambda. \tag{24}$$

[3] R. V. Churchill, Introduction to Complex Variables and Applications, p. 32.

In figure 4 it is seen that $\tau=c_1$, $\lambda=c_2$, which are lines parallel to the coordinate axes in the z-plane as shown, define the point P at their intersection. The values $\tau=c_1$, $\lambda=c_2$ placed in (24) give the corresponding point P' in the w-plane as the intersection of the circle $x^2+y^2=e^{2c_1}$ and line $y=+x \tan c_2$. That is, for every line $\tau=c_1$ in the z-plane (parallel to the λ-axis) we have a circle $x^2+y^2=e^{2c_1}$ in the w-plane. For every line $\lambda=c_2$ in the z-plane (parallel to the τ-axis) we have a line $y=+x \tan c_2$ in the w-plane. It is clear that every line $y=+x \tan c_2$ is orthogonal to every circle $x^2+y^2=e^{2c_1}$ since the lines coincide with radii of the circles.

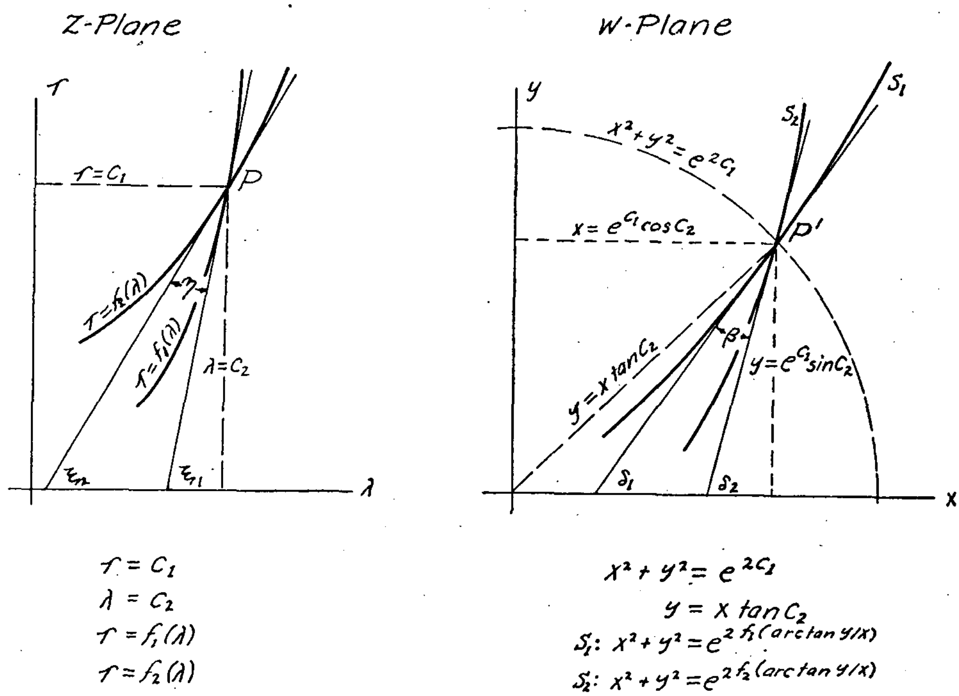

FIGURE 4.—Corresponding curves in conformal mapping.

We will now show that angles are preserved in the mapping of the z-plane upon the w-plane. In the z-plane of figure 4 we have the curves $\tau=f_1(\lambda)$, $\tau=f_2(\lambda)$ which pass through the point P if $c_1=f_1(c_2)=f_2(c_2)$. The tangents to these curves at the point P make the angles ξ_1 and ξ_2 respectively with the λ-axis, and we have $\tan \xi_1=f_1'(c_2)$, $\tan \xi_2=f_2'(c_2)$. The angle between these tangents is η and $\eta=\xi_1-\xi_2$, whence

$$\tan \eta=\frac{\tan \xi_1-\tan \xi_2}{1+\tan \xi_1 \tan \xi_2}=\frac{f_1'(c_2)-f_2'(c_2)}{1+f_1'(c_2)f_2'(c_2)}. \tag{25}$$

Now in the w-plane, the curves s_1 and s_2 correspond respectively to the curves $\tau=f_1(\lambda)$, $\tau=f_2(\lambda)$ in the z-plane, and the parametric equations of the curves are obtained from (23) by replacing τ by $f_1(\lambda)$ and $f_2(\lambda)$ respectively, that is,

$$\begin{aligned} s_1&: x=e^{f_1(\lambda)}\cos \lambda, y=+e^{f_1(\lambda)}\sin \lambda,\\ s_2&: x=e^{f_2(\lambda)}\cos \lambda, y=+e^{f_2(\lambda)}\sin \lambda. \end{aligned} \tag{26}$$

To show that the curves s_1 and s_2 pass through P' in the w-plane when the curves $\tau=f_1(\lambda)$, $\tau=f_2(\lambda)$ pass through the point P in the z-plane we have only to recall our condition for these last two curves to pass through the point P, namely $f_1(c_2)=f_2(c_2)=c_1$.

If we place $\lambda=c_2$ and $f_1(c_2)=f_2(c_2)=c_1$ in equations (26) we obtain in each case $x=e^{c_1}\cos c_2$, $y=+e^{c_1}\sin c_2$ which are the coordinates of the point P'.

If δ_1 and δ_2 are the angles which the tangents to s_1 and s_2 in the w-plane make with the x-axis then

$$\tan\delta_1=\frac{dy}{d\lambda}\Big/\frac{dx}{d\lambda}\ s_1(\lambda=c_2),\qquad \tan\delta_2=\frac{dy}{d\lambda}\Big/\frac{dx}{d\lambda}\ s_2(\lambda=c_2). \tag{27}$$

From (26) we have

$$s_1:\ \frac{dy}{d\lambda}=+e^{f_1(\lambda)}f_1'(\lambda)\sin\lambda+e^{f_1(\lambda)}\cos\lambda,$$

$$\frac{dx}{d\lambda}=e^{f_1(\lambda)}f_1'(\lambda)\cos\lambda-e^{f_1(\lambda)}\sin\lambda.$$

$$s_2:\ \frac{dy}{d\lambda}=+e^{f_2(\lambda)}f_2'(\lambda)\sin\lambda+e^{f_2(\lambda)}\cos\lambda,$$

$$\frac{dx}{d\lambda}=e^{f_2(\lambda)}f_2'(\lambda)\cos\lambda-e^{f_2(\lambda)}\sin\lambda.$$

Forming from these values the ratios in (27) and placing $\lambda=c_2$ to evaluate at the point P' we find that equations (27) become

$$\tan\delta_1=+\frac{1+f_1'(c_2)\tan c_2}{f_1'(c_2)-\tan c_2},\ \tan\delta_2=+\frac{1+f_2'(c_2)\tan c_2}{f_2'(c_2)-\tan c_2}. \tag{28}$$

Now if β is the angle between the tangents to s_1 and s_2 then $\beta=\delta_2-\delta_1$ and $\tan\beta=\frac{\tan\delta_2-\tan\delta_1}{1+\tan\delta_1\tan\delta_2}$. With the values of $\tan\delta_1$, $\tan\delta_2$ from (28) this becomes

$$\tan\beta=\frac{+\frac{1+f_2'(c_2)\tan c_2}{f_2'(c_2)-\tan c_2}-\frac{1+f_1'(c_2)\tan c_2}{f_1'(c_2)-\tan c_2}}{1+\frac{[1+f_1'(c_2)\tan c_2][1+f_2'(c_2)\tan c_2]}{[f_1'(c_2)-\tan c_2][f_2'(c_2)-\tan c_2]}}$$

$$=\frac{f_1'(c_2)-f_2'(c_2)}{1+f_1'(c_2)f_2'(c_2)}\cdot\frac{1+\tan^2 c_2}{1+\tan^2 c_2}$$

$$=\frac{f_1'(c_2)-f_2'(c_2)}{1+f_1'(c_2)f_2'(c_2)}. \tag{29}$$

From (25) and (29) we have that $\tan\eta=\tan\beta$, or $\eta=\beta$ and corresponding angles are thus preserved in the mapping.

Figure 5 is a numerical example of the general case treated in figure 4. We have chosen $f_1(\lambda)$ and $f_2(\lambda)$ to be $f_1(\lambda)=\frac{32}{\pi^2}\lambda^2-1$, $f_2(\lambda)=\frac{8}{\pi^2}\lambda^2+\frac{1}{2}$, that is, we have chosen for the point P in the z-plane the intersection of $\tau=c_1=1$, $\lambda=c_2=\frac{\pi}{4}$ and the two parabolas $\tau=\frac{32}{\pi^2}\lambda^2-1$, $\tau=\frac{8}{\pi^2}\lambda^2+\frac{1}{2}$, through the point P. From these we have $\tau_1'=f_1'(\lambda)=\frac{64}{\pi^2}\lambda$, $\tau_2'=f_2'(\lambda)=\frac{16}{\pi^2}\lambda$ and at $\lambda=c_2=\frac{\pi}{4}$, these become respectively $\tau_1'=f_1'\left(\frac{\pi}{4}\right)=\frac{16}{\pi}=\tan\xi_1$,

$\tau_2' = f_2'\left(\frac{\pi}{4}\right) = \frac{4}{\pi} = \tan \xi_2$. Hence from (25)

$$\tan \eta = \frac{\frac{16}{\pi} - \frac{4}{\pi}}{1 + \frac{16}{\pi} \cdot \frac{4}{\pi}} = \frac{12\pi}{\pi^2 + 64} = 0.51035.$$

From (24) the corresponding point P' in the w-plane is the intersection of the circle $x^2 + y^2 = e^2$, and the line $y = x$.

From (28)
$$\tan \delta_1 = \frac{1 + \frac{16}{\pi}}{\frac{16}{\pi} - 1} = \frac{16 + \pi}{16 - \pi}, \quad \tan \delta_2 = \frac{1 + \frac{4}{\pi}}{\frac{4}{\pi} - 1} = \frac{4 + \pi}{4 - \pi},$$

and
$$\tan \beta = \frac{\tan \delta_2 - \tan \delta_1}{1 + \tan \delta_1 \tan \delta_2} = \frac{\frac{4+\pi}{4-\pi} - \frac{16+\pi}{16-\pi}}{1 + \frac{4+\pi}{4-\pi} \cdot \frac{16+\pi}{16-\pi}} = \frac{12\pi}{\pi^2 + 64} = \tan \eta,$$

whence $\beta = \eta = 27°02'15''$ as was to be shown.

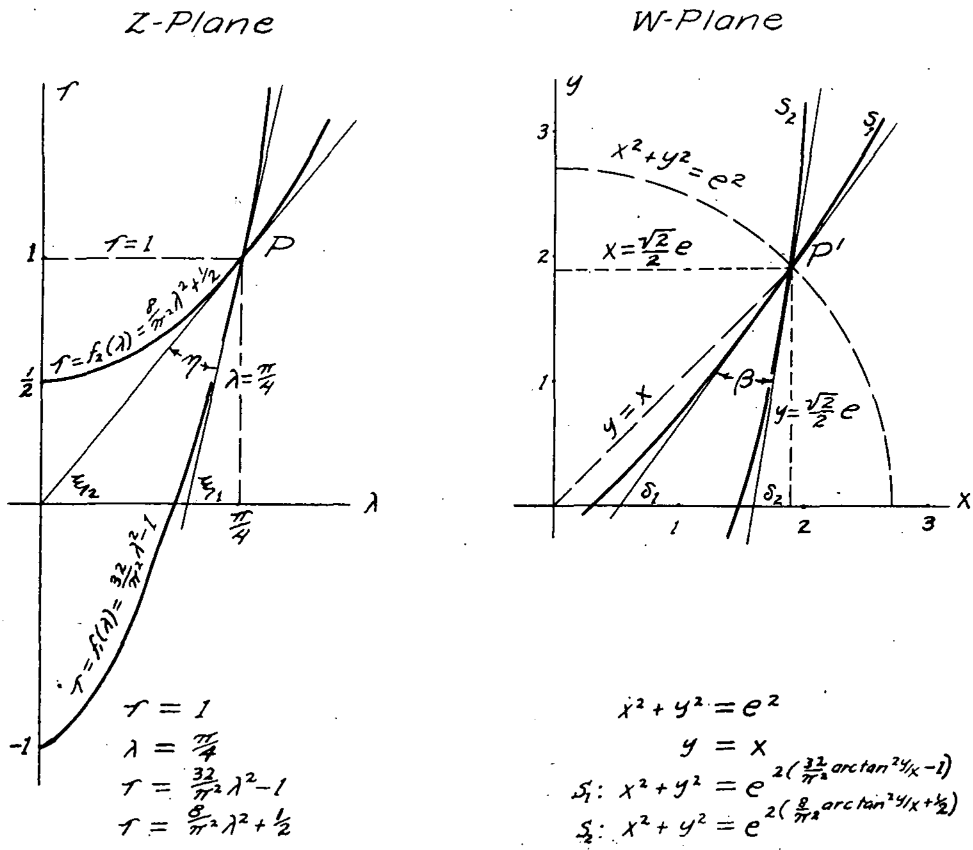

FIGURE 5.—Corresponding curves in conformal mapping.

We could have mapped both corresponding families of curves on the same complex plane. The use of two planes is for convenience in depicting the point-to-point correspondence in the conformal transformation or mapping.

Finally we can exhibit the autogonal property by considering the respective formulas for arc length in the two planes. In the w-plane the element of arc is $ds_1^2=dx^2+dy^2$. In the z-plane it is $ds_2^2=d\lambda^2+d\tau^2$. Now from equations (23), using the formulas $dx=\frac{\partial x}{\partial\lambda}d\lambda+\frac{\partial x}{\partial\tau}d\tau$, $dy=\frac{\partial y}{\partial\lambda}d\lambda+\frac{\partial y}{\partial\tau}d\tau$ with the values already obtained for the partial derivatives, $\frac{\partial x}{\partial\lambda}=-\frac{\partial y}{\partial\tau}=-e^\tau\sin\lambda$, $\frac{\partial x}{\partial\tau}=+\frac{\partial y}{\partial\lambda}=e^\tau\cos\lambda$, we have $dx=e^\tau(-\sin\lambda\, d\lambda+\cos\lambda\, d\tau)$, $dy=+e^\tau(\cos\lambda\, d\lambda+\sin\lambda\, d\tau)$ whence $ds_1^2=dx^2+dy^2=e^{2\tau}(d\lambda^2+d\tau^2)$. Now forming the ratio ds_1^2/ds_2^2 we have

$$\frac{ds_1^2}{ds_2^2}=\frac{e^{2\tau}(d\lambda^2+d\tau^2)}{d\lambda^2+d\tau^2}=e^{2\tau},\text{ or }\frac{ds_1}{ds_2}=e^\tau. \tag{30}$$

Since the right member of (30) is free of the direction in which ds_2 is measured and has a unique value for each value of τ the mapping is autogonal. The ratio ds_1/ds_2 is called the magnification or the scale of the projection. The method by which (30) was derived will be essentially the one used in obtaining the scale for the autogonal mapping of the spheroid.

PARAMETRIC REPRESENTATION OF SURFACES AND CURVES

A surface in three dimensions is given by an equation of the form $F(x,y,z)=0$, or $z=f(x,y)$. It may be given a parametric representation in terms of two parameters and in many ways. That is, we may write

$$x=x(\tau,\lambda),\ y=y(\tau,\lambda),\ z=z(\tau,\lambda), \tag{31}$$

where τ,λ are the arbitrary parameters. But we may change to other parameters by writing $\tau=\tau(\xi)$, $\lambda=\lambda(\delta)$, etc. The two parameters, of course, when eliminated among the three equations (31) must leave the equation of the original surface in the form $F(x,y,z)=0$ or $z=f(x,y)$.

For example, consider the sphere $x^2+y^2+z^2-r^2=0$ which is in the form $F(x,y,z)=0$. From figure 6 we have clearly that

$$x=r\cos\phi\cos\lambda,\ y=r\cos\phi\sin\lambda,\ z=r\sin\phi, \tag{32}$$

where the parameters are the latitude, ϕ, and longitude, λ. Squaring respective members of (32) and adding we have again $x^2+y^2+z^2-r^2=0$. Now in equations (32) place $\phi=\tan^{-1}\xi$, $\lambda=\cos^{-1}\delta$ and obtain $x=r\delta/(1+\xi^2)^{1/2}$, $y=r(1-\delta^2)^{1/2}/(1+\xi^2)^{1/2}$, $z=r\xi/(1+\xi^2)^{1/2}$, which is a new parametric representation of the sphere, since squaring and adding respective terms again produces $x^2+y^2+z^2-r^2=0$. Again in (32) place $\cos\phi=e^u$, $\cos\lambda=e^v$, whence $\sin\lambda=\sqrt{1-e^{2v}}$, $\sin\phi=\sqrt{1-e^{2u}}$ and we have $x=re^{u+v}$, $y=re^u\sqrt{1-e^{2v}}$, $z=r\sqrt{1-e^{2u}}$ which is still another parametric representation of the sphere. We could continue this process indefinitely, obtaining each time a different parametric representation of the sphere.

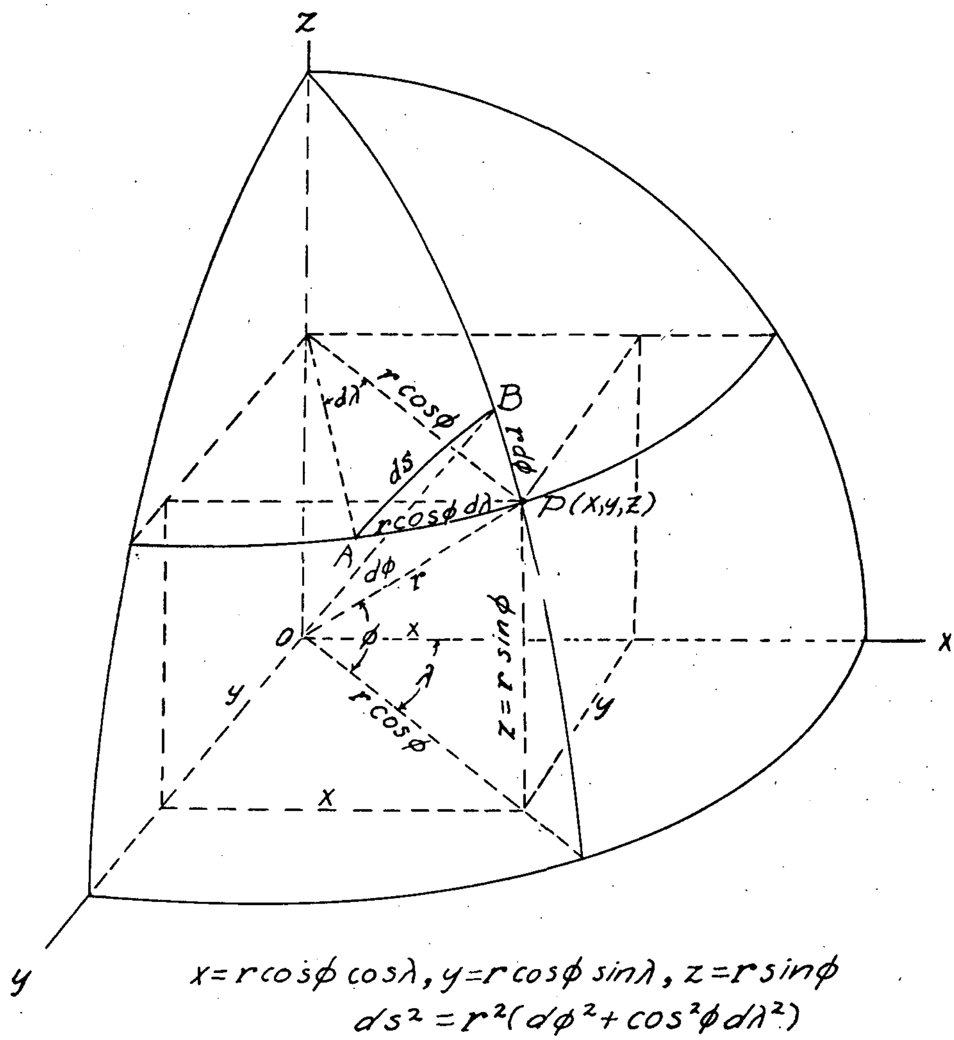

FIGURE 6.—Derivation of parametric equations for the sphere.

In the equation $z=f(x,y)$, place $y=c$ (constant). We have then the curve $z=f(x,c)$ which is the intersection of the plane $y=c$ and the surface $z=f(x,y)$. Any point on this plane curve has coordinates $x=x,\ y=c,\ z=f(x,c)$ as shown in figure 7.

If we place $y=u(x)$, where $u(x)$ is an arbitrary function of x, we have a curve on the surface which is the intersection of the cylinder $y=u(x)$ and the surface $z=f(x,y)$. (By a cylinder is meant the locus of a straight line which intersects a given fixed curve and moves always parallel to a given fixed straight line. In this case the given fixed line is the z-axis and the fixed curve is $y=u(x),\ z=0$ as shown in fig. 8.) The coordinates of any point on this curve, which is clearly a twisted or space curve since it does not lie in any plane, are

$$x=x,\ y=u(x),\ z=f[x,u(x)]. \tag{33}$$

From (33) we see that the coordinates are expressed in terms of the single parameter x.

The curve of intersection of any two surfaces $z=f_1(x,y)$, $z=f_2(x,y)$ may be considered as the intersection of a cylinder and a surface from the following considerations:

We have along their common curve of intersection $z=f_1(x,y)=f_2(x,y)$, or $f_1(x,y)-f_2(x,y)=\Phi(x,y)=0$, or $y=u(x)$. Then $x=x$, $y=u(x)$ and with either $z=f_1[x,u(x)]=f_2[x,u(x)]$ we have the coordinates of the curve of intersection in the form (33). For

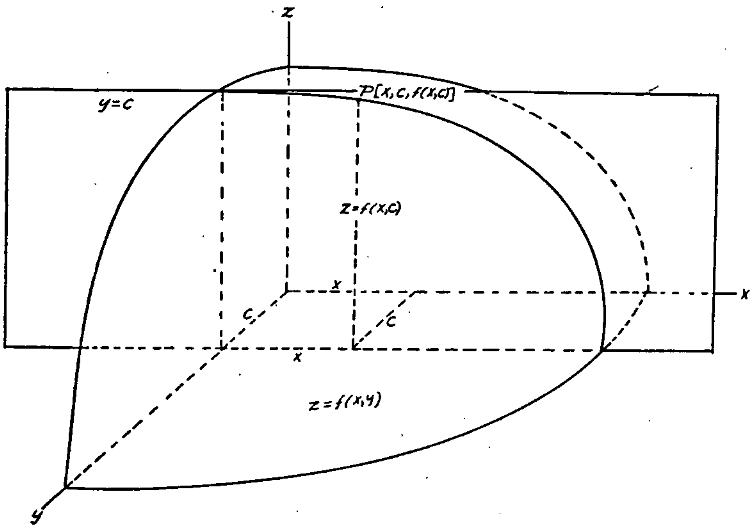

FIGURE 7.—Parametric equations of a plane curve on a surface.

example, consider the surfaces $z=2x^2+y^2-y$, $z=x^2+y^2$. We have $2x^2+y^2-y-x^2-y^2=x^2-y=0$, or $y=x^2$. Hence the coordinates of the curve of intersection are $x=x$, $y=x^2$, $z=2x^2+x^4-x^2=x^2+x^4=x^2(1+x^2)$, the curve being evidently a space quartic.

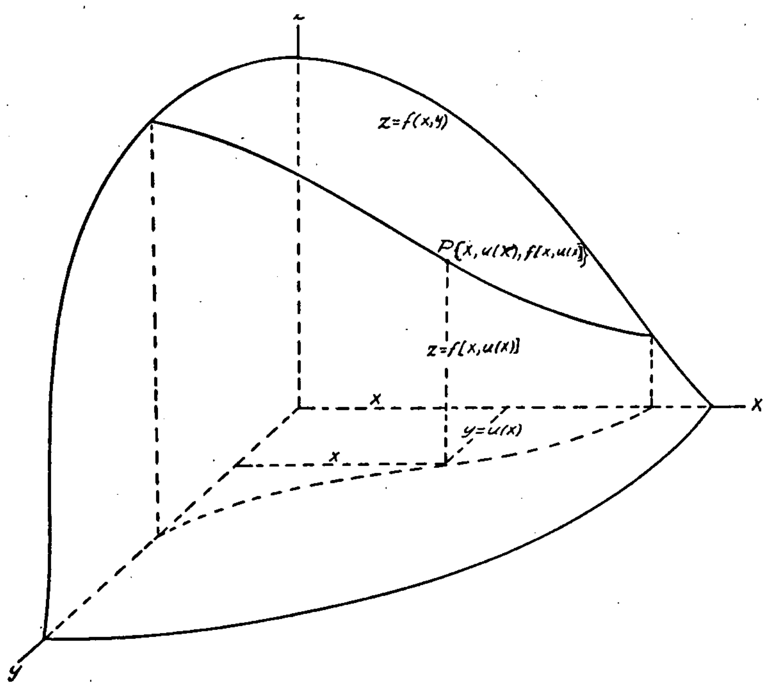

FIGURE 8.—Parametric equations of a space curve on a surface.

Now in equations (31) if we place $\lambda=c$, then the coordinates are a function of the single parameter τ, and similarly for $\tau=c$.

From figure 6 and equations (32) with $\lambda=c$ we have the coordinates as a function of ϕ or latitude only, whence the curve is the meridian section in longitude $\lambda=c$, or the intersection of a plane through the z-axis with the sphere.

Similarly with $\phi=c$, we have a parallel of latitude, or the intersection of the plane $z=r \sin c$ with the sphere. Hence $\phi=c$, $\lambda=c$ represent intersections of planes with the sphere giving curves on the surface which are called curvilinear coordinates. If we place $\lambda=\lambda(\phi)$ or $f(\lambda,\phi)=0$ we get a twisted or space curve on the surface passing through the intersection of the curves $\phi=c$, $\lambda=c$. Analogously for the general surface given by (31), $\tau=c_1$, $\lambda=c_2$ are parametric curves of the surface, whence $\tau=\tau(\lambda)$ or $\Phi(\tau,\lambda)=0$ represents a curve on the surface through the point $P(x,y,z)$, P being the intersection of the parametric curves $\tau=c_1$, $\lambda=c_2$ as shown in figure 9. Note that the parametric

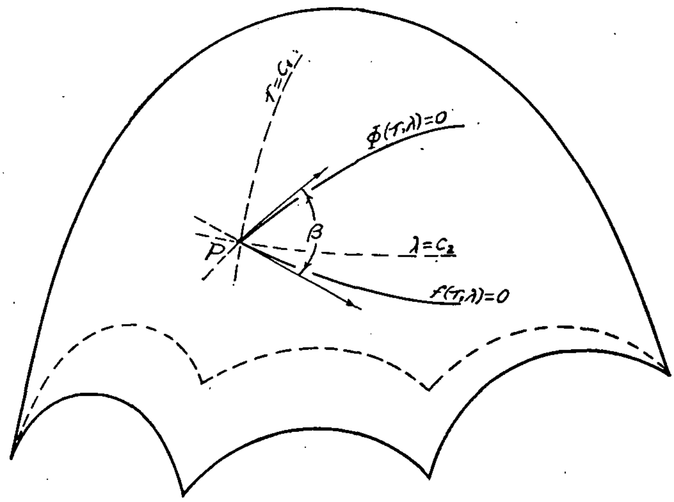

FIGURE 9.—The angle between two curves on a surface.

curves are not necessarily plane curves. In fact they seldom are. We may discuss the geometry of the surface with reference only to the parametric curves, knowing of course that the space coordinates x,y,z are functions of the parametric curves or parameters. We are familiar with this from the concept of latitude and longitude—a point is uniquely determined (except for the poles) by the intersection of the meridian in longitude, λ, and the parallel in latitude, ϕ. Such a point has also a rectangular representation, the coordinates being functions of the latitude and longitude. (See equations 32.)

We often express the coordinates of a curve on a surface in terms of arc length along the curve. This is especially convenient in the development of the differential geometry of curves and surfaces.

THE LINEAR ELEMENT OF A SURFACE

The linear element or differential of arc length of a curve $\Phi(\tau,\lambda)=0$ on a surface through a given point $P(x,y,z)$ of the surface, where x,y,z are obtained from (31) by placing $\tau=c_1$, $\lambda=c_2$ is

$$ds^2=dx^2+dy^2+dz^2 \tag{34}$$

From figure 10 we see that the chord $\Delta l=PQ$ of the curve (c) is given by $\Delta l^2=\Delta x^2+\Delta y^2+\Delta z^2$, hence

$$\frac{\Delta l^2}{\Delta s^2}=\left(\frac{\Delta x}{\Delta s}\right)^2+\left(\frac{\Delta y}{\Delta s}\right)^2+\left(\frac{\Delta z}{\Delta s}\right)^2.$$

In the limit as $Q \to P$, the chord Δl becomes the tangent at P, $\Delta l \to \Delta s$, so that we have

$$1=\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2,$$

which states that $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ are direction cosines of the tangent to the curve (c) at P. Multiplying through by ds^2 we have the differential of arc length of the arbitrary curve

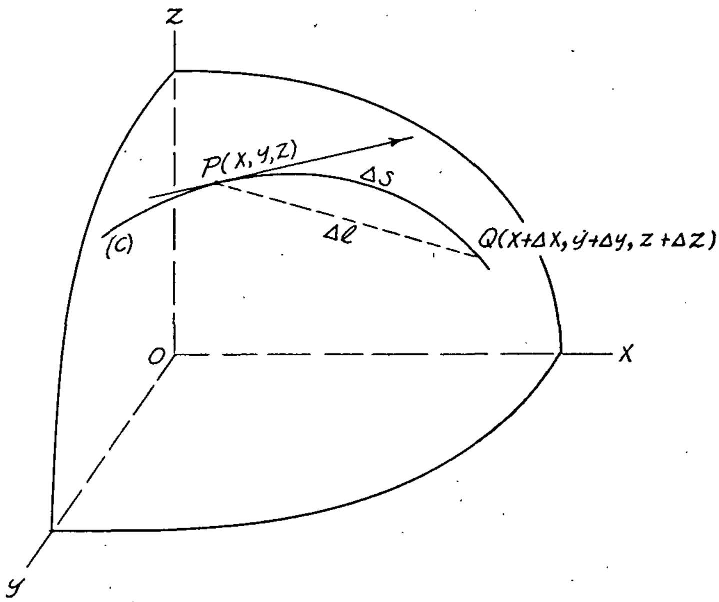

FIGURE 10.—The linear element of a surface.

(c) through P on the surface as given by (34) which is also called the linear element of the surface.

From (31), since the coordinates are functions of two parameters, we have

$$dx=\frac{\partial x}{\partial \tau}d\tau+\frac{\partial x}{\partial \lambda}d\lambda,\ dy=\frac{\partial y}{\partial \tau}d\tau+\frac{\partial y}{\partial \lambda}d\lambda,\ dz=\frac{\partial z}{\partial \tau}d\tau+\frac{\partial z}{\partial \lambda}d\lambda. \tag{35}$$

If the expressions in (35) are squared and placed in (34) there results the equation

$$ds^2=Ed\tau^2+2Fd\tau\, d\lambda+Gd\lambda^2, \tag{36}$$

where

$$\begin{aligned} E&=\left(\frac{\partial x}{\partial \tau}\right)^2+\left(\frac{\partial y}{\partial \tau}\right)^2+\left(\frac{\partial z}{\partial \tau}\right)^2,\\ F&=\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \lambda}+\frac{\partial y}{\partial \tau}\frac{\partial y}{\partial \lambda}+\frac{\partial z}{\partial \tau}\frac{\partial z}{\partial \lambda},\\ G&=\left(\frac{\partial x}{\partial \lambda}\right)^2+\left(\frac{\partial y}{\partial \lambda}\right)^2+\left(\frac{\partial z}{\partial \lambda}\right)^2. \end{aligned} \tag{37}$$

Equation (36) gives the linear element of the surface in terms of the curvilinear coordinates τ,λ. The quadratic differential form given by the right member of (36) is called the first fundamental form of the surface, and the quantities E, F, G the fundamental coefficients of the first order.

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THE ANGLE BETWEEN THE PARAMETRIC CURVES

Consider the parametric curve $\tau = c_1$. We have then $d\tau = 0$ and (36) becomes $ds_\tau = \sqrt{G}d\lambda$. For the parametric curve $\lambda = c_2, d\lambda = 0$ and (36) gives $ds_\lambda = \sqrt{E}\,d\tau$, or the elements of arc for the parametric curves $\tau = c_1$, $\lambda = c_2$ are respectively

$$ds_\tau = \sqrt{G}d\lambda,\ ds_\lambda = \sqrt{E}d\tau. \tag{38}$$

From (35) with $\tau = c_1$, $\lambda = c_2$ we have respectively

$$\begin{aligned} dx &= \frac{\partial x}{\partial \lambda}\,d\lambda,\ dy = \frac{\partial y}{\partial \lambda}\,d\lambda,\ dz = \frac{\partial z}{\partial \lambda}\,d\lambda, \\ dx &= \frac{\partial x}{\partial \tau}\,d\tau,\ dy = \frac{\partial y}{\partial \tau}\,d\tau,\ dz = \frac{\partial z}{\partial \tau}\,d\tau. \end{aligned} \tag{39}$$

If (L_1, M_1, N_1), (L_2, M_2, N_2) are direction cosines of the tangents to the curves $\tau = c_1$, $\lambda = c_2$ then

$$\begin{aligned} L_1 &= \frac{dx}{ds_\tau},\quad M_1 = \frac{dy}{ds_\tau},\quad N_1 = \frac{dz}{ds_\tau} \\ L_2 &= \frac{dx}{ds_\lambda},\quad M_2 = \frac{dy}{ds_\lambda},\quad N_2 = \frac{dz}{ds_\lambda}. \end{aligned} \tag{40}$$

Forming the ratios from (38) and (39) we find that equations (40) become

$$L_1, M_1, N_1 = \frac{\dfrac{\partial x}{\partial \lambda}, \dfrac{\partial y}{\partial \lambda}, \dfrac{\partial z}{\partial \lambda}}{\sqrt{G}},\ L_2, M_2, N_2 = \frac{\dfrac{\partial x}{\partial \tau}, \dfrac{\partial y}{\partial \tau}, \dfrac{\partial z}{\partial \tau}}{\sqrt{E}}. \tag{41}$$

If θ is the angle between the parametric curves $\tau = c_1$, $\lambda = c_2$ we have from (41) that

$$\cos\theta = L_1L_2 + M_1M_2 + N_1N_2 = \frac{\dfrac{\partial x}{\partial \lambda}\dfrac{\partial x}{\partial \tau} + \dfrac{\partial y}{\partial \lambda}\dfrac{\partial y}{\partial \tau} + \dfrac{\partial z}{\partial \lambda}\dfrac{\partial z}{\partial \tau}}{\sqrt{EG}}.$$

But by (37) the numerator of this fraction is F, hence

$$\cos\theta = F/\sqrt{EG};\ \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{EG-F^2}/\sqrt{EG}. \tag{42}$$

From (42) it is seen that the parametric curves are orthogonal if $F=0$.

For an example let us continue with the sphere. From equations (32) we have $\dfrac{\partial x}{\partial \phi} = -r\sin\phi\cos\lambda$, $\dfrac{\partial x}{\partial \lambda} = -r\cos\phi\sin\lambda$, $\dfrac{\partial y}{\partial \phi} = -r\sin\phi\sin\lambda$, $\dfrac{\partial y}{\partial \lambda} = r\cos\phi\cos\lambda$, $\dfrac{\partial z}{\partial \phi} = r\cos\phi$, $\dfrac{\partial z}{\partial \lambda} = 0$.

Forming the quantities E, F, G from equations (37) we have

$$\begin{aligned} E &= r^2\sin^2\phi\cos^2\lambda + r^2\sin^2\phi\sin^2\lambda + r^2\cos^2\phi = r^2, \\ F &= r^2\sin\phi\cos\phi\sin\lambda\cos\lambda - r^2\sin\phi\cos\phi\sin\lambda\cos\lambda = 0, \\ G &= r^2\cos^2\phi\sin^2\lambda + r^2\cos^2\phi\cos^2\lambda = r^2\cos^2\phi. \end{aligned} \tag{43}$$

Since $F=0$, the parametric curves are orthogonal. This we knew since the parametric curves are meridians and parallels on the sphere.

With the values of E, F, G from (43) the linear element as given by (36) is

$$\begin{aligned} ds^2 &= r^2(d\phi^2+\cos^2\phi\, d\lambda^2) \\ &= r^2\cos^2\phi\,(\sec^2\phi\, d\phi^2+d\lambda^2). \end{aligned} \tag{44}$$

Note in figure 6 (p. 28) that from the differential triangle PAB, considering it to be a plane right triangle with hypotenuse equal to ds, we have at once $ds^2=r^2d\phi^2+r^2\cos^2\phi d\lambda^2$.

THE ANGLE BETWEEN TWO CURVES ON A SURFACE

The direction cosines of the tangents to two curves $\Phi(\tau,\lambda)=0, f(\tau,\lambda)=0$ through $P(x,y,z)$, the intersection of the parametric curves $\tau=c_1, \lambda=c_2$, are respectively $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}; \frac{dx_1}{ds_1}, \frac{dy_1}{ds_1}, \frac{dz_1}{ds_1}$ where differentiation is with respect to arc length. (See the discussion following equation (34).) If β is the angle between the tangents then

$$\cos\beta=\frac{dx\cdot dx_1+dy\cdot dy_1+dz\cdot dz_1}{ds\cdot ds_1}. \tag{45}$$

With the values from (35) placed in (45) we have by (37)

$$\begin{aligned} \cos\beta &= \frac{Ed\tau\cdot d\tau_1+F(d\tau\cdot d\lambda_1+d\tau_1\cdot d\lambda)+Gd\lambda\cdot d\lambda_1}{ds\cdot ds_1}, \\ \sin\beta &= \sqrt{EG-F^2}\frac{(d\tau_1\cdot d\lambda-d\tau\cdot d\lambda_1)}{ds\cdot ds_1}. \end{aligned} \tag{46}$$

THE ANGLES BETWEEN A CURVE ON THE SURFACE AND THE PARAMETRIC CURVES

If the curve $f(\tau,\lambda)=0$ is the parametric curve $\lambda=c_2$ through P, then $d\lambda_1=0$ and, from (36) or (38), $ds_1=\sqrt{E}d\tau_1$. With these values of $d\lambda_1$ and ds_1, equations (46)

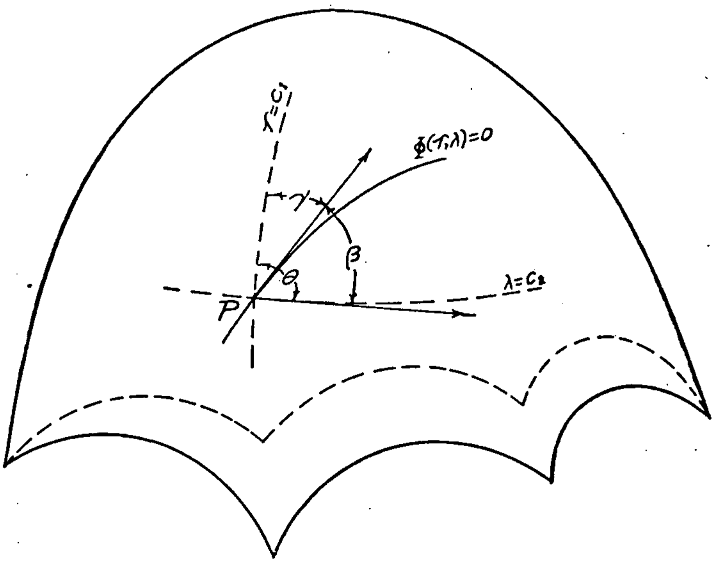


FIGURE 11.—The angles made by a curve with the parametric curves.

become

$$\cos\beta=\frac{1}{\sqrt{E}}\left(E\frac{d\tau}{ds}+F\frac{d\lambda}{ds}\right), \quad \sin\beta=\frac{\sqrt{EG-F^2}}{\sqrt{E}}\frac{d\lambda}{ds}, \tag{47}$$

where β is now the angle between the curve $\Phi(\tau,\lambda)=0$ and the parametric curve $\lambda=c_2$. See figure 11.

If γ is the angle which the curve $\Phi(\tau,\lambda)=0$ makes with the parametric curve $\tau=c_1$ and θ is the angle between the parametric curves (see fig. 11) we have $\gamma=\theta-\beta$, whence

$$\begin{aligned}\cos\gamma&=\cos(\theta-\beta)=\cos\theta\cos\beta+\sin\theta\sin\beta,\\ \sin\gamma&=\sin(\theta-\beta)=\sin\theta\cos\beta-\cos\theta\sin\beta.\end{aligned}\tag{48}$$

With the values of $\sin\theta$, $\cos\theta$, $\sin\beta$, and $\cos\beta$ from (42) and (47) placed in (48) we have

$$\cos(\theta-\beta)=\frac{1}{\sqrt{G}}\left(F\frac{d\tau}{ds}+G\frac{d\lambda}{ds}\right),\ \sin(\theta-\beta)=\frac{\sqrt{EG-F^2}}{\sqrt{G}}\frac{d\tau}{ds}.\tag{49}$$

FAMILIES OF CURVES ON A SURFACE

We have seen that $f(\tau,\lambda)=0$ represents a curve on a surface. A family of curves on a surface is given by

$$f(\tau,\lambda)=c,\tag{50}$$

where c is an arbitrary constant.

Now (50) is the solution of the ordinary differential equation of first order and first degree

$$M(\tau,\lambda)\,d\tau+N(\tau,\lambda)\,d\lambda=0.\tag{51}$$

To show that any solution of (51) defines the same family of curves we suppose that $f_1(\tau,\lambda)=c_1$ is also a solution. Now if both (50) and $f_1(\tau,\lambda)=c_1$ define the same family of curves on the surface, f_1 must be an arbitrary function of f.

From (50) and $f_1(\tau,\lambda)=c_1$ we have by differentiation

$$\frac{\partial f}{\partial\tau}d\tau+\frac{\partial f}{\partial\lambda}d\lambda=0,\quad \frac{\partial f_1}{\partial\tau}d\tau+\frac{\partial f_1}{\partial\lambda}d\lambda=0.\tag{52}$$

From (51) and (52), solving for $d\lambda/d\tau$ in each case we have

$$\frac{d\lambda}{d\tau}=-\frac{\partial f}{\partial\tau}\Big/\frac{\partial f}{\partial\lambda}=-\frac{\partial f_1}{\partial\tau}\Big/\frac{\partial f_1}{\partial\lambda}=-\frac{M}{N}.\tag{53}$$

From (53) we have $\frac{\partial f}{\partial\tau}\frac{\partial f_1}{\partial\lambda}-\frac{\partial f_1}{\partial\tau}\frac{\partial f}{\partial\lambda}=0$, or $J\left(\frac{f,f_1}{\tau,\lambda}\right)=0$, which is the condition that f_1 should be a function of f and hence all the solutions of (51) define the same family of curves on the surface.

ORTHOGONAL TRAJECTORIES

If a curve is orthogonal to every curve of a given family of curves, (50), it is called an orthogonal trajectory of the given family. (Two curves on a surface are orthogonal to each other if, at each point of intersection of the curves, the corresponding tangents to the curves are orthogonal.)

Let the given family of curves be defined by (51). From (46) the condition that two curves on a surface be orthogonal is

$$E+F\left(\frac{d\lambda}{d\tau}+\frac{d\lambda_1}{d\tau_1}\right)+G\frac{d\lambda}{d\tau}\frac{d\lambda_1}{d\tau_1}=0.\tag{54}$$

Placing the value of $d\lambda_1/d\tau_1=-M/N$ from (53) in (54) we obtain

$$(NE-MF)d\tau+(FN-GM)d\lambda=0, \tag{55}$$

which is the differential equation of the orthogonal trajectories of (51).

A differential equation of second degree of the form

$$R(\tau,\lambda)+2\,S(\tau,\lambda)\frac{d\lambda}{d\tau}+T(\tau,\lambda)\left(\frac{d\lambda}{d\tau}\right)^2=0, \tag{56}$$

may be solved as a quadratic in $\frac{d\lambda}{d\tau}$, giving then two differential equations of first degree. The integrals of these two differential equations will give then two distinct families of curves on the surface provided the discriminant of (56) does not vanish. If $\frac{d\lambda}{d\tau}$, $\frac{d\lambda_1}{d\tau_1}$ are the roots of (56), we have $\frac{d\lambda}{d\tau}+\frac{d\lambda_1}{d\tau_1}=-\frac{2S}{T}$, $\frac{d\lambda}{d\tau}\frac{d\lambda_1}{d\tau_1}=\frac{R}{T}$. With these values placed in (54) we obtain

$$ET-2SF+GR=0, \tag{57}$$

which is the condition that one of the families of curves given by (56) shall be the orthogonal trajectories of the other.

CONFORMAL REPRESENTATION OF ONE SURFACE UPON ANOTHER

A surface has conformal representation on another if a one-to-one correspondence is established between their points in such a way that the angles between corresponding lines on the surface are equal.

To obtain the condition for this we assume that both surfaces, S and S_1, are referred to a corresponding system of real lines in terms of the same parameters τ, λ and that corresponding points have the same curvilinear coordinates. The respective linear elements may then be written

$$\begin{aligned}&(S)\ \ ds^2=E\,d\tau^2+2\,F d\tau\,d\lambda+G\,d\lambda^2,\\&(S_1)\ ds_1^2=E_1\,d\tau^2+2\,F_1 d\tau\,d\lambda+G_1\,d\lambda^2.\end{aligned} \tag{58}$$

From (42) the cosines of the angles θ, θ_1 between the respective parametric curves on the two surfaces are $\frac{F}{\sqrt{EG}}$, $\frac{F_1}{\sqrt{E_1G_1}}$ and if the representation is to be conformal then

$$\frac{F}{\sqrt{EG}}=\frac{F_1}{\sqrt{E_1G_1}}. \tag{59}$$

In figure 12, β, β_1 are the angles which corresponding curves $\Phi(\tau,\lambda)=0$, $\Phi_1(\tau,\lambda)=0$ on S and S_1 respectively make with the parametric curve $\lambda=c_2$ at corresponding points P and P_1. From (47) and (49) we have

$$\begin{aligned}&\sin\beta=\frac{\sqrt{EG-F^2}}{\sqrt{E}}\frac{d\lambda}{ds},\ \sin(\theta-\beta)=\frac{\sqrt{EG-F^2}}{\sqrt{G}}\frac{d\tau}{ds},\\&\sin\beta_1=\frac{\sqrt{E_1G_1-F_1^2}}{\sqrt{E_1}}\frac{d\lambda}{ds_1},\ \sin(\theta_1-\beta_1)=\frac{\sqrt{E_1G_1-F_1^2}}{\sqrt{G_1}}\frac{d\tau}{ds_1}.\end{aligned} \tag{60}$$

If the representation is to be conformal we must have $\theta = \pm\theta_1$, $\beta = \pm\beta_1$ according as the angles have the same or the opposite sense. From (60) we have then

$$\frac{\sqrt{EG-F^2}}{\sqrt{E}}\frac{d\lambda}{ds} = \pm\frac{\sqrt{E_1G_1-F_1^2}}{\sqrt{E_1}}\frac{d\lambda}{ds_1},\quad \frac{\sqrt{EG-F^2}}{\sqrt{G}}\frac{d\tau}{ds} = \pm\frac{\sqrt{E_1G_1-F_1^2}}{\sqrt{G_1}}\frac{d\tau}{ds_1} \tag{61}$$

where the signs are chosen according to the sense of the angles in the correspondence. From (61) we have

$$\frac{(E_1G_1-F_1^2)^{1/2}}{(EG-F^2)^{1/2}}\frac{ds}{ds_1} = \pm\frac{\sqrt{G_1}}{\sqrt{G}} = \pm\frac{\sqrt{E_1}}{\sqrt{E}}. \tag{62}$$

From (59) we have $\frac{F_1}{F} = \frac{\sqrt{E_1}}{\sqrt{E}}\frac{\sqrt{G_1}}{\sqrt{G}}$ which with (62) gives

$$\frac{E_1G_1-F_1^2}{EG-F^2}\frac{ds^2}{ds_1^2} = \frac{E_1}{E} = \frac{F_1}{F} = \frac{G_1}{G}. \tag{63}$$

Solving (63) for E_1, F_1, G_1 and combining with equations (58) we have $ds_1^4 = \frac{E_1G_1-F_1^2}{EG-F^2}ds^4$, or finally

$$\frac{ds_1^2}{ds^2} = k^2,\ \text{where } k^2 = \left(\frac{E_1G_1-F_1^2}{EG-F^2}\right)^{1/2}. \tag{64}$$

Since E, F, G, E_1, F_1, G_1 are, in general, functions of τ, λ then k^2 may be a function of τ, λ.

From the above it is seen that (64) is the condition to be satisfied by the linear elements of the two surfaces in order that the representation shall be conformal. Note that we have used this condition in the discussion of the conformal representation of one plane upon another by means of the complex variable. See equation (30).

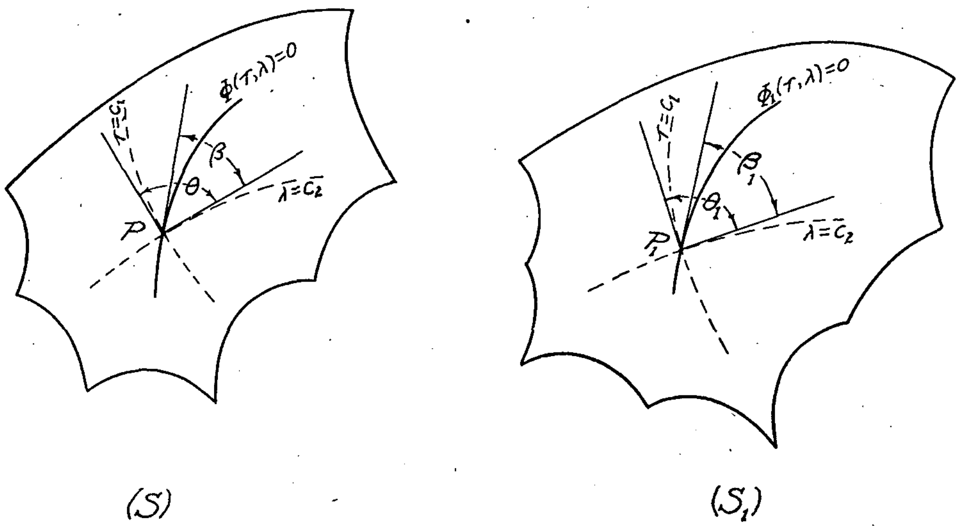


FIGURE 12.—Corresponding angles at corresponding points of two surfaces having the same curvilinear coordinates.

The conformal representation is direct or inverse according as the relative positions of the positive half-tangents to the parametric curves on the surface are the same or symmetric.

Referring to figure 12, note that the elements of area on the surfaces S and S_1 are respectively $dA=\sin\theta\, ds_\lambda\, ds_\tau$, $dA_1=\sin\theta_1\, ds_\lambda\, ds_\tau$ which become from (38) and (42) $dA=\sqrt{EG-F^2}\,d\lambda\, d\tau$, $dA_1=\sqrt{E_1G_1-F_1^2}\,d\lambda\, d\tau$. Hence $\frac{dA}{dA_1}=\sqrt{\frac{EG-F^2}{E_1G_1-F_1^2}}=\frac{1}{k^2}$, that is, $\frac{1}{k^2}$ is the ratio of the elements of area on the two surfaces.

CONFORMAL REPRESENTATION OF A SURFACE UPON A PLANE

If we consider the surface (S), equations (58), to be a plane, then the linear element, or differential of arc, must be given by

$$ds^2=d\tau^2+d\lambda^2, \tag{65}$$

where τ, λ are rectangular coordinates in a plane. We have thus $E=G=1$, $F=0$. Then because of equation (64) we must have the linear element of the surface (S_1) in the form

$$ds_1^2=m(d\tau^2+d\lambda^2),\text{ where now } m=k^2. \tag{66}$$

This means that the parametric curves must be orthogonal since $F_1=0$. Suppose the linear element of the given surface (S_1) is then $ds_1^2=E_1du^2+G_1dv^2$. If E_1 is a function of u alone and G_1 is a function of v alone we may place $d\tau^2=E_1du^2$, $d\lambda^2=G_1dv^2$ and the linear element becomes $ds_1^2=d\tau^2+d\lambda^2$, that is, $m=1$, whence there is no distortion in the representation. Surfaces having such linear elements are developable surfaces—surfaces such as cones or cylinders which can be "cut" along a linear element or "generator" and made to coincide with a plane by "rolling out" without stretching or tearing.

Suppose the linear element $ds_1^2=E_1du^2+G_1dv^2$ can be written

$$ds_1^2=m\left(\frac{du^2}{U}+\frac{dv^2}{V}\right),\quad\text{where } E_1=\frac{m}{U},\ G_1=\frac{m}{V}, \tag{67}$$

and we place

$$d\tau^2=\frac{du^2}{U},\ d\lambda^2=\frac{dv^2}{V}. \tag{68}$$

The linear element will then be in the required form (66) but in order to integrate equations (68), U must be a function of u alone and V must be a function of v alone. From (67) we have

$$m=UE_1=VG_1 \text{ or } \frac{U}{V}=\frac{G_1}{E_1}. \tag{69}$$

The conditions then for the conformal representation on a plane of a surface which is not developable are that the parametric curves must be orthogonal, $F_1=0$, and that E_1 and G_1 must satisfy a relation of the form (69) where U is a function of u alone and V is a function of v alone.

ISOMETRIC ORTHOGONAL SYSTEMS

When the linear element is in the form (66) we have seen that the parametric curves are orthogonal. Note also from (66) that the elements of arc of the parametric curves $\tau=c_1$, $\lambda=c_2$ are respectively $\sqrt{m}\,d\lambda$, $\sqrt{m}\,d\tau$. Hence when the increments $d\tau$, $d\lambda$ are taken equal, the four points (τ, λ), $(\tau+d\tau, \lambda)$, $(\tau, \lambda+d\lambda)$, $(\tau+d\tau, \lambda+d\lambda)$ are the vertices of a small square as shown in figure 13. Hence the parametric curves divide the surface into a network of small squares, not necessarily all of the same size. On this account these curves are called isometric curves and τ, λ isometric parameters.

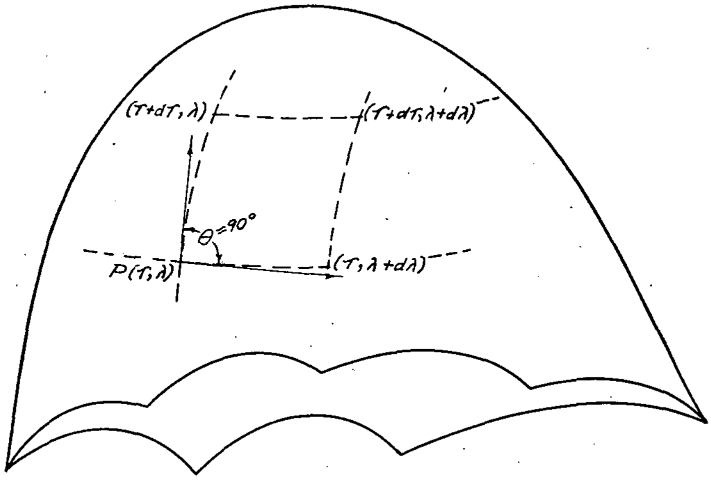


FIGURE 13.—Isometric orthogonal net on a surface.

Thus it is seen that a surface to be mapped conformally upon a plane must be referred to an isometric orthogonal system.

DIFFERENTIAL GEOMETRY OF A CURVE ON A SURFACE

In figure 14 we have a curve (c) on a surface (S). At any point $P(x,y,z)$ of this curve (c) we have always associated three mutually perpendicular lines, namely the tangent PT, the principal normal PN, and the binormal PU.

If Q is a neighboring point of P on (c), then the chord PQ and the tangent PT determine a plane which as $Q \rightarrow P$ assumes a limiting position at P, the chord PQ coinciding with the tangent PT. This plane is called the osculating plane of the curve (c) at the point P.

A plane perpendicular to the tangent PT at P is called the normal plane, and its intersection with the osculating plane determines the principal normal PN. The binormal PU orthogonal to PN and lying in the normal plane determines with the tangent the rectifying plane as shown in figure 14.

The curve (c), considered a space curve, has two radii of curvature associated with it at every point. ρ_1, the first radius of curvature, lies along the principal normal. The second radius of curvature, ρ_2, called the radius of torsion, lies along the binormal.

In order to find the equation of the osculating plane we need to find under what conditions a curve and surface have contact of a given order.

CONTACT OF A CURVE AND SURFACE

If P, P_1, P_2, . . . P_n are points of a given curve which also lie on a given surface and the points P_1, P_2, . . . P_n tend to P, then in the limit, when P_1, P_2, . . . P_n coincide with P, the curve and surface have contact of the nth order at P.

Assuming that the coordinates of P are $x=x(s)$, $y=y(s)$, $z=z(s)$ and that the equation of the surface is of the form $f(X,Y,Z)=0$, so that at a point P, common to the surface and curve, $f(x,y,z)=f(s)=0$, then the roots of $f(s)=0$ are values of s which correspond to the points of intersection of the curve and surface. If the curve and surface have contact of the first order at the point for which $s=s_1$, the equation $f(s)=0$ has two roots equal to s_1, and therefore $f(s_1)=0$, and $\frac{df}{ds_1}=\frac{\partial f}{\partial x}\frac{dx}{ds_1}+\frac{\partial f}{\partial y}\frac{dy}{ds_1}+\frac{\partial f}{\partial z}\frac{dz}{ds_1}=0$. This may be extended. That is, if the contact is of the second order, the equation $f(s)=0$ has three roots equal to s_1, therefore we must have $f(s_1)=0$, $\frac{df}{ds_1}=0$, $\frac{d^2f}{ds_1^2}=0$. In general, if the contact is of nth order, then $f(s_1)=0, f'(s_1)=f''(s_1)= \ . \ . \ . \ =f^n(s_1)=0$. (Note that we are using here essentially the theorem that an $n+1$ fold zero of a function

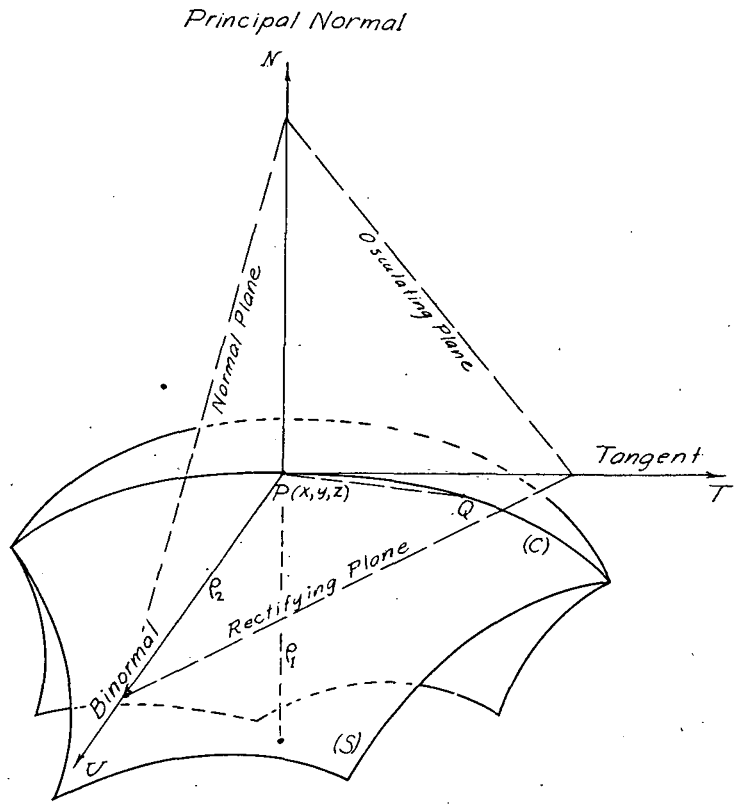


FIGURE 14.—The three planes and three axes associated with a space curve at each of its points.

is also a zero of its nth derivative. This is easily seen from the fact that if s_1 is an $n+1$ fold root of $f(s)=0$, then $s-s_1$ is an $n+1$ fold factor, or $f(s)=g(s)(s-s_1)^{n+1}$. Thus $s-s_1$ will be in every term of every derivative up to and including the nth derivative.)

THE OSCULATING PLANE

In figure 14, the osculating plane is determined when the chord PQ coincides with the tangent PT. This implies that the curve and plane have contact of second order at $P(x,y,z)$, since the tangent requires contact of first order. Therefore if $AX+BY+CZ+D=0$ is a general plane we must have

$$f(s)=Ax+By+Cz+D=0$$

$$f'(s)=Ax'+By'+Cz'=0$$

$$f''(s)=Ax''+By''+Cz''=0.$$

Eliminating A, B, C, D from these four equations by writing the eliminant we have

$$\begin{vmatrix} X & Y & Z & 1 \\ x & y & z & 1 \\ x' & y' & z' & 0 \\ x'' & y'' & z'' & 0 \end{vmatrix} = \begin{vmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = \begin{aligned} &(X-x)(y'z''-z'y'') \\ &+(Y-y)(z'x''-x'z'') \\ &+(Z-z)(x'y''-y'x'')=0, \end{aligned} \tag{70}$$

which is the equation of the osculating plane at $P(x,y,z)$.

THE NORMAL PLANE

From the discussion following equation (34), the direction cosines of the tangent PT were noted to be $x'=\frac{dx}{ds}$, $y'=\frac{dy}{ds}$, $z'=\frac{dz}{ds}$. Hence a plane through $P(x,y,z)$ normal to the tangent would have the equation

$$(X-x)x'+(Y-y)y'+(Z-z)z'=0. \tag{71}$$

Since x',y',z' are direction cosines we have also $x'^2+y'^2+z'^2=1$, and by differentiation $x'x''+y'y''+z'z''=0$.

DIRECTION COSINES OF THE TANGENT, PRINCIPAL NORMAL, BINORMAL

We now develop the formulas for some of the fundamental differential relations among the associated geometric elements of the curve in the neighborhood of the point $P(x, y, z)$.

Let l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 be the direction cosines of the tangent, principal normal, and binormal respectively. Then these direction cosines must satisfy the following two sets of conditions:

$$\begin{aligned} l_1^2+m_1^2+n_1^2&=1 \\ l_2^2+m_2^2+n_2^2&=1 \\ l_3^2+m_3^2+n_3^2&=1, \end{aligned} \tag{72}$$

$$\begin{aligned} l_1l_2+m_1m_2+n_1n_2&=0 \\ l_1l_3+m_1m_3+n_1n_3&=0 \\ l_2l_3+m_2m_3+n_2n_3&=0. \end{aligned} \tag{73}$$

From these we have the following relations among the direction cosines:

$$\begin{aligned} l_1&=m_2n_3-n_2m_3, & m_1&=n_2l_3-l_2n_3, & n_1&=l_2m_3-m_2l_3 \\ l_2&=m_3n_1-n_3m_1, & m_2&=n_3l_1-l_3n_1, & n_2&=l_3m_1-m_3l_1 \\ l_3&=m_1n_2-n_1m_2, & m_3&=n_1l_2-l_1n_2, & n_3&=l_1m_2-m_1l_2. \end{aligned} \tag{74}$$

PRINCIPAL CURVATURE OF A SPACE CURVE

Since the principal normal, PN in figure 14, is the intersection of the osculating plane and the normal plane we may obtain the direction cosines of the normal directly from the coefficients of equations (70) and (71). That is,

$$l_2:m_2:n_2=\begin{vmatrix} z'x''-x'z'' & x'y''-y'x'' \\ y' & z' \end{vmatrix} : \begin{vmatrix} x'y''-y'x'' & y'z''-z'y'' \\ z' & x' \end{vmatrix} : \begin{vmatrix} y'z''-z'y'' & z'x''-x'z'' \\ x' & y' \end{vmatrix}. \tag{75}$$

By means of the relations following equation (71) we may write l_2,m_2,n_2 from (75) as

$$\begin{aligned} l_2=\rho_1(z'^2x''-z'z''x'-x'y'y''+x''y'^2)&=\rho_1[-x'(y'y''+z'z'')+x''(y'^2+z'^2)] \\ &=\rho_1[x'^2x''+(y'^2+z'^2)x''] \\ &=\rho_1x''(x'^2+y'^2+z'^2) \\ &=\rho_1x'', \end{aligned}$$

$$m_2=\rho_1y'',$$
$$n_2=\rho_1z''.$$

Squaring and adding we have $l_2^2+m_2^2+n_2^2=\rho_1^2(x''^2+y''^2+z''^2)=1$, whence

$$\frac{1}{\rho_1^2}=x''^2+y''^2+z''^2=(l_1')^2+(m_1')^2+(n_1')^2, \tag{76}$$

that is, ρ_1 is the first radius of curvature of a space curve.

From (72) and (74) we must have for the binormal

$$l_3^2+m_3^2+n_3^2=(m_1n_2-n_1m_2)^2+(n_1l_2-l_1n_2)^2+(l_1m_2-m_1l_2)^2=1. \tag{77}$$

With the values of $l_1=x'$, $m_1=y'$, $n_1=z'$; $l_2=\rho_1x''$, $m_2=\rho_1y''$, $n_2=\rho_1z''$equation (77) becomes $(y'z''-z'y'')^2+(z'x''-x'z'')^2+(x'y''-y'x'')^2=\frac{1}{\rho_1^2}$, which may be written

$$(x'^2+y'^2+z'^2)\,(x''^2+y''^2+z''^2)-(x'\,x''+y'\,y''+z'\,z'')^2=\frac{1}{\rho_1^2}. \tag{78}$$

From (73), (74), and (76) it is seen that (78) becomes $1\cdot\frac{1}{\rho_1^2}-0=\frac{1}{\rho_1^2}$, so that ρ_1 is the factor of proportionality for the direction cosines of the binormal which may now be written

$$l_3=\rho_1(y'\,z''-z'\,y''),\ m_3=\rho_1(z'\,x''-x'\,z''),\ n_3=\rho_1(x'\,y''-y'\,x''). \tag{79}$$

SECOND CURVATURE OR TORSION OF A SPACE CURVE

The torsion, $1/\rho_2$, or second curvature is defined analogously as the first curvature, that is,

$$\frac{1}{\rho_2^2}=(l_3')^2+(m_3')^2+(n_3')^2. \tag{80}$$

THE FRENET-SERRET FORMULAS

We may express the derivatives of the direction cosines of the tangent, principal normal, and binormal as functions of the direction cosines and the two radii of curvature, ρ_1 and ρ_2, as follows:

We have $l_1=x'$, $m_1=y'$, $n_1=z'$ whence $l_1'=x''$, $m_1'=y''$, $n_1'=z''$.
But $l_2=\rho_1x''$, $m_2=\rho_1y''$, $n_2=\rho_1z''$ whence

$$l_1'=l_2/\rho_1,\ m_1'=m_2/\rho_1,\ n_1'=n_2/\rho_1. \tag{81}$$

From the third of equations (72) and the second of equations (73) we have by differentiating

$$
\begin{aligned}
l_3 l_3' + m_3 m_3' + n_3 n_3' &= 0, \\
(l_1 l_3' + m_1 m_3' + n_1 n_3') + (l_1' l_3 + m_1' m_3 + n_1' n_3) &= 0.
\end{aligned}
\tag{82}
$$

By placing the values of l_1', m_1', n_1' from (81) in the second term of the second equation of (82) we get for that term $\frac{1}{\rho_1}(l_2 l_3 + m_2 m_3 + n_2 n_3)$ which is zero by the third of equations (73). Hence we have

$$
\begin{aligned}
l_1 l_3' + m_1 m_3' + n_1 n_3' &= 0, \\
l_3 l_3' + m_3 m_3' + n_3 n_3' &= 0,
\end{aligned}
\tag{83}
$$

from which to determine l_3', m_3', n_3'. We find

$$l_3' : m_3' : n_3' = (n_1 m_3 - m_1 n_3) : (l_1 n_3 - n_1 l_3) : (m_1 l_3 - l_1 m_3)$$

and from (80) the factor of proportionality is ρ_2, that is,

$$l_3' = \frac{n_1 m_3 - m_1 n_3}{\rho_2},\ m_3' = \frac{l_1 n_3 - n_1 l_3}{\rho_2},\ n_3' = \frac{m_1 l_3 - l_1 m_3}{\rho_2}.$$

From (74) it is seen that the numerators of these last equations are respectively l_2, m_2, n_2 so that

$$l_3' = \frac{l_2}{\rho_2},\ m_3' = \frac{m_2}{\rho_2},\ n_3' = \frac{n_2}{\rho_2}. \tag{84}$$

To obtain the derivatives of l_2, m_2, n_2 we differentiate the expressions in (74) for them as follows:

$$l_2' = n_1' m_3 + n_1 m_3' - m_1' n_3 - m_1 n_3'. \tag{85}$$

In (85) place the values m_1', n_1'; m_3', n_3' from (81) and (84) to obtain

$$l_2' = -\frac{m_2 n_3 - n_2 m_3}{\rho_1} - \frac{m_1 n_2 - n_1 m_2}{\rho_2}. \tag{86}$$

But the numerators of (86) are, by (74), l_1 and l_3 so that $l_2' = -\left(\frac{l_1}{\rho_1} + \frac{l_3}{\rho_2}\right)$. We find similar expressions for m_2' and n_2' and group all these results together for easy reference.

Direction cosines:

Tangent: $l_1 = x',\ m_1 = y',\ n_1 = z'$

Principal Normal: $l_2 = \rho_1 x'',\ m_2 = \rho_1 y'',\ n_2 = \rho_1 z''$ (87)

Binormal: $l_3 = \rho_1 (y'z'' - y''z'),\ m_3 = \rho_1 (z'x'' - z''x'),\ n_3 = \rho_1 (x'y'' - x''y')$.

First derivatives of the direction cosines of the tangent, principal normal, and binormal (known as the Frenet-Serret formulas):

Tangent: $l_1' = \frac{l_2}{\rho_1},\ m_1' = \frac{m_2}{\rho_1},\ n_1' = \frac{n_2}{\rho_1}$

Principal Normal: $l_2' = -\left(\frac{l_1}{\rho_1} + \frac{l_3}{\rho_2}\right),\ m_2' = -\left(\frac{m_1}{\rho_1} + \frac{m_3}{\rho_2}\right),\ n_2' = -\left(\frac{n_1}{\rho_1} + \frac{n_3}{\rho_2}\right)$ (88)

Binormal: $l_3'=\dfrac{l_2}{\rho_2}$, $m_3'=\dfrac{m_2}{\rho_2}$, $n_3'=\dfrac{n_2}{\rho_2}$,

where ρ_1 and ρ_2 are the first and second radii of curvature of the given curve (c) at the point P.

We now derive a formula for the torsion in terms of the first radius of curvature and derivatives of the coordinates of the point $P(x,y,z)$.

In (87) take the derivatives of the direction cosines l_3,m_3,n_3 of the binormal to get

$$\begin{aligned} l_3'&=\rho_1'(y'z''-z'y'')+\rho_1(y'z'''-z'y''') \\ m_3'&=\rho_1'(z'x''-x'z'')+\rho_1(z'x'''-x'z''') \\ n_3'&=\rho_1'(x'y''-y'x'')+\rho_1(x'y'''-y'x'''). \end{aligned} \tag{89}$$

With the values of the derivatives l_3',m_3',n_3' from (88) placed in the left members of (89) and the factors $(y'z''-z'y'')$, etc. of the second members replaced by their values in terms of l_3, etc. from (87) we may write (89) as

$$\begin{aligned} \frac{l_2}{\rho_2}&=\frac{\rho_1'}{\rho_1}\,l_3+\rho_1(y'z'''-z'y''') \\ \frac{m_2}{\rho_2}&=\frac{\rho_1'}{\rho_1}\,m_3+\rho_1(z'x'''-x'z''') \\ \frac{n_2}{\rho_2}&=\frac{\rho_1'}{\rho_1}\,n_3+\rho_1(x'y'''-y'x'''). \end{aligned} \tag{90}$$

If we multiply the first of equations (90) by l_2, the second by m_2, the third by n_2 and add respective members we obtain

$$\begin{aligned} \frac{l_2^2+m_2^2+n_2^2}{\rho_2}=\frac{\rho_1'}{\rho_1}\,&(l_2l_3+m_2m_3+n_2n_3)+\rho_1l_2(y'z'''-z'y''')+\rho_1m_2(z'x'''-x'z''')+ \\ &\rho_1n_2(x'y'''-y'x'''). \end{aligned} \tag{91}$$

The numerator of the left side of (91) is unity because of the second of equations (72). The first member of the right side of (91) is zero because of the third of equations (73). In the last three members of (91) replace l_2, m_2, n_2 by their values from (87) and we may write (91) as

$$\frac{1}{\rho_2}=\rho_1^2[x''(y'z'''-z'y''')+y''(z'x'''-x'z''')+z''(x'y'''-y'x''')], \tag{92}$$

or in determinant form

$$\frac{1}{\rho_2}=-\rho_1^2\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}.$$

Equation (92) gives the torsion, or second curvature in terms of the radius of first curvature and the first three derivatives of the coordinates of the point $P(x,y,z)$ of the curve (c) on the surface.

Note that in obtaining the differential formulas of this section the parameter was the arc length along the given curve (c) measured from the point $P(x,y,z)$ on the surface (S), figure 14.

EQUATIONS OF THE TANGENT PLANE AND NORMAL TO A SURFACE

To develop these equations we will consider the surface given by the parametric representation (31). In figure 15 we have the tangents t_1 and t_2 to the parametric curves $\tau=c_1$ and $\lambda=c_2$ at the point $P(x,y,z)$ or $P(\tau,\lambda)$ of the surface (S). These tangents determine the plane tangent to the surface at the point P.

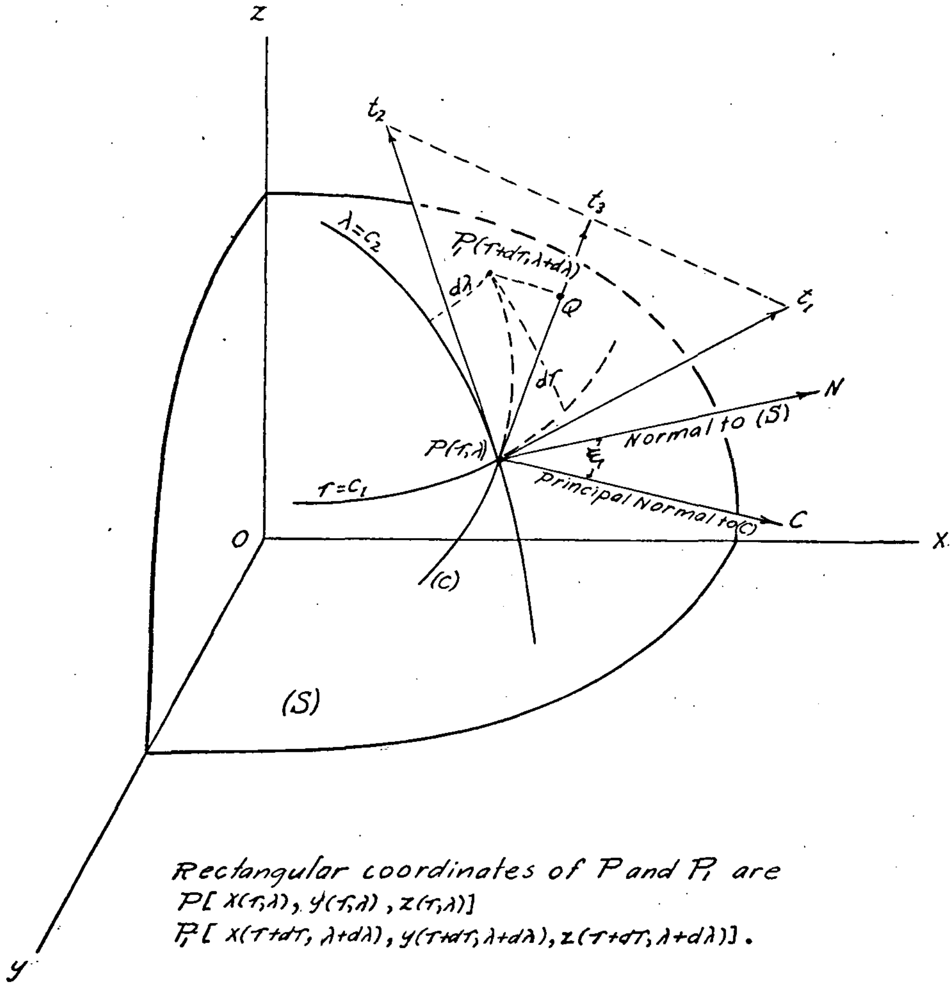


FIGURE 15.—The distance from a neighboring point on a surface to the tangent plane at a given point.

From (41) the direction cosines L_1, M_1, N_1 and L_2, M_2, N_2 of the parametric curves $\tau=c_1$, $\lambda=c_2$ are respectively

$$\frac{\frac{\partial x}{\partial \lambda}, \frac{\partial y}{\partial \lambda}, \frac{\partial z}{\partial \lambda}}{\sqrt{G}} \text{ and } \frac{\frac{\partial x}{\partial \tau}, \frac{\partial y}{\partial \tau}, \frac{\partial z}{\partial \tau}}{\sqrt{E}}.$$

From a well-known theorem of solid analytical geometry, the plane containing the two lines

$$\frac{X-x}{L_1}=\frac{Y-y}{M_1}=\frac{Z-z}{N_1}, \frac{X-x}{L_2}=\frac{Y-y}{M_2}=\frac{Z-z}{N_2}$$

having the common point $P(x,y,z)$ is given by the determinant

$$\begin{vmatrix} X-x & Y-y & Z-z \\ L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \end{vmatrix}=0.$$

Hence the tangent plane at $P(x,y,z)$ is given by

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \frac{\partial x}{\partial \lambda} & \frac{\partial y}{\partial \lambda} & \frac{\partial z}{\partial \lambda} \\ \frac{\partial x}{\partial \tau} & \frac{\partial y}{\partial \tau} & \frac{\partial z}{\partial \tau} \end{vmatrix}=0, \tag{93}$$

or

$$(X-x)\begin{vmatrix} \frac{\partial y}{\partial \lambda} & \frac{\partial z}{\partial \lambda} \\ \frac{\partial y}{\partial \tau} & \frac{\partial z}{\partial \tau} \end{vmatrix}+(Y-y)\begin{vmatrix} \frac{\partial z}{\partial \lambda} & \frac{\partial x}{\partial \lambda} \\ \frac{\partial z}{\partial \tau} & \frac{\partial x}{\partial \tau} \end{vmatrix}+(Z-z)\begin{vmatrix} \frac{\partial x}{\partial \lambda} & \frac{\partial y}{\partial \lambda} \\ \frac{\partial x}{\partial \tau} & \frac{\partial y}{\partial \tau} \end{vmatrix}=0.$$

From (93) it is seen that the direction cosines l,m,n of the normal to the surface at $P(x,y,z)$ are proportional to the three determinants in (93), the square of the factor of proportionality, μ, being of course the sum of the squares of the three determinants.

If we compute μ, we find that

$$\frac{1}{\mu^2}=\left(\frac{\partial y}{\partial \lambda}\frac{\partial z}{\partial \tau}-\frac{\partial y}{\partial \tau}\frac{\partial z}{\partial \lambda}\right)^2+\left(\frac{\partial z}{\partial \lambda}\frac{\partial x}{\partial \tau}-\frac{\partial z}{\partial \tau}\frac{\partial x}{\partial \lambda}\right)^2+\left(\frac{\partial x}{\partial \lambda}\frac{\partial y}{\partial \tau}-\frac{\partial x}{\partial \tau}\frac{\partial y}{\partial \lambda}\right)^2 \tag{94}$$

$$=\left[\left(\frac{\partial x}{\partial \lambda}\right)^2+\left(\frac{\partial y}{\partial \lambda}\right)^2+\left(\frac{\partial z}{\partial \lambda}\right)^2\right]\left[\left(\frac{\partial x}{\partial \tau}\right)^2+\left(\frac{\partial y}{\partial \tau}\right)^2+\left(\frac{\partial z}{\partial \tau}\right)^2\right]-\left(\frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \tau}+\frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial \tau}+\frac{\partial z}{\partial \lambda}\frac{\partial z}{\partial \tau}\right)^2,$$

and from (37) and (94) we have $\frac{1}{\mu^2}=EG-F^2$, or $\frac{1}{\mu}=\sqrt{EG-F^2}$; hence the direction cosines of the normal to the surface are

$$l=\mu\begin{vmatrix} \frac{\partial y}{\partial \lambda} & \frac{\partial z}{\partial \lambda} \\ \frac{\partial y}{\partial \tau} & \frac{\partial z}{\partial \tau} \end{vmatrix},\quad m=\mu\begin{vmatrix} \frac{\partial z}{\partial \lambda} & \frac{\partial x}{\partial \lambda} \\ \frac{\partial z}{\partial \tau} & \frac{\partial x}{\partial \tau} \end{vmatrix},\quad n=\mu\begin{vmatrix} \frac{\partial x}{\partial \lambda} & \frac{\partial y}{\partial \lambda} \\ \frac{\partial x}{\partial \tau} & \frac{\partial y}{\partial \tau} \end{vmatrix} \tag{95}$$

where $\mu=1/\sqrt{EG-F^2}$.

In figure 15, t_3 is the tangent to an arbitrary curve (c) through $P(x,y,z)$ on the surface, and the direction cosines, being $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, are given by equations (35) since x,y,z are functions of the parameters τ, λ. Hence the equations of the tangent line are

$$\frac{X-x}{\frac{dx}{ds}}=\frac{Y-y}{\frac{dy}{ds}}=\frac{Z-z}{\frac{dz}{ds}}. \tag{96}$$

To show that the tangent line (96) lies in the plane tangent to the surface at P, we have only to substitute the values $X-x=\nu\frac{dx}{ds}, Y-y=\nu\frac{dy}{ds}, Z-z=\nu\frac{dz}{ds}$ from (96) with the values of $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ from (35) in the determinant (93) which will vanish, since the elements of the first row are then the sums of the elements of the second and third rows after the latter have been multiplied by $\frac{d\lambda}{ds}, \frac{d\tau}{ds}$ respectively.

In figure 15, P_1Q is the distance from a point $P_1(\tau+d\tau, \lambda+d\lambda)$ on the surface to the tangent plane determined by the tangents t_1 and t_2 to the parametric curves $\tau=c_1$, $\lambda=c_2$ at the point $P(\tau,\lambda)$.

To approximate this distance we express the rectangular coordinates of the point P_1 in terms of power series in τ and λ.

In any standard treatise on the calculus [4] one may find the derivation of Taylor's formula for functions of two variables which may be written in the following form to correspond to our notation:

$$f(\tau+d\tau,\lambda+d\lambda)-f(\tau,\lambda)=df+\frac{1}{2}\left(\frac{\partial^2 f}{\partial\tau^2}\,d\tau^2+2\,\frac{\partial^2 f}{\partial\tau\partial\lambda}\,d\tau\,d\lambda+\frac{\partial^2 f}{\partial\lambda^2}\,d\lambda^2\right)+\ldots, \tag{97}$$

$$\text{where } df=\frac{\partial f}{\partial\tau}\,d\tau+\frac{\partial f}{\partial\lambda}\,d\lambda.$$

From (31) the rectangular coordinates of P_1 on the surface in terms of τ, λ, $d\tau$, $d\lambda$ are $x=x(\tau+d\tau,\lambda+d\lambda)$, $y=y(\tau+d\tau,\lambda+d\lambda)$, $z=z(\tau+d\tau,\lambda+d\lambda)$ and by (97) we may write the difference of the coordinates of the points $P(\tau,\lambda)$, $P_1(\tau+d\tau,\lambda+d\lambda)$ in the form

$$\begin{aligned}
x(\tau+d\tau,\lambda+d\lambda)-x(\tau,\lambda)&=dx+\frac{1}{2}\left(\frac{\partial^2 x}{\partial\tau^2}\,d\tau^2+2\,\frac{\partial^2 x}{\partial\tau\partial\lambda}\,d\tau d\lambda+\frac{\partial^2 x}{\partial\lambda^2}\,d\lambda^2\right)+\ldots\\
y(\tau+d\tau,\lambda+d\lambda)-y(\tau,\lambda)&=dy+\frac{1}{2}\left(\frac{\partial^2 y}{\partial\tau^2}\,d\tau^2+2\,\frac{\partial^2 y}{\partial\tau\partial\lambda}\,d\tau d\lambda+\frac{\partial^2 y}{\partial\lambda^2}\,d\lambda^2\right)+\ldots\\
z(\tau+d\tau,\lambda+d\lambda)-z(\tau,\lambda)&=dz+\frac{1}{2}\left(\frac{\partial^2 z}{\partial\tau^2}\,d\tau^2+2\,\frac{\partial^2 z}{\partial\tau\partial\lambda}\,d\tau d\lambda+\frac{\partial^2 z}{\partial\lambda^2}\,d\lambda^2\right)+\ldots.
\end{aligned} \tag{98}$$

SECOND FUNDAMENTAL QUADRATIC DIFFERENTIAL FORM OF A SURFACE

The normal form of the tangent plane (93) is

$$p=l(X-x)+m(Y-y)+n(Z-z), \tag{99}$$

where l, m, n are the direction cosines of the normal as given by (95). Hence the distance P_1Q from the point P_1 on the surface to the plane tangent to the surface at the point P is obtained by substituting the values of the left members of (98) in (99) respectively for the terms $X-x$, $Y-y$, $Z-z$. We obtain thus

$$p=P_1Q=\left(l\,\frac{dx}{ds}+m\,\frac{dy}{ds}+n\,\frac{dz}{ds}\right)ds+\frac{1}{2}\,(D\,d\tau^2+2\,D'd\tau d\lambda+D''d\lambda^2)+\ldots, \tag{100}$$

where

$$D=l\frac{\partial^2 x}{\partial\tau^2}+m\,\frac{\partial^2 y}{\partial\tau^2}+n\,\frac{\partial^2 z}{\partial\tau^2},\ D'=l\,\frac{\partial^2 x}{\partial\tau\partial\lambda}+m\,\frac{\partial^2 y}{\partial\tau\partial\lambda}+n\,\frac{\partial^2 z}{\partial\tau\partial\lambda},\ D''=l\frac{\partial^2 x}{\partial\lambda^2}+m\,\frac{\partial^2 y}{\partial\lambda^2}+n\,\frac{\partial^2 z}{\partial\lambda^2}. \tag{101}$$

[4] I. S. Sokolnikoff, Advanced Calculus, pp. 317-320.

Now the direction cosines of a line in the tangent plane are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$; and l, m, n are the direction cosines of the normal to the surface. Hence the first term of (100) is zero since the normal is perpendicular to any line in the tangent plane. Hence we have from (100) to terms of second order in $d\lambda$ and $d\tau$

$$\Phi = 2p = D\,d\tau^2 + 2D'd\tau\,d\lambda + D''d\lambda^2, \tag{102}$$

where D, D', and D'' are given by (101).

The quadratic differential form (102) is called the second fundamental form of the surface and the functions D, D', D'' the fundamental coefficients of the second order.

In figure 15, it is seen that the direction of the tangent t_3 to the arbitrary curve (c) through P may be considered to be determined by $d\tau/d\lambda$. The angle between PN (the normal to the surface) and PC (the principal normal to (c)) is ξ. From (87) and (95) we have

$$\cos \xi = ll_2 + mm_2 + nn_2 = \rho_1(lx'' + my'' + nz''). \tag{103}$$

Differentiating equations (35) we find

$$\begin{aligned}
x'' &= \frac{d^2x}{ds^2} = \frac{\partial^2 x}{\partial\tau^2}\left(\frac{d\tau}{ds}\right)^2 + 2\frac{\partial^2 x}{\partial\lambda\partial\tau}\frac{d\lambda}{ds}\frac{d\tau}{ds} + \frac{\partial^2 x}{\partial\lambda^2}\left(\frac{d\lambda}{ds}\right)^2 + \frac{\partial x}{\partial\tau}\frac{d^2\tau}{ds^2} + \frac{\partial x}{\partial\lambda}\frac{d^2\lambda}{ds^2}\\
y'' &= \frac{d^2y}{ds^2} = \frac{\partial^2 y}{\partial\tau^2}\left(\frac{d\tau}{ds}\right)^2 + 2\frac{\partial^2 y}{\partial\lambda\partial\tau}\frac{d\lambda}{ds}\frac{d\tau}{ds} + \frac{\partial^2 y}{\partial\lambda^2}\left(\frac{d\lambda}{ds}\right)^2 + \frac{\partial y}{\partial\tau}\frac{d^2\tau}{ds^2} + \frac{\partial y}{\partial\lambda}\frac{d^2\lambda}{ds^2}\\
z'' &= \frac{d^2z}{ds^2} = \frac{\partial^2 z}{\partial\tau^2}\left(\frac{d\tau}{ds}\right)^2 + 2\frac{\partial^2 z}{\partial\lambda\partial\tau}\frac{d\lambda}{ds}\frac{d\tau}{ds} + \frac{\partial^2 z}{\partial\lambda^2}\left(\frac{d\lambda}{ds}\right)^2 + \frac{\partial z}{\partial\tau}\frac{d^2\tau}{ds^2} + \frac{\partial z}{\partial\lambda}\frac{d^2\lambda}{ds^2}.
\end{aligned} \tag{104}$$

Now multiply equations (104) through by l, m, n respectively and add—ignoring differentials of second order in λ and τ—obtaining by means of (101)

$$lx'' + my'' + nz'' = \frac{D\,d\tau^2 + 2\,D'd\tau d\lambda + D''d\lambda^2}{ds^2}. \tag{105}$$

With the value of ds^2 from (36) and the right member of (105) placed in (103) we have

$$\frac{\cos \xi}{\rho_1} = \frac{D\,d\tau^2 + 2\,D'd\tau\,d\lambda + D''d\lambda^2}{E\,d\tau^2 + 2F d\tau\,d\lambda + G d\lambda^2}, \tag{106}$$

where ρ_1 is the principal radius of curvature of the curve (c).

The right member of (106) depends only on the curvilinear coordinates τ, λ and the direction of the tangent t_3, hence it is the same for all curves on the surface having the same tangent, t_3. Consider the plane curve which is the intersection with the surface of the plane determined by the normal PN to the surface and the tangent t_3 to the curve (c). This plane curve or normal section will be tangent to the curve (c) at P since they have the same tangent at P_1 and its curvature will be given by the left member of (106) with $\xi = 0$, that is, we have

$$\frac{\cos \xi}{\rho_1} = \frac{1}{\rho_n}, \tag{107}$$

where ρ_n is the radius of curvature of the normal section. Equation (107) written as $\rho_1 = \rho_n \cos \xi$, states that the first radius of curvature of a curve (c) through P is the projection of the radius of normal curvature upon the osculating plane at P of the curve (c), as shown in figure 16.

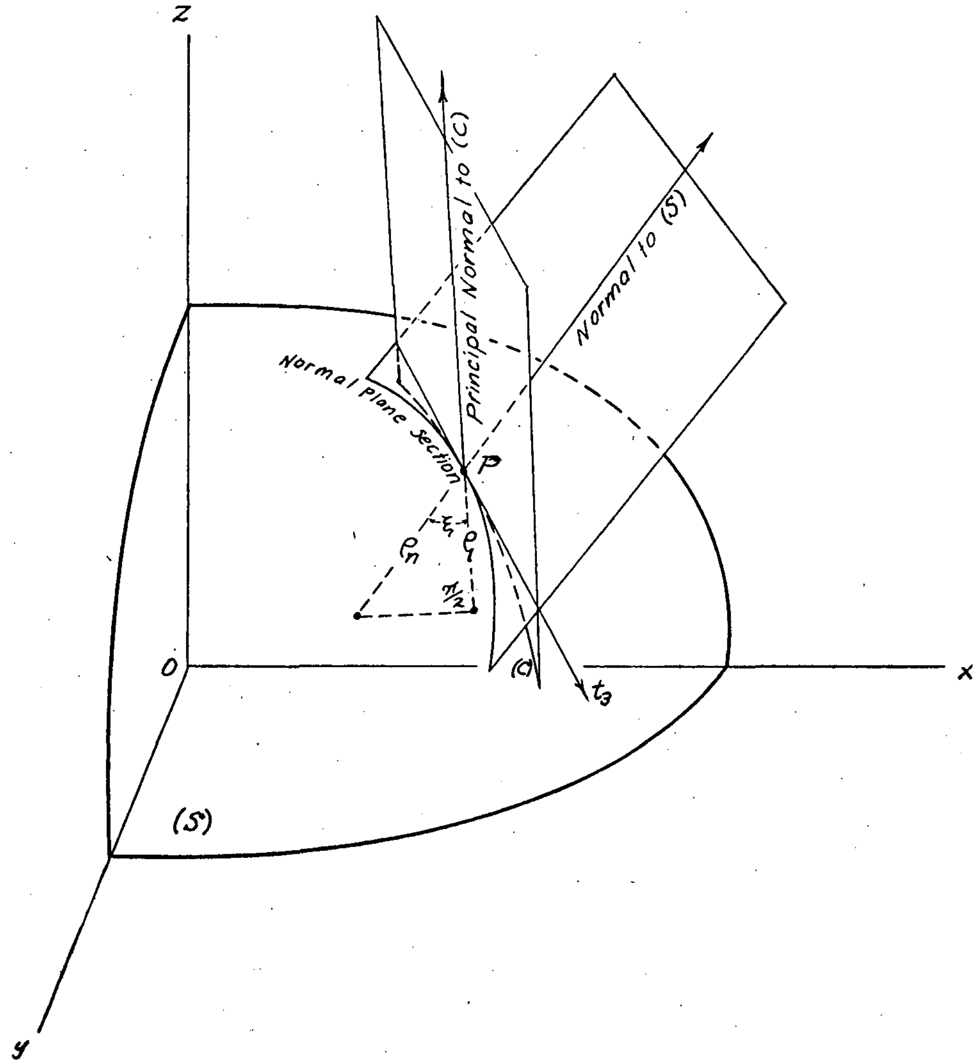


FIGURE 16.—Illustration of the relation $\rho_1 = \rho_n \cos \xi$, and its special case, Meusnier's theorem.

MEUSNIER'S THEOREM

For the plane curve, which is the plane section of the surface determined by the tangent t_3 and the normal to the curve (c) at P, equation (107) gives its curvature. That is, $\rho_1 = \rho_n \cos \xi$ gives the radius of curvature, ρ_1, of the plane section which makes an angle ξ with the normal section whose radius of curvature is ρ_n. This is usually known as Meusnier's theorem.

PRINCIPAL RADII OF NORMAL CURVATURE OF A SURFACE

Clearly there are many normal sections of a surface at a point P, generated by a variable plane containing the normal to the surface at P. We now determine the two among these for which $\frac{1}{\rho_n}$ is respectively maximum and minimum.

From (106) and (107) we have

$$\frac{1}{\rho_n}=\frac{D+2\,D'u+D''u^2}{E+2\,Fu+Gu^2}, \tag{108}$$

where $u=d\lambda/d\tau$.

Differentiating (108) with respect to u and placing the result equal to zero gives

$$(D'+D''u)(E+2\,Fu+Gu^2)-(F+Gu)(D+2\,D'u+D''u^2)=0.$$

or

$$(FD''-GD')u^2+(ED''-GD)u+(ED'-FD)=0. \tag{109}$$

Comparing (56) and (109) we see that

$$T=FD''-GD',\ 2S=ED''-GD,\ R=ED'-FD. \tag{110}$$

The values of (110) placed in (57) give $EFD''-EGD'-EFD''+GDF+GED'-GFD=0$, which shows that the two families of curves given by (109) are orthogonal. That is, the two normal sections given by (109) at each point of the surface, and whose curvatures are respectively maximum and minimum, are orthogonal to each other.

The radii of curvature of these two normal sections are called the principal radii of normal curvature of the surface at a given point. They are equal to each other for the trivial cases of the plane and sphere. (For these two trivial cases the discriminant of (109) vanishes, otherwise it is positive and the equation has two real and distinct roots provided $E\neq 0$.)

TOTAL AND MEAN CURVATURE OF A SURFACE

From (108) and (109) we have

$$\frac{1}{\rho_n}=\frac{D+D'u+u(D'+D''u)}{E+Fu+u(F+Gu)}=\frac{D+D'u}{E+Fu}=\frac{D'+D''u}{F+Gu},$$

whence

$$\begin{aligned} E+Fu&=\rho_n(D+D'u)\\ F+Gu&=\rho_n(D'+D''u). \end{aligned} \tag{111}$$

Eliminating u between equations (111) we obtain the equation

$$\frac{1}{\rho_n{}^2}(EG-F^2)-\frac{1}{\rho_n}(ED''+GD-2\,FD')+(DD''-D'^2)=0. \tag{112}$$

If $\frac{1}{R}$ and $\frac{1}{N}$ are the roots of (112) then

$$\begin{aligned} R_M&=\frac{1}{R}+\frac{1}{N}=\frac{ED''+GD-2\,FD'}{EG-F^2},\\ R_T&=\frac{1}{R}\cdot\frac{1}{N}=\frac{DD''-D'^2}{EG-F^2} \end{aligned} \tag{113}$$

R_T is called the total curvature of the surface at the given point and R_M the mean curvature of the surface at the given point.

From (36) and (102) it is seen that R_T in (113) is the ratio of the negatives of the discriminants of the first and second fundamental quadratic forms.

From (108) the normal curvature is zero if $D+2D'u+D''u^2=0$, where $u=d\lambda/d\tau$. In the directions determined by this differential equation we have from (102) that the distances of nearby points of the surface from the tangent plane are at least of the third order in $d\lambda$, $d\tau$. Hence the lines $D+2D'u+D''u^2=0$ are the tangents at a given point to the curve in which the tangent plane at that point meets the surface. Now according to the discussion, equation (56), the integral curves of the differential equation $D+2D'u+D''u^2=0$ are two distinct families of curves. These integral curves are called the asymptotic lines of the surface. From (113), the numerator of R_T is the negative of the discriminant of the quadratic $D+2D'u+D''u^2=0$. We now consider the values of this discriminant and its characterization of the asymptotic curves and of the surface.

For $DD''-D'^2>0$ at every point of the surface, R_T is positive and there are two distinct families of imaginary asymptotic curves on the surface. But from (113) we must then have both R and N positive at every point of the surface, which means that both centers of principal curvature lie on the same side of the tangent plane at each point. Such surfaces of positive curvature at each point are exemplified by the ellipsoid and the elliptic paraboloid. This will be illustrated later when R_T is computed for the spheroid.

If $DD''-D'^2<0$ at every point of the surface, R_T is negative and there are two distinct families of real asymptotic curves on the surface. From (113), R and N must differ in sign and therefore the surface lies on both sides of the tangent plane. Surfaces of such negative curvature at each point are exemplified by the hyperbolic paraboloid and the hyperboloid of one sheet.

If $DD''-D'^2=0$ at every point of the surface, then R_T is zero at every point and the differential equation of the asymptotic lines $D+2D'u+D''u^2=0$ becomes $(\sqrt{D}+\sqrt{D''}\,u)^2=0$ and there is only one family of real asymptotic curves. Since there is no change of sign as u passes through the value $u=-\frac{\sqrt{D}}{\sqrt{D''}}$, the surface lies on one side of the tangent plane and is tangent to it along the direction $\sqrt{D}+\sqrt{D''}\,u=0$. From (113), if R_T is zero at each point, then R or N must be infinite, which means that one family of lines of curvature on the surface must be straight lines. Such surfaces are called developable surfaces, such as cylinders or cones which by cutting along a straight line element can be brought into coincidence with a plane without stretching or tearing. We discussed the linear element of such surfaces following equation (66). We shall show that R_T is zero at each point of a developable surface after we have expressed R_T in terms of the first fundamental coefficients E, F, G and their derivatives.

LINES OF CURVATURE ON A SURFACE

Equation (109) is the differential equation of a pair of orthogonal curves, called the lines of curvature on the surface at each point, the directions of whose tangents are those for which the radii of normal curvature have their maximum and minimum values. If these lines of curvature are to be the parametric curves, $\tau=c_1$, $\lambda=c_2$, then from (108) we have, replacing u by $d\lambda/d\tau$ and then placing $d\tau=0$, $d\lambda=0$ in turn,

$$\frac{1}{N}=\frac{D''}{G},\quad \frac{1}{R}=\frac{D}{E}. \tag{114}$$

From (113) and (114) it is seen that this is equivalent to placing $D'=F=0$. That is, the lines of curvature are the parametric curves for the surface if $D'=F=0$.

From (47), β is the angle between the curve $\Phi\ (\tau,\lambda)=0$ and the parametric curve $\lambda=c_2$, but since the lines of curvature are to be parametric, we have $F=0$ and therefore

$$\cos\beta=\sqrt{E}\,\frac{d\tau}{ds},\quad \sin\beta=\sqrt{G}\,\frac{d\lambda}{ds}$$

or

$$E\left(\frac{d\tau}{ds}\right)^2=\cos^2\beta,\quad G\left(\frac{d\lambda}{ds}\right)^2=\sin^2\beta. \tag{115}$$

EULER'S FORMULA FOR THE CURVATURE OF A NORMAL SECTION OF A SURFACE

Now with $D'=F=0$, we may write (108), remembering the denominator is ds^2, as

$$\frac{1}{\rho_n}=D\left(\frac{d\tau}{ds}\right)^2+D''\left(\frac{d\lambda}{ds}\right)^2. \tag{116}$$

From (114) and (115) we have

$$D\left(\frac{d\tau}{ds}\right)^2=\frac{E}{R}\left(\frac{d\tau}{ds}\right)^2=\frac{\cos^2\beta}{R},\quad D''\left(\frac{d\lambda}{ds}\right)^2=\frac{G}{N}\left(\frac{d\lambda}{ds}\right)^2=\frac{\sin^2\beta}{N}. \tag{117}$$

From (116) and (117) we have finally

$$\frac{1}{\rho_n}=\frac{\cos^2\beta}{R}+\frac{\sin^2\beta}{N}. \tag{118}$$

Equation (118), known as Euler's equation, gives the curvature $\frac{1}{\rho_n}$ of any normal section at a given point in terms of the curvatures of the principal normal sections, β being the angle which the arbitrary normal section makes with the parametric curve $\lambda=c_2$, the lines of curvature being parametric.

THE GEODESIC CURVATURE OF A CURVE ON A SURFACE

In figure 17 we suppose that a variable line, meeting the curve (c) on (S), moves parallel to the normal to (S) at P generating a cylinder which is met by the plane tangent to (S) at P in the curve (c') as shown. The curve (c') is then a normal section of the cylinder. Obviously the curve (c) lies also on this cylinder and the curves (c) and (c') are tangent to each other at P as shown. Hence we may use Meusnier's theorem with respect to the cylinder, where ρ_g is the radius of curvature at P of the plane curve (c') and ρ_1 is the principal radius of curvature of (c) at P. From figure 17 and equation (107) we have then

$$\frac{1}{\rho_g}=\frac{\cos\psi}{\rho_1}. \tag{119}$$

If ξ is the angle between the normal to the surface and the principal normal to (c) as shown in figures 16 and 17, then $\psi=\frac{\pi}{2}-\xi$ and

$$\frac{1}{\rho_g}=\frac{\cos\left(\frac{\pi}{2}-\xi\right)}{\rho_1}=\frac{\sin\xi}{\rho_1}. \tag{120}$$

Clearly the principal normal to (c) may lie exterior to the plane quadrant between the normal to the surface and the normal to the cylinder, but is always coplanar with these normals. Hence we make the convention that the positive directions of the tangent, the normal to (c'), and the normal to the surface shall have the same mutual orientations as the positive x,y, and z axes as shown in figure 17.

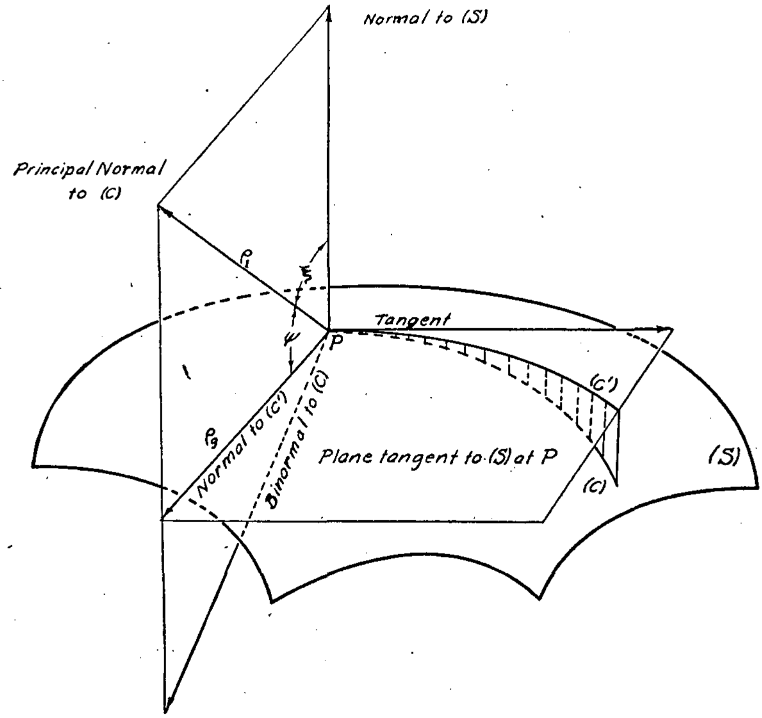


FIGURE 17.—Geodesic curvature of a curve on a surface.

A geodesic on a surface may be defined as a curve such that at each of its points the principal normal to the curve coincides with the normal to the surface. $1/\rho_g$ as defined by (120) is called the geodesic curvature of the curve (c), ρ_g being then the radius of geodesic curvature. If the curve (c) is a geodesic, then the angle ξ, as shown in figure 17, is 0 and from (120) the geodesic curvature is 0. We might then equivalently define a geodesic as a curve for which the geodesic curvature is 0 at each of its points.

We have from equation (87) that $l_1=\frac{dx}{ds}$, $m_1=\frac{dy}{ds}$, $n_1=\frac{dz}{ds}$ are direction cosines of the tangent to (c) at P. The direction cosines of the normal to (S) are l, m, n as given by (95). If l_g, m_g, n_g are the direction cosines of the normal to (c'), then from the last of equations (74) with the direction cosines of the tangent and normal, l_1, m_1, n_1 and l, m, n we have

$$l_g=mn_1-nm_1,\ m_g=nl_1-ln_1,\ n_g=lm_1-ml_1. \tag{121}$$

From figure 17, ψ is the angle between the principal normal to (c) and the normal to (c') and we have therefore from (87) and (121)

$$\begin{aligned}\cos\psi&=l_2l_g+m_2m_g+n_2n_g\\&=l_2(mn_1-nm_1)+m_2(nl_1-ln_1)+n_2(lm_1-ml_1)\\&=\rho_1x''(mz'-ny')+\rho_1y''(nx'-lz')+\rho_1z''(ly'-mx'),\end{aligned}$$

or from (119) and this last equation

$$\frac{1}{\rho_g}=\frac{\cos\psi}{\rho_1}=x''(mz'-ny')+y''(nx'-lz')+z''(ly'-mx'). \tag{122}$$

To save space in the developments to follow, we will shorten our notation by writing sums in the form

$$\sum\left(\frac{\partial x}{\partial\tau}\right)^2=\left(\frac{\partial x}{\partial\tau}\right)^2+\left(\frac{\partial y}{\partial\tau}\right)^2+\left(\frac{\partial z}{\partial\tau}\right)^2, \text{etc.}$$

From (37) we have

$$E=\sum\left(\frac{\partial x}{\partial\tau}\right)^2,\quad F=\sum\frac{\partial x}{\partial\tau}\frac{\partial x}{\partial\lambda},\quad G=\sum\left(\frac{\partial x}{\partial\lambda}\right)^2.$$

From these by partial differentiation we have

$$\begin{aligned}
&\frac{\partial E}{\partial\tau}=2\sum\frac{\partial x}{\partial\tau}\frac{\partial^2 x}{\partial\tau^2},\quad \frac{\partial E}{\partial\lambda}=2\sum\frac{\partial x}{\partial\tau}\frac{\partial^2 x}{\partial\tau\partial\lambda}\\
&\frac{\partial G}{\partial\tau}=2\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda\partial\tau},\quad \frac{\partial G}{\partial\lambda}=2\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda^2}\\
&\frac{\partial F}{\partial\tau}=\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\tau^2}+\sum\frac{\partial x}{\partial\tau}\frac{\partial^2 x}{\partial\lambda\partial\tau}=\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\tau^2}+\frac{1}{2}\frac{\partial E}{\partial\lambda}\\
&\frac{\partial F}{\partial\lambda}=\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\tau\partial\lambda}+\sum\frac{\partial x}{\partial\tau}\frac{\partial^2 x}{\partial\lambda^2}=\sum\frac{\partial x}{\partial\tau}\frac{\partial^2 x}{\partial\lambda^2}+\frac{1}{2}\frac{\partial G}{\partial\tau}.
\end{aligned} \tag{123}$$

From (37) and (95) we find that

$$\begin{aligned}
&\mu\left(E\frac{\partial x}{\partial\lambda}-F\frac{\partial x}{\partial\tau}\right)=\left(m\frac{\partial z}{\partial\tau}-n\frac{\partial y}{\partial\tau}\right)\\
&\mu\left(F\frac{\partial x}{\partial\lambda}-G\frac{\partial x}{\partial\tau}\right)=\left(m\frac{\partial z}{\partial\lambda}-n\frac{\partial y}{\partial\lambda}\right).
\end{aligned} \tag{124}$$

Identities analogous to (124) are found by permuting the letters $x, y, z; l, m, n$.

Placing the values of $x', y', z'; x'', y'', z''$ from (35) and (104) in (122) and reducing by means of (123) and (124) we find

$$\frac{1}{\rho_g}=\frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} E\dfrac{d\tau}{ds}+F\dfrac{d\lambda}{ds} & U\\[2ex] F\dfrac{d\tau}{ds}+G\dfrac{d\lambda}{ds} & V\end{vmatrix}, \tag{125}$$

where

$$U=\frac{1}{2}\frac{\partial E}{\partial\tau}\left(\frac{d\tau}{ds}\right)^2+\frac{\partial E}{\partial\lambda}\frac{d\tau}{ds}\frac{d\lambda}{ds}+\left(\frac{\partial F}{\partial\lambda}-\frac{1}{2}\frac{\partial G}{\partial\tau}\right)\left(\frac{d\lambda}{ds}\right)^2+E\frac{d^2\tau}{ds^2}+F\frac{d^2\lambda}{ds^2},$$

$$V=\left(\frac{\partial F}{\partial\tau}-\frac{1}{2}\frac{\partial E}{\partial\lambda}\right)\left(\frac{d\tau}{ds}\right)^2+\frac{\partial G}{\partial\tau}\frac{d\tau}{ds}\frac{d\lambda}{ds}+\frac{1}{2}\frac{\partial G}{\partial\lambda}\left(\frac{d\lambda}{ds}\right)^2+F\frac{d^2\tau}{ds^2}+G\frac{d^2\lambda}{ds^2}.$$

From (125) we note that the geodesic curvature of a curve depends upon the fundamental quantities of first order E, F, G and their partial derivatives.

From (38), the elements of arc length of the parametric curves are $ds_\tau=\sqrt{G}\,d\lambda$, $ds_\lambda=\sqrt{E}\,d\tau$. From (42) the parametric curves are orthogonal if $F=0$. With these values placed in (125) we find the geodesic curvatures $1/\rho_{g\lambda}$, $1/\rho_{g\tau}$ of the parametric curves $\tau=c_1$, $\lambda=c_2$ to be

$$\frac{1}{\rho_{g\lambda}}=\frac{1}{\sqrt{EG}}\begin{vmatrix} 0 & -\frac{1}{2G}\frac{\partial G}{\partial \tau} \\ \sqrt{G} & V \end{vmatrix}=\frac{1}{\sqrt{EG}}\cdot\frac{1}{2\sqrt{G}}\cdot\frac{\partial G}{\partial \tau}=\frac{1}{\sqrt{EG}}\frac{\partial\sqrt{G}}{\partial \tau},$$

$$\frac{1}{\rho_{g\tau}}=\frac{1}{\sqrt{EG}}\begin{vmatrix} \sqrt{E} & U \\ 0 & -\frac{1}{2E}\frac{\partial E}{\partial \lambda} \end{vmatrix}=-\frac{1}{\sqrt{EG}}\cdot\frac{1}{2\sqrt{E}}\cdot\frac{\partial E}{\partial \lambda}=-\frac{1}{\sqrt{EG}}\frac{\partial\sqrt{E}}{\partial \lambda}. \tag{126}$$

We have noted that a geodesic may be defined as a curve for which the geodesic curvature at each of its points is zero. From (126) we note that $\frac{\partial\sqrt{E}}{\partial\lambda}$, $\frac{\partial\sqrt{G}}{\partial\tau}$ are each zero if E is a function of τ alone and if G is a function of λ alone. That is, when the parametric curves form an orthogonal system then $\tau=c_1$ or $\lambda=c_2$ are geodesics if G is a function of λ alone or if E is a function of τ alone.

THE GAUSS CHARACTERISTIC EQUATION

We will now express the total curvature as given by (113) in terms of the fundamental quantities of first order E, F, G and their partial derivatives. To do this we will express the numerator, $DD''-D'^2$ of R_τ as given by (113) in terms of E, F, G and their partial derivatives. The resulting equation for $DD''-D'^2$ is called the Gauss characteristic equation.

From (123) we may write

$$\begin{aligned} M&=\sum\frac{\partial x}{\partial\tau}\frac{\partial^2 x}{\partial\tau^2}=\frac{1}{2}\frac{\partial E}{\partial\tau} & J&=\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\tau^2}=\frac{\partial F}{\partial\tau}-\frac{1}{2}\frac{\partial E}{\partial\lambda}\\ M'&=\sum\frac{\partial x}{\partial\tau}\frac{\partial^2 x}{\partial\tau\partial\lambda}=\frac{1}{2}\frac{\partial E}{\partial\lambda} & J'&=\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\tau\partial\lambda}=\frac{1}{2}\frac{\partial G}{\partial\tau}\\ M''&=\sum\frac{\partial x}{\partial\tau}\frac{\partial^2 x}{\partial\lambda^2}=\frac{\partial F}{\partial\lambda}-\frac{1}{2}\frac{\partial G}{\partial\tau} & J''&=\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda^2}=\frac{1}{2}\frac{\partial G}{\partial\lambda}, \end{aligned} \tag{127}$$

and if we place

$$\begin{aligned} A&=\mu^2(MG-JF) & B&=\mu^2(JE-MF)\\ A'&=\mu^2(M'G-J'F) & B'&=\mu^2(J'E-M'F)\\ A''&=\mu^2(M''G-J''F) & B''&=\mu^2(J''E-M''F), \end{aligned} \tag{128}$$

then

$$\begin{aligned} M&=EA+FB & J&=FA+GB\\ M'&=EA'+FB' & J'&=FA'+GB'\\ M''&=EA''+FB'' & J''&=FA''+GB''. \end{aligned} \tag{129}$$

From (101) we have

$$D=\sum l\frac{\partial^2 x}{\partial\tau^2},\ D'=\sum l\frac{\partial^2 x}{\partial\tau\partial\lambda},\ D''=\sum l\frac{\partial^2 x}{\partial\lambda^2}. \tag{130}$$

From (127) and (130) let us solve the three equations

$$D=\sum l\frac{\partial^2 x}{\partial\tau^2},\ M=\sum\frac{\partial x}{\partial\tau}\frac{\partial^2 x}{\partial\tau^2},\ J=\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\tau^2}$$

for

$$\frac{\partial^2 x}{\partial\tau^2},\ \frac{\partial^2 y}{\partial\tau^2},\ \frac{\partial^2 z}{\partial\tau^2}.$$

The determinant of the coefficients of $\frac{\partial^2 x}{\partial\tau^2},\ \frac{\partial^2 y}{\partial\tau^2},\ \frac{\partial^2 z}{\partial\tau^2}$ in these three equations is

$$\Delta=\begin{vmatrix} l & m & n \\ \frac{\partial x}{\partial\tau} & \frac{\partial y}{\partial\tau} & \frac{\partial z}{\partial\tau} \\ \frac{\partial x}{\partial\lambda} & \frac{\partial y}{\partial\lambda} & \frac{\partial z}{\partial\lambda}\end{vmatrix}=l\begin{vmatrix}\frac{\partial y}{\partial\tau} & \frac{\partial z}{\partial\tau} \\ \frac{\partial y}{\partial\lambda} & \frac{\partial z}{\partial\lambda}\end{vmatrix}+m\begin{vmatrix}\frac{\partial z}{\partial\tau} & \frac{\partial x}{\partial\tau} \\ \frac{\partial z}{\partial\lambda} & \frac{\partial x}{\partial\lambda}\end{vmatrix}+n\begin{vmatrix}\frac{\partial x}{\partial\tau} & \frac{\partial y}{\partial\tau} \\ \frac{\partial x}{\partial\lambda} & \frac{\partial y}{\partial\lambda}\end{vmatrix}.$$

From (95) it is seen that this may be written $\Delta=\frac{1}{\mu}(l^2+m^2+n^2)=\frac{1}{\mu}$, where $l^2+m^2+n^2=1$ since l, m, n are direction cosines of the normal to the surface. Hence the solutions for $\frac{\partial^2 x}{\partial\tau^2},\ \frac{\partial^2 y}{\partial\tau^2},\ \frac{\partial^2 z}{\partial\tau^2}$ may be written

$$\frac{\frac{\partial^2 x}{\partial\tau^2}}{\begin{vmatrix} D & m & n \\ M & \frac{\partial y}{\partial\tau} & \frac{\partial z}{\partial\tau} \\ J & \frac{\partial y}{\partial\lambda} & \frac{\partial z}{\partial\lambda}\end{vmatrix}}=\frac{\frac{\partial^2 y}{\partial\tau^2}}{\begin{vmatrix} l & D & n \\ \frac{\partial x}{\partial\tau} & M & \frac{\partial z}{\partial\tau} \\ \frac{\partial x}{\partial\lambda} & J & \frac{\partial z}{\partial\lambda}\end{vmatrix}}=\frac{\frac{\partial^2 z}{\partial\tau^2}}{\begin{vmatrix} l & m & D \\ \frac{\partial x}{\partial\tau} & \frac{\partial y}{\partial\tau} & M \\ \frac{\partial x}{\partial\lambda} & \frac{\partial y}{\partial\lambda} & J\end{vmatrix}}=\mu. \tag{131}$$

From the first of equations (131) we have

$$\frac{\partial^2 x}{\partial\tau^2}=\mu\begin{vmatrix} D & m & n \\ M & \frac{\partial y}{\partial\tau} & \frac{\partial z}{\partial\tau} \\ J & \frac{\partial y}{\partial\lambda} & \frac{\partial z}{\partial\lambda}\end{vmatrix}=\mu D\begin{vmatrix}\frac{\partial y}{\partial\tau} & \frac{\partial z}{\partial\tau} \\ \frac{\partial y}{\partial\lambda} & \frac{\partial z}{\partial\lambda}\end{vmatrix}+\mu M\begin{vmatrix} n & m \\ \frac{\partial z}{\partial\lambda} & \frac{\partial y}{\partial\lambda}\end{vmatrix}+\mu J\begin{vmatrix} m & n \\ \frac{\partial y}{\partial\tau} & \frac{\partial z}{\partial\tau}\end{vmatrix}$$

or

$$\frac{\partial^2 x}{\partial\tau^2}=\mu D\left(\frac{\partial y}{\partial\tau}\frac{\partial z}{\partial\lambda}-\frac{\partial y}{\partial\lambda}\frac{\partial z}{\partial\tau}\right)+\mu M\left(n\frac{\partial y}{\partial\lambda}-m\frac{\partial z}{\partial\lambda}\right)+\mu J\left(m\frac{\partial z}{\partial\tau}-n\frac{\partial y}{\partial\tau}\right). \tag{132}$$

From (95) and (124), equation (132) becomes

$$\frac{\partial^2 x}{\partial\tau^2}=lD+\mu^2 M\left(G\frac{\partial x}{\partial\tau}-F\frac{\partial x}{\partial\lambda}\right)+\mu^2 J\left(E\frac{\partial x}{\partial\lambda}-F\frac{\partial x}{\partial\tau}\right)$$

$$=lD+\mu^2(MG-JF)\frac{\partial x}{\partial\tau}+\mu^2(JE-MF)\frac{\partial x}{\partial\lambda}. \tag{133}$$

From (128) we may write (133) finally as

$$\frac{\partial^2 x}{\partial \tau^2}=lD+A\frac{\partial x}{\partial \tau}+B\frac{\partial x}{\partial \lambda}. \tag{134}$$

If we solve the second and third of equations (131) for $\frac{\partial^2 y}{\partial \tau^2}$, $\frac{\partial^2 z}{\partial \tau^2}$ we find expressions similar to (134) and group them together for reference.

$$\begin{aligned}\frac{\partial^2 x}{\partial \tau^2}&=lD+A\frac{\partial x}{\partial \tau}+B\frac{\partial x}{\partial \lambda}\\ \frac{\partial^2 y}{\partial \tau^2}&=mD+A\frac{\partial y}{\partial \tau}+B\frac{\partial y}{\partial \lambda}\\ \frac{\partial^2 z}{\partial \tau^2}&=nD+A\frac{\partial z}{\partial \tau}+B\frac{\partial z}{\partial \lambda}.\end{aligned} \tag{135}$$

From (127) and (130) if we solve the equations

$$D'=\sum l\frac{\partial^2 x}{\partial \tau \partial \lambda},\quad M'=\sum \frac{\partial x}{\partial \tau}\frac{\partial^2 x}{\partial \tau \partial \lambda},\quad J'=\sum \frac{\partial x}{\partial \lambda}\frac{\partial^2 x}{\partial \tau \partial \lambda}$$

for $\frac{\partial^2 x}{\partial \tau \partial \lambda}$, $\frac{\partial^2 y}{\partial \tau \partial \lambda}$, $\frac{\partial^2 z}{\partial \tau \partial \lambda}$ we find analogously as above

$$\begin{aligned}\frac{\partial^2 x}{\partial \tau \partial \lambda}&=lD'+A'\frac{\partial x}{\partial \tau}+B'\frac{\partial x}{\partial \lambda}\\ \frac{\partial^2 y}{\partial \tau \partial \lambda}&=mD'+A'\frac{\partial y}{\partial \tau}+B'\frac{\partial y}{\partial \lambda}\\ \frac{\partial^2 z}{\partial \tau \partial \lambda}&=nD'+A'\frac{\partial z}{\partial \tau}+B'\frac{\partial z}{\partial \lambda}.\end{aligned} \tag{136}$$

From (127) and (130), if the equations

$$D''=\sum l\frac{\partial^2 x}{\partial \lambda^2},\quad M''=\sum \frac{\partial x}{\partial \tau}\frac{\partial^2 x}{\partial \lambda^2},\quad J''=\sum \frac{\partial x}{\partial \lambda}\frac{\partial^2 x}{\partial \lambda^2}$$

are solved for $\frac{\partial^2 x}{\partial \lambda^2}$, $\frac{\partial^2 y}{\partial \lambda^2}$, $\frac{\partial^2 z}{\partial \lambda^2}$ as above we find

$$\begin{aligned}\frac{\partial^2 x}{\partial \lambda^2}&=lD''+A''\frac{\partial x}{\partial \tau}+B''\frac{\partial x}{\partial \lambda}\\ \frac{\partial^2 y}{\partial \lambda^2}&=mD''+A''\frac{\partial y}{\partial \tau}+B''\frac{\partial y}{\partial \lambda}\\ \frac{\partial^2 z}{\partial \lambda^2}&=nD''+A''\frac{\partial z}{\partial \tau}+B''\frac{\partial z}{\partial \lambda}.\end{aligned} \tag{137}$$

Squaring respective members of (136) and adding we obtain

$$\begin{aligned}\sum\left(\frac{\partial^2 x}{\partial \tau \partial \lambda}\right)^2&=(l^2+m^2+n^2)D'^2+A'^2\sum\left(\frac{\partial x}{\partial \tau}\right)^2+B'^2\sum\left(\frac{\partial x}{\partial \lambda}\right)^2+2A'B'\sum\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \lambda}+\\ &2A'D'\left(l\frac{\partial x}{\partial \tau}+m\frac{\partial y}{\partial \tau}+n\frac{\partial z}{\partial \tau}\right)+2B'D'\left(l\frac{\partial x}{\partial \lambda}+m\frac{\partial y}{\partial \lambda}+n\frac{\partial z}{\partial \lambda}\right).\end{aligned} \tag{138}$$

Since l, m, n are direction cosines of the normal to the surface and this normal is orthogonal to the tangents to the parametric curves whose direction cosines are given by (41), we have $l^2+m^2+n^2=1$; and the last two terms of (138) are zero.

From (37), $\sum\left(\frac{\partial x}{\partial \tau}\right)^2=E$, $\sum\left(\frac{\partial x}{\partial \lambda}\right)^2=G$, $\sum\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \lambda}=F$ so that we may write (138) as

$$\begin{aligned}\sum\left(\frac{\partial^2 x}{\partial \tau \partial \lambda}\right)^2 &= D'^2+A'^2E+B'^2G+2A'B'F\\ &= D'^2+A'(A'E+B'F)+B'(B'G+A'F).\end{aligned} \tag{139}$$

From (129) we see that (139) may be written finally as

$$\sum\left(\frac{\partial^2 x}{\partial \tau \partial \lambda}\right)^2=D'^2+A'M'+B'J'. \tag{140}$$

If we multiply respective members of (135) and (137) together and add the products we obtain

$$\begin{aligned}\sum\left(\frac{\partial^2 x}{\partial \lambda^2}\cdot\frac{\partial^2 x}{\partial \tau^2}\right) &= (l^2+m^2+n^2)DD''+AA''\sum\left(\frac{\partial x}{\partial \tau}\right)^2+BB''\sum\left(\frac{\partial x}{\partial \lambda}\right)^2+\\ &\quad (BA''+AB'')\sum\frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \tau}+(AD''+A''D)\left(l\frac{\partial x}{\partial \tau}+m\frac{\partial y}{\partial \tau}+n\frac{\partial z}{\partial \tau}\right)+\\ &\quad (BD''+B''D)\left(l\frac{\partial x}{\partial \lambda}+m\frac{\partial y}{\partial \lambda}+n\frac{\partial z}{\partial \lambda}\right).\end{aligned} \tag{141}$$

Analogously as for (138), we may write (141) as

$$\begin{aligned}\sum\left(\frac{\partial^2 x}{\partial \lambda^2}\cdot\frac{\partial^2 x}{\partial \tau^2}\right) &= DD''+AA''E+BB''G+(BA''+AB'')F\\ &= DD''+A''(AE+BF)+B''(AF+BG).\end{aligned} \tag{142}$$

By (129) we may write (142) as

$$\sum\left(\frac{\partial^2 x}{\partial \lambda^2}\cdot\frac{\partial^2 x}{\partial \tau^2}\right)=DD''+MA''+JB''. \tag{143}$$

Now from (140) and (143) we have

$$DD''-D'^2=\sum\left(\frac{\partial^2 x}{\partial \lambda^2}\cdot\frac{\partial^2 x}{\partial \tau^2}\right)-\sum\left(\frac{\partial^2 x}{\partial \tau \partial \lambda}\right)^2+A'M'+B'J'-MA''-JB''. \tag{144}$$

From the second and last of equations (123), by partial differentiation we have

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 E}{\partial \lambda^2} &= \sum\left(\frac{\partial^2 x}{\partial \tau \partial \lambda}\right)^2+\sum\frac{\partial x}{\partial \tau}\frac{\partial^3 x}{\partial \tau \partial \lambda^2},\\ \frac{\partial^2 F}{\partial \lambda \partial \tau}-\frac{1}{2}\frac{\partial^2 G}{\partial \tau^2} &= \sum\frac{\partial^2 x}{\partial \lambda^2}\cdot\frac{\partial^2 x}{\partial \tau^2}+\sum\frac{\partial x}{\partial \tau}\frac{\partial^3 x}{\partial \lambda^2 \partial \tau}.\end{aligned} \tag{145}$$

From (145) we have

$$\sum\frac{\partial^2 x}{\partial \lambda^2}\frac{\partial^2 x}{\partial \tau^2}-\sum\left(\frac{\partial^2 x}{\partial \tau \partial \lambda}\right)^2=\frac{1}{2}\left(2\frac{\partial^2 F}{\partial \lambda \partial \tau}-\frac{\partial^2 G}{\partial \tau^2}-\frac{\partial^2 E}{\partial \lambda^2}\right). \tag{146}$$

From (127), (128), and (129) we find

$$A'M'+B'J'=\frac{1}{4}\mu^2\left[E\left(\frac{\partial G}{\partial\tau}\right)^2-2F\frac{\partial E}{\partial\lambda}\frac{\partial G}{\partial\tau}+G\left(\frac{\partial E}{\partial\lambda}\right)^2\right],$$

$$MA''+JB''=\frac{1}{4}\mu^2\left(2G\frac{\partial E}{\partial\tau}\frac{\partial F}{\partial\lambda}-G\frac{\partial E}{\partial\tau}\frac{\partial G}{\partial\tau}-F\frac{\partial G}{\partial\lambda}\frac{\partial E}{\partial\tau}+2E\frac{\partial F}{\partial\tau}\frac{\partial G}{\partial\lambda}-4F\frac{\partial F}{\partial\tau}\frac{\partial F}{\partial\lambda}+\right.$$
$$\left.2F\frac{\partial F}{\partial\tau}\frac{\partial G}{\partial\tau}-E\frac{\partial E}{\partial\lambda}\frac{\partial G}{\partial\lambda}+2F\frac{\partial E}{\partial\lambda}\frac{\partial F}{\partial\lambda}-F\frac{\partial E}{\partial\lambda}\frac{\partial G}{\partial\tau}\right). \tag{147}$$

With the values from (146) and (147) placed in (144) and with the value of $\mu^2=1/(EG-F^2)$ we have finally the Gauss characteristic equation

$$DD''-D'^2=\frac{1}{2}\left(2\frac{\partial^2F}{\partial\lambda\partial\tau}-\frac{\partial^2G}{\partial\tau^2}-\frac{\partial^2E}{\partial\lambda^2}\right)+\frac{1}{4(EG-F^2)}\left[E\left(\frac{\partial G}{\partial\tau}\right)^2-F\frac{\partial E}{\partial\lambda}\frac{\partial G}{\partial\tau}+\right.$$
$$G\left(\frac{\partial E}{\partial\lambda}\right)^2-2G\frac{\partial E}{\partial\tau}\frac{\partial F}{\partial\lambda}+G\frac{\partial E}{\partial\tau}\frac{\partial G}{\partial\tau}+F\frac{\partial G}{\partial\lambda}\frac{\partial E}{\partial\tau}-2E\frac{\partial F}{\partial\tau}\frac{\partial G}{\partial\lambda}+$$
$$\left.4F\frac{\partial F}{\partial\tau}\frac{\partial F}{\partial\lambda}-2F\frac{\partial F}{\partial\tau}\frac{\partial G}{\partial\tau}+E\frac{\partial E}{\partial\lambda}\frac{\partial G}{\partial\lambda}-2F\frac{\partial E}{\partial\lambda}\frac{\partial F}{\partial\lambda}\right]. \tag{148}$$

Equation (148) may be expressed equivalently as

$$DD''-D'^2=\frac{\sqrt{EG-F^2}}{2}\left[\frac{\partial}{\partial\tau}\left(\frac{F}{E\sqrt{EG-F^2}}\cdot\frac{\partial E}{\partial\lambda}-\frac{1}{\sqrt{EG-F^2}}\frac{\partial G}{\partial\tau}\right)+\right.$$
$$\left.\frac{\partial}{\partial\lambda}\left(\frac{2}{\sqrt{EG-F^2}}\frac{\partial F}{\partial\tau}-\frac{1}{\sqrt{EG-F^2}}\frac{\partial E}{\partial\lambda}-\frac{F}{E\sqrt{EG-F^2}}\frac{\partial E}{\partial\tau}\right)\right]. \tag{149}$$

The Gauss characteristic equation (149) is significant in the theory of the differential geometry of surfaces, since it is the condition that the quantities E, F, G, D, D', D'' must satisfy in order to be the fundamental quantities for a surface.

Now the linear element of a developable surface may be reduced to the form $ds^2=d\tau^2+d\lambda^2$. (See the discussion following equation 66.) Hence $E=G=1$, $F=0$. From (149) we have therefore that $DD''-D'^2=0$ and from (113) that $R_\tau=0$ at every point of a developable surface.

We shall need equation (149) subsequently to prove an important property of the spheroid, namely that the only geodesic isometric orthogonal system on the spheroid is that formed by the meridians and parallels.

THE SPHEROID

The theorems we have discussed apply to surfaces in general, hence to surfaces of revolution, and therefore to the sphere and spheroid. We avoid discussion of surfaces of revolution as such but proceed directly to the spheroid.

In figure 18, $AP=N$ is the normal to the meridian ellipse at $P(r, z)$ and clearly $r=N\cos\phi$. The equation of the meridian ellipse is $r^2(1-\epsilon^2)+z^2=a^2(1-\epsilon^2)$, where a is the semimajor axis, ϵ the eccentricity. a, ϵ, and b, the semiminor axis, are connected by the relation $b^2=a^2(1-\epsilon^2)$. Since the slope of the tangent at P is $\frac{dz}{dr}=-\frac{r}{z}(1-\epsilon^2)$, the slope of the normal at P, being the negative reciprocal of the slope of the tangent, is $-\frac{dr}{dz}=\frac{z}{r(1-\epsilon^2)}=\tan\phi$, hence

$$z=r(1-\epsilon^2)\tan\phi=N(1-\epsilon^2)\sin\phi. \tag{150}$$

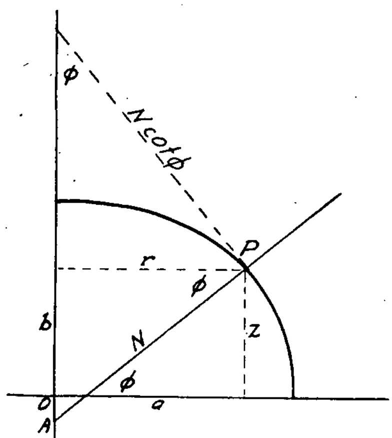


FIGURE 18.—Meridian ellipse of the spheroid.

Returning the value of z in terms of r given by (150) to the equation of the ellipse we have $r^2+r^2(1-\epsilon^2)\tan^2\phi=a^2$, or $r^2(1-\epsilon^2\sin^2\phi)=a^2\cos^2\phi$, whence

$$r=a\cos\phi/\sqrt{1-\epsilon^2\sin^2\phi}=N\cos\phi. \tag{151}$$

PRINCIPAL RADII OF NORMAL CURVATURE OF THE SPHEROID

From the latter of the two equalities of (151) we have

$$N=a/\sqrt{1-\epsilon^2\sin^2\phi}. \tag{152}$$

The radius of curvature of the meridian ellipse may be found from the usual formula for the radius of curvature of a plane curve,

$$R=\left|\frac{(1+z'^2)^{3/2}}{z''}\right|. \tag{153}$$

From the equation of the ellipse, $r^2(1-\epsilon^2)+z^2=a^2(1-\epsilon^2)$, we have $z'=-\frac{r}{z}(1-\epsilon^2)$, $z''=-\frac{(1+z'^2)-\epsilon^2}{z}$. Since the slope of the normal is $\tan\phi$, that of the tangent is $-\cot\phi$. Hence $z'=-\cot\phi$, and $z''=-\frac{1+\cot^2\phi-\epsilon^2}{z}=-\frac{\csc^2\phi-\epsilon^2}{z}$. From (150) and (152) we have $z=N(1-\epsilon^2)\sin\phi=\frac{a(1-\epsilon^2)\sin\phi}{\sqrt{1-\epsilon^2\sin^2\phi}}$. With these values of z', z'', and z placed in (153) we have

$$R=\left|\frac{a(1-\epsilon^2)\sin\phi\csc^3\phi}{\sqrt{1-\epsilon^2\sin^2\phi}\,(\csc^2\phi-\epsilon^2)}\right|=\left|\frac{a(1-\epsilon^2)}{(1-\epsilon^2\sin^2\phi)^{3/2}}\right|. \tag{154}$$

Now in figure 19, the ellipse of figure 18 has been revolved about its minor axis through an angle λ, the point P moving to the point P', and the spheroid being generated has been referred to the x, y, z coordinate system as shown. It is seen that $r=$

$N \cos \phi$ is the radius of the parallel in latitude ϕ and we have $x=r \cos \lambda = N \cos \phi \cos \lambda$, $y=r \sin \lambda = N \cos \phi \sin \lambda$, and z is still given by (150), so that we have the parametric

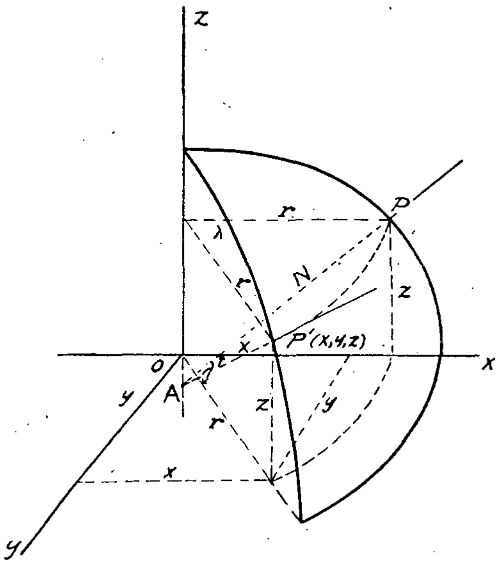


FIGURE 19.—Generation of the spheroid from the rotation of the meridian ellipse.

representation of the spheroid in terms of geodetic latitude, ϕ, and longitude, λ, namely

$$x=N \cos \phi \cos \lambda,\ y=N \cos \phi \sin \lambda,\ z=N(1-\epsilon^2) \sin \phi \tag{155}$$

where N is given by (152).

If we divide the members of (155) by a, a, and b, respectively, then square and add, making use of the relation $b^2=a^2(1-\epsilon^2)$, we obtain the well-known rectangular equation of the spheroid,

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1. \tag{156}$$

MEAN RADIUS OF THE SPHEROID AT A GIVEN POINT

From the reciprocal of equation (118) we have the radius of curvature of a normal section in given latitude ϕ for any azimuth α, namely

$$\rho_n=f(\alpha)=\frac{RN}{R \sin^2\alpha+N \cos^2\alpha}. \tag{157}$$

To find the mean value of ρ_n about a point in latitude ϕ, we make use of the theorem of the mean for a function. The theorem is easily demonstrated by means of figure 20. The slope of the tangent to the curve $y=f(x)$ is given by $f'(x)$, where the prime denotes differentiation, and at the point $Q[\xi, f(\xi)]$, the slope is $f'(\xi)$. The slope of the chord PS is $\frac{f(d)-f(c)}{d-c}$ and there exists a point Q as shown such that the slope of the tangent at Q is equal to the slope of the chord PS where $c<\xi<d$. That is, a value ξ can be found such that

$$f'(\xi)=\frac{f(d)-f(c)}{d-c}. \tag{158}$$

At such a point $f(\xi)$ is defined as the mean value of the function $f(x)$, $c \leqq x \leqq d$. By definition of the definite integral we have

$$\int_c^d f'(x)dx = f(d) - f(c).$$

With this value of $f(d) - f(c)$ placed in (158) we have

$$f'(\xi) = \frac{1}{d-c}\int_c^d f'(x)dx, \tag{159}$$

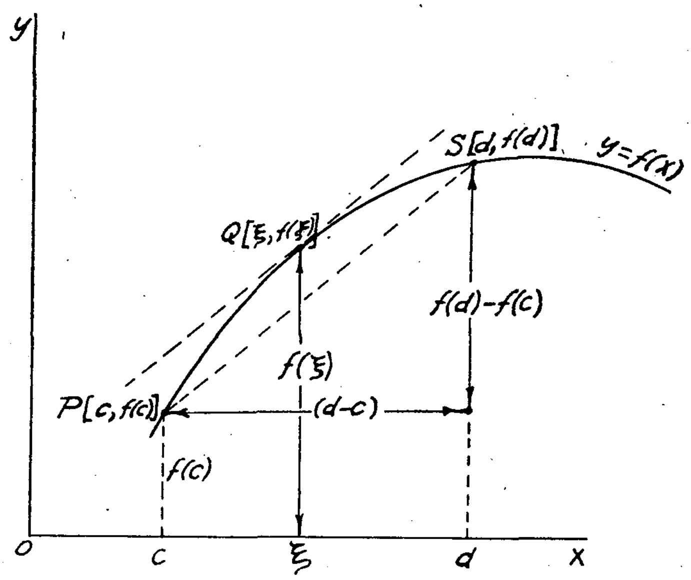


FIGURE 20.—The mean value of a function.

which allows us to compute the mean value of the function $f(x)$, $c \leqq x \leqq d$ by evaluating the definite integral of this function with limits c and d. (The primes denoting differentiation may be omitted in equation 159.)

Denoting $f(\xi)$ by R_m and the limits of α by $c=0$, $d=2\pi$ we have from (157) and (159)

$$\begin{aligned} R_m &= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)d\alpha = \frac{1}{2\pi}\int_0^{2\pi} \frac{RN\,d\alpha}{R\sin^2\alpha + N\cos^2\alpha} \\ &= \frac{2}{\pi}\int_0^{\pi/2} \frac{R\sec^2\alpha\,d\alpha}{1+\frac{R}{N}\tan^2\alpha} \\ &= \frac{2}{\pi}\sqrt{RN}\left[\text{arc tan}\left(\sqrt{\frac{R}{N}}\tan\alpha\right)\right]_0^{\pi/2} \\ &= \frac{2}{\pi}\sqrt{RN}\left(\frac{\pi}{2}-0\right) = \sqrt{RN}. \end{aligned} \tag{160}$$

Thus the mean value of the radius of a spheroid at one of its points is the geometric mean of the principal radii of curvature at the given point, or from (113) it is the square root of the radius of total curvature of the surface at the given point. Placing from (152) and (154) the values of R and N in equation (160) we have

$$R_m = \frac{a\sqrt{1-\epsilon^2}}{1-\epsilon^2\sin^2\phi}. \tag{161}$$

LINEAR ELEMENT OF THE SPHEROID

Let us compute the fundamental quantities E, F, G and D, D', D'' as given by (37) and (101), namely

$$\begin{aligned} E&=\left(\frac{\partial x}{\partial \phi}\right)^2+\left(\frac{\partial y}{\partial \phi}\right)^2+\left(\frac{\partial z}{\partial \phi}\right)^2, & D&=l\frac{\partial^2 x}{\partial \phi^2}+m\frac{\partial^2 y}{\partial \phi^2}+n\frac{\partial^2 z}{\partial \phi^2},\\ F&=\frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \lambda}+\frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \lambda}+\frac{\partial z}{\partial \phi}\frac{\partial z}{\partial \lambda}, & D'&=l\frac{\partial^2 x}{\partial \phi\partial\lambda}+m\frac{\partial^2 y}{\partial \phi\partial\lambda}+n\frac{\partial^2 z}{\partial \phi\partial\lambda},\\ G&=\left(\frac{\partial x}{\partial \lambda}\right)^2+\left(\frac{\partial y}{\partial \lambda}\right)^2+\left(\frac{\partial z}{\partial \lambda}\right)^2, & D''&=l\frac{\partial^2 x}{\partial \lambda^2}+m\frac{\partial^2 y}{\partial \lambda^2}+n\frac{\partial^2 z}{\partial \lambda^2}, \end{aligned} \tag{162}$$

where l, m, n as given by (95) are

$$l=\mu\left(\frac{\partial y}{\partial \lambda}\frac{\partial z}{\partial \phi}-\frac{\partial y}{\partial \phi}\frac{\partial z}{\partial \lambda}\right),\quad m=\mu\left(\frac{\partial z}{\partial \lambda}\frac{\partial x}{\partial \phi}-\frac{\partial z}{\partial \phi}\frac{\partial x}{\partial \lambda}\right),\quad n=\mu\left(\frac{\partial x}{\partial \lambda}\frac{\partial y}{\partial \phi}-\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \lambda}\right),$$

$$\mu=1/\sqrt{EG-F^2}.$$

From (155), by partial differentiation with respect to ϕ and λ, we have

$$\frac{\partial x}{\partial \phi}=-R\sin\phi\cos\lambda,\ \frac{\partial^2 x}{\partial \phi^2}=-\cos\lambda\,(R\sin\phi)'=-\frac{R\cos\lambda}{N\cos\phi}\,[(2N-3R)\sin^2\phi+N],$$

$$\frac{\partial x}{\partial \lambda}=-N\cos\phi\sin\lambda=-y,\ \frac{\partial^2 x}{\partial \lambda^2}=-N\cos\phi\cos\lambda=-x,\ \frac{\partial^2 x}{\partial \phi\partial\lambda}=R\sin\phi\sin\lambda=-\frac{\partial y}{\partial \phi},$$

$$\frac{\partial y}{\partial \phi}=-R\sin\phi\sin\lambda=\frac{\partial x}{\partial \phi}\tan\lambda,\ \frac{\partial^2 y}{\partial \phi^2}=-\sin\lambda\,(R\sin\phi)'=\frac{\partial^2 x}{\partial \phi^2}\tan\lambda,\ \frac{\partial y}{\partial \lambda}=N\cos\phi\cos\lambda=x,$$

$$\frac{\partial^2 y}{\partial \lambda^2}=-N\cos\phi\sin\lambda=-y=\frac{\partial x}{\partial \lambda},\ \frac{\partial^2 y}{\partial \phi\partial\lambda}=-R\sin\phi\cos\lambda=\frac{\partial x}{\partial \phi},\ \frac{\partial z}{\partial \phi}=R\cos\phi,$$

$$\frac{\partial^2 z}{\partial \phi^2}=(R\cos\phi)'=\frac{R\sin\phi}{N}\,(2N-3R),\ \frac{\partial z}{\partial \lambda}=\frac{\partial^2 z}{\partial \lambda^2}=\frac{\partial^2 z}{\partial \phi\partial\lambda}=0. \tag{163}$$

From (162) and (163) we find

$$E=R^2,\ F=0,\ G=N^2\cos^2\phi,\ l=\frac{x}{N},\ m=\frac{y}{N},\ n=\sin\phi,\ D=-R,\ D'=0,\ D''=-N\cos^2\phi. \tag{164}$$

Since $F=0$, we know from (42) that the parametric system is orthogonal. We knew this anyway since the meridians are orthogonal to the parallels. Since $D'=0$, we know from (113) and (114) that the parametric curves, the meridians and parallels, are also the lines of curvature for the surface, $D'=F=0$ being the required condition. We knew this also from an elementary property of the lines of curvature, which we have not proved in general here, namely that consecutive normals along lines of curvature intersect. Hence N and R, as given by (152) and (154), are the principal radii of curvature of the spheroid at a point P in latitude ϕ. We will find N and R entering all the mapping formulas to be obtained. As has been shown, R is the radius of curvature in latitude ϕ of the meridian ellipse, while N is the distance along the normal from the point P in latitude ϕ to the minor axis of the spheroid.

With the values of E, F, G from (164) placed in (36) we have the linear element of the spheroid,

$$ds^2=R^2d\phi^2+N^2\cos^2\phi d\lambda^2=N^2\cos^2\phi\left(\frac{R^2}{N^2}\sec^2\phi d\phi^2+d\lambda^2\right). \tag{165}$$

CURVES ON THE SPHEROID

Curves on the spheroid will be represented by the integrals of differential equations as given by (51) and (56), where τ has been replaced by ϕ. That is, any integral curve on the spheroid, expressed in terms of the curvilinear parameters ϕ and λ, is of the form $f(\phi,\lambda)=0$. We will be interested here in the following three curves on the ellipsoid and in their projections on a plane: The geodesic, or the geodetic line; the loxodrome, or the rhumb line; the curve of alinement.

THE GEODESIC

The geodesic is fundamentally defined as the curve of shortest distance between two points on a surface. From the integral for arc length we may, by the calculus of variations, determine the conditions on the integrand for the arc length to be a minimum. From these conditions may be deduced the property that the osculating plane at each point of a geodesic contains the normal to the surface, or equivalently that at each point of a geodesic the principal normal to the curve coincides with the normal to the surface. We will adopt this last property as the definition of the geodesic on the spheroid, find the differential equation of the curves and show that the integral curves depend on the evaluation of an elliptic integral.

If the principal normal to a curve on a surface is to coincide with the normal to the surface at each point of the curve, then the corresponding direction cosines of the two normals must be equal.

From (87) and (164) the direction cosines of the principal normal and of the normal to the surface are respectively $l_2=\rho_1 x''$, $m_2=\rho_1 y''$, $n_2=\rho_1 z''$ and $l=x/N$, $m=y/N$, $n=\sin\phi$. Hence we must have

$$\frac{\rho_1 x''}{x/N}=\frac{\rho_1 y''}{y/N}=\frac{\rho_1 z''}{\sin\phi}. \tag{166}$$

Now the two equations (166) are not independent as can be easily shown. From the first two members of (166) we have the differential equation $xy''-yx''=0$, a first integral being at once

$$xy'-yx'=c. \tag{167}$$

Since the derivatives of (167) are with respect to arc length, s, then

$$x'=\frac{dx}{ds}=\frac{\partial x}{\partial\phi}\frac{d\phi}{ds}+\frac{\partial x}{\partial\lambda}\frac{d\lambda}{ds},\ y'=\frac{\partial y}{\partial\phi}\frac{d\phi}{ds}+\frac{\partial y}{\partial\lambda}\frac{d\lambda}{ds}.$$

With the values of $\frac{\partial x}{\partial\phi}, \frac{\partial x}{\partial\lambda}, \frac{\partial y}{\partial\phi}, \frac{\partial y}{\partial\lambda}$ from (163) we have

$$x'=-R\sin\phi\cos\lambda\,\frac{d\phi}{ds}-N\cos\phi\sin\lambda\,\frac{d\lambda}{ds},$$

$$y'=-R\sin\phi\sin\lambda\,\frac{d\phi}{ds}+N\cos\phi\cos\lambda\,\frac{d\lambda}{ds}$$

and these values of x' and y' with those of x and y from (155) placed in (167) give

$$N^2\cos^2\phi\,\frac{d\lambda}{ds}=c, \tag{168}$$

where, as noted before, $r=N\cos\phi$ is the radius of the parallel of the spheroid in latitude ϕ.

Eliminating ds between (165) and (168) we obtain the differential equation of the geodesics on the spheroid,

$$c^2R^2d\phi^2+N^2\cos^2\phi(c^2-N^2\cos^2\phi)d\lambda^2=0. \tag{169}$$

From (169) we have

$$\frac{d\phi}{d\lambda}=\frac{\pm N\cos\phi}{cR}\sqrt{N^2\cos^2\phi-c^2}. \tag{170}$$

In figure 21, if α is the angle which the element of arc length, ds, makes with the meridian, then $N\cos\phi d\lambda=ds\sin\alpha$, or $\frac{d\lambda}{ds}=\frac{\sin\alpha}{N\cos\phi}$. With this value of $\frac{d\lambda}{ds}$ placed in (168) we obtain

$$N\cos\phi\sin\alpha=c, \tag{171}$$

which is the fundamental characteristic of the geodesic on the spheroid or on any surface of revolution. That is, at each point of a geodesic the product of the radius

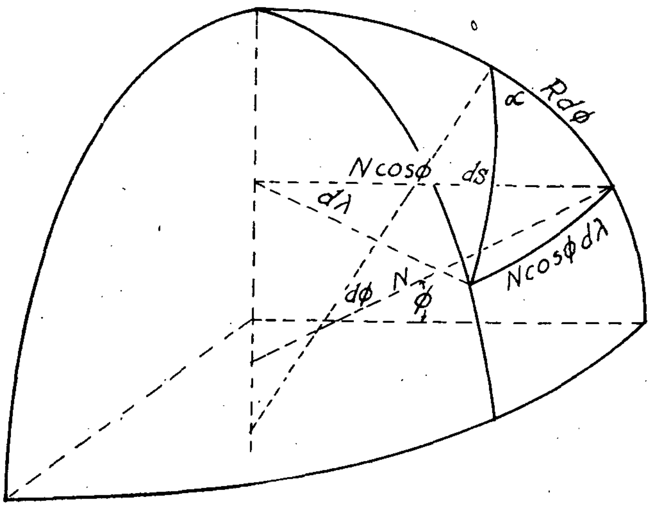


FIGURE 21.—The linear element of the spheroid as obtained from a differential right triangle.

of the parallel and the sine of the angle which the geodesic makes with the meridian is constant. When $\alpha=90°$, the geodesic is orthogonal to the meridian and $c=r_0=N_0\cos\phi_0$. When the geodesic crosses the Equator, $\phi=0$, and $r=a$, so that $c=a\sin\alpha_0$, where α_0 is the angle which the geodesic makes at the Equator with the meridian.

In (170) we note that for the geodesics to be real $N^2\cos^2\phi-c^2\geqq 0$ or $N\cos\phi\geqq c=N_0\cos\phi_0$ and that $c=r_0=N_0\cos(\pm\phi_0)=N_0\cos\phi_0$. This means that the geodesic oscillates between two parallels which are symmetric with respect to the Equator, the geodesic being tangent alternately to each parallel as shown in figure 22.

From (170) we have

$$\lambda-\lambda_0=\pm c\int_0^\phi\frac{Rd\phi}{N\cos\phi(N^2\cos^2\phi-c^2)^{1/2}}. \tag{172}$$

The transformation $\sin\phi=k\sin\theta$, with $\frac{1}{k^2}=\frac{a^2-c^2\epsilon^2}{a^2-c^2}$ and with the values of N and

R from (152) and (154), reduces (172) to

$$\lambda-\lambda_0=\pm\frac{c(1-\epsilon^2)}{(a^2-\epsilon^2c^2)^{1/2}}\int_0^\theta\frac{d\theta}{(1-k^2\sin^2\theta)(1-\epsilon^2k^2\sin^2\theta)^{1/2}},$$

or

$$\lambda-\lambda_0=\pm\frac{c(1-\epsilon^2)}{(a^2-\epsilon^2c^2)^{1/2}}\Pi(-k^2,\ \epsilon k,\ \theta), \tag{173}$$

where Π is the elliptic integral of the third kind in Legendre's notation.

From (169), if $c=0$, and $\phi<\frac{\pi}{2}$, we have $d\lambda=0$, or $\lambda=c_2$. But $c=a\sin\alpha_0$, and $c=0$ when $\alpha_0=0$ which is the condition on α_0 if the meridian is to be a geodesic. Thus the meridians on the spheroid are geodesics. We knew this from geometrical

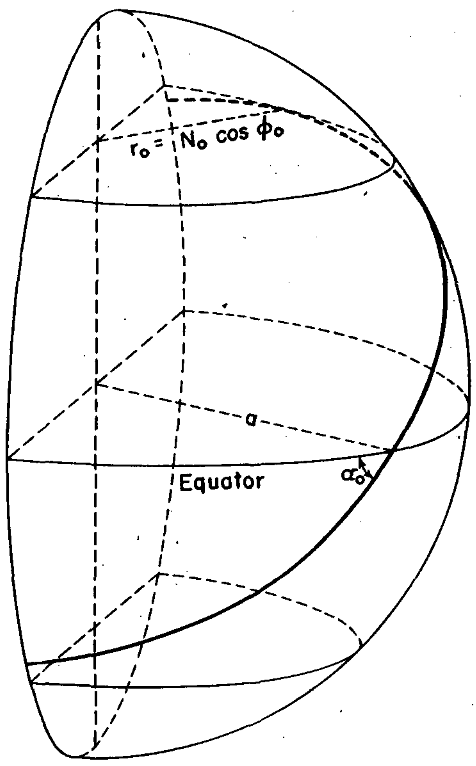


FIGURE 22.—A nonmeridian geodesic of the spheroid.

considerations. That is, the shortest path between the ends of a diameter of the Equator would obviously be the plane elliptic path through the poles. Again we knew this from equations (126) since $E=R^2$ is a function of ϕ alone. If we eliminate $d\lambda$ between (165) and (168) we obtain $\frac{d\phi}{ds}=\frac{\sqrt{r^2-a^2}}{Rr}$, where $r=N\cos\phi$ is the radius of the parallel in latitude ϕ and we have placed $c=a\sin\frac{\pi}{2}=a$, for $\alpha_0=\frac{\pi}{2}$ is the required condition on α_0 if a parallel is to be a geodesic. If a parallel is a geodesic then $\phi=c_1$, $\frac{d\phi}{ds}=0$ and we have $\sqrt{r^2-a^2}=0$, since $Rr\neq 0$ for $\phi<\frac{\pi}{2}$. This gives $r=a$. That is, the only parallel which is a geodesic is the Equator. Again this was clear geometrically since the normal (the radius) of a parallel makes the angle ϕ with the normal to the surface in latitude ϕ except at the Equator where $\phi=0$. Thus the only plane geodesics on the spheroid are the meridians and the Equator.

THE RHUMB LINE OR LOXODROME

This curve on the spheroid is such that it meets consecutive meridians at the same angle. From figure 21, we have

$$\tan \alpha=\frac{N \cos \phi \, d\lambda}{R d\phi}. \tag{174}$$

With α constant, equation (174) is the differential equation of the family of curves. Writing (174) in the form $d\lambda=\tan\alpha\cdot \frac{R}{N}\sec \phi \, d\phi$, the integral curves are

$$\lambda-\lambda_0=I \tan \alpha, \tag{175}$$

where

$$I=\int_0^\phi \frac{R}{N}\sec \phi \, d\phi=\ln\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-\epsilon \sin \phi}{1+\epsilon \sin \phi}\right)^{\epsilon/2}\right]. \tag{176}$$

The integral, I, of (176), as will soon be shown, is the key to the conformal representation of the spheroid upon the plane.

THE CURVE OF ALINEMENT

The curve of alinement is the locus of a point on the spheroid which moves so that the plane through it and two fixed points on the spheroid is normal to the surface at the moving point.

If the point $P(x, y, z)$ on the spheroid lies in a general plane, its coordinates satisfy an equation of the form

$$Ax+By+Cz=1. \tag{177}$$

If the plane (177) is to be normal to the surface then A, B, C are proportional to the direction cosines of a tangent to the surface at P, and since the normal to the surface at P is orthogonal to every tangent to the surface at P, we must have

$$lA+mB+nC=0. \tag{178}$$

From (156), placing $b^2=a^2(1-\epsilon^2)$, we have $f(x, y, z)=(1-\epsilon^2)(x^2+y^2)+z^2-a^2(1-\epsilon^2)=0$, as the equation of the spheroid. Since the direction cosines of the normal to the surface are proportional to $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ we have $l=\nu\frac{\partial f}{\partial x}$, $m=\nu\frac{\partial f}{\partial y}$, $n=\nu\frac{\partial f}{\partial z}$. Now $\frac{\partial f}{\partial x}=2(1-\epsilon^2)x$, $\frac{\partial f}{\partial y}=2(1-\epsilon^2)y$, $\frac{\partial f}{\partial z}=2z$. Hence $l=2\nu(1-\epsilon^2)x$, $m=2\nu(1-\epsilon^2)y$, $n=2\nu z$. We could have obtained these directly from (164). Since $z=N(1-\epsilon^2)\sin \phi$, we have $\sin \phi=z/N(1-\epsilon^2)$ and then from (164) we have $l=x/N$, $m=y/N$, $n=z/N(1-\epsilon^2)$. Whence multiplying these last through by $2N\nu(1-\epsilon^2)$ we have $l=2\nu(1-\epsilon^2)x$, $m=2\nu(1-\epsilon^2)y$, $n=2\nu z$. These values of l, m, n placed in (178) give

$$(1-\epsilon^2)x\,A+(1-\epsilon^2)y\,B+zC=0, \tag{179}$$

which is the condition that the plane (177) at the point $P(x, y, z)$ shall be normal to the surface. But the plane (177) must also pass through two arbitrary points, $P_1(x_1, y_1, z_1)$, and $P_2(x_2, y_2, z_2)$, on the surface and the conditions for this are

$$\begin{aligned} Ax_1+By_1+Cz_1&=1, \\ Ax_2+By_2+Cz_2&=1. \end{aligned} \tag{180}$$

From (177), (179), and (180) we have enough equations to eliminate A, B, and C. We accomplish this by writing the eliminant of the four equations as follows:

$$\begin{vmatrix} x & y & z & 1 \\ (1-\epsilon^2)x & (1-\epsilon^2)y & z & 0 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix}=0. \tag{181}$$

Expanding (181) we obtain the equation

$$Cxz-Hyz-Ux-Vy-Wz=0, \tag{182}$$

where $C=\epsilon^2(y_2-y_1)$, $H=\epsilon^2(x_2-x_1)$, $U=(1-\epsilon^2)(y_1z_2-y_2z_1)$,
$V=(1-\epsilon^2)(z_1x_2-z_2x_1)$, $W=(x_1y_2-x_2y_1)$.

Equation (182) represents a hyperbolic paraboloid and is the envelope of the plane (177) under the given conditions. That is, the curve traced by $P(x,y,z)$ on the surface, the curve of alinement between the points P_1 and P_2, is the intersection of the hyperbolic paraboloid (182) and the spheroid.

By means of (155) the coefficients C, H, U, V, W of (182) may be expressed in terms of the latitude and longitude of the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ since

$$x_1=N_1 \cos\phi_1 \cos\lambda_1,\ y_1=N_1 \cos\phi_1 \sin\lambda_1,\ z_1=N_1(1-\epsilon^2)\sin\phi_1$$

$$x_2=N_2 \cos\phi_2 \cos\lambda_2,\ y_2=N_2 \cos\phi_2 \sin\lambda_2,\ z_2=N_2(1-\epsilon^2)\sin\phi_2$$

and (182) may be written as

$$N(1-\epsilon^2)(C\cos\lambda-H\sin\lambda)\sin\phi-U\cos\lambda-V\sin\lambda-W(1-\epsilon^2)\tan\phi=0. \tag{183}$$

If we place $\cos\lambda=\sqrt{1-\sin^2\lambda}$ in (183) we obtain the quadratic $(P^2+Q^2)\sin^2\lambda+2QS\sin\lambda+S^2-P^2=0$, whose solution is

$$\sin\lambda=\frac{-QS\pm P\sqrt{P^2+Q^2-S^2}}{P^2+Q^2}, \tag{184}$$

where P, Q, S are functions of ϕ alone, given by the relations $P=[CN(1-\epsilon^2)\sin\phi-U]$, $Q=[HN(1-\epsilon^2)\sin\phi+V]$, $S=W(1-\epsilon^2)\tan\phi$. (185)

The curve of alinement may be described physically as the path of a theodolite, in adjustment, which is placed so that the plane of its vertical circle always passes through two fixed points. It is very near the geodesic between the two points.

CONFORMAL PROJECTION OF THE SPHEROID UPON A PLANE

We have already found that in order for a surface to be mapped conformally upon a plane, we must have, from (66), $ds_1^2=m(d\tau^2+d\lambda^2)$. From (165) we have

$$ds_2^2=N^2\cos^2\phi\left(\frac{R^2}{N^2}\sec^2\phi d\phi^2+d\lambda^2\right). \tag{186}$$

Hence we must have $d\tau=\frac{R}{N}\sec\phi d\phi$, and $m=N^2\cos^2\phi$, that is, m is the square of the radius of the parallel in latitude ϕ. Note that $d\tau$ may be written $d\tau=\frac{Rd\phi}{N\cos\phi}$ which is the ratio of the element of arc along the meridian to the radius of the parallel in latitude ϕ. (See fig. 21 on p. 64.) The value of the integral, $\tau=\int_0^\phi\frac{R}{N}\sec\phi\, d\phi$, as noted before, is given by (176), and we mentioned in connection with (176) that this integral was the key to the conformal projection of the spheroid upon a plane. That is, the representation $\lambda=\lambda$, $\tau=\int_0^\phi\frac{R}{N}\sec\phi\, d\phi$ maps the spheroid conformally upon the $\tau\lambda$-plane. Since τ is a function of ϕ alone (Both R and N are functions of ϕ alone.) we have for ϕ a constant (which gives a parallel on the spheroid) the straight line $\tau=c$, $\lambda=\lambda$ parallel to the λ-axis. Similarly when λ is a constant (which gives a meridian on the spheroid) we have the straight line $\lambda=c$, $\tau=\tau(\phi)$ which is parallel to the τ-axis. That is, the parallels and meridians on the spheroid are mapped into straight lines parallel to the λ- and τ-axes. (See fig. 4 on p. 24.) This is actually the Mercator projection of the spheroid upon a plane as will be demonstrated later. But the point λ,τ has the complex representation $\lambda+i\tau$ in the superimposed complex plane or z-plane as discussed before, and we have already seen that the analytic function (5) maps the z- or $\lambda\tau$-plane conformally upon the w- or xy-plane. Now from (9) we have

$$ds_1^2=dx^2+dy^2=f'(\lambda-i\tau)f'(\lambda+i\tau)(d\tau^2+d\lambda^2). \tag{187}$$

From (186) and (187), analogously as we had for (30), we have

$$\frac{ds_1^2}{ds_2^2}=\frac{f'(\lambda-i\tau)f'(\lambda+i\tau)(d\tau^2+d\lambda^2)}{N^2\cos^2\phi\left(\frac{R^2}{N^2}\sec^2\phi d\phi^2+d\lambda^2\right)}$$

$$=\frac{f'(\lambda-i\tau)f'(\lambda+i\tau)(d\tau^2+d\lambda^2)}{N^2\cos^2\phi(d\tau^2+d\lambda^2)}$$

$$=\frac{f'(\lambda-i\tau)f'(\lambda+i\tau)}{N^2\cos^2\phi}, \tag{188}$$

where we know from (16) that the product $f'(\lambda-i\tau)f'(\lambda+i\tau)$ is a real function.

We can finally state that the analytic function

$$x+iy=f(\lambda\pm i\tau), \tag{189}$$

where, from (176),

$$\tau=\int_0^\phi\frac{R}{N}\sec\phi d\phi=\ln\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-\epsilon\sin\phi}{1+\epsilon\sin\phi}\right)^{\epsilon/2}\right],$$

represents all conformal mapping of the spheroid upon a plane. The form of the function $f(\lambda\pm i\tau)$ is determined by the initial required conditions of the desired projection, that is, by which line or lines in the projection are to be held true to scale, and by the required geometric form of the map elements corresponding to meridians and parallels. The mapping equations will then be given by equating real and imaginary parts in (189) to obtain the real mapping coordinates $x=x(\lambda,\tau)$, $y=y(\lambda,\tau)$ which must satisfy the Cauchy-Riemann equations (15).

From (16) and (188) we have

$$k=\frac{ds_1}{ds_2}=\frac{\sqrt{f'(\lambda-i\tau)f'(\lambda+i\tau)}}{N\cos\phi}=\frac{\sqrt{\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2}}{N\cos\phi}=\frac{\sqrt{\left(\frac{\partial x}{\partial\tau}\right)^2+\left(\frac{\partial y}{\partial\tau}\right)^2}}{N\cos\phi}=\frac{\sqrt{J\left(\frac{x,y}{\lambda,\tau}\right)}}{N\cos\phi}, \quad (190)$$

which is the magnification or the scale at any point of the projection.

THE GEODESIC ISOMETRIC SYSTEM ON THE SPHEROID

From the discussion of isometric orthogonal systems, following equation (69), we see that equation (186) with $d\tau=\frac{R}{N}\sec\phi\, d\phi$ establishes such a system on the spheroid in the form $ds^2=r^2(d\tau^2+d\lambda^2)$, where $r=N\cos\phi$ is the radius of a parallel in latitude ϕ. The parametric curves are the same and the meridians, $\lambda=c_2$, are geodesics on the spheroid. Thus we have a geodesic isometric orthogonal system of curves on the surface. We will now show that this system formed by the meridians and parallels is the only such system possible on the spheroid.

From (113), (152) and (154) we have

$$R_T=\frac{1}{R}\cdot\frac{1}{N}=\frac{(1-\epsilon^2\sin^2\phi)^2}{a^2(1-\epsilon^2)}=f(\phi). \quad (191)$$

Since $\epsilon<1$, $R_T>0$ for all values of ϕ. That is, the spheroid is a surface of positive curvature.

Now from (126) we see that the condition for the parametric curves $\lambda=c_2$ to be geodesics is for E to be a function of ϕ alone. From (42) the parametric curves are orthogonal if $F=0$. The linear element is then of the form

$$ds^2=R^2d\phi^2+r^2d\lambda^2 \quad (192)$$

where $E=R^2$ is a function of ϕ alone and $G=r^2$ is, in general, a function of both λ and ϕ. The parametric curves are not changed if we replace $Rd\phi$ by $d\phi$ in (192) but E is now unity, that is, (192) becomes

$$ds^2=d\phi^2+r^2d\lambda^2. \quad (193)$$

From (69), an orthogonal system of parametric curves is isometric if E and G satisfy an equation of the form $\frac{U}{V}=\frac{G}{E}$, where U is a function of ϕ alone and V is a function of λ alone. If we write this condition in terms of logarithms as $\log\frac{G}{E}=\log U-\log V$, then by partial differentiation we have $\frac{\partial^2}{\partial\phi\partial\lambda}\left(\log\frac{G}{E}\right)=\frac{\partial^2}{\partial\lambda\partial\phi}\left(\log\frac{G}{E}\right)=0$, hence either of the latter equations is equivalent to $\frac{U}{V}=\frac{G}{E}$. From (193) $E=1$, $G=r^2$ whence we have $\frac{\partial^2}{\partial\phi\partial\lambda}(\log r^2)=0$, whence $\log r=\log p+\log q$, or $r=pq$ where p is a function of ϕ alone, and q is a function of λ alone, and (193) becomes then

$$ds^2=d\phi^2+p^2q^2d\lambda^2. \quad (194)$$

With the values of $E=1$, $F=0$, $G=p^2q^2$ placed in equation (149), remembering that p is a function of ϕ alone and q a function of λ alone, we have $DD''-D'^2=-pq^2\frac{\partial^2 p}{\partial\phi^2}$. With this value of $DD''-D'^2$ placed in (113) we have

$$R_T+\frac{1}{p}\frac{\partial^2 p}{\partial\phi^2}=0. \tag{195}$$

Equation (195) is an ordinary differential equation for which solutions exist if $R_T=0$, $R_T=$a constant, or if R_T is a function of ϕ alone. We have seen that if $R_T=0$, the surface is developable; if R_T is a constant, the surface is one of constant curvature (for example, a sphere). On surfaces of revolution, particularly upon the spheroid as seen from equation (191), R_T is a function of the latitude alone, whence equation (195) has solutions. Therefore the only geodesic isometric system on the spheroid is the graticule formed by the meridians and parallels. Note that the linear element (192), with $r=N\cos\phi$, is identical to that of the spheroid as given by (165).

SURFACES OF CONSTANT CURVATURE, THE APOSPHERE

Surfaces whose total curvature, R_T as given by equation (113), is the same at all points are called surfaces of constant curvature. Now we can write equation (194) as

$$ds^2=d\phi^2+Gd\lambda^2, \tag{196}$$

and the differential equation (195) as

$$R_T+\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial\phi^2}=0, \tag{197}$$

where $p^2=G$ is a function of ϕ alone. This is true for the spheroid as seen from equation (193) where $p=r=N\cos\phi$, and it is true of any surface of revolution.

Let $R_T=1/a^2$ where a is a real constant. Then equation (197) may be written as an ordinary differential equation of second order, namely

$$\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial\phi^2}=\frac{1}{2}\left[\frac{1}{G}\frac{d^2G}{d\phi^2}-\frac{1}{2G^2}\left(\frac{dG}{d\phi}\right)^2\right]=-R_T=-\frac{1}{a^2}. \tag{198}$$

If we let $\frac{dG}{d\phi}=h$, then $\frac{d^2G}{d\phi^2}=\frac{dh}{d\phi}=\frac{dh}{dG}\cdot\frac{dG}{d\phi}=h\frac{dh}{dG}$, and equation (198) may be written $\frac{2h}{G}dh-h^2\frac{dG}{G^2}=-\frac{4}{a^2}dG$, which integrates at once to give $\frac{h^2}{G}=-\frac{4}{a^2}G+C$, whence

$$h=\frac{dG}{d\phi}=\pm\sqrt{G}\sqrt{C-4G/a^2}. \tag{199}$$

Equation (199) may be written $-\dfrac{\dfrac{dG}{a\sqrt{C}\sqrt{G}}}{\sqrt{1-4G/a^2C}}=\dfrac{d\phi}{a}$, whose integral is $\cos^{-1}\dfrac{2\sqrt{G}}{a\sqrt{C}}=\dfrac{\phi}{a}+d$, or

$$\sqrt{G}=c\cos\left(\frac{\phi}{a}+d\right), \tag{200}$$

where we have placed $c=a\sqrt{C}/2=$a real constant.

The surfaces given by (200) are called spherical surfaces and they depend upon the values of the constants of integration c and d. A change in d means only a different choice of the parallel $\phi=0$, so let us take $d=0$. The linear element (196) becomes then with the value of G from (200),

$$ds^2=d\phi^2+c^2\cos^2\frac{\phi}{a}d\lambda^2, \tag{201}$$

the radius of the parallel being $r=\sqrt{G}=c\cos\frac{\phi}{a}=r(\phi)$.

To obtain the equation of the meridian curve we refer again to figures 18 and 19 (pp. 59 and 60). From figure 18, the equation of the meridians is $z=f(r)$, where $r=r(\phi)$. From figure 19 we have $x=r\cos\lambda, y=r\sin\lambda$. Hence from equations (37) and (201) we have

$$E=\left(\frac{\partial x}{\partial\phi}\right)^2+\left(\frac{\partial y}{\partial\phi}\right)^2+\left(\frac{\partial z}{\partial\phi}\right)^2=r'^2\cos^2\lambda+r'^2\sin^2\lambda+f'^2(r)r'^2$$

$$=r'^2+f'^2(r)r'^2=1,$$

whence

$$z'=f'(r)r'=\sqrt{1-r'^2},$$

or

$$dz=f'(r)dr=\sqrt{1-r'^2}d\phi. \tag{202}$$

From $r=c\cos\frac{\phi}{a}$, we have $r'=-\frac{c}{a}\sin\frac{\phi}{a}$ and with this value of r' placed in (202) we have

$$r=c\cos\frac{\phi}{a},\ z=\int\sqrt{1-\frac{c^2}{a^2}\sin^2\frac{\phi}{a}}d\phi, \tag{203}$$

which are the parametric equations of the meridian curve in terms of ϕ as parameter.

The parametric equations of the surface in terms of ϕ and λ are then

$$x=c\cos\frac{\phi}{a}\cos\lambda,\ y=c\cos\frac{\phi}{a}\sin\lambda,\ z=\int\sqrt{1-\frac{c^2}{a^2}\sin^2\frac{\phi}{a}}d\phi. \tag{204}$$

There are three types of surfaces given by equations (203) or (204), according as $c=a$, $c>a$, $c<a$.

1. $c=a$. From equations (204) we have $x=a\cos\frac{\phi}{a}\cos\lambda$, $y=a\cos\frac{\phi}{a}\sin\lambda$, $z=\int\cos\frac{\phi}{a}\,d\phi=a\sin\frac{\phi}{a}$, which, by comparison with equations (32), or by squaring and adding respective members, is seen to be a sphere of radius a.

2. $c>a$. For z to be real we must have from equations (203) or (204), $\frac{c^2}{a^2}\sin^2\frac{\phi}{a}\leqq 1$, or $\sin\frac{\phi}{a}\leqq\frac{a}{c}$ and hence $r=c\cos\frac{\phi}{a}>0$ for all allowable values of $\frac{\phi}{a}$. When $\phi=0$, we have $r=c$, $z=0$. When $\sin\frac{\phi}{a}=\frac{a}{c}$, $r_0=c\cos\frac{\phi}{a}=c\sqrt{1-a^2/c^2}=\sqrt{c^2-a^2}$, and $z_0=\int_0^{\sin^{-1}a/c}\sqrt{1-\frac{c^2}{a^2}\sin^2\frac{\phi}{a}}\,d\phi=d_0$. Thus the surface is made up of zones bounded by the minimum parallels $r_0=\sqrt{c^2-a^2}$, the greatest parallel of each zone being of radius c as shown in figure 23.

3. $c<a$. In this case we have $0\leqq\left(r=c\cos\frac{\phi}{a}\right)\leqq c$. For $r=0$ we must have $\phi=ua\,\pi/2$ where u is any odd integer. If v is the angle which the tangent to the

$c>a$

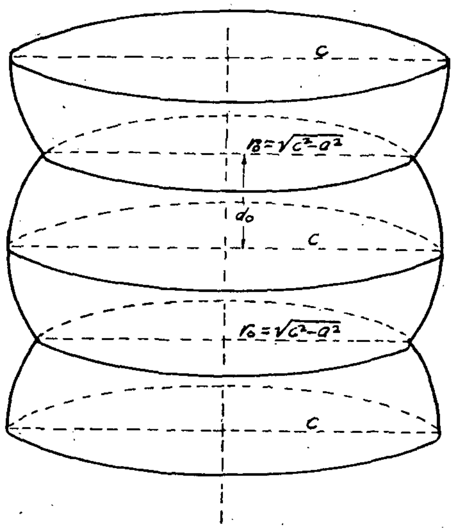


FIGURE 23.—The aposphere, zonal type.

meridian makes with the axis of rotation (the z-axis in fig. 18, p. 59), then $v+\pi/2$ is the angle which the tangent makes with the r-axis, hence $\tan(v+\pi/2)=\frac{dz}{dr}=\frac{dz}{d\phi}\Big/\frac{dr}{d\phi}=$ $-\sqrt{1-\frac{c^2}{a^2}\sin^2\frac{\phi}{a}}\Big/\frac{c}{a}\sin\frac{\phi}{a}$.

With the value of $\phi=ua\,\pi/2$, u an odd integer, for which $r=0$, we have $\sin\frac{\phi}{a}=1$, whence $\tan(v_0+\pi/2)=-\cot v_0=-\sqrt{1-\frac{c^2}{a^2}}\Big/\frac{c}{a}$, or $\sin v_0=\frac{c}{a}$, $v_0=\sin^{-1}\frac{c}{a}$, and $z_0=\int_0^{ua\frac{\pi}{2}}\sqrt{1-\frac{c^2}{a^2}\sin^2\frac{\phi}{a}}\,d\phi=d_0$. Thus it is seen that the surface is made up of a series of spindles as shown in figure 24.

The integral for z in equations (204) where $c\neq a$, may be expressed in terms of elliptic functions.

We will now show that the two surfaces, as given by equations (204) where $c\neq a$, are applicable to the sphere with the meridians and parallels of each in correspondence, that is, developable upon the sphere in the same manner as cones and cylinders upon the plane—small lengths are equal as well as corresponding angles. Or stated in another way, the ratio of their linear elements about any common point must be unity.

If we write the linear element (201) with $c=a$, replacing ϕ and λ by $\overline{\phi}$ and $\overline{\lambda}$ respectively we have $ds^2=d\overline{\phi}^2+a^2\cos^2\frac{\overline{\phi}}{a}\,d\overline{\lambda}^2$, and this is identical with the linear element (201) if we have $\overline{\phi}=\phi$, $\overline{\lambda}=\frac{c}{a}\lambda$ which establishes the applicability and the correspondence of parallels and meridians.

Let us consider a zone between the parallels ϕ_0 and ϕ_1 on the surface whose linear element is given by (201). A point of the zone is determined by values of ϕ and λ such that $\phi_1 \geqq \phi \geqq \phi_0$, $2\pi \geqq \lambda \geqq 0$. The parametric values of the corresponding point on the sphere are such that $\phi_1 \geqq \bar{\phi} \geqq \phi_0$, $2\pi \frac{c}{a} \geqq \bar{\lambda} \geqq 0$, since $\bar{\phi}=\phi$, $\bar{\lambda}=\frac{c}{a}\lambda$.

Hence when $c<a$, the given zone on the surface does not cover the zone on the sphere between the parallels ϕ_0 and ϕ_1. When $c>a$, it not only covers it, but overlaps it.

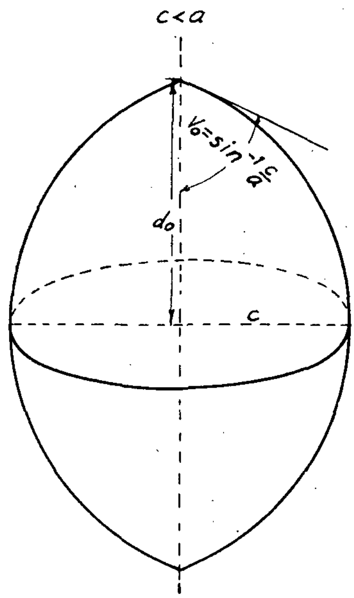


FIGURE 24.—The aposphere, spindle type.

We note that with $R_T=-1/a^2$, the differential equation (198) is integrable, the resulting family of surfaces being called pseudospherical surfaces. They are of three types, hyperbolic, elliptic, and parabolic and are of interest but not very useful for conformal mapping of the spheroid since they are not applicable to one another with meridians in correspondence.

The spherical surfaces represented by the linear element (201) and parametric representation (204) where $c \neq a$, have been employed in the conformal mapping of the spheroid by Brigadier M. Hotine. (See Orthomorphic Projection of the Spheroid, Brigadier M. Hotine, Empire Survey Review, Vols. VIII and IX, Nos. 62–65, 1946–1947.) Hotine obtains the equation of the surfaces in the form

$$p=A \operatorname{sech} B(\tau+C),$$

where p is the radius of the parallel and τ is the isometric latitude, B, C are arbitrary constants and $A=B/P$ where $P^2=1/RN=R_T$=real constant. To show that this equation and equation (200) are equivalent we have from (189) with $\epsilon=0$, $\tau=\ln \tan \left(\frac{\pi}{4}+\frac{\phi}{2}\right)$ which is the expression for the isometric latitude on the sphere. From this we have $e^\tau=\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)=\sqrt{\frac{1+\sin\phi}{1-\sin\phi}}$, or $\sin\phi=\frac{e^\tau-e^{-\tau}}{e^\tau+e^{-\tau}}=\tanh \tau$, whence $\cos^2\phi=1-\tanh^2 \tau=\operatorname{sech}^2 \tau$, or $\cos\phi=\operatorname{sech}\tau$. Hence we have $r=p=c\cos\left(\frac{\phi}{a}+d\right) \asymp A \operatorname{sech} B(\tau+C)$.

Hotine calls these surfaces where $B \neq 1$ ($B=1$ gives a sphere) apospheres, attributing the name to C. J. Sisson of London University. The spheroid is projected conformally upon the plane by the series of conformal projections, spheroid to aposphere, aposphere to sphere, sphere to plane.

The basic idea in connection with oblique projections of the spheroid is not new and such a projection may be found in Jordan-Eggert, Handbuch der Vermessungskunde, Vol. III, Second part, Chapter V. The development by Hotine with closed formulas involving hyperbolic functions and the aposphere which reproduces the surface of the spheroid to a high degree of accuracy over a considerable area, gives much simpler working formulas after certain functions involved have been tabulated.

The method is also useful for the horizon stereographic projection of the spheroid, and complete formulas for this and several other conformal projections through the aposphere are presented by Hotine in the work cited above.

MAP ELEMENTS

We have seen that an arc element of the spheroid may be expressed in the form $dS^2=r^2\,(d\tau^2+d\lambda^2)$, where $r=N\cos\phi$, $d\tau=\frac{R}{r}\,d\phi$ and it must be in this form if the spheroid is to be developed conformally on a plane. The map coordinates will in general be functions of ϕ and λ, that is, $x=x\,(\phi,\lambda)$, $y=y\,(\phi,\lambda)$. The map arc element will then be

$$ds^2=r^2\left(\frac{E}{R^2}\,d\tau^2+\frac{2F}{rR}\,d\tau d\lambda+\frac{G}{r^2}\,d\lambda^2\right)$$

where $E=\left(\frac{\partial x}{\partial\tau}\frac{d\tau}{d\phi}\right)^2+\left(\frac{\partial y}{\partial\tau}\frac{d\tau}{d\phi}\right)^2=\left(\frac{d\tau}{d\phi}\right)^2\left[\left(\frac{\partial x}{\partial\tau}\right)^2+\left(\frac{\partial y}{\partial\tau}\right)^2\right]=\frac{R^2}{r^2}\left[\left(\frac{\partial x}{\partial\tau}\right)^2+\left(\frac{\partial y}{\partial\tau}\right)^2\right],$

$$F=\frac{\partial x}{\partial\lambda}\frac{\partial x}{\partial\tau}\frac{d\tau}{d\phi}+\frac{\partial y}{\partial\lambda}\frac{\partial y}{\partial\tau}\frac{d\tau}{d\phi}=\left(\frac{d\tau}{d\phi}\right)\left(\frac{\partial x}{\partial\lambda}\frac{\partial x}{\partial\tau}+\frac{\partial y}{\partial\lambda}\frac{\partial y}{\partial\tau}\right)=\frac{R}{r}\left(\frac{\partial x}{\partial\lambda}\frac{\partial x}{\partial\tau}+\frac{\partial y}{\partial\lambda}\frac{\partial y}{\partial\tau}\right),\qquad(205)$$

$$G=\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2.$$

In order for the mapping to be conformal we must have from equation (64)

$$\frac{ds^2}{dS^2}=\frac{r^2\left(\frac{E}{R^2}\,d\tau^2+2\,\frac{F\,d\tau\,d\lambda}{rR}+\frac{G}{r^2}\,d\lambda^2\right)}{r^2(d\tau^2+d\lambda^2)}=k^2(\tau,\lambda),$$

and necessarily then

$$F=0,\ \frac{E}{R^2}=\frac{G}{r^2}.\qquad(206)$$

Note that from (206) with the values of E, F, G from (205) we have the conditions $\frac{\partial x}{\partial\lambda}\frac{\partial x}{\partial\tau}+\frac{\partial y}{\partial\lambda}\frac{\partial y}{\partial\tau}=0,\left(\frac{\partial x}{\partial\tau}\right)^2+\left(\frac{\partial y}{\partial\tau}\right)^2=\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2$, which are equivalent to

$$\frac{\partial x}{\partial\tau}=\mp\frac{\partial y}{\partial\lambda},\ \frac{\partial x}{\partial\lambda}=\pm\frac{\partial y}{\partial\tau}.\qquad(207)$$

These are again the Cauchy-Riemann equations (15) as was to be expected.

Now $dx=\frac{\partial x}{\partial\tau}\,d\tau+\frac{\partial x}{\partial\lambda}\,d\lambda$, $dy=\frac{\partial y}{\partial\tau}\,d\tau+\frac{\partial y}{\partial\lambda}\,d\lambda$. In figure 25, ds is the projection of

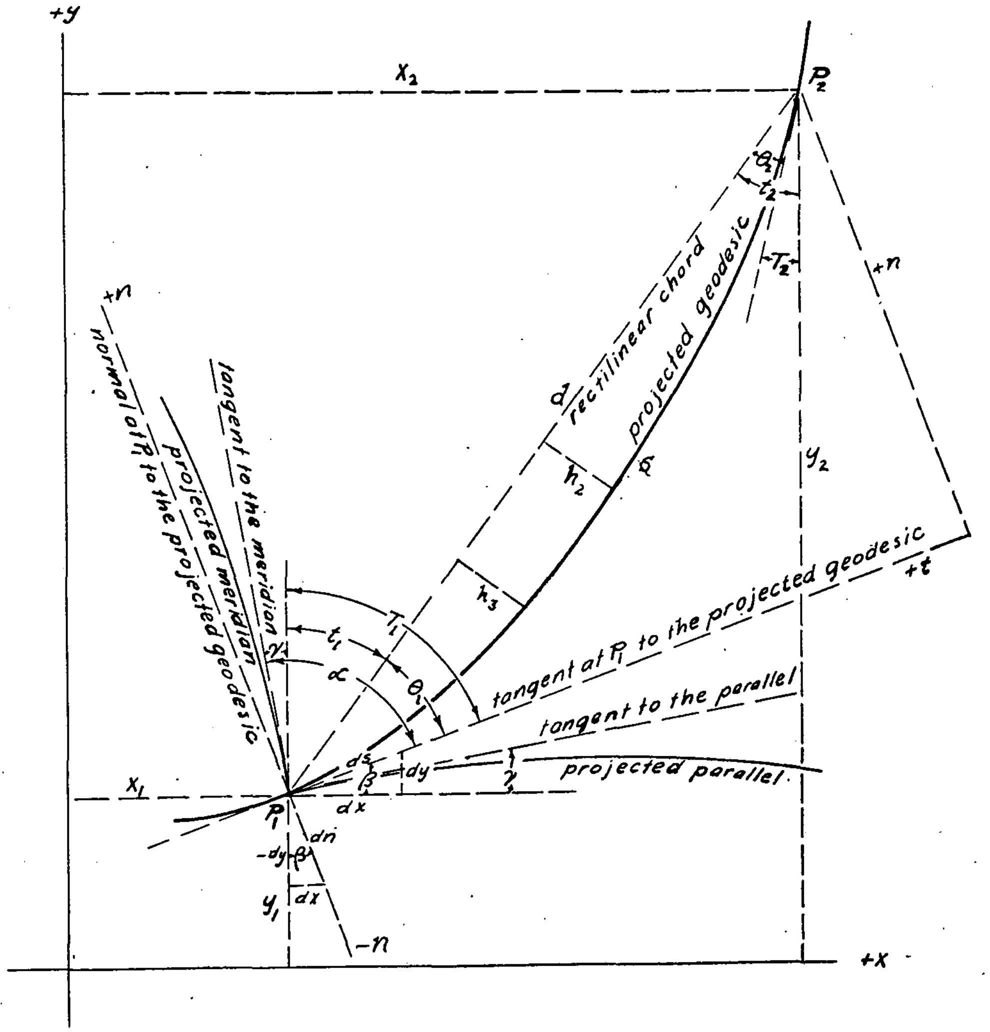


FIGURE 25.—Elements of the projected geodesic and the rectilinear chord.

the spheroidal arc length dS as shown in figure 21 (p. 64). If β is the angle which the projected arc ds makes with the map x-axis, then

$$\tan\beta=\frac{dy}{dx}=\frac{\frac{\partial y}{\partial\tau}\,d\tau+\frac{\partial y}{\partial\lambda}\,d\lambda}{\frac{\partial x}{\partial\tau}\,d\tau+\frac{\partial x}{\partial\lambda}\,d\lambda}=\frac{\frac{\partial y}{\partial\tau}+\frac{\partial y}{\partial\lambda}\frac{d\lambda}{d\tau}}{\frac{\partial x}{\partial\tau}+\frac{\partial x}{\partial\lambda}\frac{d\lambda}{d\tau}}. \tag{208}$$

From equation (174) $\tan\alpha=\frac{N\cos\phi}{R}\frac{d\lambda}{d\phi}$, or $\frac{d\lambda}{d\phi}=\frac{R}{r}\tan\alpha$.

Now $\frac{d\lambda}{d\tau}=\frac{d\lambda}{d\phi}\cdot\frac{d\phi}{d\tau}=\frac{R}{r}\tan\alpha\cdot\frac{r}{R}=\tan\alpha$. Hence (208) may be written

$$\tan\beta=\frac{dy}{dx}=\frac{\frac{\partial y}{\partial\tau}+\frac{\partial y}{\partial\lambda}\tan\alpha}{\frac{\partial x}{\partial\tau}+\frac{\partial x}{\partial\lambda}\tan\alpha}. \tag{209}$$

Solving (209) for tan α we have

$$\tan\alpha = -\frac{\dfrac{\partial y}{\partial\tau} - \dfrac{\partial x}{\partial\tau}\tan\beta}{\dfrac{\partial y}{\partial\lambda} - \dfrac{\partial x}{\partial\lambda}\tan\beta}. \tag{210}$$

Equations (209) and (210) give the relations between the azimuth, α, of a spheroidal arc element and the direction, β, of its projection on a conformal map.

CURVATURE OF PROJECTED MERIDIANS AND PARALLELS

From equation (153) writing $\frac{dy}{dx}$ for z' we have the usual formula for the curvature of a plane curve given by

$$\frac{1}{R} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}. \tag{211}$$

For the projected curve, the coordinates x, y of any point on it will be functions of a single parameter say s, the arc length along the curve.

Then

$$\frac{dy}{dx} = \frac{dy}{ds}\Big/\frac{dx}{ds} = y'/x' = f(s).$$

$$\frac{d^2y}{dx^2} = \frac{df}{ds}\cdot\frac{ds}{dx} = \frac{x'y''-y'x''}{x'^2}\cdot\frac{1}{x'} = \frac{x'y''-y'x''}{x'^3}.$$

With these values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ placed in (211) we have then

$$\frac{1}{R} = \frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}, \tag{212}$$

where x and y are functions of the same parameter and differentiation is with respect to that parameter.

Now when λ is constant, we may replace the derivatives in (212) by partial derivatives with respect to τ, that is, for a meridian we may write

$$\frac{1}{R_\lambda} = \frac{\dfrac{\partial x}{\partial\tau}\dfrac{\partial^2 y}{\partial\tau^2} - \dfrac{\partial y}{\partial\tau}\dfrac{\partial^2 x}{\partial\tau^2}}{\left[\left(\dfrac{\partial x}{\partial\tau}\right)^2+\left(\dfrac{\partial y}{\partial\tau}\right)^2\right]^{3/2}}. \tag{213}$$

From (205) and (206) it is seen that the denominator of (213) is $G^{3/2}$ so that

$$\frac{1}{R_\lambda} = \frac{1}{G^{3/2}}\left(\frac{\partial x}{\partial\tau}\frac{\partial^2 y}{\partial\tau^2} - \frac{\partial y}{\partial\tau}\frac{\partial^2 x}{\partial\tau^2}\right). \tag{214}$$

By differentiating the equations (207) we obtain $\frac{\partial^2 x}{\partial\tau^2} = \mp\frac{\partial^2 y}{\partial\lambda\partial\tau}$; $\frac{\partial^2 x}{\partial\tau\partial\lambda} = \mp\frac{\partial^2 y}{\partial\lambda^2}$;

$\frac{\partial^2 x}{\partial\lambda^2}=\pm\frac{\partial^2 y}{\partial\tau\partial\lambda};\ \frac{\partial^2 x}{\partial\lambda\partial\tau}=\pm\frac{\partial^2 y}{\partial\tau^2}$; whence

$$\frac{\partial^2 x}{\partial\tau^2}+\frac{\partial^2 x}{\partial\lambda^2}=0,\ \frac{\partial^2 y}{\partial\tau^2}+\frac{\partial^2 y}{\partial\lambda^2}=0. \tag{215}$$

Equations (215) are the well-known Laplace equations which analytic functions must satisfy.

With equations (207) and (215) we may write (214) as $\frac{1}{R_\lambda}=\frac{1}{G^{3/2}}\left(\frac{\partial y}{\partial\lambda}\frac{\partial^2 y}{\partial\lambda^2}+\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda^2}\right)$. By differentiating G in (205) we obtain $\frac{1}{2}\frac{\partial G}{\partial\lambda}=\frac{\partial y}{\partial\lambda}\frac{\partial^2 y}{\partial\lambda^2}+\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda^2}$, hence we have finally $\frac{1}{R_\lambda}=\frac{1}{2G^{3/2}}\frac{\partial G}{\partial\lambda}=-\frac{\partial G^{-\frac{1}{2}}}{\partial\lambda}$. Similarly for a parallel, equation (213) becomes

$$\frac{1}{R_\tau}=\frac{\frac{\partial x}{\partial\lambda}\frac{\partial^2 y}{\partial\lambda^2}-\frac{\partial y}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda^2}}{\left[\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2\right]^{3/2}}=\frac{1}{G^{3/2}}\left(\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda\partial\tau}+\frac{\partial y}{\partial\lambda}\frac{\partial^2 y}{\partial\lambda\partial\tau}\right)=\frac{1}{2G^{3/2}}\frac{\partial G}{\partial\tau}=-\frac{\partial G^{-\frac{1}{2}}}{\partial\tau}.$$

Hence the curvatures of the projected meridians and parallels in a conformal projection are given by

$$\frac{1}{R_\lambda}=-\frac{\partial G^{-\frac{1}{2}}}{\partial\lambda},\ \frac{1}{R_\tau}=-\frac{\partial G^{-\frac{1}{2}}}{\partial\tau}=-\frac{\partial G^{-\frac{1}{2}}}{\partial\phi}\cdot\frac{d\phi}{d\tau}, \tag{216}$$

where, from (187), $G=f'(\lambda+i\tau)f'(\lambda-i\tau)$, G being formed from the map coordinates $x=x(\phi,\lambda), y=y(\phi,\lambda)$ of the conformal projection according to (205).

CONVERGENCE OF MAP MERIDIANS

The convergence of the meridian through any point on the map is defined to be the angle between the tangent to the meridian at the point and the y-axis. Since the projection is conformal, this angle is also equal to that between the tangent to the parallel and the x-axis.

From (209), with $\alpha=0$, $\alpha=90°$, if we let γ be the convergence, we have

$$\tan\gamma=\frac{\partial y}{\partial\tau}\Big/\frac{\partial x}{\partial\tau}=\frac{\partial y}{\partial\lambda}\Big/\frac{\partial x}{\partial\lambda}=\left(\frac{dy}{dx}\right),\quad \tau=\text{constant}. \tag{217}$$

CURVATURE OF THE PROJECTED GEODESIC

From figure 25, $\frac{dx}{ds}=\cos\beta$, $\frac{dy}{ds}=\sin\beta$, $\frac{d^2x}{ds^2}=-\sin\beta\frac{d\beta}{ds}$, $\frac{d^2y}{ds^2}=\cos\beta\frac{d\beta}{ds}$. Hence from (212) we have

$$\frac{1}{R}=\left(\cos^2\beta\frac{d\beta}{ds}+\sin^2\beta\frac{d\beta}{ds}\right)=\frac{d\beta}{ds}. \tag{218}$$

Now from (206) we may write, considering k to be a function of x and y,

$$dS=\frac{1}{k}\,ds=\frac{1}{k}\sqrt{1+y'^2}\,dx=I\,dx, \text{ or}$$

$$S=\int_{x_1}^{x_2}I\,dx,\ I=\frac{1}{k}\sqrt{1+y'^2}=I(x,y,y'). \tag{219}$$

Equation (219) gives the arc length of the spheroidal arc S corresponding to the projected arc s of figure 25, but expressed as a function of the rectangular coordinates x, y and the slope y' of the projected curve.

If S is a geodesic arc of the spheroid, then the integral of (219) must be minimized. The integrand, I, must therefore satisfy the equation

$$\frac{\partial I}{\partial y}-\frac{d}{dx}\left(\frac{\partial I}{\partial y'}\right)=0. \tag{220}$$

The differential equation (220), known as Euler's equation, is obtained in considering the simplest case of the calculus of variations of which (219) is an example. The derivation of this equation may be found in treatises on advanced calculus.[5]

With $y'=dy/dx=\tan\beta$, $\frac{dx}{ds}=\cos\beta$, $\frac{dy}{ds}=\sin\beta$ and $I=\frac{1}{k}\sqrt{1+y'^2}$ we have

$$\frac{\partial I}{\partial y}=\sqrt{1+y'^2}\cdot\frac{\partial}{\partial y}\left(\frac{1}{k}\right)=\sec\beta\cdot\frac{\partial}{\partial y}\left(\frac{1}{k}\right), \tag{221}$$

$$\frac{\partial I}{\partial y'}=\frac{1}{k}\frac{y'}{\sqrt{1+y'^2}}=\frac{1}{k}\frac{\tan\beta}{\sec\beta}=\frac{\sin\beta}{k},$$

$$\frac{d}{dx}\left(\frac{\partial I}{\partial y'}\right)=\frac{d}{dx}\left(\frac{\sin\beta}{k}\right)=\sin\beta\left[\frac{\partial}{\partial x}\left(\frac{1}{k}\right)+\frac{\partial}{\partial y}\left(\frac{1}{k}\right)\cdot\frac{dy}{dx}\right]+\frac{1}{k}\cos\beta\cdot\frac{d\beta}{ds}\cdot\frac{ds}{dx}.$$

With the values of $\frac{dy}{dx}=\tan\beta$, $\frac{ds}{dx}=\sec\beta$ this last equation becomes

$$\frac{d}{dx}\left(\frac{\partial I}{\partial y'}\right)=\sin\beta\left[\frac{\partial}{\partial x}\left(\frac{1}{k}\right)+\frac{\partial}{\partial y}\left(\frac{1}{k}\right)\tan\beta\right]+\frac{1}{k}\cdot\frac{d\beta}{ds}. \tag{222}$$

The values of (221) and (222) placed in (220) give

$$-\frac{1}{k}\frac{d\beta}{ds}=\frac{\partial}{\partial x}\left(\frac{1}{k}\right)\cdot\sin\beta-\frac{\partial}{\partial y}\left(\frac{1}{k}\right)\cdot(\sec\beta-\sin\beta\tan\beta)=\frac{\partial}{\partial x}\left(\frac{1}{k}\right)\cdot\sin\beta-\frac{\partial}{\partial y}\left(\frac{1}{k}\right)\cdot\cos\beta=$$

$$-\frac{1}{k^2}\frac{\partial k}{\partial x}\cdot\sin\beta+\frac{1}{k^2}\frac{\partial k}{\partial y}\cos\beta, \text{ and from (218)}$$

$$\sigma=\frac{d\beta}{ds}=\frac{1}{k}\left(\frac{\partial k}{\partial x}\cdot\sin\beta-\frac{\partial k}{\partial y}\cdot\cos\beta\right). \tag{223}$$

Equation (223) is the expression for the curvature, σ, of the projected geodesic at a given point in terms of the scale factor at that point and the angle β.

If the projected geodesic is referred to the normal and tangent at a point, say at P_1, as shown in figure 25, then $\frac{dx}{dn}=\sin\beta$ and $-\frac{dy}{dn}=\cos\beta$ so that (223) becomes

$$\sigma=\frac{d\beta}{ds}=\frac{1}{k}\left(\frac{\partial k}{\partial x}\cdot\frac{dx}{dn}+\frac{\partial k}{\partial y}\cdot\frac{dy}{dn}\right)=\frac{1}{k}\frac{\partial k}{\partial n}, \tag{224}$$

where $\frac{\partial k}{\partial n}$ is the derivative of the scale factor in a direction normal to the curve.

[5] F. S. Woods, Advanced Calculus, p. 319.

PARAMETRIC EQUATIONS OF THE PROJECTED GEODESIC

In figure 25, consider the projected geodetic curve, s, to be referred to the normal, n, and tangent, t, as coordinate axes and suppose both n and t to be functions of the arc length s. That is, $n=n(s)$, $t=t(s)$. We may expand n and t in Maclaurin series about the point P_1, that is,

$$\begin{aligned} t&=st'(0)+\frac{s^2}{2!}t''(0)+\frac{s^3}{3!}t'''(0)+\frac{s^4}{4!}t^{\text{iv}}(0)+\frac{s^5}{5!}t^{\text{v}}(0)+\ldots \\ n&=sn'(0)+\frac{s^2}{2!}n''(0)+\frac{s^3}{3!}n'''(0)+\frac{s^4}{4!}n^{\text{iv}}(0)+\frac{s^5}{5!}n^{\text{v}}(0)+\ldots \end{aligned} \tag{225}$$

Now the differential of arc length is $ds^2=dn^2+dt^2$, and hence $\left(\frac{dn}{ds}\right)^2+\left(\frac{dt}{ds}\right)^2=n'^2+t'^2=1$. The slope of the tangent to the curve at any point is $\frac{dn}{ds}\Big/\frac{dt}{ds}=n'/t'$ and since the curve is tangent to the t-axis at the origin we must have $n'(0)/t'(0)=0$, which is true if $n'(0)=0$, $t'(0)\neq 0$. From $n'^2+t'^2=1$ we have with $n'(0)=0$ that $t'(0)=1$.

From (212) and figure 25, the radius of curvature is

$$\sigma(s)=\frac{1}{R_c}=\frac{t'n''-n't''}{(t'^2+n'^2)^{3/2}}=t'n''-n't''. \tag{226}$$

With $n'(0)=0$ and $t'(0)=1$ we have from (226) that $n''(0)=\sigma_0$.

If we differentiate $n'^2+t'^2=1$, $\sigma=t'n''-n't''$ successively with respect to s we obtain:

$$\begin{aligned} A:\ & n'n''+t't''=0 \\ & n'n'''+t't'''+n''^2+t''^2=0 \\ & n'n^{\text{iv}}+t't^{\text{iv}}+3(n''n'''+t''t''')=0 \\ & n'n^{\text{v}}+t't^{\text{v}}+4(n''n^{\text{iv}}+t''t^{\text{iv}})+3(n'''^2+t'''^2)=0 \\ B:\ & \sigma'=t'n'''-n't''' \\ & \sigma''=t''n'''-n''t'''+t'n^{\text{iv}}-n't^{\text{iv}} \\ & \sigma'''=t'n^{\text{v}}-n't^{\text{v}}-2(n''t^{\text{iv}}-t''n^{\text{iv}}). \end{aligned} \tag{227}$$

From groups A and B of equations (227) with $n'(0)=0$, $t'(0)=1$, $n''(0)=\sigma_0$ we have finally

$$\begin{aligned} & n'(0)=0,\ n''(0)=\sigma_0,\ n'''(0)=\sigma_0',\ n^{\text{iv}}(0)=\sigma_0''-\sigma_0^3,\ n^{\text{v}}(0)=\sigma_0'''-6\sigma_0^2\sigma_0' \\ & t'(0)=1,\ t''(0)=0,\ t'''(0)=-\sigma_0^2,\ t^{\text{iv}}(0)=-3\sigma_0\sigma_0',\ t^{\text{v}}(0)=-(4\sigma_0\sigma_0''+3\sigma_0'^2-\sigma_0^4). \end{aligned} \tag{228}$$

The values from (228) placed in (225) give

$$\begin{aligned} t&=s-\frac{s^3}{6}\sigma_0^2-\frac{s^4}{8}\sigma_0\sigma_0'-\frac{s^5}{120}(4_0\sigma_0\sigma''+3\sigma_0'^2-\sigma_0^4)-\ldots \\ n&=\frac{s^2}{2}\sigma_0+\frac{s^3}{6}\sigma_0'+\frac{s^4}{24}(\sigma_0''-\sigma_0^3)+\frac{s^5}{120}(\sigma_0'''-6\sigma_0^2\sigma_0')+\ldots \end{aligned} \tag{229}$$

Since the curvature, σ, corresponding to the point t, n on the curve is also a function of s as seen from (226), it may be expressed also as a Maclaurin series in s, namely

$$\sigma=\sigma_0+\sigma_0' s+\sigma_0''\frac{s^2}{2}+\sigma_0'''\frac{s^3}{6}+\dots \tag{230}$$

THE DIFFERENCE IN LENGTH OF THE PROJECTED GEODESIC AND ITS RECTILINEAR CHORD

From figure 25, d is the length of the chord of the projected geodesic s. If the curve is referred to the normal, n, and tangent, t, as shown in figure 25, then $d^2=n^2+t^2$ and from equations (229) by squaring respective members and retaining terms in s^6, we have

$$d^2=n^2+t^2=s^2-s^4\frac{\sigma_0^2}{12}-s^5\frac{\sigma_0\sigma_0'}{12}-\frac{s^6}{360}(9\sigma_0\sigma_0''+8\sigma_0'^2-\sigma_0^4)-\dots, \tag{231}$$

whence

$$d=\sqrt{n^2+t^2}=s\sqrt{1-\frac{s^2}{12}\left[\sigma_0^2+\sigma_0\sigma_0's+\frac{s^2}{30}(9\sigma_0\sigma_0''+8\sigma_0'^2-\sigma_0^4)+\dots\right]}.$$

Expanding the radical by the binomial formula and retaining terms in s^5 we have finally

$$d=s-s^3\frac{\sigma_0^2}{24}-s^4\frac{\sigma_0\sigma_0'}{24}-\frac{s^5}{5,760}(72\sigma_0\sigma_0''+64\sigma_0'^2-3\sigma_0^4)-\dots$$

or

$$s-d=s^3\frac{\sigma_0^2}{24}+s^4\frac{\sigma_0\sigma_0'}{24}+\frac{s^5}{5,760}(72\sigma_0\sigma_0''+64\sigma_0'^2-3\sigma_0^4)+\dots \tag{232}$$

From equation (230), if σ_2 is the curvature at the midpoint of the geodesic arc s, we have, replacing s by $s/2$, $\sigma_2=\sigma_0+\sigma_0'\frac{s}{2}+\sigma_0''\frac{s^2}{8}+\dots$, or $\sigma_0=\sigma_2-\sigma_0'\frac{s}{2}-\sigma_0''\frac{s^2}{8}-\dots$ and this value of σ_0 placed in (232) gives

$$s-d=\frac{s^3}{24}\sigma_2^2+\frac{s^5}{5,760}(12\sigma_2\sigma_0''+4\sigma_0'^2-3\sigma_2^4)+\dots \tag{233}$$

THE ANGLE BETWEEN THE PROJECTED GEODESIC AND THE RECTILINEAR CHORD

In figure 25, it is seen that the angle between the chord, d, and the projected geodesic, s, is $T_1-t_1=\theta_1$ and $\tan\theta_1=n/t$. With the values of n and t from (229), we have

$$\tan\theta_1=\left[\frac{s}{2}\sigma_0+\frac{s^2}{6}\sigma_0'+\frac{s^3}{24}(\sigma_0''-\sigma_0^3)+\frac{s^4}{120}(\sigma_0'''-6\sigma_0^2\sigma_0')+\dots\right]\cdot$$

$$\left[1+\frac{s^2}{6}\sigma_0^2+\frac{s^3}{8}\sigma_0\sigma_0'+\dots\right]$$

$$=\frac{s}{2}\sigma_0+\frac{s^2}{6}\sigma_0'+\frac{s^3}{24}(\sigma_0''+\sigma_0^3)+\frac{s^4}{720}(6\sigma_0'''+29\sigma_0^2\sigma_0')+\dots \tag{234}$$

From the series for $\tan^{-1} u = u - \frac{u^3}{3} + \ldots$, with $\theta_1 = \tan^{-1} u$, we have

$$\theta_1 = \tan\,\theta_1 - \frac{\tan^3\theta_1}{3} + \ldots \tag{235}$$

From (234) we have to terms in s^4, $\frac{\tan^3\theta_1}{3} = \frac{s^3}{24}\sigma_0^3 + \frac{s^4}{24}\sigma_0^2\sigma_0'$, and with this value and that of $\tan\,\theta_1$ from (234) placed in (235) we have to terms in s^4,

$$\theta_1 = T_1 - t_1 = \frac{s}{2}\sigma_0 + \frac{s^2}{6}\sigma_0' + \frac{s^3}{24}\sigma_0'' + \frac{s^4}{720}(6\sigma_0''' - \sigma_0^2\sigma_0') + \ldots \tag{236}$$

Let us replace s by $s/3$ in equation (230) and indicate by σ_3 the curvature one-third the distance along the arc s in figure 25. Then $\sigma_3 = \sigma_0 + \frac{s}{3}\sigma_0' + \frac{s^2}{18}\sigma_0'' + \frac{s^3}{162}\sigma_0''' + \ldots$, or $\sigma_0 = \sigma_3 - \frac{s}{3}\sigma_0' - \frac{s^2}{18}\sigma_0'' - \frac{s^3}{162}\sigma_0''' - \ldots$, and this value of σ_0 placed in (236) gives to terms in s^4,

$$\theta_1 = T_1 - t_1 = \frac{s}{2}\sigma_3 - \frac{s^2}{6}\sigma_0' - \frac{s^3}{36}\sigma_0'' - \frac{s^4}{324}\sigma_0''' + \frac{s^2}{6}\sigma_0' + \frac{s^3}{24}\sigma_0'' + \frac{s^4}{720}\cdot(6\sigma_0''' - \sigma_3^2\sigma_0')$$

$$= \frac{s}{2}\sigma_3 + \frac{s^3}{72}\sigma_0'' + \frac{s^4}{6{,}480}(34\sigma_0''' - 9\sigma_3^2\sigma_0') + \ldots \tag{237}$$

THE DISPLACEMENT OF THE PROJECTED GEODESIC FROM THE RECTILINEAR CHORD

In figure 26 we have taken a portion s_1 of the projected geodesic s of figure 25 and drawn the perpendicular, h, from the point $Q(t, n)$ of the geodesic upon the rectilinear chord. h is thus the displacement of the projected geodesic from the rectilinear chord

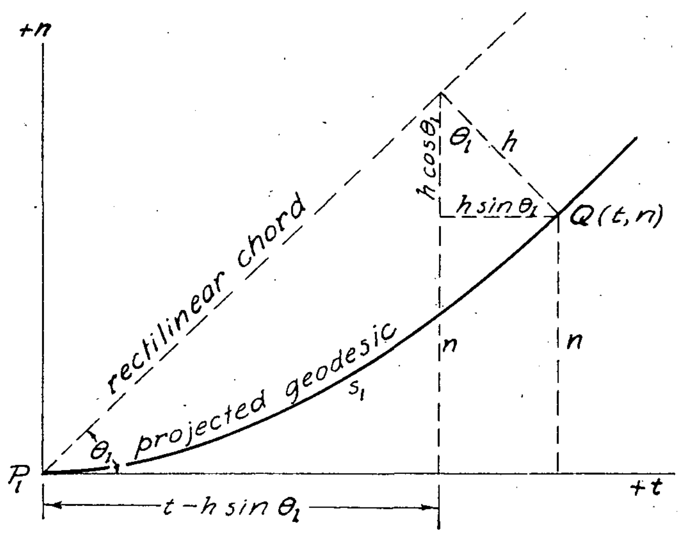


FIGURE 26.—The displacement of the projected geodesic from the rectilinear chord.

and we have from figure 26, $\frac{n+h\cos\theta_1}{t-h\sin\theta_1}=\tan\theta_1$. Solving for h we find that

$$h=t\sin\theta_1-n\cos\theta_1. \tag{238}$$

Now the series expansions for $\sin\theta_1$ and $\cos\theta_1$ are $\sin\theta_1=\theta_1-\frac{\theta_1^3}{3!}+\ldots$, $\cos\theta_1=1-\frac{\theta_1^2}{2!}+\frac{\theta_1^4}{4!}-\ldots$, and with the value of θ_1, from (236) these become

$$\begin{aligned}\sin\theta_1&=\frac{s}{2}\sigma_0+\frac{s^2}{6}\sigma_0'+\frac{s^3}{48}(2\sigma_0''-\sigma_0^3)+\ldots\\ \cos\theta_1&=1-\frac{s^2}{8}\sigma_0^2-\frac{s^3}{12}\sigma_0\sigma_0'-\ldots\end{aligned} \tag{239}$$

From equations (229) we have with s replaced by s_1

$$\begin{aligned}t&=s_1-\frac{s_1^3}{6}\sigma_0^2-\ldots\\ n&=\frac{s_1^2}{2}\sigma_0+\frac{s_1^3}{6}\sigma_0'+\frac{s_1^4}{24}(\sigma_0''-\sigma_0^3)+\ldots\end{aligned} \tag{240}$$

With the values from (239) and (240) placed in (238) we have to terms of 4th order in s_1 and s,

$$h=\frac{s_1}{2}\sigma_0(s-s_1)+\frac{s_1}{6}\sigma_0'(s^2-s_1^2)+\frac{s_1}{24}\sigma_0''(s^3-s_1^3)+\frac{\sigma_0^3}{48}\cdot(3s_1^2s^2+2s_1^4-s_1s^3-4s_1^3s)+\ldots \tag{241}$$

For the middle point of the arc s as shown in figure 25, we have $s_1=\frac{s}{2}$ and with this value (241) gives

$$h_2=\frac{s^2}{8}\sigma_0+\frac{s^3}{16}\sigma_0'+\frac{s^4}{384}(7\sigma_0''-\sigma_0^3)\ldots \tag{242}$$

We found in obtaining (233) that the curvature at the middle point of the arc s is $\sigma_2=\sigma_0+\sigma_0'\frac{s}{2}+\sigma_0''\frac{s^2}{8}+\ldots$ or $\sigma_0=\sigma_2-\sigma_0'\frac{s}{2}-\sigma_0''\frac{s^2}{8}-\ldots$ and this value of σ_0 placed in (242) gives to terms in s^4

$$h_2=\frac{s^2}{8}\sigma_2+\frac{s^4}{384}(\sigma_0''-\sigma_2^3)+\ldots \tag{243}$$

For a point one-third the distance from P_1 to P_2 as shown in figure 25 we have from (241) with $s_1=\frac{s}{3}$

$$h_3=\frac{s^2}{9}\sigma_0+\frac{4s^3}{81}\sigma_0'+\frac{s^4}{1{,}944}(26\sigma_0''-5\sigma_0^3)+\ldots \tag{244}$$

In obtaining (237) we found the curvature for a point one-third the distance from P_1 to P_2 to be $\sigma_3=\sigma_0+\frac{s}{3}\sigma_0'+\frac{s^2}{18}\sigma_0''+\ldots$ or $\sigma_0=\sigma_3-\frac{s}{3}\sigma_0'-\frac{s^2}{18}\sigma_0''-\ldots$, and this value of σ_0 placed in (244) gives to terms in s^4

$$h_3=\frac{s^2}{9}\sigma_3+\frac{s^3}{81}\sigma_0'+\frac{s^4}{1{,}944}(14\sigma_0''-5\sigma_3^3)+\ldots \tag{245}$$

THE DIFFERENCE IN LENGTH OF THE GEODESIC ARC AND ITS PROJECTION ON THE MAP

From (206) we have $ds=kdS$, or $s=\int_0^S kdS$ where S is the arc length on the spheroid, s is the projected arc, and k is the scale ratio.

Considering k to be a function of S we may expand $k(S)$ in a series, namely,

$$k(S)=k_0+\left(\frac{\partial k}{\partial S}\right)_0 S+\frac{1}{2}\left(\frac{\partial^2 k}{\partial S^2}\right)_0 S^2+\frac{1}{6}\left(\frac{\partial^3 k}{\partial S^3}\right)_0 S^3+\ldots$$

Now k may be expressed as a function of the map coordinates and the map coordinates may be expressed as functions of the arc s of the projected curve.

Hence $\frac{\partial k}{\partial S}=\frac{dk}{ds}\cdot\frac{ds}{dS}=k'k$. Similarly we have

$$\frac{\partial^2 k}{\partial S^2}=\frac{d^2k}{ds^2}\cdot\left(\frac{ds}{dS}\right)^2+\frac{dk}{ds}\cdot\frac{d}{ds}\left(\frac{ds}{dS}\right)\cdot\frac{ds}{dS}. \quad \text{But } \frac{dk}{ds}=\frac{d}{ds}\left(\frac{ds}{dS}\right)=k'$$

and hence

$$\frac{\partial^2 k}{\partial S^2}=k''k^2+k'^2k.$$

Continuing we find $\frac{\partial^3 k}{\partial S^3}=k^3k'''+4k^2k'k''+kk'^3$, so that we may write

$$s=\int_0^S kdS=\int_0^S\Big[k_0+k_0k_0'S+\frac{1}{2}(k_0^2k_0''+k_0k_0'^2)S^2+$$
$$\frac{1}{6}(k_0^3k_0'''+4k_0^2k_0'k_0''+k_0k_0'^3)S^3+\ldots\Big]dS.$$

Integrating we have finally

$$s=k_0S+\frac{1}{2}k_0k_0'S^2+\frac{1}{6}(k_0^2k_0''+k_0k_0'^2)S^3+$$
$$\frac{1}{24}(k_0^3k_0'''+4k_0^2k_0'k_0''+k_0k_0'^3)S^4+\ldots \tag{246}$$

where k_0', k_0'', k_0''' are the derivatives of the scale ratio, k, with respect to s and evaluated at $s=0$.

If we neglect terms greater than S^2 in (246) and write k_3k_3' for k_0k_0' which corresponds to the point one-third the distance along the projected curve, we obtain

$$s=k_0S+\frac{1}{2}k_3k_3'S^2+\ldots \tag{247}$$

We may also write from (206), $dS=\frac{1}{k}ds$, or $S=\int_0^s\frac{1}{k}ds$. Since k is a function of s we may write $\frac{1}{k}=\frac{1}{k_0}+\left(\frac{1}{k}\right)_0' s+\left(\frac{1}{k}\right)_0''\frac{s^2}{2}+\left(\frac{1}{k}\right)_0'''\frac{s^3}{6}+\ldots$,

whence

$$S=\int_0^s\left[\frac{1}{k_0}+\left(\frac{1}{k}\right)_0' s+\left(\frac{1}{k}\right)_0''\frac{s^2}{2}+\left(\frac{1}{k}\right)_0'''\frac{s^3}{6}+\ldots\right]ds,$$

and integrating we have

$$S=\frac{1}{k_0}s+\left(\frac{1}{k}\right)_0'\frac{s^2}{2}+\left(\frac{1}{k}\right)_0''\frac{s^3}{6}+\left(\frac{1}{k}\right)_0'''\frac{s^4}{24}+\ldots,$$

or

$$S-s=\left(\frac{1}{k_0}-1\right)s+\left(\frac{1}{k}\right)_0'\frac{s^2}{2}+\left(\frac{1}{k}\right)_0''\frac{s^3}{6}+\left(\frac{1}{k}\right)_0'''\frac{s^4}{24}+\ldots \qquad (248)$$

If we refer to the midpoint of the projected curve we may write

$$S=\int_{-\frac{s}{2}}^{+\frac{s}{2}}\left[\frac{1}{k_2}+\left(\frac{1}{k}\right)_2's+\left(\frac{1}{k}\right)_2''\frac{s^2}{2}+\left(\frac{1}{k}\right)_2'''\frac{s^3}{6}+\ldots\right]ds,$$

whence

$$S=\frac{1}{k_2}s+\left(\frac{1}{k}\right)_2''\frac{s^3}{24}+\ldots,\text{ or } S-s=\left(\frac{1}{k_2}-1\right)s+\left(\frac{1}{k}\right)_2''\frac{s^3}{24}+\ldots, \qquad (249)$$

where $\frac{1}{k_2}$, $\left(\frac{1}{k}\right)_2''$ are evaluated at the midpoint of the projected curve.

Since $k=k\,[x(s),\,y(s)]$, then $k'=\frac{dk}{ds}=\frac{\partial k}{\partial x}\frac{dx}{ds}+\frac{\partial k}{\partial y}\frac{dy}{ds}$. But we have seen (fig. 25, p. 75) that $\frac{dx}{ds}=\cos\beta$, $\frac{dy}{ds}=\sin\beta$, hence $k'=\frac{\partial k}{\partial x}\cos\beta+\frac{\partial k}{\partial y}\sin\beta$. Differentiating this last equation we have $k''=\frac{\partial^2 k}{\partial x^2}\cos\beta\cdot\frac{dx}{ds}+\frac{\partial^2 k}{\partial x\partial y}\cos\beta\cdot\frac{dy}{ds}-\frac{\partial k}{\partial x}\sin\beta\cdot\frac{d\beta}{ds}+\frac{\partial^2 k}{\partial y^2}\sin\beta\cdot\frac{dy}{ds}+\frac{\partial^2 k}{\partial y\partial x}\sin\beta\cdot\frac{dx}{ds}+\frac{\partial k}{\partial y}\cos\beta\cdot\frac{d\beta}{ds}$. With $\frac{dx}{ds}=\cos\beta$, $\frac{dy}{ds}=\sin\beta$ and the value of $\frac{d\beta}{ds}$ from (223) we have $k''=\frac{\partial^2 k}{\partial x^2}\cos^2\beta+\frac{\partial^2 k}{\partial y^2}\sin^2\beta+\frac{\partial^2 k}{\partial x\partial y}\sin 2\beta-k\sigma^2$, where σ is the curvature of the projected geodesic as given by equation (223).

$$\left(\frac{1}{k}\right)'=-\frac{k'}{k^2},\ \left(\frac{1}{k}\right)''=-\frac{k^2k''-2k\,k'^2}{k^4}=\frac{2k'^2-k\,k''}{k^3}, \qquad (250)$$

$$k'=\frac{\partial k}{\partial x}\cos\beta+\frac{\partial k}{\partial y}\sin\beta,$$

$$k''=\frac{\partial^2 k}{\partial x^2}\cos^2\beta+\frac{\partial^2 k}{\partial y^2}\sin^2\beta+\frac{\partial^2 k}{\partial x\partial y}\sin 2\beta-k\sigma^2.$$

In equation (249), k_2, k_2', etc. are evaluated at the midpoint of the projected geodesic. If we desire these to be evaluated for the midpoint of the rectilinear chord a correction term, ΔS, must be applied to equation (249). To derive this correction term we assume that h_2 as shown in figure 25 (p. 75) is coincident with the normal to the projected geodesic at its midpoint which introduces no appreciable error. Then

$\Delta S=d\left(\frac{1}{k_2}s\right)=-\frac{1}{k_2^2}\frac{\partial k_2}{\partial n}dn\cdot s$. From (224) and (243) we have $\sigma_2=\frac{1}{k_2}\frac{\partial k_2}{\partial n}$, $h_2=dn=\frac{s^2}{8}\sigma_2+\ldots$, hence $\Delta S=d\left(\frac{1}{k_2}s\right)=-\frac{1}{k_2^2}\frac{\partial k_2}{\partial n}\cdot dn\cdot s=-\frac{1}{8k_2}\sigma_2^2s^3$. Applying this correction to (249) we have

$$S-s=\left(\frac{1}{k_2}-1\right)s+\left(\frac{1}{k}\right)_2''\frac{s^3}{24}-\frac{1}{8k_2}\cdot\sigma_2^2\cdot s^3. \qquad (251)$$

where $k_2, k_2'', \ldots$ refer to the midpoint of the rectilinear chord.

THE MERCATOR CONFORMAL PROJECTION

Mercator probably arrived at his parallel spacings about 1550 by empirical methods in attempting to reduce the rhumb line on the globe to a straight line on the map, Edward Wright giving the correct mathematical formulation about 40 years later.

The historical account is perhaps best summarized in the following quotation from "A Short Dissertation on Maps and Charts" by M. Mountaine which was published in the Philosophical Transactions of the Royal Society in 1758.

"Rectilinear were therefore very early adopted, on which the meridians were described parallel to each other, and the degrees of latitude and longitude everywhere equal; the rhumbs were consequently right lines; and hereby it was thought that the courses or bearings of places would be more easily determined. But these were found also insufficient and erroneous, the meridians being parallel, which ought to converge: and no method or device used to accommodate that parallelism.—However, the errors in this were sooner discovered than corrected, both by mathematicians and mariners, as by Martin Cortese, Petrus Nonius, Coigniet, and some say by Ptolemy himself.

"The first step towards the improvement of this chart was made by Gerard Mercator, who published a map about the year 1550, in which the degrees of latitude were increased from the equator towards each pole; but on what principles this was constructed, he did not show.

"About the year 1590, Mr. Edward Wright discovered the true principles on which such a chart should be constructed; and communicated the same to one Jodocus Hondius, an engraver, who, contrary to his engagement, published the same as his own invention: this occasioned Mr. Wright, in 1599, to show his method of construction in his book, entitled, Correction of Errors in Navigation; in the preface of which may be seen his charge and proof against Hondius; and also how far Mercator has any right to share in the honour due for this great improvement in geography and navigation."

For a more detailed modern historical account of this projection see the Story of Maps by Lloyd A. Brown and Elements of Map Projection by C. H. Deetz and O. S. Adams (Coast and Geodetic Survey Special Publication No. 68). The latter treatise includes a complete development with tables. Other sources are indicated in the bibliography. We will include here the derivation of the mapping equations and the scale by means of the function of a complex variable as an easy introduction to the application of equations (189), and also to show that all autogonal projections of the spheroid on the plane are actually given by a function of the coordinates of the Mercator conformal projection.

DERIVATION OF FORMULAS

For the Mercator projection it is required that the scale shall be true along the Equator. Hence for $\phi=0$, we will have $\tau=0$, $y=0$, $x=a\lambda$. From equations (189) we may write, then, the linear analytic function $x+iy=a(\lambda+i\tau)$, whence equating real and imaginary parts

$$
\begin{aligned}
&x=a\lambda \\
&y=a\tau=a\int_0^{\phi}\frac{R}{N}\sec\phi\,d\phi=\frac{a}{M}\log\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-\epsilon\sin\phi}{1+\epsilon\sin\phi}\right)^{\epsilon/2}\right],
\end{aligned}
\tag{252}
$$

where M is the modulus of common logarithms.

From (44) let us write the linear element of the sphere as

$$ds_1^2=r^2\cos^2\chi(\sec^2\chi d\chi^2+d\lambda^2). \tag{253}$$

From (165) we have the linear element of the ellipsoid

$$ds_2^2=N^2\cos^2\phi\left(\frac{R^2}{N^2}\sec^2\phi\, d\phi^2+d\lambda^2\right). \tag{254}$$

In order for the ellipsoid to be mapped conformally upon the sphere the condition (66) must be satisfied, whence we must have

$$d\tau^2=\sec^2\chi d\chi^2=\frac{R^2}{N^2}\sec^2\phi\, d\phi^2. \tag{255}$$

But by (66) the first equality of (255) is the condition that the sphere be mapped conformally upon the plane. The second equality is the condition that the spheroid should be mapped conformally upon the sphere. From the second equality of (255) we have

$$\sec\chi d\chi=\frac{R}{N}\sec\phi\, d\phi,$$

$$\text{or } \ln\tan\left(\frac{\pi}{4}+\frac{\chi}{2}\right)=\ln\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-\epsilon\sin\phi}{1+\epsilon\sin\phi}\right)^{\epsilon/2}\right],$$

$$\text{or } \tan\left(\frac{\pi}{4}+\frac{\chi}{2}\right)=\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-\epsilon\sin\phi}{1+\epsilon\sin\phi}\right)^{\epsilon/2}. \tag{256}$$

CONFORMAL AND ISOMETRIC LATITUDES

The latitude χ, as determined from the geodetic latitude ϕ by (256), is called the conformal latitude.

The function $\tau=\ln\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-\epsilon\sin\phi}{1+\epsilon\sin\phi}\right)^{\epsilon/2}\right]$, which with the longitude, λ, determines a pair of isometric parameters on the spheroid is properly named the isometric latitude.

If a direct conformal projection of the spheroid is derived by substitution in the formulas for the projection of the sphere, the isometric latitude on the sphere is replaced by τ, or the geodetic latitude on the sphere is replaced by the conformal latitude, χ, as obtained from equation (256).

It should be noted that O. S. Adams in his special publications for the Coast and Geodetic Survey uses the designations conformal latitude and isometric latitude interchangeably for the quantity χ. However, the term isometric latitude is more appropriate to the parameter τ. No harm is done in practice as long as one knows that his tabular values of χ are conformal latitudes and not isometric latitudes.

THE CONFORMAL SPHERE

The sphere whose linear element is given by equation (253) is called the conformal sphere. From (253), (254), and (255) we have $k_1=\frac{ds_1}{ds_2}=\frac{r\cos\chi}{N\cos\phi}$, the scale factor for the conformal representation of the spheroid on the sphere. If we demand that the scale be true at the Equator, then $\phi=\chi=0$, $N=a$, and $k_1=1$, whence the radius of the conformal sphere is $r=a$. That is, for projections centered on the Equator this is the best value for the radius of the conformal sphere. If, for projections centered in latitude ϕ_0, we desire to hold the scale at this latitude, we have $k_1=1$, hence

$$\frac{r\cos\chi_0}{N_0\cos\phi_0}=1, \text{ whence } r=\frac{N_0\cos\phi_0}{\cos\chi_0}. \tag{257}$$

By substituting from (256) in (252) we obtain

$$\begin{aligned} x&=a\lambda \\ y&=\frac{a}{M}\log\tan\left(\frac{\pi}{4}+\frac{\chi}{2}\right), \end{aligned} \tag{258}$$

which represents the Mercator projection of the conformal sphere upon the plane, but actually accomplishing the projection of the ellipsoid upon the plane, since the conformal latitudes are computed from the geodetic latitudes by means of (256).

It is customary to replace χ and ϕ by their colatitudes z and p respectively, i. e. $\chi=\frac{\pi}{2}-z, \phi=\frac{\pi}{2}-p$. The relation (256) becomes then $\cot\frac{z}{2}=\cot\frac{p}{2}\cdot\left(\frac{1-\epsilon\cos p}{1+\epsilon\cos p}\right)^{\epsilon/2}$, or reciprocally $\tan\frac{z}{2}=\tan\frac{p}{2}\cdot\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\epsilon/2}$ and equations (258) become

$$x=a\lambda, y=\frac{a}{M}\log\cot\frac{z}{2}. \tag{259}$$

From (190) the magnification or scale at any point is

$$k=\frac{a}{N}\sec\phi. \tag{260}$$

Comparing equations (189) and (252) it is seen that all other conformal projections of the spheroid upon the plane are given by a function of the mapping coordinates (252) for the Mercator conformal projection. This is due to the linear function for the Mercator projection, i. e., from (189), $x+iy=f(\lambda+i\tau)$.

But for the Mercator projection, equations (252), we have $\lambda=x/a$, $\tau=y/a$ hence any other orthomorphic projection is given by

$$X+iY=f\left(\frac{x}{a}+i\frac{y}{a}\right). \tag{261}$$

This obviously may be generalized still further, that is, through the Mercator autogonal projection any conformal projection can be expressed in terms of any other.

We may put equations (259) in form for computing as follows:

The radius a is usually expressed in units of minutes on the Equator, $a=\frac{60\times 180}{\pi}=$ 3,437.′7467708. $M=0.4342944819$, hence $\frac{a}{M}=7{,}915.'704468$. With λ expressed in radians we have then $x=\frac{10{,}800}{\pi}\lambda$ (rad.), $y=7{,}915.'704468\log\cot\frac{z}{2}$. If λ is expressed in minutes of arc we have

$$x=\lambda', y=7{,}915.'704468\cdot\log\cot\frac{z}{2}. \tag{262}$$

Now the conformal latitudes for (262), which were computed for the Clarke spheroid of 1866, are given for every half degree of geodetic latitude in U. S. Coast and Geodetic Survey Special Publication No. 67. They have been tabulated more extensively for several spheroids by the War Department, Corps of Engineers, U. S. Lake Survey, Military Grid Unit. (See the bibliography.)

The series expansion for y in terms of ϕ can be obtained by writing, from (252),

$$y=\frac{a}{M}\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)-a\frac{\epsilon}{2}\ln\left(\frac{1+\epsilon\sin\phi}{1-\epsilon\sin\phi}\right). \tag{263}$$

Since

$$\frac{\epsilon}{2}\ln\left(\frac{1+\epsilon\sin\phi}{1-\epsilon\sin\phi}\right)=\epsilon^2\sin\phi+\frac{\epsilon^4\sin^3\phi}{3}+\frac{\epsilon^6\sin^5\phi}{5}+\frac{\epsilon^8\sin^7\phi}{7}+\ldots,$$

we may write (263) as

$$y=\frac{a}{M}\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)-a\left(\epsilon^2\sin\phi+\frac{\epsilon^4\sin^3\phi}{3}+\frac{\epsilon^6\sin^5\phi}{5}+\frac{\epsilon^8\sin^7\phi}{7}+\ldots\right),$$

or placing the values of $\frac{a}{M}$ and a in this last equation

$$y=7{,}915'.704468\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)-$$
$$3{,}437'.747\left(\epsilon^2\sin\phi+\frac{\epsilon^4\sin^3\phi}{3}+\frac{\epsilon^6\sin^5\phi}{5}+\frac{\epsilon^8\sin^7\phi}{7}+\ldots\right). \tag{264}$$

By use of the identities $\frac{1}{3}\sin^3\phi=\frac{1}{4}\sin\phi-\frac{1}{12}\sin 3\phi$, $\frac{1}{5}\sin^5\phi=\frac{1}{8}\sin\phi-\frac{1}{16}\sin 3\phi+\frac{1}{80}\sin 5\phi$, $\frac{1}{7}\sin^7\phi=\frac{5}{64}\sin\phi-\frac{3}{64}\sin 3\phi+\frac{1}{64}\sin 5\phi-\frac{1}{448}\sin 7\phi$, we may write (264) in the form

$$y=7{,}915'.704468\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)-$$
$$3{,}437'.747\left[\left(\epsilon^2+\frac{\epsilon^4}{4}+\frac{\epsilon^6}{8}+\frac{5\epsilon^8}{64}+\ldots\right)\sin\phi-\left(\frac{\epsilon^4}{12}+\frac{\epsilon^6}{16}+\frac{3\epsilon^8}{64}+\ldots\right)\sin 3\phi+\right.$$
$$\left.\left(\frac{\epsilon^6}{80}+\frac{\epsilon^8}{64}+\ldots\right)\sin 5\phi-\left(\frac{\epsilon^8}{448}+\ldots\right)\sin 7\phi\right]. \tag{265}$$

The flattening or compression of the spheroid is defined in terms of the semiaxes of the meridian ellipse by the equation $f=\frac{a-b}{a}=1-\frac{b}{a}$ and since $\frac{b}{a}=\sqrt{1-\epsilon^2}$ we have

$$f=1-\sqrt{1-\epsilon^2}=\frac{\epsilon^2}{2}+\frac{\epsilon^4}{8}+\frac{\epsilon^6}{16}+\frac{5}{128}\epsilon^8+\ldots, \tag{266}$$

where the radical has been expanded by the binomial formula.

From (266) we have $(1-\epsilon^2)=(1-f)^2$, or

$$\epsilon^2=2f-f^2. \tag{267}$$

Hence if the spheroid to be used is defined by its flattening we may use (267) to compute ϵ^2 and successive powers.

We may express (265) directly in terms of the flattening and eccentricity as follows: From (266) we have

$$2f=\epsilon^2+\frac{\epsilon^4}{4}+\frac{\epsilon^6}{8}+\frac{5\,\epsilon^8}{64}+\ldots. \tag{268}$$

From (268), cubing both sides, we have, retaining terms in ϵ^{10},

$$8f^3=\epsilon^6+\frac{3}{4}\,\epsilon^8+\frac{9}{16}\,\epsilon^{10}+\ldots,$$

or

$$\frac{2f^3}{3\,\epsilon^2}=\frac{\epsilon^4}{12}+\frac{\epsilon^6}{16}+\frac{3}{64}\,\epsilon^8+\ldots. \tag{269}$$

Again from (268), raising both sides to the fifth power, we have, retaining terms in ϵ^{12},

$$32f^5=\epsilon^{10}+\frac{5}{4}\,\epsilon^{12}+\ldots$$

or

$$\frac{2f^5}{5\,\epsilon^4}=\frac{\epsilon^6}{80}+\frac{\epsilon^8}{64}+\ldots. \tag{270}$$

From (268), raising both sides to the seventh power and retaining terms in ϵ^{14}, we have

$$128f^7=\epsilon^{14}+\ldots$$

or

$$\frac{2f^7}{7\,\epsilon^6}=\frac{\epsilon^8}{448}+\ldots, \tag{271}$$

With the values of the coefficients from (268), (269), (270), (271) placed in (265) we have finally

$$y=7{,}915.'704468\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)-3{,}437.'747\left(2f\sin\phi-\frac{2f^3}{3\,\epsilon^2}\sin 3\phi+\right.$$
$$\left.\frac{2f^5}{5\,\epsilon^4}\sin 5\phi-\frac{2f^7}{7\,\epsilon^6}\sin 7\phi+\ldots\right), \tag{272}$$

or since $\epsilon^2=f(2-f)$, $\epsilon^4=f^2(2-f)^2$, $\epsilon^6=f^3(2-f)^3$ this last equation may be written

$$y=7{,}915.'704468\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)-$$
$$6{,}875.'494\left(f\sin\phi-\frac{f^2}{3(2-f)}\sin 3\phi+\frac{f^3}{5(2-f)^2}\sin 5\phi-\frac{f^4}{7(2-f)^3}\sin 7\phi+\ldots\right).$$

From equations (175),(176), and (252) we have $I=\tau=\int_0^\phi\frac{R}{N}\sec\phi\,d\phi$, $\lambda=\tau\tan\alpha+\lambda_0$, where $x=a\lambda=a\tau\tan\alpha+a\lambda_0$, $y=a\tau$. Eliminating τ between these last two equations gives $x=y\tan\alpha+a\lambda_0$, which is clearly the equation of a straight line and the loxodrome on the Mercator projection.

With the value of λ from (173) placed in (252) we have the equations of the geodesic on the Mercator projection, namely

$$x=\pm ac\,\frac{(1-\epsilon^2)}{(a^2-\epsilon^2c^2)^{1/2}}\,\Pi(-k^2,\epsilon k,\theta)+a\lambda_0,\; y=a\tau=a\int_{\phi_0}^{\phi}\frac{R}{N}\sec\phi\,d\phi, \tag{273}$$

where $\theta=\sin^{-1}\frac{\sin\phi}{k}$, $\frac{1}{k^2}=\frac{a^2-c^2\epsilon^2}{a^2-c^2}$, $c=a\sin\alpha_0$, α_0 being the angle at which the geodesic crosses the Equator. x may be evaluated from tables of elliptic integrals or from a series expansion. y may be computed from the formulas (262), (265), (272), or obtained from tables of meridional parts for the Mercator projection.

THE TRANSVERSE MERCATOR PROJECTION

This projection, which has become of great importance in modern cartography and geodesy, was invented by Johann Heinrich Lambert, to whom modern cartography is also eternally indebted for his conformal conic projection. It seems proper that we should include here an account of the life and accomplishments of one who has contributed so much to modern cartography.

BIOGRAPHICAL SKETCH OF JOHANN HEINRICH LAMBERT

Lambert was born at Mülhausen in Alsace on August 26, 1728. He was the son of a poor tailor and his education was entirely the product of his own exertions, expended in a systematic course of reading which kept him up the greater part of each night. This sacrifice was probably a factor in his untimely death from consumption on September 25, 1777.

At the age of 16 Lambert discovered, in computations for the comet of 1744, the so-called Lambert's theorem. This theorem is actually an extension to the ellipse of a theorem for the parabola published by the Swiss mathematician Euler in 1743. As published by Lambert in his "Insigniores orbitae cometarum proprietates", 1761, the theorem states that the area of any focal sector of an ellipse can be expressed in terms of the focal distances of its extremities, of the chord which joins them, and of the axes of the curve. More specifically if t is the time of describing any arc PP' of an ellipse and k is the chord of the arc, then $nt=(\phi-\sin\phi)-(\phi'-\sin\phi')$, where $\sin\frac{1}{2}\phi=\frac{1}{2}\sqrt{(r+r'+k)/a}$, $\sin\frac{1}{2}\phi'=\frac{1}{2}\sqrt{(r+r'-k)/a}$, r and r' being the focal distances of P and P', n the mean angular velocity, and a the semimajor axis of the ellipse. His attempts to simplify the computation of cometary orbits led him to some remarkable theorems on conics such as the following: "If in two ellipses having a common major axis we take two arcs such that their chords are equal, and that also the sums of the radius vectors, drawn respectively from the foci to the extremities of these arcs, are equal to each other, then the sectors formed in each ellipse by the arc and the two radius vectors are to each other as the square roots of the parameters of the ellipses".

When Lambert was 30, he became a private tutor to a Swiss family and secured leisure to continue his studies. In his travels with his pupils through Europe he became acquainted with the leading mathematicians. In 1764 he settled in Berlin. He was elected a member of the Royal Academy of Sciences of Berlin and received many favors, including a small pension, from Frederick the Great. He later became editor of the Berlin Ephemeris.

Lambert's first research in pure mathematics developed in an infinite series the root x of the equation $x^m+px=q$. Since each equation of the form $ax^r+bx^s=d$ can be reduced to $x^m+px=q$ in two ways, one or the other of the two resulting series was always found to be convergent, and to give a value of x. This paper was a stimulus to both Euler and Lagrange, both of whom succeeded in extending Lambert's results.

In 1761 Lambert communicated to the Berlin Academy a memoir, in which he proved π is irrational. His paper on trigonometry, read in 1768, introduced into trigonometry the hyperbolic functions, which he designated by the notation existing today, sinh x, cosh x, etc. Also included were the developments of DeMoivre's theorems on the trigonometry of complex variables. His researches on descriptive geometry published in "Die freie Perspectiv," 1759 and 1773, were a stimulus to the great geometer Monge.

The earliest attempt to form functional equations by expressing the given properties in the language of the differential calculus and then integrating is found in an essay entitled "Analytic observations," published by Lambert in 1771. In his paper on vis viva, published posthumously in 1783, Newton's second law of motion was expressed for the first time in the notation of the differential calculus.

Astronomy was enriched by Lambert's investigations. In his "cosmological letters" he made some remarkable prophecies regarding the stellar system. For instance he aptly denominated the Milky Way the Ecliptic of the Fixed Stars. But he was also active in the physical sciences, being best known in this field for his work in optics where he developed photometry on theoretical lines. His work on optics was published as "Photometria," Augsburg, 1760.

It was in the application of mathematical analysis to the practical problems of life that Lambert especially excelled. He was the first mathematician to make general investigations in the field of map projections.

His predecessors in this work had limited themselves to the development of a single method of projection, principally the perspective, but Lambert considered the problem of the representation of a sphere upon a plane from a higher standpoint and he stated certain general conditions that the representation was to satisfy, the most important of these being preservation of angles or conformality, and equal-area or equivalence, both being, of course, unattainable in the same projection.

Although Lambert did not fully develop the theory of these two methods of projection (conformal and equal area), yet he was the first to express clearly the ideas regarding them. The former, conformality, has become of the greatest importance to pure mathematics, but both of them are of exceeding importance to the cartographer. It is no more than just, therefore, to date the beginning of a new epoch in the science of map making from the appearance of Lambert's work. What he accomplished is of importance because of the generality of his underlying ideas and for his successful application of them in methods of projection.

Lambert's treatment of the so-called transverse Mercator projection was published in his "Beiträge zum Gebrauche der Mathematik und deren Anwendung," Berlin 1772. He pointed out that it was applicable to a country of great extent in latitude but of small longitudinal width. The first known appearance of the name "Transverse Mercator" is found in Germain's "Traité des Projections", Paris 1865. Germain also called it the "Projection Cylindrique Orthomorphe de Lambert."

Lambert's development was from elementary considerations as shown by Germain, Gauss giving the analytic derivation 50 years later in a paper presented to the Academy of Sciences—Copenhagen 1822 (published by Schumacher in 1825). Gauss showed that it is a particular case of his general theory of the conformal representation of one surface upon another. Gauss also included the theory in a later publication, "Untersuchungen über Gegenstände der höheren Geodäsie," 1843.

In 1866, eleven years after the death of Gauss, General Oscar Schreiber published an account of the use by Gauss of this projection in the Survey of Hannover, "Theorie der Projectionsmethode der Hannoverschen Landesvermessung," and in 1878 published the developments essentially in use today.

In 1912, L. Krüger published a comprehensive treatise of the projection entitled "Konforme Abbildung des Erdellipsoids in der Ebene", in which the formulas were developed in a form suitable for numerical calculation, and in 1919 a second work called "Formeln zur Konformen Abbildung des Erdellipsoids in der Ebene" was published. In 1927, the system was adopted for the whole of Germany and called the "Gauss-Krüger" projection.

There are some notable differences in the transverse Mercator projections obtained by modifying the abscissa of the Cassini spherical coordinates to make it conformal and that obtained by means of the analytic function of a complex variable. In the projection obtained by modifying the Cassini abscissa, the ordinate is assumed the same for both projections. This is not true for the spheroid, but the error introduced is usually negligible. Technically, the projection thus obtained for the spheroid is not conformal, since the coordinates will not satisfy the Cauchy-Riemann equations.

The curves orthogonal to the central meridian on the modified Cassini projection are assumed to be geodesics, while in the projection obtained by means of complex variable theory it is known that they cannot be geodesics since the only geodesic-isometric system on the spheroid is that formed by the meridians and parallels. That is, non-meridian geodesics on the spheroid cannot be members of an isometric system. (See equation 195.)

The transverse Mercator projection is used officially in Great Britain, Egypt, Sweden, Poland, Portugal, Russia, Bulgaria, Finland, Germany, Yugoslavia, Norway, British African Colonies, South Africa, Australia, U. S. Army Map Service, and in the plane coordinate systems of many States of the United States. The machine method of computing geographic positions in the U. S. Coast and Geodetic Survey is based on it. The transverse Mercator system is now more extensively used for geodetic computations than the Lambert conformal conic or any other projection for the following two reasons:

(1) On the Lambert conformal conic projection, when the abscissa is large, the convergence is also large which leads to considerable divergence between the grid lines and the true north line for map sheets lying a considerable distance east or west of the central meridian.

(2) Lambert conical orthomorphic coordinates are not quite as suitable for point-to-point working as transverse Mercator coordinates are.

DERIVATION OF FORMULAS FOR THE SPHEROID

The requirement for the transverse Mercator projection is that the scale shall be true along the central map meridian. Hence when $\lambda=0$, we must have $x=0$ and from (189) if we write the analytic function $x+iy=f(\lambda+i\tau)$ we must then have

$$iy=f(i\tau)=i\,S_\phi \tag{274}$$

where S_ϕ is the arc length along the elliptic meridian of the spheroid from the Equator to latitude ϕ. But $S_\phi=\int_0^\phi R d\phi$ and from $\tau=\int_0^\phi \frac{R}{N}\sec\phi\, d\phi$, equation (189), we have $R d\phi=N\cos\phi\, d\tau$ so that we may write

$$S_\phi=\int_0^\phi N\cos\phi\, d\tau=f(\tau). \tag{275}$$

If we expand [6] $x+iy=f(\lambda+i\tau)$ about the point $z=i\tau$ by Taylor's theorem we obtain

$$x+iy=f(\lambda+i\tau)=f(i\tau)+\lambda f'(i\tau)+\frac{\lambda^2}{2!}f''(i\tau)+\frac{\lambda^3}{3!}f'''(i\tau)+\frac{\lambda^4}{4!}f^{\text{iv}}(i\tau)+\frac{\lambda^5}{5!}f^{\text{v}}(i\tau)+$$
$$\frac{\lambda^6}{6!}f^{\text{vi}}(i\tau)+\frac{\lambda^7}{7!}f^{\text{vii}}(i\tau)+\frac{\lambda^8}{8!}f^{\text{viii}}(i\tau)+\ldots. \tag{276}$$

From (274) and (275) it is seen that $f(i\tau)=i\,S_\phi=i\,f(\tau)$. Hence, differentiating this equation with respect to z, we have $\frac{d}{dz}f(i\tau)=\frac{d}{dz}[if(\tau)]$ or $f'(i\tau)=\frac{d}{d\tau}[if(\tau)]\frac{d\tau}{dz}$. Since $\frac{dz}{d\tau}=i$, $f'(i\tau)=if'(\tau)\cdot\frac{1}{i}=f'(\tau)$, where $f'(i\tau)\equiv\frac{d}{dz}f(i\tau)$ and $f'(\tau)\equiv\frac{d}{d\tau}f(\tau)$. Continuing $f''(i\tau)=-i\,f''(\tau)$, $f'''(i\tau)=-f'''(\tau)$, $f^{\text{iv}}(i\tau)=if^{\text{iv}}(\tau)$, $f^{\text{v}}(i\tau)=f^{\text{v}}(\tau)$, $f^{\text{vi}}(i\tau)=-i\,f^{\text{vi}}(\tau)$, etc. With these values placed in (276) we have

$$x+iy=if(\tau)+\lambda f'(\tau)-\frac{\lambda^2}{2!}if''(\tau)-\frac{\lambda^3}{3!}f'''(\tau)+\frac{\lambda^4}{4!}if^{\text{iv}}(\tau)+\frac{\lambda^5}{5!}f^{\text{v}}(\tau)-$$
$$\frac{\lambda^6}{6!}if^{\text{vi}}(\tau)-\frac{\lambda^7}{7!}f^{\text{vii}}(\tau)+\frac{\lambda^8}{8!}if^{\text{viii}}(\tau)+\ldots. \tag{277}$$

Equating real and imaginary parts in (277) one obtains

$$x=\lambda f'(\tau)-\frac{\lambda^3}{3!}f'''(\tau)+\frac{\lambda^5}{5!}f^{\text{v}}(\tau)-\frac{\lambda^7}{7!}f^{\text{vii}}(\tau)+\ldots.$$
$$y=f(\tau)-\frac{\lambda^2}{2!}f''(\tau)+\frac{\lambda^4}{4!}f^{\text{iv}}(\tau)-\frac{\lambda^6}{6!}f^{\text{vi}}(\tau)+\frac{\lambda^8}{8!}f^{\text{viii}}(\tau)+\ldots. \tag{278}$$

In obtaining the successive derivatives of $f(\tau)$, and other derivatives, we will need the values of N', R', $\left(\frac{N}{R}\right)'$, $(N\cos\phi)'$, $(N\sin\phi)'$, the value of $d\phi/d\tau$ from (189), and some trigonometric identities. We group them all together here for convenience:

$$N'=(N-R)\tan\phi;\ R'=3\frac{R}{N}(N-R)\tan\phi;\ \left(\frac{N}{R}\right)'=-\frac{2(N-R)}{R}\tan\phi;$$
$$\frac{d\phi}{d\tau}=\frac{N}{R}\cos\phi;\ (N\cos\phi)'=-R\sin\phi;\ (N\sin\phi)'=\sec\phi\,(N-R\sin^2\phi)=(R\cos\phi)/(1-\epsilon^2). \tag{279}$$

$$2\sin n\phi\cos\phi=\sin(n+1)\phi+\sin(n-1)\phi,$$
$$2\cos n\phi\cos\phi=\cos(n+1)\phi+\cos(n-1)\phi,$$
$$2\cos n\phi\sin\phi=\sin(n+1)\phi-\sin(n-1)\phi,$$
$$2\sin n\phi\sin\phi=\cos(n-1)\phi-\cos(n+1)\phi.$$

From (275) we have

$$f'(\tau)=N\cos\phi. \tag{280}$$

Differentiating again $f''(\tau)=(N'\cos\phi-N\sin\phi)\frac{d\phi}{d\tau}$, and with relations (279) this becomes

$$f''(\tau)=-\frac{N}{2}\sin 2\phi. \tag{281}$$

[6] Since $f(\lambda+i\tau)$ is an analytic function, the series expansion is valid—Churchill, loc. cit., p. 98.

Continuing $f'''(\tau)=-\frac{1}{2}(N'\sin 2\phi+2N\cos 2\phi)\frac{d\phi}{d\tau}$ which with relations (279) becomes

$$f'''(\tau)=-\frac{N}{4}\left[\left(3\frac{N}{R}-1\right)\cos\phi+\left(\frac{N}{R}+1\right)\cos 3\phi\right]. \tag{282}$$

Now

$$f^{\text{iv}}(\tau)=-\frac{1}{4}\left\{\begin{array}{l} N'\left[\left(3\frac{N}{R}-1\right)\cos\phi+\left(\frac{N}{R}+1\right)\cos 3\phi\right]+3N\left(\frac{N}{R}\right)'\cos\phi \\ -N\left(3\frac{N}{R}-1\right)\sin\phi+N\left(\frac{N}{R}\right)'\cos 3\phi-3N\left(\frac{N}{R}+1\right)\sin 3\phi \end{array}\right\}\frac{d\phi}{d\tau},$$

and reducing by means of (279) we have

$$f^{\text{iv}}(\tau)=\frac{N}{8}\left[2\left(-1+\frac{N}{R}+4\frac{N^2}{R^2}\right)\sin 2\phi+\left(1+\frac{N}{R}+4\frac{N^2}{R^2}\right)\sin 4\phi\right]. \tag{283}$$

Differentiating (283) we have

$$f^{\text{v}}(\tau)=\frac{1}{8}\left\{\begin{array}{l} N'\left[2\left(-1+\frac{N}{R}+4\frac{N^2}{R^2}\right)\sin 2\phi+\left(1+\frac{N}{R}+4\frac{N^2}{R^2}\right)\sin 4\phi\right] \\ +2N\left(\frac{N}{R}\right)'\left(1+8\frac{N}{R}\right)\sin 2\phi+4N\left(-1+\frac{N}{R}+4\frac{N^2}{R^2}\right)\cos 2\phi \\ +N\left(\frac{N}{R}\right)'\left(1+8\frac{N}{R}\right)\sin 4\phi+4N\left(1+\frac{N}{R}+4\frac{N^2}{R^2}\right)\cos 4\phi \end{array}\right\}\frac{d\phi}{d\tau}$$

which becomes by means of (279)

$$f^{\text{v}}(\tau)=\frac{N}{16}\left\{\begin{array}{l} 2\left(1-2\frac{N}{R}+13\frac{N^2}{R^2}-4\frac{N^3}{R^3}\right)\cos\phi \\ +\left(-3+2\frac{N}{R}-3\frac{N^2}{R^2}+44\frac{N^3}{R^3}\right)\cos 3\phi \\ +\left(1+2\frac{N}{R}-7\frac{N^2}{R^2}+28\frac{N^3}{R^3}\right)\cos 5\phi \end{array}\right\}. \tag{284}$$

From (284) differentiating again we have

$$f^{\text{vi}}(\tau)=\frac{1}{16}\left\{\begin{array}{l} 2\left(1-2\frac{N}{R}+13\frac{N^2}{R^2}-4\frac{N^3}{R^3}\right)(N\cos\phi)' \\ -4\left(\frac{N}{R}\right)'\left(1-13\frac{N}{R}+6\frac{N^2}{R^2}\right)N\cos\phi \\ +\left(-3+2\frac{N}{R}-3\frac{N^2}{R^2}+44\frac{N^3}{R^3}\right)(N\cos 3\phi)' \\ +2\left(\frac{N}{R}\right)'\left(1-3\frac{N}{R}+66\frac{N^2}{R^2}\right)N\cos 3\phi \\ +\left(1+2\frac{N}{R}-7\frac{N^2}{R^2}+28\frac{N^3}{R^3}\right)(N\cos 5\phi)' \\ +2\left(\frac{N}{R}\right)'\left(1-7\frac{N}{R}+42\frac{N^2}{R^2}\right)N\cos 5\phi \end{array}\right\}\frac{d\phi}{d\tau}$$

and after reducing by means of relations (279) we have finally

$$f^{\mathrm{vi}}(\tau)=-\frac{N}{32}\left\{\begin{array}{l}\left(5-6\frac{N}{R}-91\frac{N^2}{R^2}+364\frac{N^3}{R^3}-136\frac{N^4}{R^4}\right)\sin 2\phi\\ +4\left(-1+\frac{N^2}{R^2}-28\frac{N^3}{R^3}+88\frac{N^4}{R^4}\right)\sin 4\phi\\ +\left(1+2\frac{N}{R}+33\frac{N^2}{R^2}-196\frac{N^3}{R^3}+280\frac{N^4}{R^4}\right)\sin 6\phi\end{array}\right\}. \quad (285)$$

Continuing as above we find

$$f^{\mathrm{vii}}(\tau)=-\frac{N}{64}\left\{\begin{array}{l}\left(-5+9\frac{N}{R}-279\frac{N^2}{R^2}+1{,}911\frac{N^3}{R^3}-2{,}044\frac{N^4}{R^4}+680\frac{N^5}{R^5}\right)\cos\phi\\ +\left(9-9\frac{N}{R}+267\frac{N^2}{R^2}-2{,}831\frac{N^3}{R^3}+6{,}076\frac{N^4}{R^4}-2{,}280\frac{N^5}{R^5}\right)\cos 3\phi\\ +\left(-5-3\frac{N}{R}+97\frac{N^2}{R^2}-293\frac{N^3}{R^3}-1{,}708\frac{N^4}{R^4}+3{,}592\frac{N^5}{R^5}\right)\cos 5\phi\\ +\left(1+3\frac{N}{R}-85\frac{N^2}{R^2}+1{,}277\frac{N^3}{R^3}-4{,}116\frac{N^4}{R^4}+3{,}640\frac{N^5}{R^5}\right)\cos 7\phi\end{array}\right\} \quad (286)$$

and

$$f^{\mathrm{viii}}(\tau)=\frac{N}{128}\left\{\begin{array}{l}2\left(-7+9\frac{N}{R}+819\frac{N^2}{R^2}-12{,}413\frac{N^3}{R^3}+36{,}984\frac{N^4}{R^4}-\right.\\ \qquad\left.33{,}648\frac{N^5}{R^5}+10{,}240\frac{N^6}{R^6}\right)\sin 2\phi\\ +2\left(7-3\frac{N}{R}-279\frac{N^2}{R^2}+7{,}243\frac{N^3}{R^3}-38{,}568\frac{N^4}{R^4}+\right.\\ \qquad\left.58{,}512\frac{N^5}{R^5}-20{,}864\frac{N^6}{R^6}\right)\sin 4\phi\\ +6\left(-1-\frac{N}{R}-91\frac{N^2}{R^2}+1{,}381\frac{N^3}{R^3}-2{,}872\frac{N^4}{R^4}-\right.\\ \qquad\left.3{,}344\frac{N^5}{R^5}+7{,}168\frac{N^6}{R^6}\right)\sin 6\phi\\ +\left(1+3\frac{N}{R}+279\frac{N^2}{R^2}-7{,}235\frac{N^3}{R^3}+44{,}136\frac{N^4}{R^4}-\right.\\ \qquad\left.90{,}384\frac{N^5}{R^5}+58{,}240\frac{N^6}{R^6}\right)\sin 8\phi\end{array}\right\}. \quad (287)$$

Placing the values of $f'(\tau)$, $f''(\tau)$, $f'''(\tau)$, $f^{\mathrm{iv}}(\tau)$, $f^{\mathrm{v}}(\tau)$, $f^{\mathrm{vi}}(\tau)$, $f^{\mathrm{vii}}(\tau)$, $f^{\mathrm{viii}}(\tau)$ from (280), (281), (282), (283), (284), (285), (286), (287) with $\sigma=\frac{N}{R}$ in equations (278) we have the mapping equations to eighth-order terms for the transverse Mercator projection, namely

$$\frac{x}{N}=\lambda\cos\phi+\frac{\lambda^3}{24}[(3\sigma-1)\cos\phi+(\sigma+1)\cos 3\phi]+\frac{\lambda^5}{1{,}920}[2(1-2\sigma+13\sigma^2-4\sigma^3)\cos\phi+$$
$$(-3+2\sigma-3\sigma^2+44\sigma^3)\cos 3\phi+(1+2\sigma-7\sigma^2+28\sigma^3)\cos 5\phi]$$

$$+\frac{\lambda^7}{322{,}560}\left\{\begin{array}{l}(-5+9\sigma-279\sigma^2+1{,}911\sigma^3-2{,}044\sigma^4+680\sigma^5)\cos\phi\\+(9-9\sigma+267\sigma^2-2{,}831\sigma^3+6{,}076\sigma^4-2{,}280\sigma^5)\cos 3\phi\\+(-5-3\sigma+97\sigma^2-293\sigma^3-1{,}708\sigma^4+3{,}592\sigma^5)\cos 5\phi\\+(1+3\sigma-85\sigma^2+1{,}277\sigma^3-4{,}116\sigma^4+3{,}640\sigma^5)\cos 7\phi\end{array}\right\},\qquad (288)$$

$$\frac{y}{N}=\frac{S_\phi}{N}+\frac{\lambda^2}{4}\sin 2\phi+\frac{\lambda^4}{192}[2(-1+\sigma+4\sigma^2)\sin 2\phi+(1+\sigma+4\sigma^2)\sin 4\phi]$$

$$+\frac{\lambda^6}{23{,}040}\left\{\begin{array}{l}(5-6\sigma-91\sigma^2+364\sigma^3-136\sigma^4)\sin 2\phi\\+4(-1+\sigma^2-28\sigma^3+88\sigma^4)\sin 4\phi\\+(1+2\sigma+33\sigma^2-196\sigma^3+280\sigma^4)\sin 6\phi\end{array}\right\}$$

$$+\frac{\lambda^8}{5{,}160{,}960}\left\{\begin{array}{l}2(-7+9\sigma+819\sigma^2-12{,}413\sigma^3+36{,}984\sigma^4-33{,}648\sigma^5\\\qquad+10{,}240\sigma^6)\sin 2\phi\\+2(7-3\sigma-279\sigma^2+7{,}243\sigma^3-38{,}568\sigma^4+58{,}512\sigma^5\\\qquad-20{,}864\sigma^6)\sin 4\phi\\+6(-1-\sigma-91\sigma^2+1{,}381\sigma^3-2{,}872\sigma^4-3{,}344\sigma^5\\\qquad+7{,}168\sigma^6)\sin 6\phi\\+(1+3\sigma+279\sigma^2-7{,}235\sigma^3+44{,}136\sigma^4-90{,}384\sigma^5\\\qquad+58{,}240\sigma^6)\sin 8\phi\end{array}\right\}.$$

If we place $\sigma=\frac{N}{R}=1+\delta\cos^2\phi$ where $\delta=\frac{\epsilon^2}{1-\epsilon^2}=\epsilon'^2$ and $\eta^2=\delta\cos^2\phi, t=\tan\phi$, we may write the mapping equations (288) as

$$\frac{x}{V}=\lambda\cos\phi+\frac{\lambda^3\cos^3\phi}{6}(1-t^2+\eta^2)+\frac{\lambda^5\cos^5\phi}{120}(5-18t^2+t^4+14\eta^2-58t^2\eta^2+13\eta^4-64t^2\eta^4+4\eta^6-24t^2\eta^6)+$$

$$\frac{\lambda^7\cos^7\phi}{5{,}040}\left[\begin{array}{l}61-479t^2+179t^4-t^6+331\eta^2-3{,}262\eta^2t^2+1{,}771\eta^2t^4+715\eta^4\\-8{,}655t^2\eta^4+6{,}080t^4\eta^4+769\eta^6-10{,}964t^2\eta^6+9{,}480t^4\eta^6+412\eta^8\\-6{,}760t^2\eta^8+6{,}912t^4\eta^8+88\eta^{10}-1{,}632t^2\eta^{10}+1{,}920t^4\eta^{10}\end{array}\right],\qquad (289)$$

$$\frac{y}{N}=\frac{S_\phi}{N}+\frac{\lambda^2}{2}\sin\phi\cos\phi+\frac{\lambda^4}{24}\sin\phi\cos^3\phi(5-t^2+9\eta^2+4\eta^4)+\frac{\lambda^6}{720}\sin\phi\cos^5\phi(61-58t^2+t^4+270\eta^2-330t^2\eta^2+445\eta^4-680t^2\eta^4+324\eta^6-600t^2\eta^6+88\eta^8-192t^2\eta^8)+$$

$$\frac{\lambda^8}{40{,}320}\sin\phi\cos^7\phi\left[\begin{array}{l}1{,}385-3{,}111t^2+543t^4-t^6+10{,}899\eta^2-32{,}802t^2\eta^2\\+9{,}219t^4\eta^2+34{,}419\eta^4-129{,}087t^2\eta^4+49{,}644t^4\eta^4\\+56{,}385\eta^6-252{,}084t^2\eta^6+121{,}800t^4\eta^6+50{,}856\eta^8\\-263{,}088t^2\eta^8+151{,}872t^4\eta^8+24{,}048\eta^{10}-140{,}928t^2\eta^{10}\\+94{,}080t^4\eta^{10}+4{,}672\eta^{12}-30{,}528t^2\eta^{12}+23{,}040t^4\eta^{12}\end{array}\right].$$

If in the coefficients of the 5th- and 6th-order terms of equations (289) we delete those terms involving powers of η above the second and in the 7th- and 8th-order terms delete from the coefficients all terms involving η, we may write them in the form

$$
\begin{aligned}
\frac{x}{N}&=\frac{\lambda}{\rho}\cos\phi+\frac{\lambda^3\cos^3\phi}{6\rho^3}(1-t^2+\eta^2)+\frac{\lambda^5\cos^5\phi}{120\rho^5}(5-18t^2+t^4+14\eta^2-58t^2\eta^2)\\
&\quad+\frac{\lambda^7}{5{,}040\rho^7}\cos^7\phi(61-479t^2+179t^4-t^6),\\
\frac{y}{N}&=\frac{S_\phi}{N}+\frac{\lambda^2}{2\rho^2}\sin\phi\cos\phi+\frac{\lambda^4}{24\rho^4}\sin\phi\cos^3\phi(5-t^2+9\eta^2+4\eta^4)\\
&\quad+\frac{\lambda^6}{720\rho^6}\sin\phi\cos^5\phi(61-58t^2+t^4+270\eta^2-330t^2\eta^2)\\
&\quad+\frac{\lambda^8}{40{,}320\rho^8}\sin\phi\cos^7\phi(1{,}385-3{,}111t^2+543t^4-t^6),
\end{aligned}
\tag{290}
$$

where $\rho=\operatorname{cosec} 1''$, $t=\tan\phi$, $\eta^2=\delta\cos^2\phi=\frac{\epsilon^2}{1-\epsilon^2}\cos^2\phi$. Equations (290) are in a more practical form for computation and are essentially as given in Jordan-Eggert, Handbuch der Vermessungskunde, Vol. III, part 2, paragraph 32 (8th enlarged edition 1941), or in the Army Map Service Technical Manual No. 19, pages 4 and 5. In actual practice the 7th- and 8th-order terms are seldom if ever needed.

For the convergence we have from equation (217)

$$\tan\gamma=\frac{\partial y}{\partial\lambda}\Big/\frac{\partial x}{\partial\lambda}. \tag{291}$$

From (289) we have

$$
\begin{aligned}
\frac{\partial x}{\partial\lambda}&=N\cos\phi\left(1+A\,\frac{\lambda^2\cos^2\phi}{2}+B\,\frac{\lambda^4\cos^4\phi}{24}+C\,\frac{\lambda^6\cos^6\phi}{720}+\dots\right),\\
\frac{\partial y}{\partial\lambda}&=\lambda N\sin\phi\cos\phi\left(1+D\,\frac{\lambda^2\cos^2\phi}{6}+E\,\frac{\lambda^4\cos^4\phi}{120}+F\,\frac{\lambda^6\cos^6\phi}{5{,}040}+\dots\right),
\end{aligned}
\tag{292}
$$

where A, B, C, D, E, F are the respective coefficients in t and η.

From (291) and (292) we have

$$
\begin{aligned}
\tan\gamma=\frac{\partial y}{\partial\lambda}\Big/\frac{\partial x}{\partial\lambda}=\lambda\sin\phi\Big[1&+(D-3A)\,\frac{\lambda^2\cos^2\phi}{6}+(E-10AD-5B+30A^2)\,\frac{\lambda^4\cos^4\phi}{120}+\\
&(F-7C-21AE-35BD+210AB+210A^2D-630A^3)\,\frac{\lambda^6\cos^6\phi}{5{,}040}\Big].
\end{aligned}
\tag{293}
$$

Placing the values of A, B, C, D, E, F from (289) in (293) and simplifying we have, deleting terms in η in the coefficient of λ^7,

$$
\tan\gamma=\lambda\sin\phi\left\{
\begin{aligned}
&1+\frac{\lambda^2\cos^2\phi}{3}(1+t^2+3\eta^2+2\eta^4)+\frac{17}{315}(1+t^2)^3\lambda^6\cos^6\phi\\
&+\frac{\lambda^4\cos^4\phi}{15}\begin{pmatrix}2+4t^2+2t^4+15\eta^2+35\eta^4-40t^2\eta^4\\+33\eta^6-60t^2\eta^6+11\eta^8-24t^2\eta^8\end{pmatrix}
\end{aligned}
\right\}. \tag{294}
$$

The series expansion for arc tan x is arc tan $x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$ and placing $\gamma=\text{arc tan } x$, we have

$$\gamma=\tan\gamma-\frac{1}{3}\tan^3\gamma+\frac{1}{5}\tan^5\gamma-\frac{1}{7}\tan^7\gamma+\ldots. \tag{295}$$

Writing (294) in the form

$$\tan\gamma=\lambda\sin\phi\left(1+a\frac{\lambda^2\cos^2\phi}{3}+b\frac{\lambda^4\cos^4\phi}{15}+c\frac{17\lambda^6\cos^6\phi}{315}\right), \tag{296}$$

where a, b, c are the corresponding coefficients in t and η, we have then, retaining terms in λ^7

$$-\frac{1}{3}\tan^3\gamma=-\frac{1}{3}\lambda^3\sin^3\phi\left[1+\lambda^2\cos^2\phi\left(a+\frac{b}{5}\lambda^2\cos^2\phi\right)+\frac{\lambda^4\cos^4\phi}{3}a^2\right],$$

$$\frac{1}{5}\tan^5\gamma=\frac{1}{5}\lambda^5\sin^5\phi\left(1+\frac{5\lambda^2\cos^2\phi}{3}a\right),$$

$$-\frac{1}{7}\tan^7\gamma=-\frac{1}{7}\lambda^7\sin^7\phi.$$

Hence by (295) we have

$$\gamma=\lambda\sin\phi\left[1+\frac{a-t^2}{3}\lambda^2\cos^2\phi+\frac{b-5at^2+3t^4}{15}\lambda^4\cos^4\phi+\frac{17c-21bt^2-35a^2t^2+105at^4-45t^6}{315}\lambda^6\cos^6\phi\right]. \tag{297}$$

Placing the values of a, b, c from (294) in (297) and deleting, as before, terms in η in the coefficient of λ^7, we have finally

$$\gamma=\lambda\sin\phi\left\{\begin{array}{l}1+\frac{\lambda^2\cos^2\phi}{3\rho^2}(1+3\eta^2+2\eta^4)+\frac{\lambda^6\cos^6\phi}{315\rho^6}(17-26t^2+2t^4)\\ +\frac{\lambda^4\cos^4\phi}{15\rho^4}\begin{pmatrix}2-t^2+15\eta^2+35\eta^4-15\eta^2t^2+33\eta^6\\ -50\eta^4t^2+11\eta^8-60t^2\eta^6-24t^2\eta^8\end{pmatrix}\end{array}\right\}. \tag{298}$$

In equation (298) the term in λ^7 is seldom if ever needed and the terms in η in the coefficient of λ^5 are usually negligible. For instance in the Universal Transverse Mercator System as given in the Army Map Service Technical Manual No. 19, equation (298) would correspondingly be

$$\gamma=\lambda\sin\phi\left[1+\frac{\lambda^2\cos^2\phi}{3\rho^2}(1+3\eta^2+2\eta^4)+\frac{\lambda^4\cos^4\phi}{15\rho^4}(2-t^2)\right].$$

From (190) we have the scale

$$k=\sqrt{\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2}\Big/N\cos\phi=\frac{1}{N\cos\phi}\cdot\frac{\partial x}{\partial\lambda}\cdot\sqrt{1+\tan^2\gamma},$$

and expanding the radical by the binomial formula

$$k=\frac{1}{N\cos\phi}\cdot\frac{\partial x}{\partial\lambda}\cdot\left(1+\frac{1}{2}\tan^2\gamma-\frac{1}{8}\tan^4\gamma+\frac{1}{16}\tan^6\gamma-\ldots\right). \tag{299}$$

From (292) we have

$$\frac{1}{N\cos\phi}\cdot\frac{\partial x}{\partial\lambda}=1+\frac{A}{2}\lambda^2\cos^2\phi+\frac{B}{24}\lambda^4\cos^4\phi+\frac{C}{720}\lambda^6\cos^6\phi+\ldots$$

and from (296) retaining terms in λ^6

$$\frac{1}{2}\tan^2\gamma=\frac{1}{2}\lambda^2\sin^2\phi\left(1+\frac{2a}{3}\lambda^2\cos^2\phi+\frac{a^2}{9}\lambda^4\cos^4\phi+\frac{2b}{15}\lambda^4\cos^4\phi\right),$$

$$-\frac{1}{8}\tan^4\gamma=-\frac{1}{8}\lambda^4\sin^4\phi\left(1+\frac{4a}{3}\lambda^2\cos^2\phi\right),$$

$$\frac{1}{16}\tan^6\gamma=\frac{1}{16}\lambda^6\sin^6\phi.$$

With these values placed in (299) we obtain

$$k=1+\frac{A+t^2}{2}\lambda^2\cos^2\phi+\frac{B+2(3A+4a)t^2-3t^4}{24}\lambda^4\cos^4\phi+$$

$$\frac{C+(15B+120aA+40a^2+48b)t^2-15(3A+8a)t^4+45t^6}{720}\lambda^6\cos^6\phi. \tag{300}$$

Placing the values of A, B, C from (289) and the values of a, b from (294) in (300), ignoring terms in η in the coefficient of λ^6, we obtain finally

$$k=1+\frac{\lambda^2}{2}\cos^2\phi\,(1+\eta^2)+\frac{\lambda^4\cos^4\phi}{24}(5-4t^2+14\eta^2+13\eta^4-28t^2\eta^2+4\eta^6-48t^2\eta^4-24t^2\eta^6)+$$

$$\frac{\lambda^6\cos^6\phi}{720}(61-148t^2+16t^4). \tag{301}$$

In equation (301) the term in λ^6 is usually deleted and the terms in η omitted in the coefficient of λ^4.

We now develop the formulas for λ, ϕ, γ, k in terms of the rectangular coordinates x and y.

Let us write equation (189) in the form

$$\lambda+i\tau=F(x+iy). \tag{302}$$

When $x=0$, $\lambda=0$, then $F(iy)=i\tau$ and from (189) and (275) we have

$$\tau=\int_0^\phi\frac{R}{N}\sec\phi\,d\phi,\ \frac{d\tau}{ds_\phi}=\frac{1}{N\cos\phi},\ \frac{d\phi}{ds_\phi}=\frac{1}{R}. \tag{303}$$

Taylor's expansion for the function $F(z)=F(x+iy)$ about the point $z_0=iy$ is

$$\lambda+i\tau=F(iy)+xF'(iy)+\frac{x^2}{2!}F''(iy)+\frac{x^3}{3!}F'''(iy)+\frac{x^4}{4!}F^{\mathrm{iv}}(iy)+\frac{x^5}{5!}F^{\mathrm{v}}(iy)+\frac{x^6}{6!}F^{\mathrm{vi}}(iy)+$$

$$\frac{x^7}{7!}F^{\mathrm{vii}}(iy)+\frac{x^8}{8!}F^{\mathrm{viii}}(iy)+\ \ldots\ . \tag{304}$$

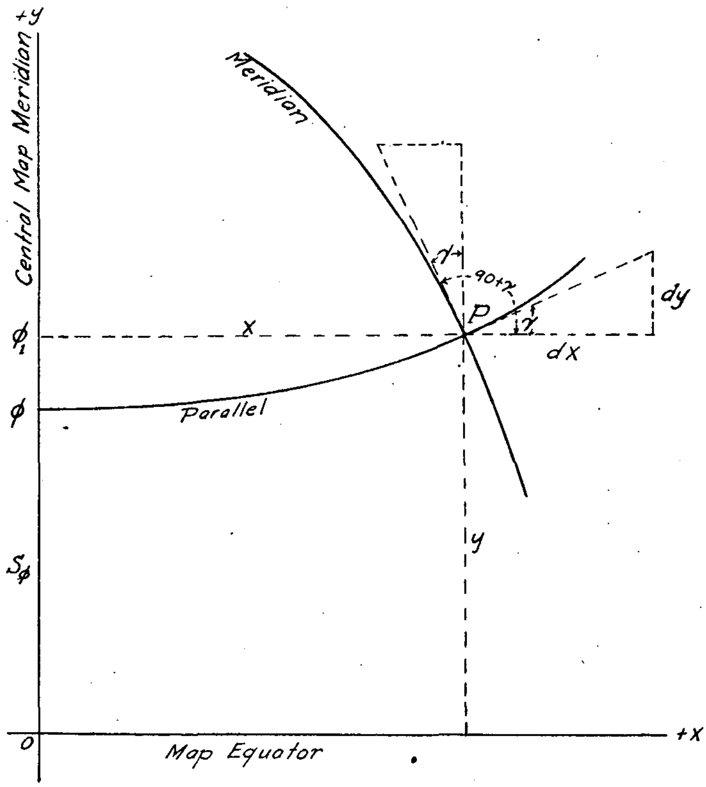
From the relation $F(iy)=i\tau$, we have $F'(iy)=\tau'$, $F''(iy)=-i\tau''$, $F'''(iy)=-\tau'''$, $F^{iv}(iy)=i\tau^{iv}$, $F^{v}(iy)=\tau^{v}$, $F^{vi}(iy)=-i\tau^{vi}$, $F^{vii}(iy)=-\tau^{vii}$, $F^{viii}(iy)=i\tau^{viii}$ and with these values placed in (304) we obtain

$$\lambda+i\tau=i\tau_1+x\tau_1'-\frac{x^2}{2!}i\tau_1''-\frac{x^3}{3!}\tau_1'''+\frac{x^4}{4!}i\tau_1^{iv}+\frac{x^5}{5!}\tau_1^{v}-\frac{x^6}{6!}i\tau_1^{vi}-\frac{x^7}{7!}\tau_1^{vii}+\frac{x^8}{8!}i\tau_1^{viii}+\dots \tag{305}$$

Equating real and imaginary parts in (305) we have

$$\begin{aligned}\lambda&=x\tau_1'-\frac{x^3}{3!}\tau_1'''+\frac{x^5}{5!}\tau_1^{v}-\frac{x^7}{7!}\tau_1^{vii}+\dots,\\ \tau&=\tau_1-\frac{x^2}{2!}\tau_1''+\frac{x^4}{4!}\tau_1^{iv}-\frac{x^6}{6!}\tau_1^{vi}+\frac{x^8}{8!}\tau_1^{viii}-\dots,\end{aligned} \tag{306}$$

where the subscript one refers to the "footpoint" latitude of a given point of the projection. (See fig. 27.)



ϕ_1 is the "footpoint" latitude.

FIGURE 27.—Convergence of map meridians and the footpoint latitude, transverse Mercator projection.

From (303) we have $\tau'=\frac{d\tau}{ds_\phi}=\frac{1}{N\cos\phi}$, whence $\tau''=\left(\frac{1}{N\cos\phi}\right)'\frac{d\phi}{ds}=-\frac{(N\cos\phi)'}{N^2\cos^2\phi}\frac{d\phi}{ds}.$

From (303), $\frac{d\phi}{ds}=\frac{1}{R}$ and from (279), $(N\cos\phi)'=-R\sin\phi$, so that

$$\tau''=\frac{\sin\phi}{N^2\cos^2\phi}=\frac{\tan\phi}{N^2\cos\phi}=\frac{t}{N^2\cos\phi}. \tag{307}$$

Continuing $\tau'''=\frac{N\cos^2\phi-2\sin\phi(N\cos\phi)'}{N^3\cos^3\phi}\frac{d\phi}{ds}$ and with $\frac{d\phi}{ds}=\frac{1}{R}$, $(N\cos\phi)'=-R\sin\phi$ this becomes $\tau'''=\frac{\frac{N}{R}\cos^2\phi+2\sin^2\phi}{N^3\cos^3\phi}=\frac{\frac{N}{R}+2\tan^2\phi}{N^3\cos\phi}=\frac{\frac{N}{R}+2t^2}{N^3\cos\phi}$, or since $\frac{N}{R}=1+\eta^2$ we may write finally

$$\tau'''=\frac{1}{N^3\cos\phi}(1+2t^2+\eta^2). \tag{308}$$

Continuing in this way we find that

$$\tau^{\text{iv}}=\frac{t}{N^4\cos\phi}(5+\eta^2+6t^2-4\eta^4), \tag{309}$$

$$\tau^{\text{v}}=\frac{1}{N^5\cos\phi}(5+6\eta^2+28t^2-3\eta^4+8t^2\eta^2+24t^4-4\eta^6+4t^2\eta^4+24t^2\eta^6), \tag{310}$$

$$\tau^{\text{vi}}=\frac{t}{N^6\cos\phi}\begin{pmatrix}61+46\eta^2+180t^2-3\eta^4+48t^2\eta^2+120t^4\\+100\eta^6-36t^2\eta^4-96t^2\eta^6+88\eta^8-192t^2\eta^8\end{pmatrix}, \tag{311}$$

$$\tau^{\text{vii}}=\frac{1}{N^7\cos\phi}\begin{bmatrix}61+662t^2+1{,}320t^4+720t^6+107\eta^2+43\eta^4+440t^2\eta^2\\+97\eta^6-234t^2\eta^4+336t^4\eta^2+188\eta^8-772t^2\eta^6-192t^4\eta^4+88\eta^{10}\\-2{,}392t^2\eta^8+408t^4\eta^6+1{,}536t^4\eta^8-1{,}632t^2\eta^{10}+1{,}920t^4\eta^{10}\end{bmatrix}, \tag{312}$$

$$\tau^{\text{viii}}=\frac{t}{N^8\cos\phi}\begin{bmatrix}1{,}385+7{,}266t^2+1{,}731\eta^2+10{,}920t^4+4{,}416t^2\eta^2-573\eta^4\\+5{,}040t^6-1{,}830t^2\eta^4+2{,}688t^4\eta^2-2{,}927\eta^6+5{,}052t^2\eta^6\\-1{,}536t^4\eta^4-8{,}808\eta^8+27{,}456t^2\eta^8+744t^4\eta^6-11{,}472\eta^{10}\\+53{,}952t^2\eta^{10}-7{,}872t^4\eta^8-4{,}672\eta^{12}+30{,}528t^2\eta^{12}\\-24{,}960t^4\eta^{10}-23{,}040t^4\eta^{12}\end{bmatrix}. \tag{313}$$

Placing the values of τ', τ'', τ''', τ^{iv}, τ^{v}, τ^{vi}, τ^{vii}, τ^{viii} from (303), (307), (308), (309), (310), (311), (312), (313) in equations (306) we have

$$\Delta\lambda=\rho\sec\phi_1\left\{\begin{aligned}&\frac{x}{N_1}-\frac{1}{6}\left(\frac{x}{N_1}\right)^3(1+2t_1^2+\eta_1^2)\\&+\frac{1}{120}\left(\frac{x}{N_1}\right)^5\begin{pmatrix}5+6\eta_1^2+28t_1^2-3\eta_1^4+8t_1^2\eta_1^2\\+24t_1^4-4\eta_1^6+4t_1^2\eta_1^4+24t_1^2\eta_1^6\end{pmatrix}\\&-\frac{1}{5{,}040}\left(\frac{x}{N_1}\right)^7\begin{bmatrix}61+662t_1^2+1{,}320t_1^4+720t_1^6+107\eta_1^2\\+43\eta_1^4+440t_1^2\eta_1^2+97\eta_1^6-234t_1^2\eta_1^4\\+336t_1^4\eta_1^2+188\eta_1^8-772t_1^2\eta_1^6-192t_1^4\eta_1^4\\+88\eta_1^{10}-2{,}392t_1^2\eta_1^8+408t_1^4\eta_1^6\\+1{,}536t_1^4\eta_1^8-1{,}632t_1^2\eta_1^{10}+1{,}920t_1^4\eta_1^{10}\end{bmatrix}\end{aligned}\right\}. \tag{314}$$

$$\Delta\tau=\tau-\tau_1=\rho t_1 \sec\phi_1\left\{\begin{aligned}&-\frac{1}{2}\left(\frac{x}{N_1}\right)^2+\frac{1}{24}\left(\frac{x}{N_1}\right)^4(5+\eta_1^2+6t_1^2-4\eta_1^4)\\&-\frac{1}{720}\left(\frac{x}{N_1}\right)^6\left[\begin{aligned}&61+46\eta_1^2+180t_1^2-3\eta_1^4\\&+48t_1^2\eta_1^2+120t_1^4+100\eta_1^6\\&-36t_1^2\eta_1^4-96t_1^2\eta_1^6+88\eta_1^8-192t_1^2\eta_1^8\end{aligned}\right]\\&+\frac{1}{40{,}320}\left(\frac{x}{N_1}\right)^8\left[\begin{aligned}&1{,}385+7{,}266t_1^2+1{,}731\eta_1^2+10{,}920t_1^4\\&+4{,}416t_1^2\eta_1^2-573\eta_1^4+5{,}040t_1^6-1{,}830t_1^2\eta_1^4\\&+2{,}688t_1^4\eta_1^2-2{,}927\eta_1^6+5{,}052t_1^2\eta_1^6\\&-1{,}536t_1^4\eta_1^4-8{,}808\eta_1^8+27{,}456t_1^2\eta_1^8\\&+744t_1^4\eta_1^6-11{,}472\eta_1^{10}+53{,}952t_1^2\eta_1^{10}\\&-7{,}872t_1^4\eta_1^8-4{,}672\eta_1^{12}+30{,}528t_1^2\eta_1^{12}\\&-24{,}960t_1^4\eta_1^{10}-23{,}040t_1^4\eta_1^{12}\end{aligned}\right]\end{aligned}\right\} \quad (315)$$

where the subscript one, on ϕ and the functions of ϕ involved, refers to the "footpoint" latitude, ϕ_1, of the point whose rectangular coordinates are given. (See fig. 27.)

In formula (314) the terms containing η_1^4 and higher powers of η_1 are usually omitted in the coefficient of $\left(\frac{x}{N_1}\right)^5$ and the term in $\left(\frac{x}{N_1}\right)^7$ is seldom needed. For example in the Army Map Service Technical Manual No. 19, the corresponding equation (314) for the universal transverse Mercator grid would be

$$\Delta\lambda=\rho \sec\phi_1\left[\frac{x}{N_1}-\frac{1}{6}\left(\frac{x}{N_1}\right)^3(1+2t_1^2+\eta_1^2)+\frac{1}{120}\left(\frac{x}{N_1}\right)^5(5+28t_1^2+24t_1^4+6\eta_1^2+8t_1^2\eta_1^2)\right].$$

Formula (315) does not give us directly the difference of the geodetic latitudes but the difference of the true isometric latitudes. In order to get the difference in geodetic latitudes, $\Delta\phi$, we expand $\Delta\phi$ into a Taylor's series in $\Delta\tau$ as follows:

$$\Delta\phi=\phi-\phi_1=\Delta\tau\frac{d\phi_1}{d\tau_1}+\frac{\Delta\tau^2}{2!}\frac{d^2\phi_1}{d\tau_1^2}+\frac{\Delta\tau^3}{3!}\frac{d^3\phi_1}{d\tau_1^3}+\frac{\Delta\tau^4}{4!}\frac{d^4\phi_1}{d\tau_1^4}+\cdots. \quad (316)$$

From (303) we have the relation $R_1 d\phi_1=N_1\cos\phi_1 d\tau_1$ or

$$\frac{d\phi_1}{d\tau_1}=\frac{N_1}{R_1}\cos\phi_1=(1+\eta_1^2)\cos\phi_1=P\cos\phi_1. \quad (317)$$

From (317) by differentiation we have $\frac{d^2\phi_1}{d\tau_1^2}=\left[\cos\phi_1\left(\frac{N_1}{R_1}\right)'-\frac{N_1}{R_1}\sin\phi_1\right]\frac{d\phi_1}{d\tau_1}$. From (279), $\left(\frac{N_1}{R_1}\right)'=-2\left(\frac{N_1}{R_1}-1\right)\tan\phi_1=-2(1+\eta_1^2-1)t_1=-2\eta_1^2 t_1$ and with the value $\frac{d\phi_1}{d\tau_1}$ from (317) we find

$$\frac{d^2\phi_1}{d\tau_1^2}=\frac{N_1}{R_1}t_1\cos^2\phi_1\left(2-3\frac{N_1}{R_1}\right)=-(1+\eta_1^2)(1+3\eta_1^2)t_1\cos^2\phi_1=-Q\cos^2\phi_1. \quad (318)$$

Continuing we find

$$\frac{d^3\phi_1}{d\tau_1{}^3}=\frac{N_1}{R_1}\cos^3\phi_1\left[4t_1^2+2\frac{N_1}{R_1}(1-9t_1^2)+3\left(\frac{N_1}{R_1}\right)^2(5t_1^2-1)\right]$$
$$=(1+\eta_1^2)\,[4t_1^2+2(1+\eta_1^2)(1-9t_1^2)+3(1+\eta_1^2)^2(5t_1^2-1)]\cos^3\phi_1$$
$$=S\cos^3\phi_1, \tag{319}$$

$$\frac{d^4\phi_1}{d\tau_1{}^4}=-\frac{N_1}{R_1}t_1\cos^4\phi_1\left[-8t_1^2+4\frac{N_1}{R_1}(21t_1^2-4)+4\left(\frac{N_1}{R_1}\right)^2(17-45t_1^2)\right.$$
$$\left.+3\left(\frac{N_1}{R_1}\right)^3(35t_1^2-19)\right]$$
$$=-t_1(1+\eta_1^2)\,[-8t_1^2+4(1+\eta_1^2)(21t^2-4)+4(1+\eta_1^2)^2(17-45t_1^2)$$
$$+3(1+\eta_1^2)^3(35t_1^2-19)]\cos^4\phi_1 \tag{320}$$
$$=-T\cos^4\phi_1.$$

We now write $\Delta\tau$ from (315) in the form

$$\Delta\tau=t_1\sec\phi_1\left[-\frac{1}{2}\left(\frac{x}{N_1}\right)^2+\frac{F}{24}\left(\frac{x}{N_1}\right)^4-\frac{G}{720}\left(\frac{x}{N_1}\right)^6+\frac{H}{40{,}320}\left(\frac{x}{N_1}\right)^8\right], \tag{321}$$

where F, G, H are the corresponding coefficients in η_1 and t_1.

By placing the values of the derivatives from (317), (318), (319), (320) and the value of $\Delta\tau$ from (321) in (316) we obtain, retaining terms in $\left(\frac{x}{N_1}\right)^8$

$$\Delta\phi=Pt_1\left[-\frac{1}{2}\left(\frac{x}{N_1}\right)^2+\frac{F}{24}\left(\frac{x}{N_1}\right)^4-\frac{G}{720}\left(\frac{x}{N_1}\right)^6+\frac{H}{40{,}320}\left(\frac{x}{N_1}\right)^8\right]$$
$$-\frac{Q}{2}t_1^2\left[\frac{1}{4}\left(\frac{x}{N_1}\right)^4-\frac{F}{24}\left(\frac{x}{N_1}\right)^6+\frac{F^2}{576}\left(\frac{x}{N_1}\right)^8+\frac{G}{720}\left(\frac{x}{N_1}\right)^8\right]$$
$$+\frac{S}{6}t_1^3\left[-\frac{1}{8}\left(\frac{x}{N_1}\right)^6+\frac{F}{32}\left(\frac{x}{N_1}\right)^8\right]-\frac{T}{24}t_1^4\left[\frac{1}{16}\left(\frac{x}{N_1}\right)^8\right], \text{ or}$$
$$\Delta\phi=-\frac{Pt_1}{2}\left(\frac{x}{N_1}\right)^2+\frac{PFt_1-3Qt_1^2}{24}\left(\frac{x}{N_1}\right)^4-\frac{PGt_1-15QFt_1^2+15St_1^3}{720}\left(\frac{x}{N_1}\right)^6$$
$$+\frac{PHt_1-Q(35F^2+28G)t_1^2+210SFt_1^3-105Tt_1^4}{40{,}320}\left(\frac{x}{N_1}\right)^8. \tag{322}$$

Placing the values of F, G, H, P, Q, S, T in (322), deleting terms in η_1 in the coefficient of $\left(\frac{x}{N_1}\right)^8$, we have finally

$$\Delta\phi=\phi-\phi_1=-\frac{\rho t_1}{2}(1+\eta_1^2)\left(\frac{x}{N_1}\right)^2+\frac{\rho t_1}{24}(1+\eta_1^2)\left(\frac{x}{N_1}\right)^4\cdot(5+3t_1^2+\eta_1^2-4\eta_1^4-9\eta_1^2t_1^2)$$
$$-\frac{\rho t_1}{720}(1+\eta_1^2)\left(\frac{x}{N_1}\right)^6\begin{pmatrix}61+90t_1^2+46\eta_1^2+45t_1^4-252t_1^2\eta_1^2-3\eta_1^4+100\eta_1^6-66t_1^2\eta_1^4\\-90t_1^4\eta_1^2+88\eta_1^8+225t_1^4\eta_1^4+84t_1^2\eta_1^6-192t_1^2\eta_1^8\end{pmatrix}$$
$$+\frac{\rho t_1}{40{,}320}(1+\eta_1^2)\left(\frac{x}{N_1}\right)^8(1{,}385+3{,}633t_1^2+4{,}095t_1^4+1{,}575t_1^6). \tag{323}$$

Since $1+\eta_1^2=\frac{N_1}{R_1}$, equation (323) may be written in the form

$$\phi=\phi_1-\frac{\rho t_1}{2R_1N_1}x^2+\frac{\rho t_1}{24R_1N_1^3}x^4(5+3t_1^2+\eta_1^2-4\eta_1^4-9\eta_1^2t_1^2)$$
$$-\frac{\rho t_1}{720R_1N_1^5}x^6\begin{pmatrix}61+90t_1^2+46\eta_1^2+45t_1^4-252t_1^2\eta_1^2-3\eta_1^4+100\eta_1^6-66t_1^2\eta_1^4\\-90t_1^4\eta_1^2+88\eta_1^8+225t_1^4\eta_1^4+84t_1^2\eta_1^6-192t_1^2\eta_1^8\end{pmatrix}$$
$$+\frac{\rho t_1}{40{,}320R_1N_1^7}x^8(1{,}385+3{,}633t_1^2+4{,}095t_1^4+1{,}575t_1^6). \tag{324}$$

Here $\rho=\text{cosec } 1''$, and the subscript one refers to the "footpoint" latitude.

To express the convergence of the meridian, γ, in terms of x we have from (217) $\tan\gamma=\frac{dy}{dx}$ for $\tau=\text{constant}$. Differentiating this equation yields $d\tau=\frac{\partial\tau}{\partial x}dx+\frac{\partial\tau}{\partial y}dy=0$, whence $\frac{dy}{dx}=-\frac{\partial\tau}{\partial x}\Big/\frac{\partial\tau}{\partial y}=-\frac{\partial\tau}{\partial x}\Big/\frac{\partial\lambda}{\partial x}$ since $\frac{\partial\tau}{\partial y}=\frac{\partial\lambda}{\partial x}$ from the Cauchy-Riemann equations. The derivatives $-\frac{\partial\tau}{\partial x}$ and $\frac{\partial\lambda}{\partial x}$ are obtained by differentiating (306) and therefore

$$\tan\gamma=-\frac{\partial\tau}{\partial x}\Big/\frac{\partial\tau}{\partial y}=\frac{x\tau''-\frac{x^3}{6}\tau^{\text{iv}}+\frac{x^5}{120}\tau^{\text{vi}}-\frac{x^7}{5{,}040}\tau^{\text{viii}}+\cdots}{\tau'-\frac{x^2}{2}\tau'''+\frac{x^4}{24}\tau^{\text{v}}-\frac{x^6}{720}\tau^{\text{vii}}+\cdots}. \tag{325}$$

Performing the division in (325) we have

$$\tan\gamma=Ax+\frac{x^3}{6}(3AB-C)+\frac{x^5}{120}(E-5AD+30AB^2-10BC)$$
$$+\frac{x^7}{5{,}040}(7AF-G+35CD+21BE-210ABD-210B^2C+630AB^3), \tag{326}$$

where $A=\frac{\tau''}{\tau'}, B=\frac{\tau'''}{\tau'}, C=\frac{\tau^{\text{iv}}}{\tau'}, D=\frac{\tau^{\text{v}}}{\tau'}, E=\frac{\tau^{\text{vi}}}{\tau'}, F=\frac{\tau^{\text{vii}}}{\tau'}, G=\frac{\tau^{\text{viii}}}{\tau'}$.

Computing the values of A, B, C, D, E, F, G from (303), (307), (308), (309), (310), (311), (312), (313) and placing them in (326), neglecting terms in η_1 in the coefficient of x^7, we have finally

$$\tan\gamma=\frac{t_1}{N_1}x-\frac{t_1}{3}\left(\frac{x}{N_1}\right)^3(1-\eta_1^2-2\eta_1^4)$$
$$+\frac{t_1}{15}\left(\frac{x}{N_1}\right)^5\cdot\begin{pmatrix}2+2\eta_1^2+9\eta_1^4+6t_1^2\eta_1^2+20\eta_1^6\\+3t_1^2\eta_1^4-27t_1^2\eta_1^6+11\eta_1^8-24t_1^2\eta_1^8\end{pmatrix}-\frac{17t_1}{315}\left(\frac{x}{N_1}\right)^7. \tag{327}$$

Writing (327) in the form

$$\tan\gamma=t_1\left(\frac{x}{N_1}\right)\left[1-\frac{a}{3}\left(\frac{x}{N_1}\right)^2+\frac{b}{15}\left(\frac{x}{N_1}\right)^4-\frac{17}{315}\left(\frac{x}{N_1}\right)^6\right], \tag{328}$$

where a, b are the corresponding coefficients in t_1 and η_1, we have, retaining terms in $\left(\frac{x}{N_1}\right)^7$

$$-\frac{1}{3}\tan^3\gamma=-\frac{1}{3}t_1^3\left(\frac{x}{N_1}\right)^3\left[1-a\left(\frac{x}{N_1}\right)^2+\frac{b}{5}\left(\frac{x}{N_1}\right)^4+\frac{a^2}{3}\left(\frac{x}{N_1}\right)^4\right]$$

$$+\frac{1}{5}\tan^5\gamma=\frac{1}{5}t_1^5\left(\frac{x}{N_1}\right)^5\left[1-\frac{5a}{3}\left(\frac{x}{N_1}\right)^2\right]$$

$$-\frac{1}{7}\tan^7\gamma=-\frac{1}{7}t_1^7\left(\frac{x}{N_1}\right)^7.$$

Substituting these values in the series for γ from (295)

$$\gamma=t_1\frac{x}{N_1}-\frac{t_1}{3}(a+t_1^2)\left(\frac{x}{N_1}\right)^3+\frac{t_1}{15}(b+5at_1^2+3t_1^4)\left(\frac{x}{N_1}\right)^5$$
$$-\frac{t_1}{315}(17+21bt_1^2+35a^2t_1^2+105at_1^4+45t_1^6)\left(\frac{x}{N_1}\right)^7. \tag{329}$$

Placing the values of a, b from (327) in (329), ignoring terms in η_1 in the coefficient of $\left(\frac{x}{N_1}\right)^7$, we have finally

$$\frac{\gamma}{\rho}=t_1\frac{x}{N_1}-\frac{t_1}{3}\left(\frac{x}{N_1}\right)^3(1+t_1^2-\eta_1^2-2\eta_1^4)+\frac{t_1}{15}\left(\frac{x}{N_1}\right)^5\cdot\begin{pmatrix}2+5t_1^2+2\eta_1^2+3t_1^4+t_1^2\eta_1^2+9\eta_1^4\\+20\eta_1^6-7t_1^2\eta_1^4-27t_1^2\eta_1^6+11\eta_1^8-24t_1^2\eta_1^8\end{pmatrix}$$
$$-\frac{t_1}{315}\left(\frac{x}{N_1}\right)^7(17+77t_1^2+105t_1^4+45t_1^6). \tag{330}$$

To express the scale, k, in terms of the abscissa, x, we write the reciprocal of k in the form

$$\frac{1}{k}=N\cos\phi\cdot\frac{d\lambda}{dx}\cdot\sqrt{1+\tan^2\gamma}. \tag{331}$$

From (314) we have

$$\frac{d\lambda}{dx}=\frac{\sec\phi_1}{N_1}\left[1-\frac{1}{2}U\left(\frac{x}{N_1}\right)^2+\frac{1}{24}V\left(\frac{x}{N_1}\right)^4-\frac{1}{720}W\left(\frac{x}{N_1}\right)^6\right], \tag{332}$$

where U, V, W are the corresponding coefficients in t_1 and η_1.

We now expand $N\cos\phi$ (the radius of a parallel on the spheroid in latitude ϕ) in a Taylor's series in $(\phi-\phi_1)$ as follows:

$$f(\phi)=N\cos\phi=f[\phi_1+(\phi-\phi_1)]=f(\phi_1)+(\phi-\phi_1)f'(\phi_1)+\frac{(\phi-\phi_1)^2}{2!}f''(\phi_1)$$
$$+\frac{(\phi-\phi_1)^3}{3!}f'''(\phi_1)+\dots \tag{333}$$

By differentiating $f(\phi)=N\cos\phi$ and using relations (279) we have

$$f'(\phi)=-R\sin\phi=-Rt\cos\phi, \tag{334}$$

$$f''(\phi)=-\frac{R^2}{N}\cos\phi(1+\eta^2+3t^2\eta^2), \tag{335}$$

$$f'''(\phi)=-\frac{R^3}{N^2}t\cos\phi(-1+7\eta^2+8\eta^4-3t^2\eta^2+12t^2\eta^4). \tag{336}$$

With the values of f', f'', f''' from (334), (335), (336) and the value of $\phi-\phi_1$ from (323) placed in (333) we find, retaining terms in $\left(\frac{x}{N_1}\right)^6$, that

$$N \cos\phi = N_1 \cos\phi_1 \left\{ \begin{aligned} &1+\frac{t_1^2}{2}\left(\frac{x}{N_1}\right)^2-\frac{t_1^2}{24}\left(\frac{x}{N_1}\right)^4(8+3t_1^2+4\eta_1^2-4\eta_1^4) \\ &+\frac{t_1^2}{720}\left(\frac{x}{N_1}\right)^6\left(\begin{aligned}&136+120t_1^2+136\eta_1^2+45t_1^4-12t_1^2\eta_1^2-48\eta_1^4\\&+40\eta_1^6-36t_1^2\eta_1^4+88\eta_1^8-96t_1^2\eta_1^6-192t_1^2\eta_1^8\end{aligned}\right)\end{aligned}\right\}. \quad (337)$$

From (337) we may write

$$N \cos\phi = N_1 \cos\phi_1 \left[1+\frac{t_1^2}{2}\left(\frac{x}{N_1}\right)^2-\frac{At_1^2}{24}\left(\frac{x}{N_1}\right)^4+\frac{Bt_1^2}{720}\left(\frac{x}{N_1}\right)^6\right], \quad (338)$$

where A, B are the corresponding coefficients in t_1 and η_1. We have also, by the binomial theorem, that

$$(1+\tan^2\gamma)^{1/2}=1+\frac{1}{2}\tan^2\gamma-\frac{1}{8}\tan^4\gamma+\frac{1}{16}\tan^6\gamma \ldots$$

and with the value of $\tan\gamma$ from (328) this becomes

$$(1+\tan^2\gamma)^{1/2}=1+\frac{t_1^2}{2}\left(\frac{x}{N_1}\right)^2-\frac{t_1^2}{24}\left(\frac{x}{N_1}\right)^4(8a+3t_1^2)+\frac{t_1^2}{720}\left(\frac{x}{N_1}\right)^6(40a^2+120at_1^2+48b+45t_1^4). \quad (339)$$

Placing the values of $\frac{d\lambda}{dx}$, $N\cos\phi$, $(1+\tan^2\gamma)^{1/2}$ from (332), (338), (339) in (331) and retaining terms in $\left(\frac{x}{N_1}\right)^6$, we obtain

$$\frac{1}{k}=1+\frac{2t_1^2-U}{2}\left(\frac{x}{N_1}\right)^2+\frac{V-t_1^2(A+8a+12U)+3t_1^4}{24}\left(\frac{x}{N_1}\right)^4 -\frac{15(A+3U)t_1^4-(B+15AU+30V+120aU+40a^2+48b)t_1^2+W}{720}\left(\frac{x}{N_1}\right)^6. \quad (340)$$

With values of U, V, W from (314); A, B from (337); a, b from (327) placed in (340), neglecting terms in η_1 in the coefficient of $\left(\frac{x}{N_1}\right)^6$, we have finally

$$\frac{1}{k}=1-\frac{1+\eta_1^2}{2}\left(\frac{x}{N_1}\right)^2+\frac{5+6\eta_1^2-3\eta_1^4-4\eta_1^6+24t_1^2\eta_1^4+24t_1^2\eta_1^6}{24}\left(\frac{x}{N_1}\right)^4-\frac{61}{720}\left(\frac{x}{N_1}\right)^6. \quad (341)$$

Taking the reciprocal of both sides of (341), we find by division that

$$k=1+\frac{1+\eta_1^2}{2}\left(\frac{x}{N_1}\right)^2+\frac{1+6\eta_1^2+9\eta_1^4+4\eta_1^6-24t_1^2\eta_1^4-24t_1^2\eta_1^6}{24}\left(\frac{x}{N_1}\right)^4+\frac{1}{720}\left(\frac{x}{N_1}\right)^6. \quad (342)$$

When the rectangular coordinates of a point are known, the "footpoint" latitude, ϕ_1, is easily found from a table of meridional distances since it is the latitude corresponding to $S_{\phi_1}=S_\phi \pm y$, where S_ϕ is the meridional distance to the origin, the plus sign applying to the northern hemisphere.

In the formulas given for x, y, γ, k the series converge rapidly and for relatively narrow belts many of the terms given will not be required. Usually the fifth- and higher-order terms may be discarded, but in high latitudes the effect of omitting any of the parts of the fourth-order terms should be examined before extensive computations are made. In many cases the omission of the spheroidal parts (those containing η or η_1) of the fourth-order term in y has more effect than the entire fifth-order term in x.

In computing, it is customary in the United States and in the British Kingdom to tabulate the coefficients in the formulas for sufficiently small intervals of latitude and to interpolate the corrections for higher-order terms from graphs (for instance the universal transverse Mercator coordinate system of the U. S. Army Map Service). However, the formulas can be expressed in terms of the latitude of the origin, so that the longer, less rapidly converging power series expansions have constant coefficients.[7]

DERIVATION OF FORMULAS FOR THE SPHERE

If we place $\epsilon=0$ in the formulas just developed for the transverse Mercator projection of the spheroid, the resulting formulas are valid for the sphere. However, the mapping coordinates for the sphere may be derived in closed form.

With $\epsilon=0$, we have from equations (255) and (259), $\tau=\ln\cot\frac{p}{2}$, or $e^\tau=\cot\frac{p}{2}$, whence $p=2\cot^{-1}(e^\tau)$. Hence a distance along the meridian from the pole is $p=2a\cot^{-1}(e^\tau)$, where a is the radius of the sphere.

Since the scale is to be true along the meridian, the analytic function which establishes the transverse Mercator projection of the sphere is given by

$$y+ix=2a\cot^{-1}e^{\tau+i\lambda}. \tag{343}$$

That is, when $\lambda=0$, $x=0$ and $y=2a\cot^{-1}(e^\tau)$, where the coordinates are referred to the pole.

With $a=1$ for simplicity in (343) we may write

$$\cot\tfrac{1}{2}(y+ix)=e^{\tau+i\lambda}=e^\tau(\cos\lambda+i\sin\lambda), \tag{344}$$

where $e^\tau=\cot\frac{p}{2}=\frac{\sin p}{1-\cos p}=\frac{\cos\phi}{1-\sin\phi}$.

Now we have

$$\cot\tfrac{1}{2}(y+ix)=\frac{\sin(y+ix)}{1-\cos(y+ix)}=\frac{1+\cos(y+ix)}{\sin(y+ix)}=\frac{1+\cos y\cos ix-\sin y\sin ix}{\sin y\cos ix+\cos y\sin ix},$$

and since $\sin ix=i\sinh x$, $\cos ix=\cosh x$, this last identity becomes

$$\cot\tfrac{1}{2}(y+ix)=\frac{1+\cos y\cosh x-i\sin y\sinh x}{\sin y\cosh x+i\cos y\sinh x},$$

which by rationalizing the denominator can be reduced to

$$\cot\tfrac{1}{2}(y+ix)=\frac{(\sin y-i\sinh x)(\cosh x+\cos y)}{\cosh^2 x-\cos^2 y}=\frac{\sin y-i\sinh x}{\cosh x-\cos y}. \tag{345}$$

From (344) and (345) we have then

$$\frac{\sin y-i\sinh x}{\cosh x-\cos y}=\frac{\cos\phi}{1-\sin\phi}(\cos\lambda+i\sin\lambda). \tag{346}$$

The coordinates x, y in (346) are referred to the pole. To refer them to the Equator we have but to replace trigonometric functions of y by cofunctions and change the sign of x. Thus (346) becomes

$$\frac{\cos y+i\sinh x}{\cosh x-\sin y}=\frac{\cos\phi}{1-\sin\phi}(\cos\lambda+i\sin\lambda). \tag{347}$$

[7] Wl. K. Hristow, Die Gauss-Krügerschen Koordinaten auf dem Ellipsoid, B. G. Teubner, Leipzig, 1943.

Equating real and imaginary parts in (347) we obtain the equations

$$\frac{\cos y}{\cosh x-\sin y}=\frac{\cos\phi\cos\lambda}{1-\sin\phi},\quad \frac{\sinh x}{\cosh x-\sin y}=\frac{\cos\phi\sin\lambda}{1-\sin\phi}. \tag{348}$$

Solving equations (348) for $\tanh x$, $\tan y$ we find $\tanh x=\cos\phi\sin\lambda$, $\tan y=\tan\phi\sec\lambda$, and reintroducing a in these last equations,

$$x=a\tanh^{-1}(\cos\phi\sin\lambda),\ y=a\tan^{-1}(\tan\phi\sec\lambda), \tag{349}$$

which are mapping equations for the transverse Mercator projection of the sphere.

Since $\tanh^{-1}U=\frac{1}{2}\ln\frac{1+U}{1-U}$, we may write x alternatively in equations (349) as $x=\frac{a}{2}\ln\left(\frac{1+\cos\phi\sin\lambda}{1-\cos\phi\sin\lambda}\right)$.

From equations (349) we may write

$$\tanh\frac{x}{a}=\cos\phi\sin\lambda,\qquad \tan\frac{y}{a}=\tan\phi\sec\lambda$$

$$\text{or}\quad \sec\phi=\sin\lambda\coth\frac{x}{a},\qquad \tan\phi=\cos\lambda\tan\frac{y}{a}, \tag{350}$$

$$\text{and}\quad \sin\lambda=\sec\phi\tanh\frac{x}{a},\qquad \cos\lambda=\tan\phi\cot\frac{y}{a}. \tag{351}$$

From equations (350) we eliminate ϕ by means of the identity $\sec^2\phi-\tan^2\phi=1$ which gives

$$\sin^2\lambda\coth^2\frac{x}{a}-\cos^2\lambda\tan^2\frac{y}{a}=1, \tag{352}$$

which is the equation of the meridians.

From equations (351) we eliminate λ by means of the identity $\sin^2\lambda+\cos^2\lambda=1$ and obtain

$$\sec^2\phi\tanh^2\frac{x}{a}+\tan^2\phi\cot^2\frac{y}{a}=1, \tag{353}$$

which is the equation of the parallels.

The scale factor for the sphere is, from (190) placing $N=a$,

$$k=\sqrt{\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2}\Big/a\cos\phi. \tag{354}$$

From equations (349) we have

$$\frac{\partial x}{\partial\lambda}=a\frac{\cos\phi\cos\lambda}{1-\cos^2\phi\sin^2\lambda},\quad \frac{\partial y}{\partial\lambda}=a\frac{\tan\phi\tan\lambda\sec\lambda}{1+\tan^2\phi\sec^2\lambda}=a\frac{\sin\lambda\sin\phi\cos\phi}{1-\cos^2\phi\sin^2\lambda}.$$

Thus

$$\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2=a^2\cos^2\phi\frac{\cos^2\lambda+\sin^2\lambda\sin^2\phi}{(1-\cos^2\phi\sin^2\lambda)^2}=\frac{a^2\cos^2\phi}{(1-\cos^2\phi\sin^2\lambda)},$$

and from (354) we have

$$k=\frac{a\cos\phi}{\sqrt{1-\cos^2\phi\sin^2\lambda}}\cdot\frac{1}{a\cos\phi}=\frac{1}{\sqrt{1-\cos^2\phi\sin^2\lambda}}. \tag{355}$$

If we replace ϕ by the conformal latitude χ to produce a conformal projection of the spheroid through the sphere then the total scale factor will be, from (190),

$$k=\frac{a\cos\chi}{N\cos\phi\sqrt{1-\cos^2\chi\sin^2\lambda}}=\frac{\cos\chi\sqrt{1-\epsilon^2\sin^2\phi}}{\cos\phi\sqrt{1-\cos^2\chi\sin^2\lambda}}. \tag{356}$$

GEODETIC CORRECTIONS FOR THE TRANSVERSE MERCATOR PROJECTION

Formulas (341) and (342) give respectively the value of $1/k$ and k for the transverse Mercator projection, and both are functions of x alone. Hence formulas (223) and (250) become

$$\begin{aligned}
\sigma&=\frac{d\beta}{ds}=\frac{1}{k}\frac{dk}{dx}\sin\beta,\\
k'&=\frac{dk}{dx}\cos\beta,\\
k''&=\frac{d^2k}{dx^2}\cos^2\beta-k\sigma^2,\\
\left(\frac{1}{k}\right)'&=\frac{d}{dx}\left(\frac{1}{k}\right)\frac{dx}{ds}=\frac{d}{dx}\left(\frac{1}{k}\right)\cos\beta,\\
\left(\frac{1}{k}\right)''&=\frac{d^2}{dx^2}\left(\frac{1}{k}\right)\cdot\frac{dx}{ds}\cdot\cos\beta-\frac{d}{dx}\left(\frac{1}{k}\right)\cdot\sin\beta\frac{d\beta}{ds}\\
&=\frac{d^2}{dx^2}\left(\frac{1}{k}\right)\cdot\cos^2\beta-\frac{d}{dx}\left(\frac{1}{k}\right)\cdot\sigma\sin\beta.
\end{aligned} \tag{357}$$

With these values we may compute rigorously the various corrections as given by equations (229) to (251). For geodetic computations, particularly the computation of plane coordinates over limited areas, approximation formulas have been developed and can be found in several sources, for instance: The South African Survey Journal, Volume V, Part 2, No. 35, January 1938, page 59; Empire Survey Review, Volume IX, No. 65, July, 1947, page 119; Jordan-Eggert, Handbuch der Vermessungskunde, Dritter Band, Zweiter Halbband, 1941, page 180; Driencourt et Laborde, Traité de Projections Cartes Géographiques, page 303; Clark, D., Plane and Geodetic Surveying, Volume II, Fourth edition, Chapter V.

The listed formulas of geodetic corrections for the transverse Mercator projection were taken from Jordan-Eggert and Clark.

THE OBLIQUE MERCATOR PROJECTION OF THE SPHERE

In figure 28, the point $O(\phi_0,\lambda_0)$ is the pole of the projection and $O'\left(0,\lambda_0-\frac{\pi}{2}\right)$ is the origin of x, y coordinates as shown. The great circle OO' is orthogonal to the meridian SOP at O. The great circle UO' is the Equator considering O to be the pole. That is,

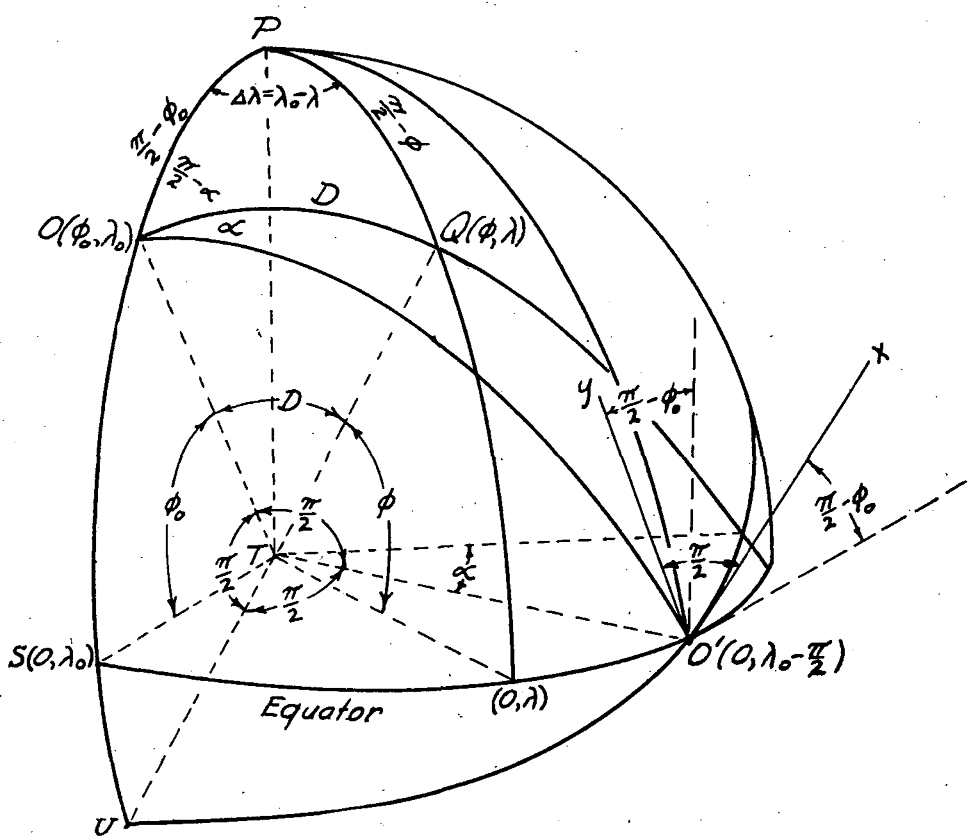


FIGURE 28.—Geometric transformations for deriving skew projections of the sphere.

if we move P to O, the equator SO' turns about the line $O'T$ through the angle $\frac{\pi}{2}-\phi_0$ and assumes the position of the great circle UO'.

Now the Mercator coordinates for the sphere in terms of colatitude and longitude are from equations (259), $x=a\lambda$, $y=a \ln \cot \frac{p}{2}$. From figure 28 with UO' as Equator and O as pole, it is seen that the corresponding colatitude is D and longitude is α, so that the corresponding Mercator coordinates at O' with respect to the pole O are then $x=a\alpha$, $y=a \ln \cot \frac{D}{2}$. But we wish the great circle OO' to be true to scale instead of the great circle UO'. Hence we interchange the values of x and y, namely

$$\begin{aligned} x&=a \ln \cot \frac{D}{2}=\frac{a}{2} \ln \frac{1+\cos D}{1-\cos D}=a \tanh^{-1} (\cos D), \\ y&=a\alpha. \end{aligned} \tag{358}$$

From the spherical triangle POQ we have

$$\begin{aligned} \cos D&=\sin \phi_0 \sin \phi+\cos \phi_0 \cos \phi \cos \Delta\lambda, \\ \sin D \cos \alpha&=\cos \phi \sin \Delta\lambda, \\ \sin D \sin \alpha&=\cos \phi_0 \sin \phi-\sin \phi_0 \cos \phi \cos \Delta\lambda, \end{aligned} \tag{359}$$

where $\Delta\lambda=\lambda_0-\lambda$.

Dividing the third by the second of equations (359) we obtain

$$\tan \alpha = \frac{\cos \phi_0 \sin \phi - \sin \phi_0 \cos \phi \cos \Delta\lambda}{\cos \phi \sin \Delta\lambda}. \tag{360}$$

From the first of equations (359) and equation (360) we may write equations (358) as

$$\begin{aligned} x &= \frac{a}{2} \ln \frac{1+\sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \sin \Delta\lambda}{1-\sin \phi_0 \sin \phi - \cos \phi_0 \cos \phi \sin \Delta\lambda} \\ &= a \tanh^{-1} (\sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \sin \Delta\lambda), \\ y &= a \tan^{-1} \frac{\sin \phi_0 \cos \phi \sin \Delta\lambda - \cos \phi_0 \sin \phi}{\cos \phi \cos \Delta\lambda}, \end{aligned} \tag{361}$$

where we have replaced $\Delta\lambda$ by $\Delta\lambda - \frac{\pi}{2}$, since, as seen from figure 28, the formulas (359) refer to the meridian SOP in longitude λ_0, while the mapping coordinates are referred to the point O' on the meridian PO' in longitude $\lambda_0 - \frac{\pi}{2}$. The formulas (361) will yield the proper values for $\phi_0 = 0, \frac{\pi}{2}$. That is, with $\phi_0 = 0$, equations (361) give equations (349) for the transverse Mercator projection. With $\phi_0 = \frac{\pi}{2}$, and interchanging x and y, we have $x = a\Delta\lambda$ and

$$y = \frac{a}{2} \ln \left(\frac{1+\sin \phi}{1-\sin \phi}\right) = \frac{a}{2} \ln \left(\frac{1+\cos p}{1-\cos p}\right) = a \ln \cot \frac{p}{2},$$

which are the coordinates of the Mercator projection of the sphere, equations (259). From equations (361) we have

$$\begin{aligned} \tanh \frac{x}{a} &= \sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \sin \Delta\lambda \\ \tan \frac{y}{a} &= \sec \phi \sec \Delta\lambda (\sin \phi_0 \cos \phi \sin \Delta\lambda - \cos \phi_0 \sin \phi). \end{aligned} \tag{362}$$

Solving equations (362) for $\sec \phi$ and $\tan \phi$ we have $\sec \phi = \coth \frac{x}{a} \sec \phi_0 \left(\sin \Delta\lambda - \sin \phi_0 \cos \Delta\lambda \tan \frac{y}{a}\right)$, and $\tan \phi = \sec \phi_0 \left(\sin \phi_0 \sin \Delta\lambda - \cos \Delta\lambda \tan \frac{y}{a}\right)$, whence by means of the identity $\sec^2\phi - \tan^2\phi = 1$, we obtain the equation for the meridians,

$$\left(\sin \Delta\lambda - \sin \phi_0 \cos \Delta\lambda \tan \frac{y}{a}\right)^2 \coth^2 \frac{x}{a} - \left(\sin \phi_0 \sin \Delta\lambda - \cos \Delta\lambda \tan \frac{y}{a}\right)^2 = \cos^2 \phi_0. \tag{363}$$

Solving equations (362) for $\sin \Delta\lambda$, $\cos \Delta\lambda$ we have $\sin \Delta\lambda = \sec \phi_0 \sec \phi \left(\tanh \frac{x}{a} - \sin\phi_0 \sin\phi\right)$, and $\cos \Delta\lambda = \sec \phi \cot \frac{y}{a}\left[\tan \phi_0 \left(\tanh \frac{x}{a} - \sin \phi_0 \sin \phi\right) - \cos \phi_0 \sin \phi\right]$, and by means of the identity $\sin^2\Delta\lambda + \cos^2\Delta\lambda = 1$, we obtain the equation of the parallels,

$$\sec^2\phi_0\left(\tanh\frac{x}{a}-\sin\phi_0\sin\phi\right)^2+\cot^2\frac{y}{a}\left[\tan\phi_0\left(\tanh\frac{x}{a}-\sin\phi_0\sin\phi\right)-\cos\phi_0\sin\phi\right]^2=\cos^2\phi. \tag{364}$$

Note that with $\phi_0=0$, equations (363) and (364) become equations (352) and (353) respectively as they should.

From the mapping equations (361) we have

$$\frac{\partial x}{\partial\Delta\lambda}=a\frac{\cos\phi_0\cos\phi\cos\Delta\lambda}{1-(\sin\phi_0\sin\phi+\cos\phi_0\cos\phi\sin\Delta\lambda)^2},\text{ and}$$

$$\frac{\partial y}{\partial\Delta\lambda}=a\cos\phi\frac{\sin\phi_0\cos\phi-\cos\phi_0\sin\phi\sin\Delta\lambda}{\cos^2\phi\cos^2\Delta\lambda+(\sin\phi_0\cos\phi\sin\Delta\lambda-\cos\phi_0\sin\phi)^2}$$

$$=a\cos\phi\frac{\sin\phi_0\cos\phi-\cos\phi_0\sin\phi\sin\Delta\lambda}{1-(\sin\phi_0\sin\phi+\cos\phi_0\cos\phi\sin\Delta\lambda)^2},$$

whence

$$\left(\frac{\partial x}{\partial\Delta\lambda}\right)^2+\left(\frac{\partial y}{\partial\Delta\lambda}\right)^2=a^2\cos^2\phi\frac{\cos^2\phi_0\cos^2\Delta\lambda+(\sin\phi_0\cos\phi-\cos\phi_0\sin\phi\sin\Delta\lambda)^2}{[1-(\sin\phi_0\sin\phi+\cos\phi_0\cos\phi\sin\Delta\lambda)^2]^2}$$

$$=\frac{a^2\cos^2\phi}{1-(\sin\phi_0\sin\phi+\cos\phi_0\cos\phi\sin\Delta\lambda)^2} \tag{365}$$

From (365) and (190) we have the scale factor for the projection of the sphere of radius a,

$$k=\frac{\sqrt{\left(\frac{\partial x}{\partial\Delta\lambda}\right)^2+\left(\frac{\partial y}{\partial\Delta\lambda}\right)^2}}{a\cos\phi}=\frac{1}{a\cos\phi}\frac{a\cos\phi}{\sqrt{1-(\sin\phi_0\sin\phi+\cos\phi_0\cos\phi\sin\Delta\lambda)^2}}$$

$$=1/\sqrt{1-(\sin\phi_0\sin\phi+\cos\phi_0\cos\phi\sin\Delta\lambda)^2}. \tag{366}$$

If the conformal latitude χ is substituted for geodetic latitude ϕ, the total scale factor for the spheroid becomes then from (190)

$$k=\frac{a\cos\chi}{N\cos\phi\sqrt{1-(\sin\chi_0\sin\chi+\cos\chi_0\cos\chi\sin\Delta\lambda)^2}}$$

$$=\frac{\cos\chi\sqrt{1-\epsilon^2\sin^2\phi}}{\cos\phi\sqrt{1-(\sin\chi_0\sin\chi+\cos\chi_0\cos\chi\sin\Delta\lambda)^2}}. \tag{367}$$

If the radius of the conformal sphere is used, we have from (257) and (190) the scale factor

$$k=\frac{N_0\cos\phi_0\cos\chi}{N\cos\phi\cos\chi_0\sqrt{1-(\sin\chi_0\sin\chi+\cos\chi_0\cos\chi\sin\Delta\lambda)^2}}. \tag{368}$$

In constructing an oblique Mercator projection where the great circle to be held true to scale is that through two given points, say Q_1 (ϕ_1, λ_1) and Q_2 (ϕ_2, λ_2), we must compute the latitude and longitude of the pole, O (ϕ_0, λ_0), on this great circle.

In figure 29, the point Q (ϕ, λ) is any point on the great circle through the points Q_1 (ϕ_1, λ_1), Q_2 (ϕ_2, λ_2). From the right spherical triangle POQ we have

$$\cos(\lambda-\lambda_0)=\tan\left(\frac{\pi}{2}-\phi_0\right)\cot\left(\frac{\pi}{2}-\phi\right)=\cot\phi_0\tan\phi, \tag{369}$$

which is the equation of the great circle OQ.

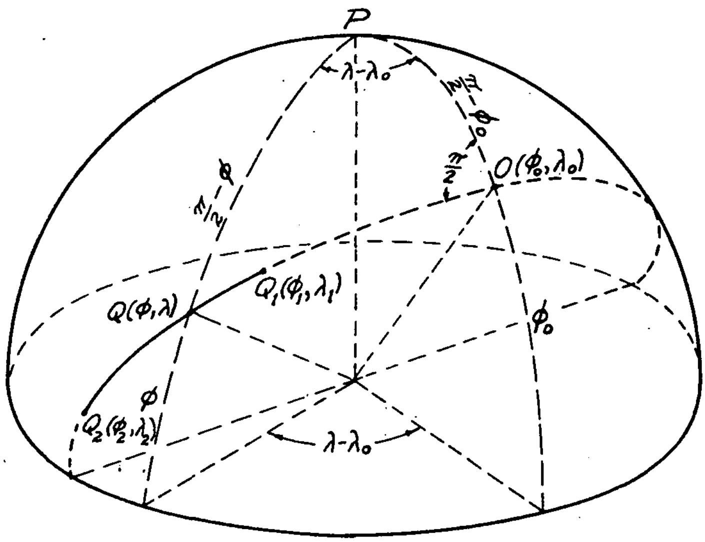


FIGURE 29.—Derivation of coordinates of the projection pole for the oblique Mercator projection of the sphere.

Now if the points Q_1 and Q_2 lie on the great circle OQ, then the coordinates ϕ_1, λ_1 and ϕ_2, λ_2 must satisfy equation (369), that is, we must have

$$\begin{aligned}\cos(\lambda_1-\lambda_0)&=\cot\phi_0\tan\phi_1,\\ \cos(\lambda_2-\lambda_0)&=\cot\phi_0\tan\phi_2,\end{aligned} \tag{370}$$

which may be written

$$\begin{aligned}\cos\lambda_1\cos\lambda_0+\sin\lambda_1\sin\lambda_0&=\cot\phi_0\tan\phi_1,\\ \cos\lambda_2\cos\lambda_0+\sin\lambda_2\sin\lambda_0&=\cot\phi_0\tan\phi_2.\end{aligned} \tag{371}$$

Solving equations (371) for $\sin\lambda_0$, $\cos\lambda_0$ we find $\sin\lambda_0=\cot\phi_0\csc(\lambda_2-\lambda_1)(\tan\phi_2\cos\lambda_1-\tan\phi_1\cos\lambda_2)$, and $\cos\lambda_0=\cot\phi_0\csc(\lambda_2-\lambda_1)(\tan\phi_1\sin\lambda_2-\tan\phi_2\sin\lambda_1)$, whence

$$\tan\lambda_0=\frac{\tan\phi_2\cos\lambda_1-\tan\phi_1\cos\lambda_2}{\tan\phi_1\sin\lambda_2-\tan\phi_2\sin\lambda_1}. \tag{372}$$

From equations (370) we have

$$\cot\phi_0=\cot\phi_1\cos(\lambda_1-\lambda_0)=\cot\phi_2\cos(\lambda_2-\lambda_0). \tag{373}$$

Hence given two points Q_1 (ϕ_1, λ_1), Q_2 (ϕ_2, λ_2) on the required great circle track we compute λ_0 from equation (372) and then ϕ_0 from either of equations (373) or from both as a check. With these values of ϕ_0 and λ_0 we may compute other points on the great circle track from equation (369).

We now derive the mapping equations of the diagonal Mercator projection in another form for direct use with the formulas just discussed. The origin of coordinates will now be at the point ϕ_0, λ_0 as determined by (372) and (373), or at the point Q (ϕ_0, λ_0) as shown in figure 30.

Analogously, as in figure 28, the point $O\left(\frac{\pi}{2}-\phi_0, \lambda_0-\pi\right)$ in figure 30 is the pole of the great circle QS. D corresponds to the colatitude of the point T referred to the

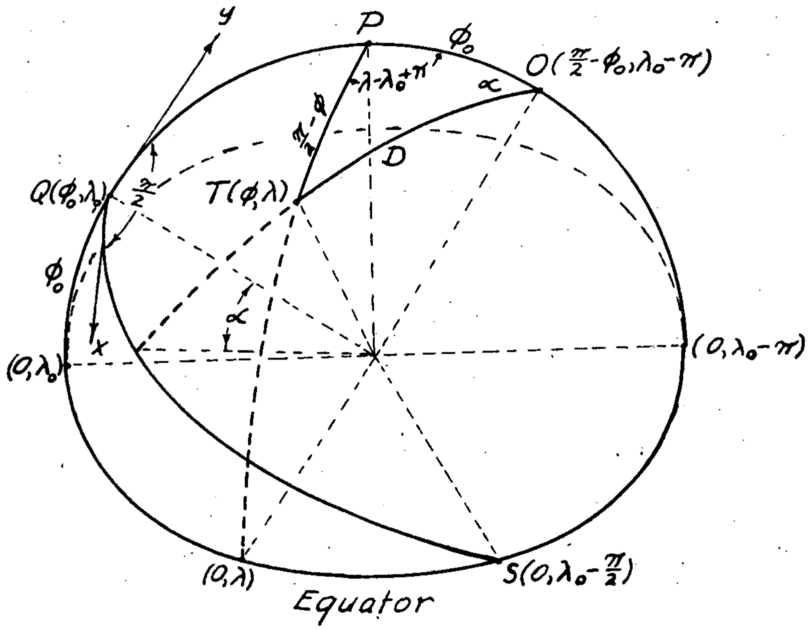


FIGURE 30.—Oblique Mercator projection of the sphere.

pole O and great circle QS; α is the measure of arc along QS corresponding to longitude along the Equator. Therefore the Mercator coordinates of T in this system are

$$x=a\alpha, y=\frac{a}{2}\ln\cot\frac{D}{2}=\frac{a}{2}\ln\frac{1+\cos D}{1-\cos D}. \tag{374}$$

From the spherical triangle POT we have the identities

$$\begin{aligned}\cos D&=\sin\phi\cos\phi_0-\cos\phi\sin\phi_0\cos\Delta\lambda\\ \tan\alpha&=\frac{\cos\phi\sin\Delta\lambda}{\sin\phi_0\sin\phi+\cos\phi_0\cos\phi\cos\Delta\lambda},\end{aligned} \tag{375}$$

where $\Delta\lambda=\lambda_0-\lambda$.

With the values of $\cos D$ and α from (375) we may write equations (374) as

$$\begin{aligned}x&=a\tan^{-1}\frac{\cos\phi\sin\Delta\lambda}{\sin\phi_0\sin\phi+\cos\phi_0\cos\phi\cos\Delta\lambda},\\ y&=\frac{a}{2}\ln\frac{1+\sin\phi\cos\phi_0-\cos\phi\sin\phi_0\cos\Delta\lambda}{1-\sin\phi\cos\phi_0+\cos\phi\sin\phi_0\cos\Delta\lambda}\\ &=a\tanh^{-1}(\sin\phi\cos\phi_0-\cos\phi\sin\phi_0\cos\Delta\lambda).\end{aligned} \tag{376}$$

To show that the point $Q(\phi_0, \lambda_0)$, as shown in figure 30, is now the origin of oblique Mercator coordinates of the sphere, place $\phi=\phi_0$, $\Delta\lambda=\lambda_0-\lambda=0$ in the mapping equations (376) which gives $x=y=0$ at the point $Q(\phi_0, \lambda_0)$.

THE OBLIQUE MERCATOR PROJECTION OF THE SPHEROID

A type of oblique Mercator projection of the spheroid, which will represent a considerable extent of spheroidal surface accurate enough for geodetic computations, is the representation through the aposphere devised by Brigadier M. Hotine. It employs hyperbolic functions and closed formulas to give simple computational forms after certain functions involved have been tabulated. The development is found in Hotine's Orthomorphic Projection of the Spheroid, Empire Survey Review, Volumes VIII and IX, 1946–47, Nos. 62–66. The particular formulas for this projection are found in No. 64, section 19, pages 66–69. In the tables which have been prepared for Malaya and Borneo—Projection Tables for British Commonwealth Territories in Borneo (Malaya), prepared by Directorate of Colonial Surveys, Teddington, Middlesex, England—the projection is called "rectified skew orthomorphic" and the publications contain examples of the use of the formulas.

Another type of diagonal, skew, or oblique Mercator projection of the spheroid is given by J. H. Cole in the Use of the Conformal Sphere for the Construction of Map Projections, Survey of Egypt, Paper No. 46, Giza 1943, where he obtains such a projection for Italy.

The projections of both Hotine and Cole are approximations. In geodetic work, we deal with the projected geodesic and Beltrami's theorem that only surfaces of constant curvature can be represented upon a plane so that all geodesics become straight lines indicates the undesirability of using geodesics to determine projections of the spheroid. By assuming the spheroid to be an aposphere, a surface of constant curvature applicable to the spheroid over a certain area, Hotine accomplishes the "rectified skew orthomorphic" projection of the spheroid by means of geodesics and within allowable error limits over a limited area of the spheroid. Cole accomplishes it through the conformal sphere, a surface of constant curvature.

THE LAMBERT CONFORMAL CONIC PROJECTION

First developed by Lambert in his "Beiträge zum Gebrauche der Mathematik," Berlin 1772, the projection was later fully discussed by Gauss. Although his "cylindrical orthomorphic" or so-called transverse Mercator projection seems destined to be the most important of the conformal projections, Lambert has already become immortal to cartographers because of his conformal conic projection.

The projection received great notice and publicity in World War I, when it was adopted for the battle maps in France. Suitable for areas of small latitudinal width but great longitudinal extent, it is used for maps of the United States and as a basis for the plane coordinate systems of many of the States of the Union. Many of the aeronautical charts published by the Coast and Geodetic Survey are based on it. It is used as a basis for most of the world aeronautical charts published by the Aeronautical Chart Service of the United States Air Force.

The Geographical Section of the General Staff in Canada uses it in connection with military surveys and the production of military maps.

It is the official projection of Venezuela and it is used by other South and Central American countries.

The following European countries use it officially: Belgium, Spain, France, Estonia, and Rumania.

On the African continent it is used officially by Algeria, Egypt, Libya, Tunisia, French and Spanish Morocco.

In Asia it is the official projection for India and Syria.

DERIVATION OF FORMULAS

The requirements for the projection are that the parallels and meridians shall be respectively arcs of concentric circles, and radii of these concentric circles.

To determine the most general form of the function in (189) we recall that the conformal mapping of the spheroid upon the plane was given by $\tau=\int_0^\phi \frac{R}{N}\sec\phi\, d\phi$, $\lambda=\lambda$ which gives for $\phi=c_1$, or $\lambda=c_2$ lines parallel to the coordinate axes in the $\tau\lambda$-plane or the Mercator projection of the spheroid. Hence the function $f(\lambda\pm i\tau)$ in (189) must be such that the parallel lines in the $\lambda\tau$-plane representing the meridians on the spheroid must be transformed into a pencil of lines in the xy-plane, and the parallel lines in the $\lambda\tau$-plane representing the parallels on the spheroid must be transformed into concentric circles in the xy-plane having the same center as the pencil of lines for the meridians. This means that x and y must be functions of τ and λ such that

$$x^2+y^2=K^2f(\tau), y=m(\lambda)\cdot x. \tag{377}$$

In (377) we see that, since τ is a function of ϕ alone, for every value of ϕ we will get a circle, and since m is a function of λ alone for every value of λ we will get a straight line with slope $m(\lambda)$.

Now solving equations (377) for x and y find

$$x=K\frac{\sqrt{f(\tau)}}{[1+m^2(\lambda)]^{½}}, y=K\frac{m(\lambda)\sqrt{f(\tau)}}{[1+m^2(\lambda)]^{½}}. \tag{378}$$

We know that for the analytic function (189) to exist the functions $x(\lambda,\tau)$, $y(\lambda,\tau)$ of (378) must satisfy the Cauchy-Riemann equations (207). From (378) we have $\frac{\partial x}{\partial\lambda}=-K\frac{\sqrt{f}mm'}{(\sqrt{1+m^2})^3}$, $\frac{\partial x}{\partial\tau}=\frac{Kf'}{2\sqrt{f}\sqrt{1+m^2}}$, $\frac{\partial y}{\partial\lambda}=\frac{K\sqrt{f}m'}{(\sqrt{1+m^2})^3}$, and $\frac{\partial y}{\partial\tau}=\frac{Kmf'}{2\sqrt{f}\sqrt{1+m^2}}$, whence the Cauchy-Riemann equations $\frac{\partial x}{\partial\lambda}=\frac{\partial y}{\partial\tau}$, $\frac{\partial x}{\partial\tau}=-\frac{\partial y}{\partial\lambda}$ both lead to the equation

$$-\frac{f'}{f}=\frac{2m'}{1+m^2}. \tag{379}$$

Since f is a function of τ alone and m a function of λ alone, the only possibility for (379) is for both ratios to be equal to the same constant, for example $2l$. We have then from (379) the two differential equations $\frac{f'}{f}=-2l$, $\frac{m'}{1+m^2}=+l$ or $\frac{df}{f}=-2ld\tau$, $\frac{dm}{1+m^2}=ld\lambda$, whence the solutions are $\ln f=-2l\tau$ and $\arctan m=l\lambda$ (considering constants of integration zero). These may be written

$$f(\tau)=e^{-2l\tau}, m(\lambda)=\tan l\lambda. \tag{380}$$

Returning the value of $f(\tau)$ and $m(\lambda)$ to (378), we have

$$x=Ke^{-l\tau}\cos l\lambda,\ y=Ke^{-l\tau}\sin l\lambda \tag{381}$$

where $r=Ke^{-l\tau}$ is the radius of the map parallels and from (189),

$$e^{l\tau}=\tan^l\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-\epsilon\sin\phi}{1+\epsilon\sin\phi}\right)^{\frac{\epsilon l}{2}}.$$

Equations (381) are sufficient since we have used the Cauchy-Riemann equations in obtaining them, but we can actually write the function (189), $x+iy=f(\lambda\pm i\tau)$, from equations (381). That is, we have $x+iy=Ke^{-l\tau}$ ($\cos l\lambda+i\sin l\lambda$) and from (17) we have $\cos\theta+i\sin\theta=e^{+i\theta}$ where $\theta=l\lambda$ so that $x+iy=Ke^{-l\tau}\cdot e^{+il\lambda}=Ke^{+il(\lambda+i\tau)}=f(\lambda+i\tau)$.

Referring now to equations (23) and figure 5 (p. 26), it is seen that the case discussed there was that given by (381) with $K=1$, $l=-1$.

Note that equations (381) can be expressed in terms of Mercator coordinates since for the Mercator projection $x_M=a\lambda$, $y_M=a\tau$.

From (381) we have $\frac{\partial x}{\partial\tau}=-Kle^{-l\tau}\cos l\lambda$, $\frac{\partial y}{\partial\tau}=-Kle^{-l\tau}\sin l\lambda$ and from (190) the scale is given by

$$k=\frac{\sqrt{\left(\frac{\partial x}{\partial\tau}\right)^2+\left(\frac{\partial y}{\partial\tau}\right)^2}}{N\cos\phi}=\frac{\sqrt{K^2l^2e^{-2l\tau}\cos^2 l\lambda+K^2l^2e^{-2l\tau}\sin^2 l\lambda}}{N\cos\phi}=\frac{Kle^{-l\tau}}{N\cos\phi}, \tag{382}$$

where, as before, $e^{l\tau}=\tan^l\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-\epsilon\sin\phi}{1+\epsilon\sin\phi}\right)^{\frac{\epsilon l}{2}}$.

We have two arbitrary constants or parameters, K and l, at our disposal in equations (381) which we may use to impose two conditions upon the projection. Let us use them to hold the length true along two parallels. From (382) if we are to hold the length exact along the parallels, ϕ_1 and ϕ_2, we have

$$k=\frac{Kle^{-l\tau_1}}{N_1\cos\phi_1}=\frac{Kle^{-l\tau_2}}{N_2\cos\phi_2}=1. \tag{383}$$

From (383) we have $\left(\frac{e^{-\tau_1}}{e^{-\tau_2}}\right)^l=\frac{N_1\cos\phi_1}{N_2\cos\phi_2}$, whence taking logarithms of both sides and solving for l we have

$$l=\frac{\ln N_1-\ln N_2+\ln\cos\phi_1-\ln\cos\phi_2}{\tau_2-\tau_1}. \tag{384}$$

Again from (383) we have

$$K=\frac{N_1\cos\phi_1}{le^{-l\tau_1}}=\frac{N_2\cos\phi_2}{le^{-l\tau_2}}. \tag{385}$$

Hence having been given ϕ_1 and ϕ_2, l is computed from (384), whence K is computed by either relation in (385) or by both as a check.

It is easier to compute the map radii if we expand $r=Ke^{-l\tau}$ into a power series, in arc length, s, of the meridian of the spheroid, about the fixed map radius, $r(\phi_0)$, corresponding to the fixed parallel, ϕ_0. That is, we expand $\Delta r=r(\phi)-r(\phi_0)$ by Taylor's theorem in a power series to be tabulated for sufficiently small intervals of ϕ. By Taylor's theorem we have

$$\Delta r = r(\phi) - r(\phi_0) = r'(\phi_0)s + r''(\phi_0)\frac{s^2}{2!} + r'''(\phi_0)\frac{s^3}{3!} + r^{\text{iv}}(\phi_0)\frac{s^4}{4!} + r^{\text{v}}(\phi_0)\frac{s^5}{5!} + r^{\text{vi}}(\phi_0)\frac{s^6}{6!} + \ldots \quad (386)$$

From $r = Ke^{-l\tau}$, we have $\ln r = \ln K - l\tau$, whence by differentiation with respect to arc length, s, of the meridian of the spheroid

$$\frac{r'}{r} = -l\frac{d\tau}{ds}. \quad (387)$$

For this derivation we choose the arc length, s, as positive with decreasing latitude ϕ to correspond to positive values of increase in r, the map radius. Hence from (186) we have

$$N \cos \phi \, d\tau = R \, d\phi = -ds. \quad (388)$$

Whence $\frac{d\tau}{ds} = -\frac{1}{N \cos \phi}$ and (387) becomes

$$\frac{r'}{r} = \frac{l}{N \cos \phi}. \quad (389)$$

Taking logarithms of both sides of (389) we have $\log r' - \log r = \log l - \log (N \cos \phi)$, and differentiating this last gives

$$\frac{r''}{r'} - \frac{r'}{r} = -\frac{(N \cos \phi)'}{N \cos \phi}\frac{d\phi}{ds}. \quad (390)$$

With the values of $(N \cos \phi)' = -R \sin \phi$ from (279), $\frac{d\phi}{ds} = -\frac{1}{R}$ from (388), and the value of $\frac{r'}{r}$ from (387) placed in (390) we have

$$\frac{r''}{r'} = \frac{l}{N \cos \phi} - \frac{\sin \phi}{N \cos \phi} = \frac{l - \sin \phi}{N \cos \phi}. \quad (391)$$

From (391) again by logarithms we have, $\log r'' - \log r' = \log (l - \sin \phi) - \log (N \cos \phi)$, and by differentiation

$$\frac{r'''}{r''} - \frac{r''}{r'} = -\left[\frac{\cos \phi}{l - \sin \phi} + \frac{(N \cos \phi)'}{N \cos \phi}\right]\frac{d\phi}{ds}. \quad (392)$$

With the values of $(N \cos \phi)' = -R \sin \phi$ from (279), $\frac{d\phi}{ds}$ from (388), $\frac{r''}{r'}$ from (391), we may write (392) as

$$\frac{r'''}{r'} = \frac{1}{RN} + \frac{(l - \sin \phi)(l - 2 \sin \phi)}{N^2 \cos^2 \phi}. \quad (393)$$

Differentiating (393) gives

$$\frac{r^{\text{iv}}}{r'} - \frac{r''}{r'} \cdot \frac{r'''}{r'} = -\left[\frac{(NR)'}{N^2R^2} + \frac{l - 2 \sin \phi}{N^2 \cos \phi} + \frac{2(l - \sin \phi)}{N^2 \cos \phi} + \right.$$

$$\left. \frac{2(l - \sin \phi)(l - 2 \sin \phi)(N \cos \phi)'}{N^3 \cos^3 \phi}\right]\frac{d\phi}{ds}. \quad (394)$$

Now $(NR)' = 4R(N - R) \tan \phi$ and with the value of $(N \cos \phi)' = -R \sin \phi$ and the other values from (388), (391), and (393), we find that (394) becomes

$$\frac{r^{\text{iv}}}{r'}=\frac{\tan\phi}{R^2N^2}(4N-5R)+\frac{4(l-\sin\phi)}{RN^2\cos\phi}+\frac{(l-\sin\phi)(l-2\sin\phi)(l-3\sin\phi)}{N^3\cos^3\phi}. \tag{395}$$

Differentiating (395) we find

$$\frac{r^{\text{v}}}{r'}-\frac{r''}{r'}\cdot\frac{r^{\text{iv}}}{r'}=\left\{\begin{array}{l}\sec^2\phi\dfrac{(4N-5R)}{R^2N^2}+\dfrac{\tan\phi}{R^2N^2}\left[(5R'-4N')+2\left(\dfrac{5N'R}{N}-\dfrac{4NR'}{R}\right)\right]\\ -\dfrac{4}{RN^2}+4(l-\sin\phi)\left(\dfrac{1}{RN^2\cos\phi}\right)'-\dfrac{(l-2\sin\phi)(l-3\sin\phi)}{N^3\cos^2\phi}\\ +(l-\sin\phi)\left[\dfrac{(l-2\sin\phi)(l-3\sin\phi)}{N^3\cos^3\phi}\right]'\end{array}\right\}\frac{d\phi}{ds}. \tag{396}$$

Simplifying with the values of N', R' from (279) and the values from (388), (391) (395) we find that (396) becomes

$$\frac{r^{\text{v}}}{r'}=\frac{9R-4N}{R^3N^2}+3\tan^2\phi\frac{(9R^2-16NR+8N^2)}{R^3N^3}+20\frac{\sin\phi(l-\sin\phi)}{R^2N^2\cos^2\phi}+\frac{5(l-\sin\phi)(2l-9\sin\phi)}{RN^3\cos^2\phi}$$
$$+\frac{(l-\sin\phi)(l-2\sin\phi)(l-3\sin\phi)(l-4\sin\phi)}{N^4\cos^4\phi}. \tag{397}$$

Continuing as before by differentiating (397) and simplifying by known relations we find that

$$\frac{r^{\text{vi}}}{r'}=-\frac{\tan\phi}{R^4N^3}(88N^2-228NR+161R^2)-\frac{12\tan^3\phi}{R^4N^4}(14R^3-41R^2N+44RN^2-16N^3)$$
$$+\frac{(l-\sin\phi)(20l^2-175l\sin\phi+432\sin^2\phi)}{RN^4\cos^3\phi}$$
$$+\frac{4(l-\sin\phi)(16\cos^2\phi-102\sin^2\phi+15l\sin\phi)}{R^2N^3\cos^3\phi}+\frac{24(l-\sin\phi)(6\sin^2\phi-\cos^2\phi)}{R^3N^2\cos^3\phi}$$
$$+\frac{(l-\sin\phi)(l-2\sin\phi)(l-3\sin\phi)(l-4\sin\phi)(l-5\sin\phi)}{N^5\cos^5\phi}. \tag{398}$$

If we hold the radius of the parallel, ϕ_0, we have from (383) that $K=\dfrac{N_0\cos\phi_0}{le^{-l\tau_0}}$, or

$$Ke^{-l\tau_0}=\frac{N_0\cos\phi_0}{l}=r(\phi_0). \tag{399}$$

From (389) $r'(\phi_0)=r(\phi_0)\dfrac{l}{N_0\cos\phi_0}$, whence with the value of $r(\phi_0)$ from (399) we have

$$r'(\phi_0)=1. \tag{400}$$

The radius of the parallel in latitude ϕ_0 is $N_0\cos\phi_0$, (fig. 21, p. 64).

Then

$$l=\frac{d(N_0\cos\phi_0)}{ds}=\frac{d(N_0\cos\phi_0)}{d\phi}\cdot\frac{d\phi}{ds}=-R_0\sin\phi_0\cdot-\frac{1}{R_0}=\sin\phi_0. \tag{401}$$

From (399) and (401) we have $r(\phi_0)=N_0\cot\phi_0$. This may also be seen geometrically from figure 18 (p. 59).

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With the value of $r'(\phi_0)$ and l from (400) and (401), equations (391), (393), (395), (397), (398) become respectively with $\phi=\phi_0$, $r''(\phi_0)=0$, $r'''(\phi_0)=\frac{1}{R_0N_0}$, $r^{\mathrm{iv}}(\phi_0)=\frac{\tan\phi_0}{R_0^2N_0^2}(4N_0-5R_0)$, $r^{\mathrm{v}}(\phi_0)=\frac{9R_0-4N_0}{R_0^3N_0^2}+3\tan^2\phi_0\,\frac{(9R_0^2-16N_0R_0+8N_0^2)}{R_0^3N_0^3}$, $r^{\mathrm{vi}}(\phi_0)=-\frac{\tan\phi_0}{R_0^4N_0^4}\cdot[N_0(88N_0^2-228N_0R_0+161R_0^2)+12\tan^2\phi_0(14R_0^3-41R_0^2N_0+44R_0N_0^2-16N_0^3)]$.

With these values of the derivatives at ϕ_0 placed in (386) we have finally

$$\begin{aligned}\Delta r=s+\frac{s^3}{6R_0N_0}-\frac{s^4(5R_0-4N_0)\tan\phi_0}{24R_0^2N_0^2}\\
+\frac{s^5[N_0(9R_0-4N_0)+3(9R_0^2-16N_0R_0+8N_0^2)\tan^2\phi_0]}{120R_0^3N_0^3}\\
-\frac{s^6\tan\phi_0}{720\,R_0^4N_0^4}[N_0(88N_0^2-228N_0R_0+161R_0^2)\\
+12(14R_0^3-41R_0^2N_0+44R_0N_0^2-16N_0^3)\tan^2\phi_0].\end{aligned}\tag{402}$$

It is customary for general use to place $N_0=R_0$ in the numerators of the terms in s^5 and s^6. Equation (402) becomes then

$$\Delta r=s+\frac{s^3}{6R_0N_0}-\frac{s^4(5R_0-4N_0)\tan\phi_0}{24R_0^2N_0^2}+\frac{s^5(5+3\tan^2\phi_0)}{120R_0N_0^3}-\frac{s^6(7+4\tan^2\phi_0)\tan\phi_0}{240\,R_0N_0^4}.\tag{403}$$

See for instance, D. Clark, Plane and Geodetic Surveying, Volume II, Fourth edition, page 372.

In (402) or (403) Δr is positive when s is positive, that is, when the point to be mapped is in latitude $\phi<\phi_0$ and the map radius is $r=r(\phi_0)+\Delta r$ with Δr given by (402) or (403). When the point to be mapped is in latitude $\phi>\phi_0$, then s is negative and Δr will be negative. That is, by replacing s by $-s$ in (402) or (403) all terms on the right will have negative signs and Δr will thus be negative. Then will $r=r(\phi_0)-\Delta r$ where Δr is obtained from (402) or (403) with all signs positive.

A SECOND METHOD OF OBTAINING THE SERIES EXPANSION FOR Δr

From (389) we have $\frac{dr}{r}=\frac{l}{N\cos\phi}\,ds$ or

$$\frac{dr}{ds}=\frac{rl}{N\cos\phi}.\tag{404}$$

But $r=r_0+\Delta r$ and $\frac{dr}{ds}=\frac{d(r_0+\Delta r)}{ds}=\frac{d(\Delta r)}{ds}$, consequently (404) may be written

$$\frac{d(\Delta r)}{ds}\cdot(N\cos\phi)-l(r_0+\Delta r)=0.\tag{405}$$

Let us assume that Δr is given by a series in the form

$$\Delta r=As+Bs^2+Cs^3+Ds^4+Es^5+Fs^6+\ldots.\tag{406}$$

Then

$$\frac{d(\Delta r)}{ds}=A+2Bs+3Cs^2+4Ds^3+5Es^4+6Fs^5+\ldots.\tag{407}$$

We next expand N cos ϕ, the radius of the ellipsoid parallel, in a Taylor's series of the form

$$N\cos\phi=N_0\cos\phi_0+(N_0\cos\phi_0)'s+(N_0\cos\phi_0)''\frac{s^2}{2!}+(N_0\cos\phi_0)'''\frac{s^3}{3!}$$
$$+(N_0\cos\phi_0)^{\text{iv}}\frac{s^4}{4!}+(N_0\cos\phi_0)^{\text{v}}\frac{s^5}{5!}+\ .\ .\ .\ . \tag{408}$$

From (401) we have $\dfrac{d(N\cos\phi)}{ds}=\sin\phi$,

whence
$$(N\cos\phi)''=\cos\phi\frac{d\phi}{ds}=-\frac{\cos\phi}{R}. \tag{409}$$

Continuing $(N\cos\phi)'''=-\dfrac{-R\sin\phi-R'\cos\phi}{R^2}\cdot-\dfrac{1}{R}=-\dfrac{R\sin\phi+R'\cos\phi}{R^3}$ and with the value of R' from (279) this becomes

$$(N\cos\phi)'''=\frac{\sin\phi}{NR^2}(3R-4N). \tag{410}$$

Continuing in this manner we find

$$(N\cos\phi)^{\text{iv}}=-\frac{N(3R-4N)\cos\phi+12(N-R)(2N-R)\sin\phi\tan\phi}{N^2R^3}, \tag{411}$$

and

$$(N\cos\phi)^{\text{v}}=\frac{\sin\phi}{N^3R^4}[N(45R^2-132RN+88N^2)$$
$$-12(N-R)(16N^2-20NR+5R^2)\tan^2\phi]. \tag{412}$$

With the values of the derivatives from (401), (409), (410), (411), (412) we may write (408) as

$$N\cos\phi=N_0\cos\phi_0+s\sin\phi_0-s^2\frac{\cos\phi_0}{2R_0}+s^3\frac{\sin\phi_0}{6N_0R_0^2}(3R_0-4N_0)$$
$$-s^4\frac{\cos\phi_0}{24N_0^2R_0^3}[N_0(3R_0-4N_0)+12(N_0-R_0)(2N_0-R_0)\tan^2\phi_0] \tag{413}$$
$$+s^5\frac{\sin\phi_0}{120N_0^3R_0^4}[N_0(45R_0^2-132R_0N_0+88N_0^2)$$
$$-12(N_0-R_0)(16N_0^2-20N_0R_0+5R_0^2)\tan^2\phi_0].$$

From (401) we have $l=\sin\phi_0$ and $r_0=r(\phi_0)=N_0\cot\phi_0$. Hence with these values and those from (406), (407), and (413) we may write equation (405) in the form

$$(A+2Bs+3Cs^2+4Ds^3+5Es^4+6Fs^5+\ldots)\begin{bmatrix} N_0\cos\phi_0+s\sin\phi_0-s^2\dfrac{\cos\phi_0}{2R_0} \\ +s^3\dfrac{\sin\phi_0}{6N_0R_0^2}(3R_0-4N_0)-C_1s^4+C_2s^5\end{bmatrix}$$
$$-\sin\phi_0(N_0\cot\phi_0+As+Bs^2+Cs^3+Ds^4+Es^5+\ldots)=0, \tag{414}$$

where C_1 and C_2 are the corresponding coefficients of s^4 and s^5 in (413).

In equation (414) we now equate to zero the sums of the coefficients of like powers of s, which will give equations in A, B, C, D, E, F to solve.

For the constant terms we have

$$AN_0 \cos \phi_0 - N_0 \cos \phi_0 = 0,$$

whence

$$A = 1. \tag{415}$$

For the terms in s we have

$$2BN_0 \cos \phi_0 + A \sin \phi_0 - A \sin \phi_0 = 0,$$

or

$$B = 0. \tag{416}$$

We now place $A=1$, $B=0$ in (414) and continue, finding for the terms in s^2

$$-\frac{\cos \phi_0}{2R_0} + 3CN_0 \cos \phi_0 = 0,$$

or

$$C = \frac{1}{6N_0R_0}. \tag{417}$$

With this value of C placed in (414) we find for terms in s^3,

$$\frac{\sin \phi_0}{6N_0R_0^2}(3R_0 - 4N_0) + \frac{3 \sin \phi_0}{6N_0R_0} + 4DN_0 \cos \phi_0 - \frac{\sin \phi_0}{6N_0R_0} = 0,$$

from which we find

$$D = \frac{\tan \phi_0}{24N_0^2R_0^2}(4N_0 - 5R_0). \tag{418}$$

Returning this value of D to (414) we find for terms in s^4,

$$-\frac{\cos \phi_0}{24N_0^2R_0^3}[N_0(3R_0 - 4N_0) + 12(N_0 - R_0)(2N_0 - R_0) \tan^2 \phi_0] - \frac{3 \cos \phi_0}{12R_0^2N_0}$$

$$+ \frac{3 \sin \phi_0 \tan \phi_0 R_0(4N_0 - 5R_0)}{24N_0^2R_0^3} + 5EN_0 \cos \phi_0 = 0,$$

and solving for E we have

$$E = \frac{9R_0 - 4N_0}{120N_0^2R_0^3} + \frac{3 \tan^2 \phi_0 (9R_0^2 - 16N_0R_0 + 8N_0^2)}{120N_0^3R_0^3}. \tag{419}$$

For the coefficients of the terms in s^5 we find the equation

$$C_2 + C\frac{\sin \phi_0}{2N_0R_0^2}(3R_0 - 4N_0) - 2\frac{D}{R_0}\cos \phi_0 + 4E \sin \phi_0 + 6FN_0 \cos \phi_0 = 0. \tag{420}$$

With the value of C_2 from (413) and the values of C, D, E from (417), (418), (419) placed in (420) we find, solving for F, that

$$F = -\frac{\tan \phi_0}{720N_0^4R_0^4}[N_0(88N_0^2 - 228N_0R_0 + 161R_0^2)$$

$$+ 12(14R_0^3 - 41N_0R_0^2 + 44R_0N_0^2 - 16N_0^3) \tan^2 \phi_0]. \tag{421}$$

Placing the values of A, B, C, D, E, F from (415), (416), (417), (418), (419), (421) in (406) we have

$$\Delta r = s + \frac{s^3}{6N_0R_0} - \frac{s^4 \tan\phi_0}{24N_0^2R_0^2}(5R_0 - 4N_0) + \frac{s^5}{120N_0^3R_0^3}[N_0(9R_0 - 4N_0)$$

$$+3(9R_0^2 - 16N_0R_0 + 8N_0^2)\tan^2\phi_0]$$

$$-\frac{s^6 \tan\phi_0}{720N_0^4R_0^4}[N_0(88N_0^2 - 228N_0R_0 + 161R_0^2)$$

$$+12(14R_0^3 - 41R_0^2N_0 + 44R_0N_0^2 - 16N_0^3)\tan^2\phi_0],$$

which is identical with the value obtained in equation (402).

In order to use the formula for Δr, as given by equation (403), to compute radii for the Lambert conformal conic projection, it is seen from equations (384) and (401) that we must have $\sin\phi_0 = l = \dfrac{\ln N_1 - \ln N_2 + \ln\cos\phi_1 - \ln\cos\phi_2}{\tau_2 - \tau_1}$,

or

$$\phi_0 = \sin^{-1} l = \sin^{-1}\frac{\ln N_1 - \ln N_2 + \ln\cos\phi_1 - \ln\cos\phi_2}{\tau_2 - \tau_1}. \tag{422}$$

With this value of ϕ_0, we have the map radius and scale at ϕ_0 given by

$$r(\phi_0) = Ke^{-\tau_0 \sin\phi_0},\ k_0 = \frac{r(\phi_0)}{N_0}\tan\phi_0, \tag{423}$$

where K is given by (385).

To determine the scale factor as a function of s we have $k_s = \dfrac{dr(s)}{ds} = \dfrac{d}{ds}[r(\phi_0) + m\Delta r] = m\,\dfrac{d(\Delta r)}{ds}$.

From equation (402) we have $\dfrac{d\Delta r}{ds} = 1 + \dfrac{s^2}{2R_0N_0} - A\dfrac{s^3}{6R_0^2N_0^2} + B\dfrac{s^4}{24R_0^3N_0^3} - C\dfrac{s^5}{120R_0^4N_0^4}$, whence $k_s = m\left(1 + \dfrac{s^2}{2R_0N_0} - A\dfrac{s^3}{6R_0^2N_0^2} + B\dfrac{s^4}{24R_0^3N_0^3} - C\dfrac{s^5}{120R_0^4N_0^4}\right)$.

Now when $s = 0$, we have from this last equation that $m = k_0$, whence

$$k_s = k_0\left(1 + \frac{s^2}{2R_0N_0} - A\frac{s^3}{6R_0^2N_0^2} + B\frac{s^4}{24R_0^3N_0^3} - C\frac{s^5}{120R_0^4N_0^4}\right), \tag{424}$$

where A, B, C are respectively the numerators of the coefficients of the terms in s^4, s^5, s^6, in equation (402).

ONE STANDARD PARALLEL

The projection just discussed with two standard parallels is the conformal conic projection discussed by Lambert and is also called a conformal secant conical projection since the cone through the two rectified parallels is a secant cone with respect to the spheroid.

If we desire to hold the scale along only one parallel, say latitude ϕ_0, we have from (382)

$$k = \frac{Kle^{-l\tau_0}}{N_0\cos\phi_0} = 1, \text{ or } Kle^{-l\tau_0} = N_0\cos\phi_0. \tag{425}$$

For a second condition let us suppose that the map radius for ϕ_0, $r(\phi_0)=Ke^{-l\tau_0}$, is equal to the length of the tangent to the meridian from the point of tangency in latitude ϕ_0 to the polar axis, or equivalently equal to the slant height of the cone touching the spheroid along the parallel ϕ_0. From figure 18 (p. 59) it is seen that the slant height is $N_0 \cot \phi_0$, hence we have

$$r(\phi_0)=Ke^{-l\tau_0}=N_0 \cot \phi_0. \tag{426}$$

By dividing the members of (425) by the respective members of (426) we find at once that $l=\sin \phi_0$, whence $K=e^{\tau_0 \sin \phi_0} N_0 \cot \phi_0$, or the constants are

$$l=\sin \phi_0,\ K=e^{\tau_0 \sin \phi_0} N_0 \cot \phi_0. \tag{427}$$

These are identical with the values obtained in equations (399) and (401) as they should be.

We may then use the value of Δr as given by equation (403) to compute the map radii according to the rules as stated in the last section but we need no scale correction for Δr. The mapping equations are given as before by equations (381). This projection is often called the conformal simple conic projection or the Lambert conformal conic projection with one standard parallel, since it is a special case of the Lambert conformal conic projection.

We obtained the analytic function (189) for the Lambert conformal conic projection by starting with the equations of the required meridians and parallels involving general functions of τ and λ. Then, after solving for x and y in terms of the arbitrary functions of τ and λ involved, we demanded that x and y satisfy the Cauchy-Riemann equations. This produced the differential equations whose solutions gave the required forms of the functions of τ and λ. See equation (379). We will now show how the same result may be produced by considering the curvature of the map meridians and map parallels.

From (216) we have the curvatures of the meridians and parallels in a conformal projection given by

$$\frac{1}{R_\lambda}=-\frac{\partial G^{-\frac{1}{2}}}{\partial \lambda},\ \frac{1}{R_\tau}=-\frac{\partial G^{-\frac{1}{2}}}{\partial \tau}, \tag{428}$$

where $E=G=f'(\lambda+i\tau)f'(\lambda-i\tau)$.

In the Lambert conformal conic projection the meridians are straight lines. Hence the radius of curvature of the meridians is infinite, that is, $R_\lambda \rightarrow \infty$ and we have $\frac{\partial G^{-\frac{1}{2}}}{\partial \lambda}=0$. This means that G is a function of τ alone. Hence we have $G=f'(\lambda+i\tau)f'(\lambda-i\tau)=F(\tau)$. If we differentiate this last equation with respect to λ, writing g for $f(\lambda-i\tau)$, we obtain $f''g'+f'g''=0$, or $\frac{f''}{f'}=-\frac{g''}{g'}$. But since the first of these ratios is a function of $\lambda+i\tau$ alone and the second of $\lambda-i\tau$ alone, the equality can only exist if each ratio is equal to a constant, for example c. Hence with $u=\lambda+i\tau$, $v=\lambda-i\tau$ we have

$$\begin{aligned} &\frac{f''(u)}{f'(u)}=c,\ \ln f'(u)=cu+\ln A,\ f'(u)=Ae^{cu},\ f(u)=\frac{A}{c}e^{cu}+B. \\ &\frac{g''(v)}{g'(v)}=-c,\ \ln g'(v)=-cv+\ln A,\ g'(v)=Ae^{-cv},\ g(v)=-\frac{A}{c}e^{-cv}+B. \end{aligned} \tag{429}$$

Since the scale ratio, (190), must be real and contain the product $f'(\lambda+i\tau)f'(\lambda-i\tau)=f'(u)g'(v)=A^2e^{c(u-v)}=A^2e^{c(2i\tau)}$, it is seen that c must be pure imaginary, for example $c=il$. Then with $B=0$, $K=A/c$ we have finally from (429), $f(u)=f(\lambda+i\tau)=Ke^{il(\lambda+i\tau)}$ which is the same as found before.

Finally, to give the complete geometric characterization, we derive the differential equation of the map radius directly from geometric properties. From figure 21 (p. 64) we have $\cot\alpha=(-Rd\phi)/N\cos\phi d\lambda$, where we have taken the arc length along the meridian to be negative so that an increase in map radius will correspond to a decrease in latitude. From figure 31, we have the corresponding angle β, and in terms of the map elements, $\cot\beta=dr/rld\lambda$.

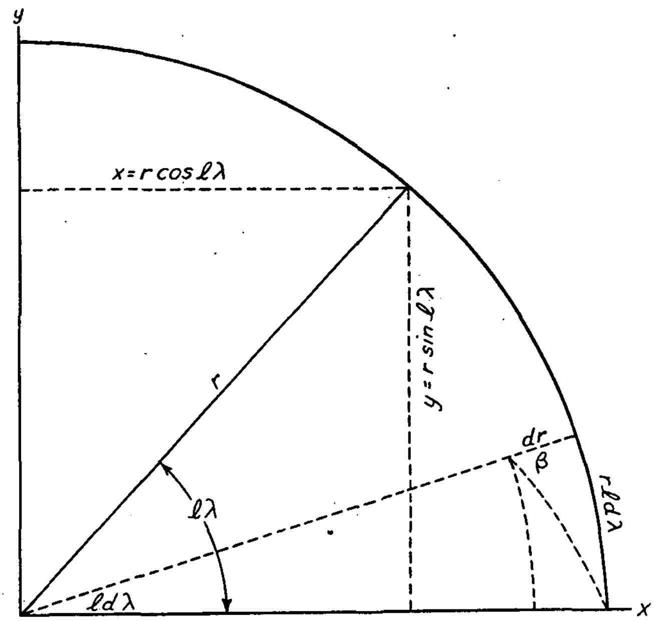

FIGURE 31.—Elements of Lambert conformal conic projection.

If the projection is to be conformal, then the angles α and β must be equal, that is, $\cot\alpha=\cot\beta=\dfrac{-Rd\phi}{N\cos\phi d\lambda}=\dfrac{dr}{rld\lambda}$, whence

$$\frac{dr}{r}=-l\frac{R}{N}\sec\phi\, d\phi, \tag{430}$$

which is the differential equation of the map radius. Integrating we have $\ln r=-l\int\frac{R}{N}\sec\phi\, d\phi+\ln K$. But from equation (252),

$$\int\frac{R}{N}\sec\phi\, d\phi=\tau=\ln\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-\epsilon\sin\phi}{1+\epsilon\sin\phi}\right)^{\frac{\epsilon}{2}}\right],$$

hence $\ln r=-l\tau+\ln K$, or $r=Ke^{-l\tau}=K\left[\cot^l\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1+\epsilon\sin\phi}{1-\epsilon\sin\phi}\right)^{\frac{l\epsilon}{2}}\right]$, as obtained before. See equations (381).

In order to reduce as far as possible the maximum scale error over the whole area to be projected by the Lambert conformal projection the mean latitude for the area should be chosen as standard parallel. The scale error at the maximum distance in the representation from this standard parallel is then determined and the coordinates are multiplied by the reciprocal of one-half of this scale error. This actually produces a Lambert conformal conic projection with two correct parallels whose latitudinal distances north and south of the central parallel will be approximately two-thirds of the latitude differences of the central parallel and the maximum north and south latitudes of the map. That is, we may specify two standard parallels to be held true to scale and then determine the central parallel from these, or we may choose a central parallel and apply an arbitrary over-all scale factor which results in the fixing of two unspecified standard parallels.

The origin is usually chosen as the intersection of the central meridian with the standard parallel in order to avoid computation of large values of the map radius, i. e. in order to use to advantage the formulas for Δr as given by equations (402) or (403).

POINT-TO-POINT WORKING ON THE LAMBERT CONFORMAL CONIC PROJECTION

E. L. M. Burns in "Point-to-Point Working for the Conical Orthomorphic Projection", Empire Survey Review, Volume II, No. 11, 1934, developed convenient formulas for corrections to bearing and distance allowing coordinates to be computed by the ordinary methods of plane trigonometry, but the formulas were intended to be used for topographical work only.

J. Clendinning investigated the formulas for point-to-point working on the conformal conic projection and extended Burns' formulas for application to more precise work. His work was published in the Empire Survey Review, Volume VII, Nos. 48, 51, 52; 1943–44.

Clendinning developed rigorously the necessary formulas which are extensive, cumbersome and devoid of terms higher than the third order although as he demonstrates for points no farther away from the origin than occur in practice, terms of the fourth order may hardly suffice to give the accuracy desired.

The investigation proved conclusively that if point-to-point working is the most important consideration when choosing a projection for a given area, then the Lambert conformal conic projection, when the formulas are expressed as functions of rectangular coordinates, is not the best projection to select, even when the area to be covered has its principal extent in longitude. Several belts of transverse Mercator coordinates give more satisfactory results than a single belt of Lambert conformal conic coordinates.

Brigadier K. M. Papworth in "The $(t—T)$ correction for the Lambert No. 2 (Conical Orthomorphic) Projection", Empire Survey Review, Volume VIII, No. 56, 1945, developed by empirical methods simplified formulas for computing the $(t—T)$ correction. B. L. Gulatee in "Angular Corrections for the Lambert Orthomorphic Conical Projection", Empire Survey Review, Volume VIII, No. 62, 1946 gave the mathematical proof of Papworth's formula and presented the correction in another simple form involving the chord of the projected geodesic and its curvature evaluated at a point one-third of the way along the arc.

In order to apply the Lambert conformal conic projection to China, J. T. Fang developed suitable formulas in terms of the vertical distance between the parallel passing through an arbitrary point of the map and the central parallel. The formulas containing third-order terms in this parameter suffice for zones of 3½° in latitude difference and 65° in longitude. These developments are found in the Empire Survey Review, Volume IX, Nos. 70 and 71, 1948–49. Fang also includes the formulas for transformation of coordinates from one zone to an adjacent one with numerical examples of their application. His derivations of formulas for azimuth and distance corrections for geodetic lines on the Lambert conformal conic projection are found in the Empire Survey Review, Volume X, No. 75, 1950. His formulas for transformation between the Lambert conformal conic and the transverse Mercator projections are given in the Empire Survey Review, Volume X, No. 74, 1949.

The derivations of the usual formulas for point-to-point working on the Lambert conformal conic projection are found in several treatises. Some of these sources are: Jordan-Eggert. Handbuch der Vermessungskunde, Dritter Band, Zweiter Halbband, Stuttgart, 1941, pages 204–218. Driencourt et Laborde. Traité des Projections des Cartes Géographiques, Paris 1932, Volume IV, pages 323–331. Clark, D. Plane and Geodetic Surveying. Volume II, Fourth edition, London, 1951, pages 370–376. Courtier, M. Exposé de la Projection de Lambert, Annales Hydrographiques, Tome Dix-septième, Paris, 1946, pages 101–114.

The formulas listed in the front of this publication for point-to-point working on the Lambert conformal conic projection have been taken from these sources..

THE STEREOGRAPHIC PROJECTION

Hipparchus (about 150 B. C.), to whom we are indebted for plane and spherical trigonometry, is also credited with the invention of the stereographic projection. It was employed in the astrolabe-planisphere for the solution of the astronomical triangle as revealed in Chaucer's treatise on the astrolabe.

The ruler and compass constructions of this projection and graphical solution of problems by means of it had and still have a fascination for geometers. The most important of these constructions and solutions may be found in U. S. Coast and Geodetic Survey Special Publication No. 57. Two papers by S. L. Penfield, "The Stereographic Projection and its Possibilities from a Graphical Standpoint," and "On the Use of the Stereographic Projection for Geographical Maps and Sailing Charts," published in the American Journal of Science for February 1901 and May 1902 give additional graphical applications of the projection. A complete treatise is found in "The Stereographic Projection" by F. W. Sohon. (See the bibliography.)

Besides its use in cartography, it is of interest to the student of the complex variable. It is also used in its various forms for the solutions of problems in crystallography, seismology, astronomy, navigation, and hydrodynamics.

As in the case of the other conformal projections already discussed, the stereographic projection of the spheroid is complicated compared to the sphere. A way of avoiding the difficulty is to employ a method which has already been explained, namely that of projecting the spheroid conformally on a sphere and then projecting the sphere on a plane. This method will be followed here in discussing the horizonal and equatorial forms.

POLAR STEREOGRAPHIC PROJECTION OF THE SPHEROID

For the polar stereographic projection, the meridians are straight lines radiating from a central point corresponding to the pole of the spheroid and the parallels are concentric circles about this central point. It is thus clear that the polar stereographic projection of the spheroid is a special case of the Lambert projection where one of the fixed parallels is taken to be the pole, or equivalently when l is placed equal to 1. Geometrically the tangent or secant cone of the Lambert projection has become the tangent plane at the pole.

With $l=1$, equations (381) become

$$x=r\cos\lambda,\ y=r\sin\lambda, \tag{431}$$

where $r=Ke^{-\tau}$, $e^{\tau}=\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-\epsilon\sin\phi}{1+\epsilon\sin\phi}\right)^{\frac{\epsilon}{2}}=\tan\left(\frac{\pi}{4}+\frac{\chi}{2}\right)=\cot\frac{z}{2}$, z being as before the colatitude of the conformal latitude, χ. See equations (256) and (259).

The scale factor is obtained from (382) by placing $l=1$, namely

$$k=\frac{Ke^{-\tau}}{N\cos\phi}=\frac{K\tan\frac{z}{2}}{N\cos\phi}. \tag{432}$$

We have the arbitrary constant K which we may use to hold the scale along a given parallel, ϕ_0. If we place $k=1$ in (432) and solve for K, it is found that

$$K=N_0\cos\phi_0\cot\frac{z_0}{2}. \tag{433}$$

To determine the value of K when $\phi_0=\frac{\pi}{2}$ (the scale is then true only at the pole), we may write equation (433) by means of (431) as $K=N\cos\phi\cot\frac{z}{2}=N\cos\phi$ $\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-\epsilon\sin\phi}{1+\epsilon\sin\phi}\right)^{\frac{\epsilon}{2}}=\frac{N\cos^2\phi}{1-\sin\phi}\left(\frac{1-\epsilon\sin\phi}{1+\epsilon\sin\phi}\right)^{\frac{\epsilon}{2}}=\frac{a(1+\sin\phi)}{\sqrt{1-\epsilon^2\sin^2\phi}}\left(\frac{1-\epsilon\sin\phi}{1+\epsilon\sin\phi}\right)^{\frac{\epsilon}{2}}.$

Placing $\phi=\frac{\pi}{2}$ in this last equation we obtain

$$K=\frac{2a}{\sqrt{1-\epsilon^2}}\left(\frac{1-\epsilon}{1+\epsilon}\right)^{\frac{\epsilon}{2}}=\frac{2a^2}{b}\left(\frac{1-\epsilon}{1+\epsilon}\right)^{\frac{\epsilon}{2}}. \tag{434}$$

The scale factor is then, from (432) and (434)

$$k=\frac{2a^2}{b\,N\cos\phi}\left(\frac{1-\epsilon}{1+\epsilon}\right)^{\frac{\epsilon}{2}}\tan\frac{z}{2}. \tag{435}$$

The mapping equations for this particular case are then

$$\left.\begin{aligned}&x=r\cos\lambda,\ y=r\sin\lambda\\ &r=k_0Ke^{-\tau}=k_0\frac{a^2}{b}\left(\frac{1-\epsilon}{1+\epsilon}\right)^{\frac{\epsilon}{2}}\tan\frac{z}{2},\end{aligned}\right\} \tag{436}$$

where k_0 is the scale factor at the pole—an arbitrary reduction applied to all geodetic lengths to reduce the maximum scale distortion of the projection.

It should be noted that the polar stereographic projection of the spheroid is not perspective. If we place $\epsilon=0$ in the mapping equations (436), z will become the polar distance, and we will then have a perspective stereographic projection of the sphere from the South Pole upon the tangent plane at the North Pole.

From (259) we have $\tan \frac{z}{2}=\tan \frac{p}{2}\left(\frac{1+\epsilon \cos p}{1-\epsilon \cos p}\right)^{\frac{\epsilon}{2}}$. Hence r in equation (436) may be expanded analogously as before for the Lambert projection into a series in the colatitude p. The series to terms in p^{11} is

$$r=\frac{k_0 a}{\sqrt{1-\epsilon^2}}\left(p+\frac{1-7\epsilon^2}{12(1-\epsilon^2)}p^3+\frac{1-2\epsilon^2+46\epsilon^4}{120(1-\epsilon^2)^2}p^5+\frac{17-93\epsilon^2-1{,}335\epsilon^4-4{,}889\epsilon^6}{20{,}160(1-\epsilon^2)^3}p^7\right.$$

$$+\frac{31-184\epsilon^2+3{,}831\epsilon^4+41{,}906\epsilon^6+53{,}641\epsilon^8}{362{,}880(1-\epsilon^2)^4}p^9 \tag{437}$$

$$\left.+\frac{691-4{,}841\epsilon^2-44{,}966\epsilon^4-2{,}420{,}926\epsilon^6-10{,}194{,}436\epsilon^8-6{,}982{,}072\epsilon^{10}}{79{,}833{,}600(1-\epsilon^2)^5}p^{11}\right).$$

If the coefficients in equation (437) are evaluated for the international spheroid we have

$$r=6{,}361{,}536.988459\ p+508{,}600.09984\ p^3$$
$$+53{,}122.087\ p^5+5{,}252.83\ p^7+539.4\ p^9+54\ p^{11}, \tag{438}$$

where r is in meters and p is in radians. If r is in meters and p is in minutes of arc multiplied by 10^{-3}, equation (437) becomes

$$r=1{,}850{,}496.09893\ p+12{,}518.57204\ p^3+110.63836\ p^5$$
$$+0.9257\ p^7+0.0080\ p^9+0.0001\ p^{11}. \tag{439}$$

In the polar stereographic projection the convergence is equal to the longitude, λ, in numerical value. In the northern polar area it has the same sign as λ. In the southern polar area it has the opposite sign. Since the pole is the center of the projection it is seen that the convergence may be any angle up to 180° E or W. That is, at times grid north may be the same direction as true south, east, or west, depending on the position on the projection.

Tables are usually constructed of r with ϕ or p as argument. Hence to compute ϕ and λ from rectangular coordinates we have from equations (431),

$$\tan \lambda=\frac{y}{x} \text{ or } \lambda=\tan^{-1}\frac{y}{x},\ r=x \sec \lambda=y \csc \lambda, \tag{440}$$

where ϕ for the corresponding value of r is interpolated from the tables.

DEVELOPMENT OF r IN SERIES

The method of undetermined coefficients will be used in obtaining the series expansion for r as given by equation (436). From equation (430) the corresponding equation for the polar stereographic projection in terms of p is

$$\frac{N}{R}\sin p\cdot\frac{dr}{dp}-r=0. \tag{441}$$

Now
$$f(p)=\frac{N}{R}\sin p=\frac{(1-\epsilon^2\cos^2 p)\sin p}{1-\epsilon^2}. \tag{442}$$

Next expand $f(p)$ in a Maclaurin series in p of the form

$$f(p)=f(0)+f'(0)p+f''(0)\frac{p^2}{2!}+\ .\ .\ .\ +f^{(n)}(0)\frac{p^n}{n!}+\ .\ .\ .\ . \tag{443}$$

We have from equation (442)

$$
\begin{aligned}
(1-\epsilon^2)f(p)&=(1-\epsilon^2\cos^2 p)\sin p, & f(0)&=0.\\
(1-\epsilon^2)f'(p)&=(1+2\epsilon^2)\cos p-3\epsilon^2\cos^3 p, & f'(0)&=1.\\
(1-\epsilon^2)f''(p)&=(9\epsilon^2\cos^2 p-2\epsilon^2-1)\sin p, & f''(0)&=0.\\
(1-\epsilon^2)f'''(p)&=27\epsilon^2\cos^3 p-(1+20\epsilon^2)\cos p, & f'''(0)&=-(1-7\epsilon^2)/(1-\epsilon^2).\\
(1-\epsilon^2)f^{\text{iv}}(p)&=(1+20\epsilon^2-81\epsilon^2\cos^2 p)\sin p, & f^{\text{iv}}(0)&=0.\\
(1-\epsilon^2)f^{\text{v}}(p)&=(1+182\epsilon^2)\cos p-243\epsilon^2\cos^3 p, & f^{\text{v}}(0)&=(1-61\epsilon^2)/(1-\epsilon^2).\\
(1-\epsilon^2)f^{\text{vi}}(p)&=(729\epsilon^2\cos^2 p-182\epsilon^2-1)\sin p, & f^{\text{vi}}(0)&=0.\\
(1-\epsilon^2)f^{\text{vii}}(p)&=2{,}187\epsilon^2\cos^3 p-(1+1{,}640\epsilon^2)\cos p, & f^{\text{vii}}(0)&=-(1-547\epsilon^2)/(1-\epsilon^2).\\
(1-\epsilon^2)f^{\text{viii}}(p)&=(1+1{,}640\epsilon^2-6{,}561\epsilon^2\cos^2 p)\sin p, & f^{\text{viii}}(0)&=0.\\
(1-\epsilon^2)f^{\text{ix}}(p)&=(1+14{,}762\epsilon^2)\cos p-19{,}683\epsilon^2\cos^3 p, & f^{\text{ix}}(0)&=(1-4{,}921\epsilon^2)/(1-\epsilon^2).\\
(1-\epsilon^2)f^{\text{x}}(p)&=(59{,}049\epsilon^2\cos^2 p-14{,}762\epsilon^2-1)\sin p, & f^{\text{x}}(0)&=0.\\
(1-\epsilon^2)f^{\text{xi}}(p)&=177{,}147\epsilon^2\cos^3 p-(1+132{,}860\epsilon^2)\cos p, & f^{\text{xi}}(0)&=-(1-44{,}287\epsilon^2)/(1-\epsilon^2).
\end{aligned}
\tag{444}
$$

With the values of the derivatives from (444), the series (443) becomes

$$
\begin{aligned}
f(p)=\frac{N}{R}\sin p=p-\frac{1-7\epsilon^2}{6(1-\epsilon^2)}p^3+\frac{1-61\epsilon^2}{120(1-\epsilon^2)}p^5-\frac{1-547\epsilon^2}{5{,}040(1-\epsilon^2)}p^7\\
+\frac{1-4{,}921\epsilon^2}{362{,}880(1-\epsilon^2)}p^9-\frac{1-44{,}287\epsilon^2}{39{,}916{,}800(1-\epsilon^2)}p^{11}.
\end{aligned}
\tag{445}
$$

If we place the series (445) in equation (441) and assume a series for r of the form $r=A+Bp+Cp^2+\ .\ .\ .$ we see at once that A and all coefficients of even powers of p vanish since only odd powers of p occur in the series (445). Also it will be seen that $B=1$. Hence we assume a series for r of the form

$$r=p+Ap^3+Bp^5+Cp^7+Dp^9+Ep^{11}, \tag{446}$$

whence

$$\frac{dr}{dp}=1+3\,Ap^2+5\,Bp^4+7\,Cp^6+9\,Dp^8+11\,Ep^{10}. \tag{447}$$

We now write the series (445) in the form

$$f(p)=\frac{N}{R}\sin p=p-Pp^3+Qp^5-Up^7+Vp^9-Tp^{11}, \tag{448}$$

where P, Q, U, V, T are the corresponding coefficients in ϵ from (445).

Placing the values of r, $\frac{dr}{dp}$, $\frac{N}{R}\sin p$ from (446), (447) and (448) in equation (441) we have

$$
\begin{aligned}
(1+3Ap^2+5Bp^4+7Cp^6+9Dp^8+11Ep^{10})\cdot(p-Pp^3+Qp^5-Up^7+Vp^9-Tp^{11})\\
-(p+Ap^3+Bp^5+Cp^7+Dp^9+Ep^{11})=0.
\end{aligned}
\tag{449}
$$

Equating to zero the sums of the coefficients of like powers of p in equation (449), returning the values of P, Q, U, V, T where needed from equation (445), we have

p^3: $$2A-P=0,\ A=\frac{1}{2}P=\frac{1-7\epsilon^2}{12(1-\epsilon^2)}. \tag{450}$$

p^5: $$4B+Q-3AP=0,\ B=\frac{1}{4}(3AP-Q)=\frac{1}{4}\left[\frac{(1-7\epsilon^2)^2}{24(1-\epsilon^2)^2}-\frac{1-61\epsilon^2}{120(1-\epsilon^2)}\right]$$

$$=\frac{1-2\epsilon^2+46\epsilon^4}{120(1-\epsilon^2)^2}. \tag{451}$$

p^7: $$6C-U+3AQ-5BP=0,\ C=\frac{1}{6}(U-3AQ+5BP)$$

$$C=\frac{1}{6}\left[\frac{1-547\epsilon^2}{5{,}040(1-\epsilon^2)}-\frac{(1-7\epsilon^2)(1-61\epsilon^2)}{480(1-\epsilon^2)^2}+\frac{(1-7\epsilon^2)(1-2\epsilon^2+46\epsilon^4)}{144(1-\epsilon^2)^3}\right]$$

$$=\frac{1}{6}\left[\frac{51-279\epsilon^2-4{,}005\epsilon^4-14{,}667\epsilon^6}{10{,}080(1-\epsilon^2)^3}\right]=\frac{17-93\epsilon^2-1{,}335\epsilon^4-4{,}889\epsilon^6}{20{,}160(1-\epsilon^2)^3}. \tag{452}$$

p^9: $$8D+V-3AU+5BQ-7PC=0,\ D=\frac{1}{8}(7PC-5BQ+3AU-V).$$

$$D=\frac{1}{8}\left[\frac{(1-7\epsilon^2)(17-93\epsilon^2-1{,}335\epsilon^4-4{,}889\epsilon^6)}{17{,}280(1-\epsilon^2)^4}-\frac{(1-61\epsilon^2)(1-2\epsilon^2+46\epsilon^4)}{2{,}880(1-\epsilon^2)^3}\right.$$

$$\left.+\frac{(1-7\epsilon^2)(1-547\epsilon^2)}{20{,}160(1-\epsilon^2)^2}-\frac{1-4{,}921\epsilon^2}{362{,}880(1-\epsilon^2)}\right]$$

$$=\frac{1}{8}\left[\frac{248-1{,}472\epsilon^2+30{,}648\epsilon^4+335{,}248\epsilon^6+429{,}128\epsilon^8}{362{,}880(1-\epsilon^2)^4}\right]$$

$$=\frac{31-184\epsilon^2+3{,}831\epsilon^4+41{,}906\epsilon^6+53{,}641\epsilon^8}{362{,}880(1-\epsilon^2)^4}. \tag{453}$$

p^{11}: $$10E-T+3AV-5BU+7CQ-9PD=0,$$

$$E=\frac{1}{10}(T-3AV+5BU-7CQ+9PD),$$

$$E=\frac{1}{10}\left[\frac{1-44{,}287\epsilon^2}{39{,}916{,}800(1-\epsilon^2)}-\frac{(1-7\epsilon^2)(1-4{,}921\epsilon^2)}{1{,}451{,}520(1-\epsilon^2)^2}+\frac{(1-2\epsilon^2+46\epsilon^4)(1-547\epsilon^2)}{120{,}960(1-\epsilon^2)^3}\right.$$

$$-\frac{(17-93\epsilon^2-1{,}335\epsilon^4-4{,}889\epsilon^6)(1-61\epsilon^2)}{345{,}600(1-\epsilon^2)^4}$$

$$\left.+\frac{(1-7\epsilon^2)(31-184\epsilon^2+3{,}831\epsilon^4+41{,}906\epsilon^6+53{,}641\epsilon^8)}{241{,}920(1-\epsilon^2)^5}\right]$$

$$=\frac{1}{10}\left[\frac{6{,}910-48{,}410\epsilon^2-449{,}660\epsilon^4-24{,}209{,}260\epsilon^6-101{,}944{,}360\epsilon^8-69{,}820{,}720\epsilon^{10}}{79{,}833{,}600(1-\epsilon^2)^5}\right]$$

$$=\frac{691-4{,}841\epsilon^2-44{,}966\epsilon^4-2{,}420{,}926\epsilon^6-10{,}194{,}436\epsilon^8-6{,}982{,}072\epsilon^{10}}{79{,}833{,}600(1-\epsilon^2)^5}. \tag{454}$$

With the values of A, B, C, D, E from (450), (451), (452), (453), (454) returned to (446), the series as given in (437) is produced.

STEREOGRAPHIC MERIDIAN PROJECTION OF THE SPHERE

The analytic function which produces this projection for the sphere may be derived analogously as for the other conformal projections already discussed and is found to be

$$x+iy=ai\,\frac{e^{\frac{1}{2}(\tau-i\lambda)}-e^{-\frac{1}{2}(\tau-i\lambda)}}{e^{\frac{1}{2}(\tau-i\lambda)}+e^{-\frac{1}{2}(\tau-i\lambda)}}, \tag{455}$$

where τ is now an isometric parameter for the sphere.

From the definitions $\cosh u=\frac{1}{2}\,(e^u+e^{-u})$, $\sinh u=\frac{1}{2}\,(e^u-e^{-u})$ we may write equation (455) as

$$x+iy=ai\,\frac{\sinh\frac{1}{2}\,(\tau-i\lambda)}{\cosh\frac{1}{2}\,(\tau-i\lambda)}=ai\,\frac{\sinh\frac{1}{2}\,(\tau-i\lambda)\cosh\frac{1}{2}\,(\tau-i\lambda)}{\cosh^2\frac{1}{2}\,(\tau-i\lambda)}$$

$$=\frac{ai\sinh(\tau-i\lambda)}{2\cosh^2\frac{1}{2}\,(\tau-i\lambda)}=\frac{ai\sinh(\tau-i\lambda)}{1+\cosh(\tau-i\lambda)}$$

$$=\frac{ai\,(\sinh\tau\cosh i\lambda-\cosh\tau\sinh i\lambda)}{1+\cosh\tau\cosh i\lambda-\sinh\tau\sinh i\lambda}. \tag{456}$$

Now $\cosh i\lambda=\cos\lambda$, $\sinh i\lambda=i\sin\lambda$, $\sinh\tau=\tan\phi$, $\cosh\tau=\sec\phi$. These values placed in equation (456) give

$$x+iy=ai\,\frac{\tan\phi\cos\lambda-i\sec\phi\sin\lambda}{1+\sec\phi\cos\lambda-i\tan\phi\sin\lambda}$$

$$=ai\,\frac{\tan\phi\cos\lambda-i\sec\phi\sin\lambda}{1+\sec\phi\cos\lambda-i\tan\phi\sin\lambda}\cdot\frac{1+\sec\phi\cos\lambda+i\tan\phi\sin\lambda}{1+\sec\phi\cos\lambda+i\tan\phi\sin\lambda}$$

$$=ai\,\frac{(\tan\phi-i\sin\lambda)\,(\sec\phi+\cos\lambda)}{(\sec\phi+\cos\lambda)^2}=\frac{a\,(\cos\phi\sin\lambda+i\sin\phi)}{1+\cos\phi\cos\lambda}.$$

Equating real and imaginary parts in this last equation we obtain the mapping equations of the stereographic meridian projection of the sphere

$$x=\frac{a\cos\phi\sin\lambda}{1+\cos\phi\cos\lambda},\ y=\frac{a\sin\phi}{1+\cos\phi\cos\lambda}. \tag{457}$$

If we use the conformal latitudes as defined by equation (256) in place of ϕ in the mapping equations (457), we will have then taken into account the spheroid. That is, we have mapped the spheroid on the sphere and the sphere in turn upon the plane. The scale factor will be the product of the scale factors in the two projections.

The total scale factor is thus obtained from (190) as follows:

In mapping equations (457) place $\phi=\chi$ and find

$$\frac{\partial x}{\partial\lambda}=a\cos\chi\,\frac{\cos\lambda+\cos\chi}{(1+\cos\chi\cos\lambda)^2},\ \frac{\partial y}{\partial\lambda}=a\cos\chi\,\frac{\sin\chi\sin\lambda}{(1+\cos\chi\cos\lambda)^2},$$

whence

$$\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2+\left(\frac{\partial y}{\partial \lambda}\right)^2}=\sqrt{\frac{a^2 \cos^2\chi}{(1+\cos \chi \cos \lambda)^2}}=\frac{a \cos \chi}{1+\cos \chi \cos \lambda}. \tag{458}$$

With $N=a/\sqrt{1-\epsilon^2 \sin^2\phi}$, we have then from (190) and (458) the total magnification

$$k=\frac{\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2+\left(\frac{\partial y}{\partial \lambda}\right)^2}}{N \cos \phi}=\frac{a \cos \chi}{N \cos \phi(1+\cos \chi \cos \lambda)}=\frac{\cos \chi\sqrt{1-\epsilon^2 \sin^2\phi}}{\cos \phi(1+\cos \chi \cos \lambda)}. \tag{459}$$

If we solve the mapping equations first for $\sin \phi$, $\cos\phi$ and then for $\sin \lambda$, $\cos \lambda$ we obtain respectively

$$\sin \phi=\frac{y \sin \lambda}{a \sin \lambda-x \cos \lambda}, \cos\phi=\frac{x}{a \sin \lambda-x \cos \lambda}, \tag{460}$$

$$\sin \lambda=\frac{x \sin\phi}{y \cos \phi}, \cos \lambda=\frac{a \sin\phi-y}{y \cos \phi}. \tag{461}$$

Eliminating ϕ between equations (460) and λ between equations (461) by squaring and adding respective members in each case we arrive at the equations of the meridians and parallels. That is, from equations (460) we have the equation of the meridians, $\frac{x^2}{(a \sin \lambda-x \cos \lambda)^2}+\frac{y^2 \sin^2 \lambda}{(a \sin \lambda-x \cos \lambda)^2}=1$, which may be written in the standard form for the equation of a circle,

$$(x+a \cot \lambda)^2+y^2=a^2 \csc^2 \lambda, \tag{462}$$

with center at $x=-a \cot \lambda$, $y=0$ and radius $r_\lambda=a \csc \lambda$. From equations (461) we have the equation of the parallels,

$$\frac{x^2 \sin^2 \phi}{y^2 \cos^2 \phi}+\frac{(a \sin \phi-y)^2}{y^2 \cos^2 \phi}=1,$$

which may be written in the standard form for the equation of a circle

$$x^2+(y-a \csc \phi)^2=a^2 \cot^2 \phi, \tag{463}$$

with center at $x=0$, $y=a \csc \phi$ and radius $r_\phi=a \cot \phi$. Thus the meridians are circles with centers on the x-axis and the parallels are circles with centers on the y-axis.

STEREOGRAPHIC HORIZON PROJECTION OF THE SPHERE

The analytic function of τ (considered an isometric parameter for the sphere) and λ which gives this projection is

$$x+iy=ai\,\frac{e^{\frac{1}{2}(\tau-i\lambda-\delta)}-e^{-\frac{1}{2}(\tau-i\lambda-\delta)}}{e^{\frac{1}{2}(\tau-i\lambda+\delta)}+e^{-\frac{1}{2}(\tau-i\lambda+\delta)}}. \tag{464}$$

We may as before transform the right member of (464) into hyperbolic functions to obtain

$$x+iy=ai\,\frac{\sinh\frac{1}{2}(\tau-i\lambda-\delta)}{\cosh\frac{1}{2}(\tau-i\lambda+\delta)}\cdot\frac{\cosh\frac{1}{2}(\tau+i\lambda+\delta)}{\cosh\frac{1}{2}(\tau+i\lambda+\delta)}. \tag{465}$$

The hyperbolic identities $\sinh u - \sinh v = 2 \sinh\frac{1}{2}(u-v) \cosh\frac{1}{2}(u+v)$, and $\cosh u + \sinh v = 2 \cosh\frac{1}{2}(u-v) \cosh\frac{1}{2}(u+v)$ applied respectively to the numerator and denominator of (465) give

$$x+iy = ai\frac{\sinh \tau - \sinh(\delta+i\lambda)}{\cosh(\tau+\delta)+\cosh i\lambda}$$

$$= ai\frac{\sinh \tau - \sinh \delta \cosh i\lambda - \cosh \delta \sinh i\lambda}{\cosh \tau \cosh \delta + \sinh \tau \sinh \delta + \cosh i\lambda}. \tag{466}$$

As before we have $\cosh i\lambda = \cos \lambda$, $\sinh i\lambda = i \sin \lambda$, $\sinh \tau = \tan \phi$, $\cosh \tau = \sec \phi$, $\sinh \delta = \tan \phi_0$, $\cosh \delta = \sec \phi_0$ and with these values placed in equation (466) we obtain

$$x+iy = ai\frac{\tan \phi - \tan \phi_0 \cos \lambda - i \sin \lambda \sec \phi_0}{\sec \phi \sec \phi_0 + \tan \phi \tan \phi_0 + \cos \lambda}.$$

Multiplying numerator and denominator of the right member of this last equation by $\cos \phi \cos \phi_0$ we obtain finally

$$x+iy = a\frac{\sin \lambda \cos \phi + i(\sin \phi \cos \phi_0 - \sin \phi_0 \cos \phi \cos \lambda)}{1+\sin \phi \sin \phi_0 + \cos \phi \cos \phi_0 \cos \lambda}. \tag{467}$$

Equating real and imaginary parts in (467) we have the mapping equations for the stereographic horizon projection of the sphere.

$$x = a\frac{\sin \lambda \cos \phi}{1+\sin \phi \sin \phi_0 + \cos \phi \cos\phi_0 \cos \lambda},$$
$$y = \frac{a(\sin \phi \cos \phi_0 - \sin \phi_0 \cos \phi \cos \lambda)}{1+\sin \phi \sin \phi_0 + \cos \phi \cos \phi_0 \cos \lambda}. \tag{468}$$

The spheroid is taken into account as before by substituting the conformal latitude χ for ϕ.

The total scale factor is obtained in the same manner as for the stereographic meridian projection. That is, from the mapping equations (468) we have with $\phi=\chi$

$$\frac{\partial x}{\partial \lambda} = a \cos \chi \frac{\cos \chi \cos \chi_0 + (1+\sin \chi \sin \chi_0)\cos \lambda}{(1+\sin \chi \sin \chi_0 + \cos \chi \cos \chi_0 \cos \lambda)^2},$$

$$\frac{\partial y}{\partial \lambda} = a \cos \chi \frac{\sin \lambda(\sin \chi + \sin \chi_0)}{(1+\sin \chi \sin \chi_0 + \cos \chi \cos \chi_0 \cos \lambda)^2},$$

whence

$$\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2} = \sqrt{\frac{a^2 \cos^2 \chi}{(1+\sin \chi \sin \chi_0 + \cos \chi \cos \chi_0 \cos \lambda)^2}}$$

$$= \frac{a \cos \chi}{1+\sin \chi \sin \chi_0 + \cos \chi \cos \chi_0 \cos \lambda}.$$

From (190) with $N=a/(1-\epsilon^2 \sin^2 \phi)^{\frac{1}{2}}$ we have then

$$k = \frac{\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}}{N \cos \phi} = \frac{a \cos \chi}{N \cos \phi(1+\sin \chi \sin \chi_0 + \cos \chi \cos \chi_0 \cos \lambda)}$$

$$= \frac{\cos \chi \sqrt{1-\epsilon^2 \sin^2 \phi}}{\cos \phi(1+\sin \chi \sin \chi_0 + \cos \chi \cos \chi_0 \cos \lambda)}. \tag{469}$$

To obtain the equations of the meridians and parallels we solve the mapping equations (468) for $\sin\phi$, $\cos\phi$ and then for $\sin\lambda$, $\cos\lambda$ analogously as was done for the stereographic meridian projection, obtaining then

$$\begin{aligned}\sin\phi&=\frac{x\sin\phi_0\cos\lambda+y\sin\lambda}{a\cos\phi_0\sin\lambda-x\cos\lambda-y\sin\phi_0\sin\lambda},\\ \cos\phi&=\frac{x\cos\phi_0}{a\cos\phi_0\sin\lambda-x\cos\lambda-y\sin\phi_0\sin\lambda},\end{aligned}\tag{470}$$

$$\begin{aligned}\sin\lambda&=\frac{x\,(\sin\phi_0+\sin\phi)}{a\sin\phi_0\cos\phi+y\cos\phi_0\cos\phi},\\ \cos\lambda&=\frac{a\cos\phi_0\sin\phi-y-y\sin\phi_0\sin\phi}{a\sin\phi_0\cos\phi+y\cos\phi_0\cos\phi}.\end{aligned}\tag{471}$$

By squaring and adding respective members of equations (470) and then of equations (471) we arrive as before at the equations of the meridians and parallels. That is, from equations (470) we have after reducing and arranging in standard form the equation of the meridians which are circles given by $(x+a\sec\phi_0\cot\lambda)^2+(y+a\tan\phi_0)^2=a^2\sec^2\phi_0\csc^2\lambda$, with centers at $x=-a\sec\phi_0\cot\lambda, y=-a\tan\phi_0$ and radii $r_\lambda=a\sec\phi_0\csc\lambda$. Clearly the centers all lie on the line $y=-a\tan\phi_0$.

Analogously from equations (471) we have the equation of the parallels which are circles given by $x^2+\left(y-\frac{a\cos\phi_0}{\sin\phi_0+\sin\phi}\right)^2=\frac{a^2\cos^2\phi}{(\sin\phi_0+\sin\phi)^2}$, with centers at $x=0$, $y=\frac{a\cos\phi_0}{\sin\phi_0+\sin\phi}$ and radii $r_\phi=\frac{a\cos\phi}{\sin\phi_0+\sin\phi}$.

If we place $\phi=-\phi_0$ in the formulas for r_ϕ and y we find that r_ϕ becomes infinite and y becomes infinite which means that the parallel for $\phi=-\phi_0$ is a straight line.

We will now show that the analytic functions (455) and (464) for the stereographic meridian and horizon projections are special cases of a more general function which may be obtained by considering the expressions for the curvature of the meridians and parallels in a conformal projection.

When the parallels are circles, $R_\tau=c$ (constant) and from (216) we have $\frac{\partial G^{-\frac{1}{2}}}{\partial\tau}=\frac{1}{c}$, whence $\frac{\partial^2G^{-\frac{1}{2}}}{\partial\tau\partial\lambda}=0$. But $\frac{1}{R_\lambda}=-\frac{\partial G^{-\frac{1}{2}}}{\partial\lambda}$ and if $\frac{\partial^2G^{-\frac{1}{2}}}{\partial\lambda\partial\tau}=0$, then $\frac{\partial G^{-\frac{1}{2}}}{\partial\lambda}$ is a function of λ alone. Hence the meridians (for which λ is constant) must also be circles. Also from (216) we have $G=f'(\lambda+i\tau)f'(\lambda-i\tau)$ and let us suppose that $G^{-\frac{1}{2}}=g(\lambda+i\tau)g(\lambda-i\tau)$.

Then $\frac{\partial^2G^{-\frac{1}{2}}}{\partial\lambda\partial\tau}=\frac{\partial^2}{\partial\lambda\partial\tau}\,[g(\lambda+i\tau)g(\lambda-i\tau)]=g''(\lambda+i\tau)g(\lambda-i\tau)-g''(\lambda-i\tau)g(\lambda+i\tau)=0.$

From this last equation we have $\frac{g''(\lambda+i\tau)}{g(\lambda+i\tau)}=\frac{g''(\lambda-i\tau)}{g(\lambda-i\tau)}$. Since the left member is a function of $\lambda+i\tau$ and the right member a function of $\lambda-i\tau$, the equality can only exist if both ratios are equal to the same constant, for example c^2. That is, $\frac{g''(\lambda+i\tau)}{g(\lambda+i\tau)}=\frac{g''(\lambda-i\tau)}{g(\lambda-i\tau)}=c^2$. Placing $\lambda+i\tau=u$ we have from these last equations $g''(u)-c^2g(u)=0$, which is clearly a linear homogeneous differential equation with constant coefficients. The auxiliary equation is $m^2-c^2=0$, whence $m=\pm c$ and the solution is then $g(u)=Ae^{cu}+Be^{-cu}$, or $g(\lambda+i\tau)=Ae^{c(\lambda+i\tau)}+Be^{-c(\lambda+i\tau)}$. But we have $G^{-\frac{1}{2}}=g(\lambda+i\tau)g\,(\lambda-i\tau)=[f'(\lambda+i\tau)\,f'(\lambda-i\tau)]^{-\frac{1}{2}}$, whence $f'(\lambda+i\tau)=g^{-2}(\lambda+i\tau)=[Ae^{c(\lambda+i\tau)}+Be^{-c(\lambda+i\tau)}]^{-2}$. Again with $u=\lambda+i\tau$ we have yet to solve the differential equation $f'(u)=(Ae^{cu}+Be^{-cu})^{-2}$. From the definitions of hyperbolic functions we have $e^{cu}=\sinh cu+\cosh cu$, $e^{-cu}=\cosh cu-\sinh cu$, so that

$$\begin{aligned} Ae^{cu}+Be^{-cu}&=(A-B)\sinh cu+(A+B)\cosh cu \\ &=2\sqrt{AB}\left[\frac{(A-B)}{2\sqrt{AB}}\sinh cu+\frac{(A+B)}{2\sqrt{AB}}\cosh cu\right] \\ &=C\cosh(cu+\delta),\text{ where } C=2\sqrt{AB},\ \delta=\tanh^{-1}\frac{A-B}{A+B}. \end{aligned}$$

Our differential equation is then $f'(u)=C^{-2}\cosh^{-2}(cu+\delta)=C^{-2}\operatorname{sech}^2(cu+\delta)$, and the solution is at once $f(u)=c^{-1}C^{-2}\tanh(cu+\delta)+D$.

In this last equation place $D=\delta=0$, $c=-i/2$, $c^{-1}C^{-2}=ai$ and we have

$$\begin{aligned} f(u)&=ai\tanh(-iu/2)=ai\tanh[-i(\lambda+i\tau)/2]=ai\tanh\frac{1}{2}(\tau-i\lambda) \\ &=ai\frac{\sinh\frac{1}{2}(\tau-i\lambda)}{\cosh\frac{1}{2}(\tau-i\lambda)}=ai\frac{e^{\frac{1}{2}(\tau-i\lambda)}-e^{-\frac{1}{2}(\tau-i\lambda)}}{e^{\frac{1}{2}(\tau-i\lambda)}+e^{-\frac{1}{2}(\tau-i\lambda)}}, \end{aligned}$$

which is the analytic function (455) for the stereographic meridian projection. Similarly we may obtain the analytic function (464) for the stereographic horizon projection.

STEREOGRAPHIC HORIZON PROJECTION OF THE SPHEROID

For the computation of triangulation by plane coordinates in large areas where the horizon stereographic projection could be applied, the distortion is so great at the boundaries that either the Lambert conformal conic or the transverse Mercator is more suitable when used in bands. Also tables for conversion from the spheroid to the projection are either already available for these latter projections or can be computed with greater ease.

Examples of the application of the horizon stereographic projection to an area of considerable extent are "Emploi des coordonnées rectangulaires stéréographiques pour le calcul de la triangulation dans un rayon de 560 kilomètres autour de l'origine" by M. H. Roussilhe and published by the Section of Geodesy of the International Union of Geodesy and Geophysics, May 1922; "De stereografische kaart projectie in hare toepassing", by Hk. J. Heurelink, Nederlandsche Rijksdriehoeksmeting, Delft, 1918.

In the development of orthomorphic projections through the aposphere, Brigadier Hotine gives formulas for a satisfactory approximation to the horizon stereographic projection of a considerable area of the spheroid for geodetic purposes. These formulas with worked examples are found in sections 20 and 26 of his treatise, "The Orthomorphic Projection of the Spheroid," Empire Survey Review, Vols. VIII and IX, Nos. 62–66, 1946–47.

For a small country whose boundaries are contained in a small circle of radius 3 to 3½ degrees, the stereographic projection in polar form as given by equations (475), (476), and (477) is useful using the conformal sphere as suggested by J. H. Cole. The scale is about 1 in 1,000 at a distance of 400 kilometers from the central point. Thus by applying an over-all scale factor the scale error can be kept less than 1 in 2,000 over a small circle of radius of 400 kilometers from the central point.

GEOMETRIC DERIVATION OF EQUATIONS FOR THE STEREOGRAPHIC PROJECTIONS OF THE SPHERE

The stereographic projections of the sphere are perspective. That is, we may project the sphere from any given point of the sphere upon any plane perpendicular to the diameter passing through the given point of perspectivity, the only difference for different choices of planes being one of scale. For the geometric demonstration of the three forms for the sphere we will use for the plane of projection a diametral plane (one passing through the center of the sphere).

In figure 32, W, the center of projection, is an arbitrary point of the sphere. U is the projection of Q from W upon the diametral plane V. The angle TWU is equal to $\frac{1}{2}D$, and if a is the radius of the sphere we have $TU=a \tan \angle TWU = a \tan \frac{1}{2}D$.

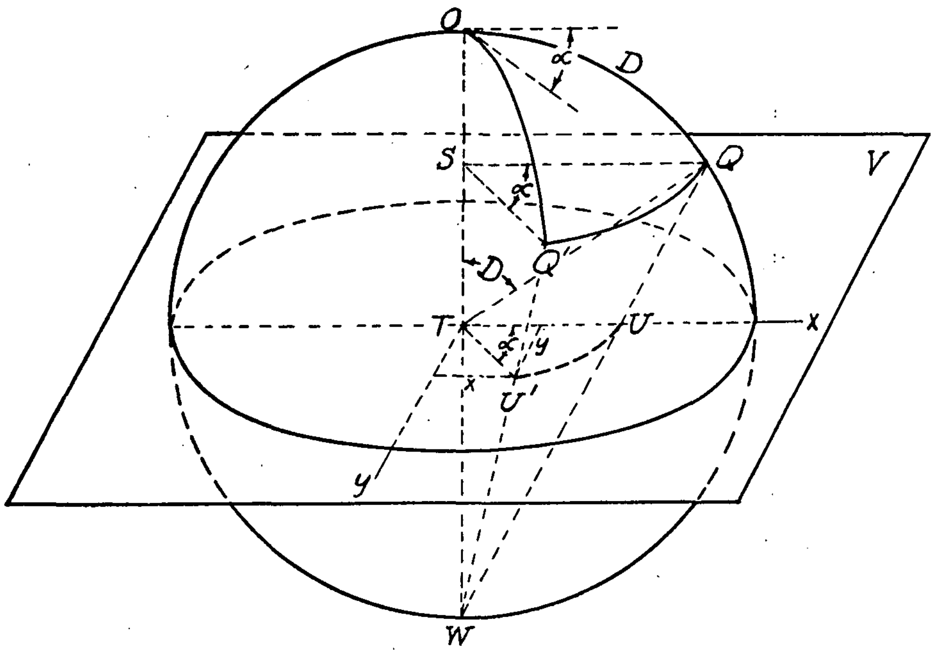

FIGURE 32.—Geometric derivation of stereographic projections.

Now for a given value of D, QS is fixed in length. Hence if QS turns about the fixed point S through an angle α then Q traces the circular arc QQ' on the sphere and therefore U describes a corresponding circular arc UU' in the plane V.

RECTANGULAR COORDINATES

If we call the line TU in the V-plane the map x-axis we have

$$x=TU' \cos \alpha = TU \cos \alpha = a \tan \frac{1}{2} D \cos \alpha = \frac{a \sin D \cos \alpha}{1+\cos D},$$
$$y=TU' \sin \alpha = TU \sin \alpha = a \tan \frac{1}{2} D \sin \alpha = \frac{a \sin D \sin \alpha}{1+\cos D}. \tag{472}$$

Now comparing figures 28 (p. 110) and 32, the points O and Q correspond in the figures and we have from equations (359)

$$\begin{aligned} \cos D &= \sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \cos \lambda, \\ \sin D \cos \alpha &= \cos \phi \sin \lambda, \\ \sin D \sin \alpha &= \cos \phi_0 \sin \phi - \sin \phi_0 \cos \phi \cos \lambda. \end{aligned} \tag{473}$$

With these values placed in equations (472) we have

$$\begin{aligned} x &= \frac{a \cos \phi \sin \lambda}{1+\sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \cos \lambda}, \\ y &= \frac{a (\cos \phi_0 \sin \phi - \sin \phi_0 \cos \phi \cos \lambda)}{1+\sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \cos \lambda}, \end{aligned} \tag{474}$$

which are identical with equations (468) as obtained before.

Note from equations (474) and figures 28 and 32 that with $\phi_0=0$ we have again equations (457) for the meridian stereographic projection of the sphere. With $\phi_0=\pi/2$ we have $x=\frac{a \cos \phi \sin \lambda}{1+\sin \phi}=\frac{a \sin p \sin \lambda}{1+\cos p}=a \tan \frac{1}{2} p \sin \lambda$, and $y=-\frac{a \cos \phi \cos \lambda}{1+\sin \phi}=-\frac{a \sin p \cos \lambda}{1+\sin p}=-a \tan \frac{1}{2} p \cos \lambda$, which give, after interchanging x and y and changing the negative sign to reverse the positive direction of x, $x=r \cos \lambda$, and $y=r \sin \lambda$, where $r=a \tan \frac{1}{2} p$, p being the colatitude. These are the equations for the polar stereographic projection of the sphere corresponding to equations (431) with $\epsilon=0$.

POLAR COORDINATES

From figure 32 and equations (472) we have at once the horizon stereographic projection in polar coordinates. That is,

$$\begin{aligned} \theta &= \alpha, \\ \rho &= TU' = a \tan \frac{1}{2} D. \end{aligned} \tag{475}$$

By dividing the members of the third by the corresponding members of the second of equations (473) we have

$$\tan \alpha = \frac{\cos \phi_0 \tan \phi - \sin \phi_0 \cos \lambda}{\sin \lambda}, \tag{476}$$

and from the second of equations (473)

$$\sin D = \frac{\cos \phi \sin \lambda}{\cos \alpha}. \tag{477}$$

To obtain polar coordinates for the meridional and polar stereographic projections we have but to place $\phi_0=0, \frac{\pi}{2}$ in equations (476) and (477).

For the spherical forms of the stereographic projections it is customary to take as the radius of the sphere, the mean radius of the spheroid at the pole of the projection.

That is, from (160) we take $r=\sqrt{R_0N_0}$, where the subscript $_0$ refers to the latitude ϕ_0 of the pole of the projection. We would use this value in place of a throughout the formulas for the sphere.

When the conformal latitude, χ, is used to take into account the spheroid, the radius of the conformal sphere is then, from (257), $r=\frac{N_0 \cos \phi_0}{\cos \chi_0}$.

For the horizon stereographic, if it is used to map a small area on a large scale, the scale will be improved by using $r=\frac{N_0 \cos \phi_0}{\cos \chi_0}$ instead of $r=a$ as shown in the above development. That is, we would replace a in the formulas for the horizon stereographic by $\frac{N_0 \cos \phi_0}{\cos \chi_0}$, the scale factor from (190) being then

$$k=\frac{N_0 \cos \phi_0 \cos \chi}{N \cos \phi \cos \chi_0(1+\sin \chi \sin \chi_0+\cos \chi \cos \chi_0 \cos \lambda)}. \tag{478}$$

For the meridian stereographic, which is centered on the Equator, we have $\phi_0=0$ and thus $r=a$, the semimajor axis of the spheroid. Hence the formulas as given for this projection are satisfactory for mapping small areas on a large scale.

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