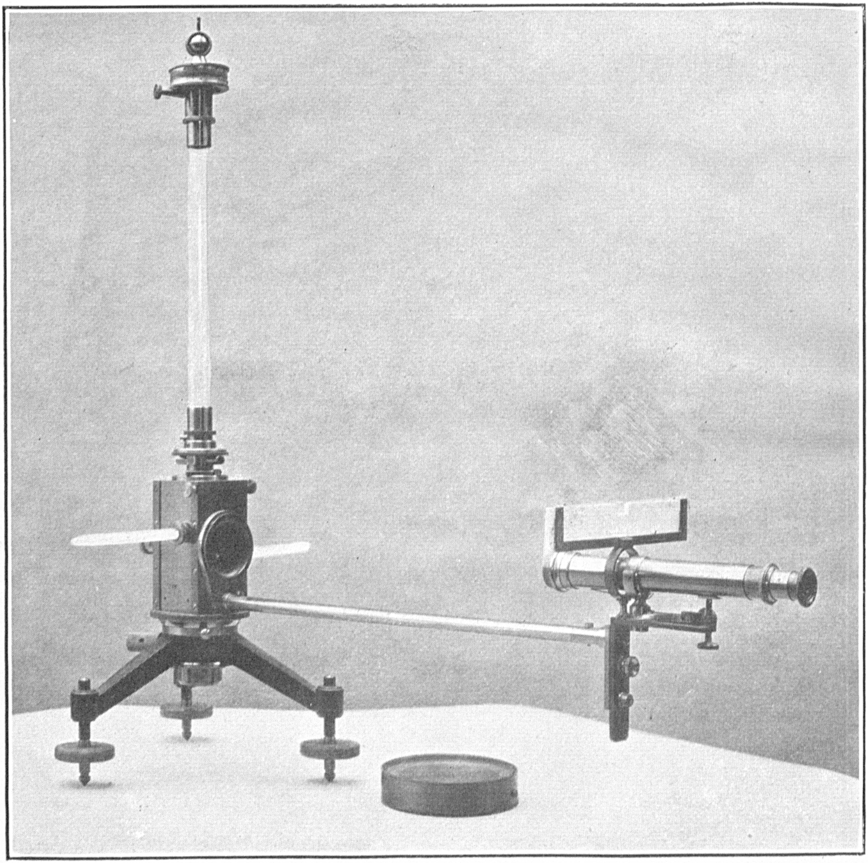

UNIFILAR *H* VARIOMETER.

Serial No. 212

DEPARTMENT OF COMMERCE

U. S. COAST AND GEODETIC SURVEY

E. LESTER JONES, DIRECTOR

TERRESTRIAL MAGNETISM

HORIZONTAL INTENSITY VARIOMETERS

By
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Special Publication No. 89

PRICE, 10 CENTS

Sold only by the Superintendent of Documents, Government Printing Office
Washington, D. C.

WASHINGTON
GOVERNMENT PRINTING OFFICE
1922

CONTENTS.

HORIZONTAL INTENSITY VARIOMETERS.

By George Hartnell, *Observer in Charge, Cheltenham Magnetic Observatory.*

INTRODUCTION.

The total intensity of the earth's magnetic field at any place may be resolved into a vertical intensity (Z) and two rectangular component intensities in a horizontal plane. Choosing a pair of horizontal rectangular axes, the X axis directed toward the geographical north, and the Y axis directed toward the geographical east, the horizontal components are X and Y. The resultant of the X and Y intensities is customarily called the *horizontal intensity* (H). Its magnitude is

$$H^2 = X^2 + Y^2$$

The angle H makes with the geographical meridian is the *declination* (D), and its magnitude is

$$\tan D = \frac{Y}{X}$$

It is the direction asumed by a suspended magnet free to turn in a horizontal plane; it is often called the *variation of the compass.* The *magnetic meridian* is the vertical plane defined by the declination.

At magnetic observatories it has become the established practice to record variations in D, H, and Z by means of a set of three instruments, a D, an H, and a Z variometer, the set being called a magnetograph. The D variometer is essentially a horizontal intensity variometer, as will be seen from the equation for tan D above; for any changes in the X and Y intensities must also change D. The choice of D and H was determined by the practical importance of the declination in surveying and navigation.

As will be shown later, a set of two horizontal intensity variometers may be arranged so as to record independently any desired pair of horizontal components, such as X and Y for example.

The magnet of the D variometer lies in the magnetic meridian and is suspended by a delicate fiber of silk or quartz, free from torsion, and just large enough to sustain the weight of the magnet in safety. The magnet of the H variometer, however, is perpendicular to the magnetic meridian, and hence the couple acting on it must be counteracted by an equal couple in the suspension. That form of suspension in which the magnet is suspended by a single fiber, the counteracting couple being produced by the torsional rigidity of the substance of the fiber, is here called the *unifilar suspension.* In the *bifilar suspension*, the magnet is sustained by two fibers a small distance apart. In practice the two fibers constitute a single continuous fiber which passes around a pulley attached to the magnet system, and is fastened at two points vertically above the magnet. The purpose of the pulley is to equalize the tension in the two halves of the fiber. In general there will be some torsion in each half of the fiber.

To the magnets are fastened mirrors which reflect the light from an illuminated slit onto a cylindrical revolving drum, covered with photographic paper, which, when developed, is the *magnetogram.*

Attached to each variometer is a fixed mirror which produces on the magnetogram a straight line—the *base line.*

In magnetographs, which record all the variation curves on one sheet, the variometers are mounted in the magnetic prime vertical either east or west of the recording apparatus. Except in special types of D variometers, the D magnet points northward. The north end of the H magnet may point east or west, depending on the choice of direction for positive ordinates on the magnetogram.

Magnetographs which record each variation curve on a separate sheet follow a somewhat different arrangement. In the Adie type, for example, the D variometer may be on the east side and the H variometer may be on the west side of the recording apparatus, or vice versa. The Z variometer may be placed either north or south of the recording apparatus.

As distinguished from intrinsic constants there are two operating constants of the H and D variometers, namely, *temperature coefficient* and *scale value.*

A correction for temperature is necessary because the magnets become weaker by an increase in temperature. Since the direction in which the D magnet points does not depend on the strength of the magnet, the temperature coefficient of the D variometer is zero. The temperature coefficient of the H variometer is, strictly speaking, the ratio of the apparent change in H per degree of change of temperature to the absolute value of H, when the variometer is subjected to a change of temperature only. Usually, however, the temperature coefficient is expressed in gammas per degree of temperature change. The commonly accepted unit of intensity in terrestrial magnetism is the gamma (γ), which equals 0.00001 of the C. G. S. unit.

The scale value of the horizontal intensity variometer is usually expressed in gammas per millimeter of ordinate on the magnetogram. The methods most frequently used to determine the H scale value consist in subjecting the variometer to a known change, ΔH, in the field intensity by means of a magnet (method of deflections), or by means of an electric solenoid placed at an assigned distance from the H magnet. The scale value is then the quotient of ΔH divided by the change in ordinate.

In the method by oscillations, the period of the magnet when in place in the variometer instrument, is compared with its period when it is removed and suspended north end north, and allowed to oscillate under the influence of the H intensity.

In another method (method by torsion angle) the scale value is derived from a measurement of the actual amount of torsion in the suspension when the magnet is in place in its recording position.

Still another method (method by weight), which applies to the bifilar suspension only, consists in placing on the magnet an extra weight of known mass, the scale value being derived from the observed deflection. The last three methods are not, in general, convenient or feasible, but when feasible, are sometimes useful as checks.

The scale values of *D* and *H* variometers depend on the distance from the recording apparatus, and on the torsion factor of the suspension. Obviously the most convenient scale value for the *D* variometer is 1 mm. of ordinate per minute of arc. The torsion factor being small, this scale value is attained by placing the variometer at such a distance that the reciprocal of the distance in millimeters is equal to 2 tan 1′, that is, at the distance of 172 cm., approximately.

When all of the variometers of the magnetograph are mounted on one side of the recording apparatus, so as to obtain a photographic record of all three variation curves on one sheet, and when the suspension of the *H* variometer is of the unifilar type, the distance of the *H* variometer from the recording apparatus is restricted to a small range. If the *H* variometer is too near the recording apparatus, its scale value will be too large, and equal increments in ordinate will not correspond to equal increments of angle at the center of the *H* magnet. If it is too near the *D* variometer it will interfere with the proper functioning of the *D* magnet. Moreover, it is seldom practicable to select a fiber of just the right size to give the desired scale value. Hence it is often necessary to use control magnets to increase or decrease the scale value. Control magnets are placed above or below the *H* magnet and parallel to it. When the north pole of the *H* magnet and the north pole of the control magnet point in the same direction, the control magnet tends to turn the *H* magnet out of the prime vertical, and so decreases the scale value and this increases the sensitiveness of the variometer. When the north pole of the *H* magnet and the north pole of the control magnet point in opposite directions, the control magnet tends to hold the *H* magnet in the prime vertical, and this increases the scale value and decreases the sensitiveness of the variometer.

The scale value of an *H* variometer having a bifilar suspension can be readily adjusted to the desired value by changing the distance between the fibers; no control magnets are necessary.

The value of *D* or *H* or *Z* corresponding to its respective base-line is readily computed when the absolute value and the ordinates of the variation curves on the magnetogram for the same time are known, together with the scale value and the temperature coefficient. Conversely, when the base-line values are known, the absolute values of *D* and *H* and *Z* can be derived from the magnetogram which thus provides a complete and continuous record of the state of the earth's magnetic field.

It is apparent that the theory of the bifilar and unifilar types of suspension is an important part of a knowledge of the working of the complete instrument.

In discussing the theory of the horizontal intensity variometer, then, the subject matter will be divided into two parts:

Part 1 will contain the theory of the horizontal intensity variometer proper.

Part 2 will contain the theory of the bifilar and unifilar suspensions.

Part 1 will be subdivided into the following sections: I. Notation and fundamental equation; II. Characteristics of the bifilar variometer; III. Characteristics of the unifilar variometer; IV. Characteristics common to both types of horizontal variometers.

Part I.—THEORY OF THE HORIZONTAL INTENSITY VARIOMETER PROPER.

I. NOTATION AND FUNDAMENTAL EQUATION.

1. **Notation—**

V = potential energy.
M = magnetic moment of suspended magnet.
M' = magnetic moment of control magnet.
M_c = magnetic moment of compensating magnet.
r = distance between suspended and control magnets or between suspended and compensating magnet.
$F=\frac{M'}{r^3}$ = field intensity of control magnet at the suspended magnet.
$F_0=\frac{2M_c}{r^3}$ = field intensity of compensating magnet at the suspended magnet.
H = value of the horizontal intensity in C. G. S. units.
H_0 = base line or standard value at the station in C. G. S. units.
H_γ = horizontal intensity in gammas.
F_γ = field intensity of control magnet in gammas.
h = torsion factor of the unifilar variometer; also torsion factor for both fibers in the bifilar suspension.
A = torsion factor of bifilar variometer.
n = ordinate in millimeters on magnetogram.
ϵ = angular value in radians of 1 mm. on magnetogram.
S = scale value in C. G. S. units per radian.
S_0 = base line scale value in C. G. S. units per radian.
s = scale value in gammas per millimeter of ordinate.
s_0 = base line scale value in gammas per millimeter of ordinate.
S' = scale value in C. G. S. units per radian of unifilar with control magnet.
s' = scale value in gammas per millimeter of unifilar having a control magnet.
S'_0 = base line scale value, in C. G. S. units, unifilar with control magnet.
s'_0 = base line scale value, in gammas, unifilar with control magnet.
a = a factor, coefficient of n in series development of scale value.
p = coefficient of magnetic loss.
q = temperature coefficient.
T_f = free half period of suspended magnet.
T_d = damped half period of magnet.
T_H = half period of magnet due to H alone.

Choosing the magnetic north for the $+X$ axis and the magnetic east for the $+Y$ axis, we have for the angles counted from the $+X$ axis toward the $+Y$ axis:

θ = angle between suspended magnet and the magnetic meridian.

θ_0 = a particular equilibrium value of θ.

When $\theta=\frac{\pi}{2}$ the recording spot of the light is on the base line, and the corresponding value of H is H_0.

ψ = a small angular displacement of the magnet from a position of equilibrium.

φ = angle between magnet and magnetic prime vertical, counted from $+Y$ axis toward $-X$ axis.

δ = angle through which the torsion head has been turned. (The fiber is assumed to be free from torsion when a marked line on the torsion head and the magnet are parallel.)

τ = angular torsion in the fiber when $\theta=\frac{\pi}{2}$.

κ = angle between control magnet and X axis.

Other symbols will be explained in the text.

2. **Constants of sample variometers.**—Some of the equations to be developed will be illustrated by a concrete case of each type of variometer having the following constants:

Constants of bifilar.	*Constants of unifilar.*
1 mm. of ordinate=1!13.	M=10.
log ϵ=6.51681.	1 mm. of ordinate=1!5.
τ=60° 01!6.	log ϵ=6.63982.
log sin τ=9.93765.	τ=4.2963=246°16.
log cos τ=9.69862.	log τ=0.63309.
s_0=3.64γ.	s_0=1.95γ.
log s_0=0.5611.	log s_0=0.29003.
q=0.00037.	q=0.000677.
δ=150° 01!6.	δ=5.8671 radians.

H_0=0.192 C. G. S. units=19200γ.

The standard position of the magnet is north end east. Increasing ordinates correspond to increasing H. The + values of the ordinate n are measured toward the top of the magnetogram.

3. **Fundamental equation.**—The fundamental equation of the horizontal intensity variometer will be expressed in the form of the potential energy of the system of magnets and suspensions. Written in full it is

$$V=-MH\cos\theta+\frac{MM'}{r^3}\cos(\theta-\kappa)-A\cos(\delta-\theta)+\frac{h}{2}(\delta-\theta)^2 \qquad (1)$$

The first term on the right is the potential energy of the suspended magnet. The second term is the mutual potential energy of the suspended and control magnets. The third term is the potential energy of the bifilar suspension. The last term is the potential energy due to torsion in the fibers.

II. CHARACTERISTICS OF THE BIFILAR VARIOMETER.

4. **Equation of the bifilar variometer.**—In discussing the characteristics of the bifilar variometer, we shall not consider control magnets, which are seldom, if ever, used on this type of variometer. Furthermore, it will be assumed, either that the fibers are free from torsion, or that the torsion factor h is so small as to be negligible. The equation of the bifilar variometer thus becomes:

$$V=-MH\cos\theta-A\cos(\delta-\theta) \qquad (2)$$

The couple tending to increase θ is

$$-\frac{dV}{d\theta}=-MH\sin\theta+A\sin(\delta-\theta) \tag{2'}$$

For equilibrium, this couple is zero, so that the equation of equilibrium is

$$MH\sin\theta=A\sin(\delta-\theta) \tag{3}$$

5. **Scale value of bifilar variometer.**—The scale value, that is, the change in H corresponding to a unit change in θ, may be found by differentiating equation (3) with respect to H, remembering that θ decreases as H increases.

$$S=-\frac{dH}{d\theta}=\frac{MH\cos\theta+A\cos(\delta-\theta)}{M\sin\theta}=H\cot\theta+\frac{A\cos(\delta-\theta)}{M\sin\theta} \tag{4}$$

From equation (3) we have

$$\frac{A}{M\sin\theta}=\frac{H}{\sin(\delta-\theta)}$$

Eliminating A from equation (4)

$$S=H[\cot\theta+\cot(\delta-\theta)] \tag{5}$$

The scale values of the bifilar variometer, expressed in the units we shall have occasion to use, are

$$S=H[\cot\theta+\cot(\delta-\theta)] \text{ (C. G. S. units per radian)} \tag{5a}$$

$$s=H_\gamma\,\epsilon[\cot\theta+\cot(\delta-\theta)] \text{ (gammas per millimeter)} \tag{5b}$$

One form may be converted to the other by introducing the appropriate factors. For example, to convert the scale value expressed in C. G. S. units per radian into the scale value expressed in gammas per millimeter, write H in gammas, and multiply by ϵ. For the base-line scale values we have, remembering that then $\theta=\frac{\pi}{2}$ and that $\delta-\theta=\tau$

$$S_0=H\cot\tau \text{ (C. G. S. units per radian)} \tag{6a}$$

$$s_0=H_\gamma\,\epsilon\cot\tau \text{ (gammas per millimeter)} \tag{6b}$$

An inspection of equation (6b) shows that the scale value can be determined by measuring the torsion angle τ. This method may be called scale value by measurement of torsion angle.

6. **Development of scale value in powers of the ordinates.**—Introduce into equation (5) a small auxiliary angle φ, which is the angle between the suspended magnet and the magnetic prime vertical. Then

$$\theta=\varphi+\frac{\pi}{2}$$

$$\delta-\theta=\tau-\varphi \tag{7}$$

We shall have frequent occasion to convert the angles φ and θ into millimeters of ordinates by means of the relations

$$\theta=\frac{\pi}{2}-n\epsilon \text{ and } \delta-\theta=\tau+n\epsilon \tag{8a}$$

also

$$-\varphi= n\epsilon \text{ and } \tau-\varphi=\tau+n\epsilon \tag{8b}$$

Substituting (7) into equation (5a),

$$S=H\,[-\tan\varphi+\cot(\tau-\varphi)] \tag{9}$$

The expression in the brackets on the right is a function of φ, and may be written:

$$f(\varphi)=-\tan\varphi+\cot(\tau-\varphi) \tag{10}$$

and may be expanded in powers of φ by Maclaurin's theorem:

$$f(\varphi)=f(\varphi)_{\varphi=0}+\varphi\frac{df}{d\varphi}_{\varphi=0}+\frac{\varphi^2}{2}\left(\frac{d^2f}{d\varphi^2}\right)_{\varphi=0} \tag{11}$$

Performing the indicated operations

$$f(\varphi)_{\varphi=0}=[-\tan\varphi+\cot(\tau-\varphi)]_{\varphi=0}=\cot\tau$$

$$\left(\frac{df}{d\varphi}\right)_{\varphi=0}=[-\sec^2\varphi+\operatorname{cosec}^2(\tau-\varphi)]_{\varphi=0}=-1+\frac{1}{\sin^2\tau}$$

$$\left(\frac{d^2f}{d\varphi^2}\right)_{\varphi=0}=[-2\sec^2\varphi\tan\varphi+2\operatorname{cosec}^2(\tau-\varphi)\cot(\tau-\varphi)]_{\varphi=0}=\frac{2\cos\tau}{\sin^3\tau}$$

When these expressions are substituted, equation (11) becomes

$$f(\varphi)=-\tan\varphi+\cot(\tau-\varphi)=\cot\tau-\varphi\left(1-\frac{1}{\sin^2\tau}\right)+\varphi^2\frac{\cos\tau}{\sin^3\tau}$$

and the scale value, equation (9), becomes

$$S=H\left[\cot\tau-\varphi\left(1-\frac{1}{\sin^2\tau}\right)+\varphi^2\frac{\cos\tau}{\sin^3\tau}\right] \tag{12}$$

Using equation (8b), we obtain for the scale value expressed in powers of the ordinates:

$$S=H\left[\cot\tau+n\epsilon\left(1-\frac{1}{\sin^2\tau}\right)+n^2\epsilon^2\frac{\cos\tau}{\sin^8\tau}\right] \tag{13}$$

and for the scale value in gammas per millimeter

$$s=s_0+H_\gamma\epsilon^2\left(1-\frac{1}{\sin^2\tau}\right)n+H_\gamma\epsilon^3\frac{\cos\tau}{\sin^3\tau}n^2 \tag{14}$$

This is of the form sought, namely:

$$s = s_0 + an + bn^2 \tag{15}$$

As an example, use the constants of the sample bifilar variometer given in paragraph 2.

$$s = 3.64\gamma - 0.0006907n + 0.000000024n^2 \tag{16}$$

7. **Characteristics of bifilar scale value.**—The term containing the square of the ordinate n^2 is of course negligible.

As seen from equations (13) and (14), the coefficient of n is minus since $\sin^2\tau$ is less than unity for scale values greater than zero, according to equation (6b).

Hence the scale value of the bifilar variometer decreases with ordinate. As the magnet grows weaker, and the recording spot of light drifts down on the magnetogram, the trend of the scale values during the course of time should be toward larger values.

It should be noted however, that the coefficient of n in equations (13) and (14) is very small, so that the scale value is practically constant for all ordinary ranges of ordinate and for all ordinary ranges of H.

In the scale value equation (5a) the expression in the parenthesis is never zero for δ and θ greater than zero. The limiting value of the angle θ is δ, as may be shown as follows:

For equilibrium,

$$MH \sin\theta = A \sin(\delta - \theta) \tag{3}$$

When the magnet is in the magnetic prime vertical

$$MH_0 = A \sin\tau \tag{17}$$

By division

$$H = \frac{H_0}{\sin\tau} \frac{\sin(\delta - \theta)}{\sin\theta} \tag{18}$$

If H could become zero then θ must equal δ. Hence we conclude that the bifilar variometer has a scale value greater than zero for all values of H.

In determining the scale value by the method of deflections, the magnetic field at the H magnet is *increased* and *decreased* by a known amount, ΔH, which is small compared with the absolute value of H, hence the original field is on the whole constant, and H, in the scale value equation (5a), is regarded as a particular constant.

The scale value is obtained from the relation

$$S = \frac{\Delta H}{n}$$

and when an auxiliary magnet is used the field strength of this magnet is determined by placing it "end on" in reference to the D magnet. The field strength is, in gammas,

$$\frac{2M}{r^3} \times 10^5 = \Delta H = H_\gamma \tan u,$$

u being the deflection of the D magnet. The scale value formula is thus:

$$s = \frac{H_\gamma \tan u}{n} \tag{19}$$

8. **Change in scale value due to turning torsion head.**—As the spot drifts down on the magnetogram, it is occasionally necessary to restore it to its normal position. Also, if the magnet is twisted on its stem, a turning of the torsion head is necessary in order to bring the spot back to its original position. The physical principles involved will be more clearly understood by dividing the adjustment into two steps. First step: Let the magnet be held in place while the torsion head is turned. The torsion in the suspension corresponding to the base-line scale value will be changed, and consequently the base-line scale value will be changed. Second step: Release the magnet. The magnet will now move to its new position of equilibrium. The change in scale value will thus consist of two parts: (1) a change in base-line scale value due to a change in the torsion corresponding to a zero ordinate; (2) a change in scale value due to the change in the position of the magnet.

To determine the amount of the change in scale value due to turning the torsion head, let the equation of equilibrium, equation (3), before the torsion head is turned, be:

$$MH \sin \theta_1 = A \sin (\delta_1 - \theta_1) \tag{20}$$

After the head is turned, the eqúation will be

$$MH \sin \theta_2 = A \sin (\delta_2 - \theta_2) \tag{21}$$

From these two equations we obtain

$$\sin (\delta_2 - \theta_2) = \sin (\delta_1 - \theta_1) \frac{\sin \theta_2}{\sin \theta_1} \tag{22}$$

Noting that, from equation (8a)

$$\theta = \frac{\pi}{2} - n\epsilon \text{ and } \delta - \theta = \tau + n\epsilon$$

we can write (22) in ordinates, as follows:

$$\sin (\tau_2 + n_2\epsilon) = \sin (\tau_1 + n_1\epsilon) \frac{\cos n_2\epsilon}{\cos n_1\epsilon} \tag{23}$$

Since the last factor on the right is nearly unity for small changes in θ, we have

$$\tau_2 = \tau_1 - (n_2 - n_1)\epsilon \tag{24}$$

That is, the amount of change of torsion in the suspension corresponding to zero ordinate is equal to the change in ordinate brought about by turning the head.

That the head and magnet, for ordinary changes in ordinate, move together practically as a rigid body, when the torsion head is turned, will be made evident by differentiating the equation of equilibrium

$$MH \sin \theta = A \sin (\delta - \theta)$$

with respect to δ and θ

$$\frac{d\delta}{d\theta} = 1 + \frac{MH \cos \theta}{A \cos (\delta - \theta)} = 1 \qquad \text{when } \theta = \frac{\pi}{2}$$

The base-line scale value after turning the head will be found by substituting τ_2 in the equation for base-line scale value

$$s_{02} = \epsilon H_\gamma \cot \tau_2 \tag{24'}$$

The change in base-line scale value due to change in base-line torsion—that is, torsion corresponding to zero ordinate, will be

$$s_{02} - s_{01} = \epsilon H_\gamma \ (\cot \tau_2 - \cot \tau_1) \tag{25}$$

The change in scale value due to change in ordinate will be

$$\Delta s = a \ (n_2 - n_1) \tag{26}$$

where a has the value determined from equations (14) and (15). The equation for the scale value after turning the torsion head will now be

$$s = s_{01} + a \ (n_2 - n_1) \tag{27}$$

Changes in scale value due to turning the torsion head may also be investigated as follows:

The scale value is

$$S = H \ [\cot \theta + \cot (\delta - \theta)] \tag{5a}$$

By substituting from equations (8a) this may be expressed in ordinates

$$S = H [\tan n\epsilon + \cot (\tau + n\epsilon)] \tag{28}$$

The scale values before and after the torsion head has been turned are, respectively,

$$S_1 = H \ [\tan n_1\epsilon + \cot \ (\tau_1 + n_1\epsilon)]$$

$$S_2 = H \ [\tan n_2\epsilon + \cot \ (\tau_2 + n_2\epsilon)]$$

and, therefore,

$$S_2 - S_1 = H \ [\tan n_2\epsilon - \tan n_1\epsilon + \cot \ (\tau_2 + n_2\epsilon) - \cot \ (\tau_1 + n_1\epsilon)]$$

By means of equation (24), this simplifies to

$$S_2 - S_1 = H \ (\tan n_2\epsilon - \tan n_1\epsilon)$$

Using the trigonometric formula,

$$\tan A - \tan B = \frac{\sin (A - B)}{\cos A \cos B}$$

and making the substitutions, remembering that $\cos n_2\epsilon$ and $\cos n_1\epsilon$ are very nearly unity, we will get approximately

$$S_2 - S_1 = H \sin [(n_2 - n_1)\epsilon]$$

or in gammas,

$$s_2 - s_1 = H_\gamma \epsilon \sin [(n_2 - n_1)\epsilon] \tag{29}$$

We shall now exemplify the use of equations (25), (26), (27), and (29) by means of our sample bifilar variometer.

The problem is: Determine the change in scale value due to turning the torsion head of the bifilar so as to bring the recording spot from a minus ordinate of 80 mm. up to the base line.

Here $n_2=0$ $\qquad n_1=-80 \qquad -n_1\epsilon=1^\circ\ 30'$.

From equation (24) the new base-line torsion is

$$\tau_2=58^\circ\ 31'.6.$$

This value substituted in (24′) gives for the new base-line scale value

$$s_o=3.86$$

The change in base-line scale value, is from (25):

$$3.86-3.64=0.22$$

The change in scale value due to change from a minus ordinate of 80 mm. to zero is, from (26) or from (14), since the a factor is practically unchanged:

$$-0.0006907\times 80=-0.05.$$

The total change in scale value is

$$+0.22-0.05=+0.17$$

The change in scale value from equation (29) is

$$s_2-s_1=H_\gamma\epsilon \sin 1^\circ\ 30'=0.17$$

The formula for the scale value now is $s=3.81-0.0007n$.

9. **Scale value of bifilar by oscillations.**—We have seen that the couple acting on the magnet is

$$-\frac{dV}{d\theta}=-MH\sin\theta+A\sin(\delta-\theta)$$

The kinetic reaction of the magnet is $K\ddot{\theta}$, K being the moment of inertia, and $\ddot{\theta}=\frac{d^2\theta}{dt^2}$

Action and reaction being equal and opposite, the equation of motion is

$$K\ddot{\theta}+MH\sin\theta-A\sin(\delta-\theta)=0 \tag{30}$$

When the magnet is displaced through a small angle ψ from a position of equilibrium θ_0, $\sin\theta$ becomes $\sin\theta_0 + \cos\theta_0\psi$ and $\sin(\delta-\theta)$ becomes $\sin(\delta-\theta_0)-\cos(\delta-\theta_0)\psi$. Substituting these expressions, the equation of motion will be:

$$K\ddot{\psi}+[MH\cos\theta_0+A\cos(\delta-\theta_0)]\psi+MH\sin\theta_0-A\sin(\delta-\theta_0)=0 \tag{31}$$

The period is

$$T^2_t=\frac{\pi^2 K}{MH\cos\theta_0+A\cos(\delta-\theta_0)} \tag{32}$$

The scale value for the position of equilibrium θ_0 is from equation (4)

$$S=\frac{MH\cos\theta_0+A\cos(\delta-\theta_0)}{M\sin\theta_0} \tag{33}$$

Combining (32) and (33)

$$S=\frac{\pi^2 K}{T^2_t M\sin\theta_0}=\frac{\pi^2 KH}{T^2_t MH\sin\theta_0} \tag{34}$$

The free period of the magnet, when it has been removed from the variometer, and suspended by a single fiber so as to oscillate under the effect of H alone, is

$$T^2_H=\frac{\pi^2 K}{MH} \tag{35}$$

Substitute in (34)

$$S=\frac{H T^2_H}{T^2_t\sin\theta_0}=\frac{H T^2_H}{T^2_t}\ \text{(for small ordinates)} \tag{36}$$

From (32) when $\theta_0=\frac{\pi}{2}$

$$T^2_t=\frac{\pi^2 K}{A\cos\tau} \tag{37}$$

When $M=0$ or when $H=0$, and so $\delta=\theta_0$

$$T^2_t=\frac{\pi^2 K}{A}$$

So that (a) when the magnet lies in the prime vertical, and (b) when no magnetic field exists or when the suspended body is not magnetized, the oscillations are due to torsion alone. From equations (17), (35), and (37),

$$\frac{T^2_H}{T^2_t}=\cot\tau$$

which equation provides a check on the scale value determined by measuring the torsion angle, or rather, a check on the measurement of τ.

10. **Correction of period of magnet for damping.**—The magnets of variometers are surrounded, as far as possible, by copper; that is, they are inclosed in copper damping boxes. The purpose of the damping box is to check the free oscillations of the magnet; for it is only the movements of the magnet due to the changes in the earth's magnetic field that are desired. The damping box also has the effect of slightly lengthening the free period of the magnet. In the equations above for determining the scale value by oscillations, the period of the magnet was assumed to be the free or undamped period; that is, the period which the magnet would have if it were not inclosed in a damping box. Hence, when the period of the magnet in place in the variometer has been determined by oscillations, it will be necessary to reduce the observed period to the free period.

When the magnet is drawn to one side, a definite amount of potential energy is imparted to it, and when it is released it will

873°—22——2

continue to oscillate until all the energy has been dissipated. Recognizing the fact that the magnet and damping box constitute an electromagnetic system, we can understand just how the energy is consumed.

Consider a small area of the damping box. Through this area there passes a certain amount of magnet flux which is changing at a rate proportional to the angular velocity of the magnet, and, therefore, induces an electromotive force (E). Thus, around the boundary of each elementary area of the damping box there whirls a closed electric current—the eddy currents of the electrical engineer. The resistance being R and the current being I the current around the area is $\frac{E}{R}=I$. The rate at which energy is dissipated is $EI=\frac{E^2}{R}$; so that the rate of dissipation of energy is proportional directly to the square of the angular velocity and inversely as the equivalent resistance of the damping box. The more massive the damping box, and so the less its resistance, the more rapidly the energy is dissipated, and the more effective the damping box in checking the free oscillations of the magnet.

The total energy, at any instant, possessed by the magnet and suspension is the sum of the kinetic energy $\frac{K\dot{\theta}^2}{2}$ and the potential energy V. Therefore the above statement may be expressed:

$$-\frac{d}{dt}\left[\frac{K\dot{\theta}^2}{2}-MH\cos\theta-A\cos(\delta-\theta)\right]=\alpha\dot{\theta}^2$$

where α is simply a factor of proportionality. Performing the differentiation, and then dividing by $\dot{\theta}$

$$K\ddot{\theta}+MH\sin\theta-A\sin(\delta-\theta)=-\alpha\dot{\theta} \qquad (39)$$

This equation means simply that a resisting force $-\alpha\theta$ is acting to hinder the free motion of the magnet. For the position of equilibrium, θ_0 such that $\theta=\theta_0+\psi$ equation (39) will become,

$$K\ddot{\psi}+\alpha\dot{\psi}+[MH\cos\theta_0+A\cos(\delta-\theta_0)]\psi=0 \qquad (39')$$

Dividing through by K, the equation may be written in the form:

$$\ddot{\psi}+2c\dot{\psi}+f^2\psi=0 \qquad (40)$$

where $\quad 2c=\frac{\alpha}{K}$ and $f^2=\frac{MH\cos\theta_0+A\cos(\delta-\theta_0)}{K}$

The equation may be further simplified by substituting another variable u such that $\psi=e^{-ct}u$ where e is the base of the natural system of logarithms.

Thus the equation (40) becomes

$$\ddot{u}+(f^2-c^2)u=0 \tag{41}$$

the solution of which is

$$u=P\cos(\omega t-\nu)$$

where P = an arbitrary constant, and ν = an arbitrary phase angle, and

$\omega=\dfrac{4\pi}{T}$

For simplicity, the time may be chosen so that

$$u=P\cos\omega t$$

The solution of equation (39) is, now,

$$\theta=\theta_0+\psi=\theta_0+e^{-ct}P\cos\omega t \tag{42}$$

or in ordinates, by means of equation (8a)

$$n\epsilon=n_0\epsilon-Pe^{-ct}\cos\omega t$$

To evaluate P, suppose $n\epsilon=n_1\epsilon$ when $t=0$. Then $-\mathrm{P}=n_1\epsilon-n_0\epsilon$ and (42) becomes

$$n=n_0+(n_1-n_0)\ e^{-ct}\cos\omega t \tag{43}$$

n_1-n_0 being the amplitude at the beginning of the motion, and n_0 the equilibrium ordinate. If n_1, n_2, n_3, etc., are the ordinates corresponding to the elongations of the oscillating magnet, which obviously follow each other at intervals of time T_d apart, we obtain from equation (43)

$$\frac{n_1-n_0}{n_0-n_2}=\frac{n_0-n_2}{n_3-n_0},\ \text{etc.,}=e^{cT_d}$$

or

$$\frac{n_1-n_2}{n_3-n_2}=\frac{n_3-n_2}{n_3-n_4}=e^{cT_d} \tag{44}$$

Now, evidently, n_1-n_2, n_3-n_2, n_3-n_4, etc., are simply the differences of the readings of the scale at the successive elongations. Denote them by Δ_0, Δ_1, Δ_2, Δ_3, etc. Then

$$\frac{\Delta_0}{\Delta_1}=\frac{\Delta_1}{\Delta_2}=\frac{\Delta_2}{\Delta_3},\ \text{etc.,}=e^{cT_d} \tag{45}$$

Taking the logarithms

$$\lambda=\log\frac{\Delta_0}{\Delta_1}=\log\frac{\Delta_1}{\Delta_2}=\log\frac{\Delta_2}{\Delta_3},\ \text{etc.,}=cT_d \tag{46}$$

Another useful relation is

$$\log\frac{\Delta_0}{\Delta_N}=N\lambda \tag{47}$$

where N is the number of half periods.

λ is the natural logarithm and is called the logarithmic decrement. In practice λ is most easily computed in common logarithms, and then converted into natural logarithms by the formula

$$\text{Nat. log } \lambda = \frac{\text{com. log } \lambda}{0.4343} \tag{48}$$

From equation (41) the damped period is

$$T^2{}_d = \frac{\pi^2}{f^2 - c^2} \tag{48a}$$

The free period is obtained from this by setting $c^2 = 0$

$$T^2{}_f = \frac{\pi^2}{f^2} \tag{48b}$$

Thus the damped period is longer than the free period by reason of the quantity c.

From (48a) and (46)

$$c^2 T^2{}_d = \frac{\pi^2 c^2}{f^2 - c^2} = \frac{\pi^2 \frac{c^2}{f^2}}{1 - \frac{c^2}{f^2}} = \lambda^2$$

or

$$\frac{c^2}{f^2} = \frac{\lambda^2}{\lambda^2 + \pi^2} \tag{49}$$

From (48a) and (48b)

$$T^2{}_f = T^2{}_d \frac{f^2}{f^2 - c^2} = T^2{}_d \frac{1}{1 - \frac{c^2}{f^2}}$$

Eliminating the ratio $\frac{c^2}{f^2}$ by equation (49), we have for reducing the damped, or observed period, to the free period,

$$T^2{}_f = T^2{}_d \frac{1}{1 + \frac{\lambda^2}{\pi^2}} \tag{50}$$

The noteworthy feature of this equation is that to obtain the free period all that is necessary is the observed period T_d and the logarithmic decrement λ, which is derived from scale readings at the successive elongations of the magnet. Substituting equation (50) in equation (36), we obtain for the scale value by method of oscillations:

$$s = H_\gamma \epsilon \frac{T^2{}_H}{T^2{}_d} \left(1 + \frac{\lambda^2}{\pi^2}\right) \tag{51}$$

In using the method of scale value by oscillations, it will be convenient to make observations for period, and separate observations for elongations.

As a concrete example, let us determine the scale value of our sample bifilar variometer by the method of oscillations. The observed periods are: $T_d = 6.36s$ and $T_H = 4.77s$.

The tabulation will show about what order of accuracy may be expected:

Scale readings.			Log Δ.	λ_{10}
397	Δ_0	211	2.3243	0.2171
186	Δ_1	128	2.1072	.2096
314	Δ_2	79	1.8976	.2104
235	Δ_3	48	1.6812	.2188
283	Δ_4	29	1.4624	.2071
254	Δ_5	18	1.2553	....
272	....	....	Mean log = .2138	

By means of equation (48) we obtain $\lambda = 0.4923$.

We get the same value for λ from the first and last differences by means of equation (47). Substituting the known quantities in equation (51), we obtain the scale value $s = 3.64$.

11. **Stability of bifilar variometer.**—We have

$$T^2{}_t = \frac{\pi^2 K}{MH \cos \theta_0 + A \cos (\delta - \theta_0)} \tag{32}$$

and from equation (4)

$$MH \cos \theta_0 + A \cos (\delta - \theta_0) = MS \sin \theta_0$$

This substituted in equation (32) transforms it to

$$T^2{}_t = \frac{\pi^2 K}{MS \sin \theta_0}$$

If for any value of θ greater than zero and less than π, and for values of H greater than zero, the scale value becomes zero, the period will be infinity, that is, the magnet will be unstable. The stability of the variometer is thus seen to depend on its scale value. At any position of the magnet where s is zero the variometer will be unstable.

In the bifilar variometer (see equation 18) θ can range only from 0 to δ, at which position H must be zero also. In this range the scale value is never zero. Hence the bifilar variometer is stable for all values of H.

We note here some points in connection with the periods of a bifilar variometer.

If no magnetic field exists, that is, if H is zero, or if the suspended body is not a magnet, that is, if M is zero, the period is

$$T^2{}_t = \frac{\pi^2 K}{A}$$

The oscillations of the magnet are then due to torsion alone.

When the magnet is in the magnetic prime vertical, $\theta_0 = \frac{\pi}{2}$

$$T^2{}_t = \frac{\pi^2 K}{A \cos \tau}$$

The oscillations are still due to torsion alone, but are longer in the ratio $1 : \cos \tau$

12. **Recording range of the bifilar variometer.**—Since the bifilar is stable for all values of H, it is capable of recording all values of H. Or, practically speaking, a bifilar variometer can record all values of H permitted by the mechanical construction of the instrument. When the upper and lower limits of θ have been determined, the actual range that can be recorded may be computed from equation (18).

III. CHARACTERISTICS OF THE UNIFILAR VARIOMETER.

13. **Equation of the unifilar variometer.**—Placing $A=0$ in the fundamental equation (1) we obtain for the equation of the unifilar variometer having a control magnet:

$$V=-MH\cos\theta+\frac{h}{2}(\delta-\theta)^2+\frac{MM'}{r^3}\cos(\theta-\kappa) \tag{53}$$

We shall first discuss the characteristics of a unifilar variometer that has no control magnet. The equation is then

$$V=-MH\cos\ \ +\frac{h}{2}(\delta-\theta)^2 \tag{54}$$

The couple tending to increase θ is

$$-\frac{dV}{d\theta}=-MH\sin\theta+h(\delta-\theta) \tag{55}$$

For equilibrium this is equal to zero, hence

$$MH\sin\theta=h(\delta-\theta) \tag{56}$$

The scale value is

$$-\frac{dH}{d\theta}=\frac{MH\cos\theta+h}{M\sin\theta}=H\cot\theta+\frac{h}{M\sin\theta} \tag{57}$$

From equation (56)

$$\frac{h}{M\sin\theta}=\frac{H}{(\delta-\theta)}$$

and substituting this in (57), we obtain for the scale value equation of the unifilar variometer

$$S=H\left[\cot\theta+\frac{1}{(\delta-\theta)}\right]\text{(C. G. S. units per radian)} \tag{58}$$

and for the scale value in gammas per millimeter

$$s=H_{\gamma}\epsilon\left(\cot\theta+\frac{1}{\delta-\theta}\right) \tag{59}$$

The corresponding base-line values are

$$S_0=\frac{H}{\tau}\text{(C. G. S. units per radian)} \tag{60}$$

$$s_0=\frac{H_\gamma \epsilon}{\tau} \text{ (gammas per millimeter)} \tag{61}$$

In these scale value equations H is regarded as a constant.

14. **Development of unifilar scale value in powers of the ordinates.**—Introducing the angle ϕ into equation (58)

$$S=H\left(-\tan\varphi+\frac{1}{\tau-\varphi}\right) \tag{62}$$

The expression on the right in equation (62) is a function of φ, and may be written

$$f(\varphi)=-\tan\varphi+\frac{1}{\tau-\varphi} \tag{63}$$

and may be expanded in powers of φ by Maclaurin's theorem:

$$f(\varphi)=f(\varphi)_{\varphi=0}+\varphi\left(\frac{df}{d\varphi}\right)_{\varphi=0}+\frac{\varphi^2}{2}\left(\frac{d^2f}{d\varphi^2}\right)_{\varphi=0} \tag{11}$$

Performing the indicated operations,

$$f(\phi)_{\varphi=0}=\left(-\tan\varphi+\frac{1}{\tau-\varphi}\right)_{\varphi=0}=\frac{1}{\tau}$$

$$\left(\frac{df}{d\varphi}\right)_{\varphi=0}=\left[-\sec^2\varphi+\frac{1}{(\tau-\varphi)^2}\right]_{\varphi=0}=-1+\frac{1}{\tau^2}$$

$$\left(\frac{d^2f}{d\varphi^2}\right)_{\varphi=0}=\left[-2\sec^2\varphi\tan\varphi+\frac{2}{(\tau-\varphi)^3}\right]_{\varphi=0}=\frac{2}{\tau^3}$$

Making the substitutions, equation (62) becomes

$$S=H\left[\frac{1}{\tau}-\varphi\left(1-\frac{1}{\tau^2}\right)+\frac{\varphi^2}{\tau^3}\right] \tag{64}$$

Changing this into ordinates, $(\varphi=-n\epsilon)$

$$S=H\left[\frac{1}{\tau}+n\epsilon\left(1-\frac{1}{\tau^2}\right)+\frac{n^2\epsilon^2}{\tau^3}\right] \tag{65}$$

In gammas per millimeter this is

$$s=s_0+H_\gamma\epsilon^2 n\left(1-\frac{1}{\tau^2}\right)+\frac{H_\gamma\epsilon^3 n^2}{\tau^3}=s_0+an+bn^2 \tag{66}$$

For a concrete case, use the constants of the sample unifilar variometer given in paragraph 2.

$$s=1.95+0.003457n+0.00000002n^2 \tag{67}$$

15. **Characteristics of unifilar scale value.**—Since for practical scale values τ is greater than unity, the coefficient of n in equations (65) and (66) is positive. Thus the scale value of the unifilar variometer increases with ordinate. Moreover, the a factor is so large

that it must be taken into account in converting the ordinates of the variation curve into absolute values of H. As the magnet grows weaker the recording spot drifts down on the sheet; the trend of the scale values during the course of time should be toward decreasing values. From the scale value equation

$$S = H\left(\cot\theta + \frac{1}{\delta - \theta}\right)$$

we see that the scale value is large for small values of θ, that is, for large ordinates; and that it becomes smaller with increasing θ, that is, with decreasing ordinate. For H greater than zero and for a value of θ greater than zero and less than π, the expression in the parenthesis becomes equal to zero when

$$\tan\theta = \theta - \delta \tag{68}$$

In the case of our sample unifilar variometer $\delta = 5.8671$. The angle at which the scale value is equal to zero is from equation (68), $\theta = 103° 51'.3$, that is, when the north end of the magnet points $13° 51'.3$ to the south of the magnetic prime vertical. That H is not zero for $\theta < \pi$ may be shown as follows: For equilibrium

$$MH\sin\theta = h(\delta - \theta) \tag{56}$$

When the magnet lies in the magnetic prime vertical

$$MH_0 = h\tau \tag{69}$$

By division

$$H = \frac{H_0}{\tau}\,\frac{\delta - \theta}{\sin\theta} \tag{70}$$

For practicable scale values, δ is always considerably larger than θ. Hence for $\theta < \pi$, H is never zero.

We thus see that the unifilar scale value becomes zero for a small ordinate corresponding to a value of θ somewhat greater than $\frac{\pi}{2}$.

16. **Unequal deflections of unifilar variometer.**—When a deflecting magnet is brought near the variometer an additional force acts upon the suspended magnet corresponding to a change of ΔH in H. When the magnet is reversed, but otherwise in the same position, the added force corresponds to $-\Delta H$. As the scale value changes with change of ordinate, the angular deflection and the distance in millimeters on the magnetogram will be different in the two cases.

Suppose that n_1 and $-n_2$ are the deflections in millimeters above and below the undeflected position of the spot which for simplicity we shall take as the base line. Then

$$\begin{aligned} \Delta H &= \int (s_0 + an_1)\,dn_1 = s_0 n_1 + \frac{a}{2}n^2{}_1 = n_1\left(s_0 + \frac{a}{2}n_1\right) \\ -\Delta H &= \int (s_0 - an_2)\,dn_2 = s_0 n_2 - \frac{a}{2}\,n^2{}_2 = n_2\left(s_0 - \frac{a}{2}n_2\right) \end{aligned} \tag{71}$$

Solving these equations for n_1 and n_2, we get,

$$n_1 = \frac{s_0}{a}\left(1 - \sqrt{1 - \frac{2a\Delta H}{s^2{}_0}}\right) \tag{72}$$

$$n_2=\frac{s_0}{a}\left(-1+\sqrt{1+\frac{2\,a\Delta H}{s^2_0}}\right) \tag{73}$$

Also $\frac{n_2}{n_1}=$ ratio of deflections.

For $a=0.003457$, $\Delta H_\gamma=100$, $s_0=1.95$, we find $n_1=49.15$ and $n_2=53.86$ and for the ratio of the deflections

$$\frac{n_2}{n_1}=1.096$$

For a simple working formula, proceed as follows: From the last members of (71)

$$\frac{n_2}{n_1}=\frac{s_0+\frac{a}{2}n_1}{s_0-\frac{a}{2}n_2}=\frac{\left(s_0+\frac{a}{2}n_1\right)^2}{s_0{}^2-\frac{a^2}{4}n^2{}_2}$$

and dropping terms in a^2

$$\frac{n_2}{n_1}=1+\frac{a}{s_0}n_1 \text{ (approximately)} \tag{74}$$

The unequal deflections of the unifilar variometer may also be investigated as follows: Equation (70) may be written

$$\frac{\delta-\theta}{\sin\theta}=\frac{H\tau}{H_0}$$

where H's are values in gammas.
Introducing the angle φ

$$\frac{\tau-\varphi}{\cos\varphi}=\frac{H\tau}{H_0}=\frac{H\epsilon}{s_0}=\frac{\tau-\varphi}{1-\frac{\varphi^2}{2}}$$

Solving for φ, we have finally

$$\varphi=\frac{s_0}{H\epsilon}\pm\sqrt{\left(\frac{s_0}{H\epsilon}\right)^2+\frac{2\Delta H}{H}}$$
$$=\frac{s_0}{H\epsilon}\left(1\pm\sqrt{\frac{1+2\Delta H\ H\epsilon^2}{s^2_0}}\right)$$

In ordinates,

$$n_1=\frac{s_0}{H_1\epsilon^2}\left(-1+\sqrt{1+\frac{2\ H_1\epsilon^2\Delta H_1}{s^2_0}}\right) \tag{75}$$

$$-n_2=\frac{s_0}{H_2\epsilon^2}\left(1-\sqrt{1-\frac{2\ H_2\epsilon^2\Delta H_2}{s^2_0}}\right) \tag{76}$$

where H and H_2 refer to values of H corresponding to deflected position of magnet. For the values $s_0=1.95$ and $\Delta H_\gamma=100$ as in the example given above

$$\frac{n_2}{n_1}=\frac{54.00}{49.02}=1.101.$$

17. **Condition for constant scale value in the unifilar variometer.**—From equations (72) and (73) it is evident that the unequal deflections depend on the a factor in the series development of the scale value, viz:

$$a = H_\gamma \epsilon^2 \left(1 - \frac{1}{\tau^2}\right) \tag{77}$$

The a factor equals zero when τ equals unity. As τ is not readily determined in practice, we may express the a factor in a different form. From equation (61)

$$\frac{H_\gamma \epsilon}{\tau} = s_0$$

When $\tau = 1$, the condition for constant scale value is

$$s_0 = H_\gamma \epsilon \tag{78}$$

In general this condition means either a larger scale value than is desired (about $s_0 = 2.5$ being the usual value) or that the variometer will have to be placed at a distance so far from the recording apparatus as to be impracticable. For example, in the case of our sample unifilar variometer we obtain:

$$s_0 = 8.38$$

Two questions now arise: (1) Can the scale value be reduced to the desired value by means of control magnets? and (2) How is the a factor affected by a control magnet? These questions lead to a consideration of control magnets.

18. **Control magnets for unifilar variometer.**—The equation of the unifilar variometer having a control magnet is:

$$V = -MH\cos\theta + \frac{h}{2}(\delta - \theta)^2 + \frac{MM'}{r^3}\cos(\theta - \kappa) \tag{53}$$

To simplify this equation somewhat, let us make use of the identities

$$F = \frac{M'}{r^3}$$

$$F_\gamma = \frac{M'}{r^3} \times 10^5$$

where F is the field intensity of the control magnet at a point on a line perpendicular to its axis, at distance r. Equation (53) is now

$$V = -MH\cos\theta + \frac{h}{2}(\delta - \theta)^2 + FM\cos(\theta - \kappa) \tag{80}$$

The equation of moments is

$$-\frac{dV}{d\theta} = -MH\sin\theta + h(\delta - \theta) + FM\sin(\theta - \kappa) \tag{81}$$

The equation of equilibrium is

$$MH\sin\theta = h(\delta - \theta) + FM\sin(\theta - \kappa) \tag{82}$$

The scale value equation is

$$-\frac{dH}{d\theta}=S'=H\cot\theta+\frac{h}{M\sin\theta}-\frac{F\cos(\theta-\kappa)}{\sin\theta} \tag{83}$$

From equation (82)

$$h=\frac{MH\sin\theta}{\delta-\theta}-FM\frac{\sin(\theta-\kappa)}{\delta-\theta}$$

When this is substituted in (83) the scale value equation becomes

$$S'=H\left(\cot\theta+\frac{1}{\delta-\theta}\right)-\frac{F\sin(\theta-\kappa)}{(\delta-\theta)\sin\theta}-\frac{F\cos(\theta-\kappa)}{\sin\theta} \tag{84}$$

The scale value expressed in gammas is

$$s'=H_\gamma\epsilon\left(\cot\theta+\frac{1}{\delta-\theta}\right)-F_\gamma\epsilon\frac{\sin(\theta-\kappa)}{(\delta-\theta)\sin\theta}-F_\gamma\epsilon\frac{\cos(\theta-\kappa)}{\sin\theta} \tag{85}$$

Corresponding to these two general scale value equations, we shall have, when the suspended magnet is in the magnetic prime vertical $\left(\theta=\frac{\pi}{2}\right)$

$$S'_0=\frac{H-F\cos\kappa}{\tau}-F\sin\kappa \tag{86}$$

$$s'_0=\frac{H_\gamma\epsilon-F_\gamma\epsilon\cos\kappa}{\tau}-F_\gamma\epsilon\sin\kappa \tag{87}$$

From equation (87) we see that the control magnet is most effective when $\kappa=\frac{\pi}{2}$, that is, when the control magnet is parallel to the suspended magnet. The scale value will then be

$$s'_0=\frac{H_\gamma\epsilon}{\tau}-F_\gamma\epsilon \tag{88}$$

Thus the scale value is decreased when suspended and control magnets point in the same direction, and it is increased when they point in opposite directions. The change in scale value due to the control magnet is

$$s_0-s'_0=F_\gamma\epsilon \tag{89}$$

Let us note the effect of placing the control magnet in the magnetic meridian $\kappa=0$, the control magnet then being perpendicular to the suspended magnet. Equation (87) will become

$$s'_0=\frac{(H_\gamma-F_\gamma)\epsilon}{\tau} \tag{90}$$

We observe that the base-line scale value is changed to the extent the earth's magnetic field at the suspended magnet is changed. The suspended magnet will also take up another position of equilibrium, as may be seen by putting $\kappa=0$ in the equation of equilibrium

$$M(H-F)\sin\theta=MH_2\sin\theta=h(\delta-\theta) \tag{91}$$

where

$$H_2=H-F$$

The approximate displacement of the magnet is $\frac{F\gamma}{8'_0}$

Now, suppose the torsion head is turned so as to bring the magnet back into the prime vertical. If s_1 and s_2 and H_1 and H_2 are the base-line scale values and intensities before and after the control magnet has been attached, we shall have:

$$s_1=\frac{H_1\epsilon}{\tau_1}$$
$$s_2=\frac{H_2\epsilon}{\tau_2} \tag{92}$$
$$\frac{s_2}{s_1}=\frac{H_2\tau_1}{H_1\tau_2}$$

But from the equation of equilibrium,

$$MH_1=h\tau_1$$
$$MH_2=h\tau_2 \tag{93}$$
$$\frac{H_2}{H_1}=\frac{\tau_2}{\tau_1}$$

Combining (92) and (93)

$$s_2=s_1$$

The scale value remains unchanged. For the same reason we may state that the scale value of a unifilar variometer is the same at whatever station on the earth it may be placed, provided the torsion head is turned so as to bring the magnet into the same position with reference to the magnetic meridian.

19. **Effect of control magnet on *a* factor.**—To answer the second question at the close of paragraph 17, it will be necessary to develop the scale value of a unifilar variometer having a control magnet in the form of a series. Introducing the angle φ into the scale value equation (84) we obtain

$$S'=H\left(-\tan\varphi+\frac{1}{\tau-\varphi}\right)-\frac{F\cos(\kappa-\varphi)}{(\tau-\varphi)\cos\varphi}-\frac{F\sin(\kappa-\varphi)}{\cos\varphi} \tag{94}$$

The first parenthesis on the right has already been dealt with (see equation 66) so that it is only necessary to develop the second and third terms, and add the results to equation (64). Moreover, we may take advantage of the fact that as the coefficient of φ^2 will clearly be negligible, the first differential coefficient in Maclaurin's theorem is sufficient. The function of φ is then

$$f(\varphi)=\frac{\cos(\kappa-\varphi)}{(\tau-\varphi)\cos\varphi}+\frac{\sin(\kappa-\varphi)}{\cos\varphi}$$

$$=\frac{\cos\kappa+\sin\kappa\tan\varphi}{\tau-\varphi}+\sin\kappa-\cos\kappa\tan\varphi$$

$$f(\varphi)_{\varphi=0}=\frac{\cos\kappa}{\tau}+\sin\kappa$$

By differentiating

$$\frac{df}{d\varphi}=\frac{(\tau-\varphi)\sin\kappa\sec^2\varphi+\cos\kappa+\sin\kappa\tan\varphi}{(\tau-\varphi)^2}-\cos\kappa\sec^2\varphi$$

and

$$\left(\frac{df}{d\varphi}\right)_{\varphi=0}=\frac{\tau\sin\kappa+\cos\kappa}{\tau^2}-\cos\kappa$$

Thus

$$S'=\frac{H}{\tau}-F\left(\frac{\cos\kappa}{\tau}+\sin\kappa\right)-\varphi\left(H-F\cos\kappa+\frac{F\sin\kappa}{\tau}+\frac{F\cos\kappa-H}{\tau^2}\right)$$

and in ordinates

$$S'=\frac{H}{\tau}-F\left(\frac{\cos\kappa}{\tau}+\sin\kappa\right)+n\epsilon\left(H-F\cos\kappa+\frac{F\sin\kappa}{\tau}+\frac{F\cos\kappa-H}{\tau^2}\right) \tag{95}$$

The scale value in gammas is

$$s'=\frac{H_\gamma\epsilon}{\tau}-F_\gamma\epsilon\left(\frac{\cos\kappa}{\tau}+\sin\kappa\right)+n\epsilon^2\left(H_\gamma-F_\gamma\cos\kappa+\frac{F_\gamma\sin\kappa}{\tau}+\frac{F_\gamma\cos\kappa-H_\gamma}{\tau^2}\right) \tag{96}$$

As the control magnet is most effective when parallel to the suspended magnet, we shall only consider the case where $\kappa=\frac{\pi}{2}$. We thus obtain for the a factor of a unifilar variometer having a control magnet

$$a'=\epsilon^2\left(H_\gamma+\frac{F_\gamma}{\tau}-\frac{H_\gamma}{\tau^2}\right)=a+\frac{F_\gamma\epsilon^2}{\tau} \tag{97}$$

Equation (97) is the answer to the question as to how the a factor is affected by the control magnet. The a factor is increased or decreased by the control magnet, depending on the direction of its field, that is, whether F is plus or minus, with reference to the suspended magnet. This equation may be put in a more convenient form:

Let

$$a'=x\epsilon^2 \tag{98}$$

Then

$$\frac{F\epsilon}{H}=\frac{\epsilon}{\tau}-\epsilon\tau+\frac{\epsilon\tau x}{H} \tag{99}$$

From equation (88)

$$\frac{F\epsilon}{H}=\frac{\epsilon}{\tau}-\frac{s'_0}{H} \tag{100}$$

Combining (99) and (100)

$$\tau=\frac{s'_0}{\epsilon(H-x)} \tag{101}$$

Solving (101) for x and multiplying by ϵ^2

$$a'=x\epsilon^2=\frac{H\epsilon^2\tau-s'_0\epsilon}{\tau}$$

$$=\frac{H^2\epsilon^2-s_0s'_0}{H} \tag{102}$$

From equation (88)

$$F=\frac{H}{\tau}-\frac{s'_0}{\epsilon}$$

$$=\frac{s_0-s'_0}{\epsilon} \tag{103}$$

Also

$$F=\frac{M'}{r^3}\times 10^5 \tag{104}$$

In equations (98–104), H, F, and s are expressed in gammas. The condition for zero a factor may be found by equating the right-hand member of (102) to zero. The condition is

$$s'_0=H\epsilon\tau \tag{105}$$

Now $s'_0=5$, $\tau=1$ and $\log\epsilon=6.63982$ are about the practicable values for these quantities and for these values $H_\gamma=11{,}459$. Hence the condition for zero a factor can not, in general, be satisfied. If, however, we can tolerate a value $a'=0.001$, which is as small as can be derived from observations, we find that for $\tau=1$ and $s'_0=5$, in equations (98) and (101), $H_\gamma=16{,}700$, approximately. Thus the answer to the first question at the close of paragraph 17 is that the condition for a negligible a factor can be satisfied only at stations where H is comparatively small, or at stations where H can be made small enough by extra magnets.

Evidently, the required scale value can be obtained in two ways: (1) By using a small fiber, giving a small scale value, in which case the control magnet will increase the scale value; (2) by using a large fiber, giving a large scale value, and reducing it to the desired value by a control magnet. The preferable method will be the one that gives the smaller value for the a factor.

It is practicable to increase or decrease the scale value to the extent of 4γ by means of control magnets.

As a practical example of the use of equations (98–104) let us determine whether a large or small fiber will be preferable at a station where $H_\gamma = 15{,}600$, and a scale value of 4.45γ (s'_0) is required. The control magnet is to make a change of $\pm\ 3\gamma$ say in the scale value.

In the case of the small fiber, we obtain from equation (102) second form, $a' = 0.0027$, s_0 being in this case 1.45γ.

In the case of the large fiber, $s_0 = 7.45$ and $a' = 0.0008$. The formulas for the scale values in the two cases will be

$$s' = 4.45 + 0.003n \text{ (small fiber)}$$

$$s' = 4.45 + 0.0008n \text{ (large fiber)}$$

so that the large fiber will give practically a constant scale value and is to be preferred. The field intensity F, of the control magnet, will be found from the second form of equation (103), and the strength and distance of the control magnet, if one or the other is given, may be found from (104).

As another example, determine whether it is preferable to use a large or a small fiber at a station where $H_\gamma = 18{,}200$, the required scale value being 2.5γ (s'_0). Assume that the large fiber alone gives a scale value $s_0 = 5.50$, and that the small fiber alone gives $s_0 = 1$. From equation (102) we obtain for the scale values

$$s' = 2.50 + 0.0035n \text{ (small fiber)}$$

$$s' = 2.50 + 0.0029n \text{ (large fiber)}$$

There is a small gain in using the large fiber. If the scale value without a control magnet was 2.5γ, we would obtain from equation (102), since $s_0 = s'_0 = 2.5$

$$s = 2.50 + .0033n \text{ (no control magnet)}$$

20. **Remark on best position of control magnet.**—In the preceding investigation the control magnet was assumed to be vertically above or below the center of the suspended magnet. If the control magnet were placed in line with the axis of the suspended magnet, it would be twice as effective, since its field intensity is twice as great at a given distance.

Equation (104) would then be replaced by the equation

$$F = \frac{2M}{r^3} \times 10^5 \tag{106}$$

By means of a control magnet so placed it would be practical to increase or decrease the scale value at least by 6.0γ and the a factor could be correspondingly reduced. For example, for a scale value of $2.5\gamma(s'_0)$ at a station where $H_\gamma = 19200$, $s_0 = 2.5 + 6.0 = 8.5$ and from equation (102) we obtain:

$$s' = 2.5 + 0.0025n$$

The a factor decreases with increase of scale value. For a scale value of 4.0γ, and $H_\gamma = 19200$, $s_0 = 10.0$ and from (102)

$$s' = 4.0 + 0.0016n$$

Furthermore, it is entirely feasible, by placing a control magnet in the magnetic meridian north or south of the suspended magnet, and in the same horizontal plane, to reduce the earth's horizontal intensity by one-fourth of its value. Substituting $H_\gamma = 14400$, $s'_0 = 2.5$, and $s_0 = 8.5$ in equation (102), we obtain:

$$s' = 2.5 + 0.0013n$$

Substituting $H_\gamma = 14400$, $s'_0 = 4.0$ and $s_0 = 10.0$, we get

$$s' = 4.0 \text{ (absolutely constant).}$$

21. **Change in scale value of unifilar due to turning torsion head.**—The equation of equilibrium of the unifilar without a control magnet before turning the torsion head is:

$$MH \sin \theta_1 = h(\delta_1 - \theta_1) \tag{107}$$

After turning the torsion head, the equation is

$$MH \sin \theta_2 = h(\delta_2 - \theta_2) \tag{108}$$

From these two equations, by division,

$$\delta_2 - \theta_2 = (\delta_1 - \theta_1) \frac{\sin \theta_2}{\sin \theta_1} \tag{109}$$

Converting (109) into ordinates by means of the relation

$$\theta = \frac{\pi}{2} - n\epsilon$$

$$\tau_2 + n_2\epsilon = (\tau_1 + n_1\epsilon) \frac{\cos n_2\epsilon}{\cos n_1\epsilon}$$

or

$$\tau_2 - \tau_1 = (n_1 - n_2)\epsilon, \text{ nearly} \tag{110}$$

Thus the change in base line torsion is equal to the angular change in ordinate. That the torsion head, fiber and magnet move together very nearly as if they were parts of the same rigid body, may be shown by differentiating the equation of equilibrium with respect to δ and θ.

$$\frac{d\delta}{d\theta} = 1 + \frac{MH \cos \theta}{h} = 1 \text{ for } \theta = \frac{\pi}{2}$$

The base line torsion after the head has been turned, is

$$\tau_2 = \tau_1 - (n_2 - n_1)\epsilon \tag{111}$$

The base line scale value now is

$$s^0_2 = \frac{H_\gamma \epsilon}{\tau_1 - (n_2 - n_1)\epsilon} \tag{112}$$

The change in base line scale value is

$$s^0_2 - s^0_1 = \frac{H_\gamma \epsilon}{\tau_1 - (n_2 - n_1)\epsilon} - s^0_1 \tag{113}$$

Also

$$a_2(n_2 - n_1) = \text{change due to ordinate.} \tag{114}$$

The total change will be the sum of (113) and (114). The a_2 factor may be computed from equation (102), but ordinarily the change is negligible. In case $(n_2-n_1)\epsilon$ is small compared with τ_1, there will be very little change in base line scale value and the total change will be given approximately by equation (114).

Changes in scale value due to turning the torsion head may also be investigated as follows: The scale value equation (59) in ordinates is

$$s=H_\gamma\epsilon\left(\tan n\epsilon+\frac{1}{\tau+n\epsilon}\right)$$

The scale value before turning the head is

$$s_1=H_\gamma\epsilon\left(\tan n_1\epsilon+\frac{1}{\tau_1+n_1\epsilon}\right)$$

After turning the head it is

$$s_2=H_\gamma\epsilon\left(\tan n_2\epsilon+\frac{1}{\tau_2+n_2\epsilon}\right)$$

The change in scale value is

$$s_2-s_1=H_\gamma\epsilon\left(\tan n_2\epsilon-\tan n_1\epsilon+\frac{1}{\tau_2+n_2\epsilon}-\frac{1}{\tau_1+n_1\epsilon}\right)$$

The last two terms on the right cancel in virtue of equation (111). So

$$s_2-s_1=H_\gamma\epsilon\ (\tan n_2\epsilon-\tan n_1\epsilon)$$

Using the trigonometric formula

$$\tan A-\tan B=\frac{\sin\ (A-B)}{\cos A\ \cos B}$$

and remembering that $\cos n_1\epsilon$ and $\cos n_2\epsilon$ are very nearly unity, we get, approximately,

$$s_2-s_1=H_\gamma\epsilon\ \sin\ (n_2-n_1)\epsilon \tag{115}$$

As a practical problem, let us determine the change in scale value due to increasing the ordinate of our sample unifilar variometer by 200 mm. $=5°=.0873$ radian. From equation (111) $\tau_2=4.209$. From equation (112) $s°_2=1.99$. The change in base line scale value is 0.04. From equation (114) the change due to change in ordinate is 0.69. The total change is $0.04+0.69=0.73$. The scale value formula now is

$$s=2.68+0.003457n.$$

The change in scale value from equation (115) is 0.73. An approximate value of the a factor may be found by putting $n_2-n_1=1$ in equation (115), inasmuch as the a factor is the change in scale value per unit change in ordinate.

22. **Scale value of unifilar by oscillations.**—Placing $\kappa=\frac{\pi}{2}$ in the equation of moments (81), equating the resulting equation to $K\ddot{\theta}$ and transposing all terms to the left side, we obtain for the equation of motion

$$K\ddot{\theta}+MH\sin\theta-h(\delta-\theta)+FM\cos\theta=0 \tag{116}$$

If the magnet is slightly displaced from a position of equilibrium θ_0, such that $\theta=\theta_0+\psi$, where ψ is a small angle

$\sin\theta$ will become $\sin\theta_0+\psi\cos\theta_0$

$\delta-\theta$ will become $\delta-\theta_0-\psi$

$\cos\theta$ will become $\cos\theta_0-\psi\sin\theta_0$

Substituting these into equation (116)

$$K\ddot{\psi}+(MH\cos\theta_0+h-FM\sin\theta_0)\psi=0 \tag{117}$$

The period is

$$T_f^2=\frac{\pi^2 K}{MH\cos\theta+h-FM\sin\theta} \tag{118}$$

θ now indicating the position of equilibrium. Placing $\kappa=\frac{\pi}{2}$ in the scale value equation (83) we may write it

$$S'=\frac{MH\cos\theta+h-FM\sin\theta}{M\sin\theta} \tag{119}$$

Substituting from (118)

$$S'=\frac{\pi^2 K}{T_f^2 M\sin\theta}=\frac{\pi^2 KH}{T_f^2 MH\sin\theta} \tag{120}$$

The period of the suspended magnet when removed from the variometer and oscillating under the effect of H alone, and corrected for torsion, is

$$T_H^2=\frac{\pi^2 K}{MH} \tag{121}$$

From (120) and (121)

$$S'=\frac{HT_H^2}{T_f^2\sin\theta}$$

For the base line scale value $\theta=\frac{\pi}{2}$

$$S_0'=\frac{HT_H^2}{T_f^2}$$

and in gammas

$$s_0'=H_\gamma\epsilon\frac{T_H^2}{T_f^2} \tag{122}$$

The observed period T_d will require a correction for damping to reduce it to the free period T_f.

From equation (118) we notice that when the control magnet is placed in the magnetic meridian ($\theta=0$) it has no effect on the period. The period then is

$$T_f^2=\frac{\pi^2 K}{MH+h} \tag{123}$$

When the suspended magnet lies in the magnetic prime vertical, $\theta=\frac{\pi}{2}$, the period is

$$T_f^2=\frac{\pi^2 K}{h-FM} \tag{124}$$

The oscillations are then due to the torsion and the control magnet. The effect of the control magnet is to decrease or increase the period, depending on whether it points in the same or in the opposite direction with reference to the suspended magnet. If there is no control magnet, the suspended magnet oscillates under the effect of torsion alone.

23. **Stability of the unifilar variometer.**—Equation (83) may be written

$$MH \cos\theta+h-FM\sin\theta=MS'\sin\theta$$

Substituting this in (117) the equation of motion becomes

$$K\ddot{\psi}+(MS'\sin\theta)\psi=0 \tag{125}$$

in which $\sin\theta$ is always regarded as positive. We thus see that the scale value determines the character of the motion of the magnet. When the scale value is plus, the coefficient of ψ is plus, and represents a restoring force which, when the magnet is displaced, tends to restore it to its original position. In this case the solution of the equation represents an oscillatory motion about a position of equilibrium. If the scale value is less than zero, or minus, the coefficient of ψ represents a displacing force which tends to carry the magnet farther and farther away from its original position. In this case, the solution of the equation is exponential in form, indicating that the value of ψ continually increases. Thus the critical value of S' is the scale value at which the magnet passes from a condition of stability to one of instability.

When the scale value is plus, the solution of equation (125) is

$$\psi=P\sin\left[\left(\frac{MS'\sin\theta}{K}\right)^{\frac{1}{2}}t+\nu\right] \tag{126}$$

where P and ν are constants depending on initial conditions. The solution for θ is

$$\theta=\theta_0+P\sin\left[\left(\frac{MS'\sin\theta}{K}\right)^{\frac{1}{2}}t+\nu\right] \tag{127}$$

where θ_0 is determined by the condition

$$MH\sin\theta_0-h(\delta-\theta_0)+FM\cos\theta_0=0$$

The period is

$$T_f^2=\frac{\pi^2 K}{MS'\sin\theta} \tag{128}$$

When the scale value is equal to zero the period becomes infinitely long, and when the scale value is minus, the period is imaginary. We thus conclude that the criterion for stability is that the scale value shall not be equal to zero for any value of H greater than zero.

To investigate the stability of the unifilar variometer, let us return to the general scale value equation

$$S' = H\left(\cot\theta + \frac{1}{\delta - \theta}\right) - \frac{F\sin(\theta - \kappa)}{(\delta - \theta)\sin\theta} - \frac{F\cos(\theta - \kappa)}{\sin\theta} \qquad (84)$$

Introducing the angle φ and assuming that the control magnet is in the prime vertical, so that $\kappa = \frac{\pi}{2}$.

$$S' = H\left(-\tan\varphi + \frac{1}{\tau - \varphi}\right) - \frac{F\tan\varphi}{\tau - \varphi} - F \qquad (129)$$

For instability, $S' = 0$. Solving for $\tan\varphi$ the critical angle of stability is found to be

$$\tan\varphi = \frac{H - F(\tau - \varphi)}{H(\tau - \varphi) + F} \qquad (130)$$

This equation can be readily solved by approximation, H, F, and τ being given. As a first approximation, assume that $\varphi = 0$ in the right hand member. Compute a value for $\tan\varphi$, then substitute the value of φ thus obtained on the right again, and repeat the process until the desired accuracy is attained. From this equation we can infer the effect of the quantities F and τ on the critical angle φ. Without a control magnet, we shall have

$$\tan\varphi = \frac{1}{\tau - \varphi} \qquad (131)$$

In this case, the critical angle varies, roughly, inversely as τ; that is, the larger the fiber the greater is the critical angle. The control magnet (F) decreases the critical angle (if the control magnet and suspended magnet point in the same direction). If the control magnet is reversed so that F is minus, the critical angle is increased. It should be observed that the scale value depends on F and τ, as shown by the equation:

$$\tau = \frac{H_\gamma \epsilon}{s_0} \qquad (61)$$

$$F_\gamma = \frac{s_0 - s_0'}{\epsilon} \qquad (89)$$

We have seen that a given scale value may be obtained either by using a fiber of just the right size to require no control magnet, or by using a small fiber and a control magnet to increase the scale value, or by using a large fiber and a control magnet to decrease the scale value.

As a practical problem, let us take 2.5γ as our required scale value, and determine from equations (130), (61), and (89), which of the combinations just mentioned gives the greatest critical angle.

In the case where no control magnet is necessary we obtain from equation (131)

$$\varphi = 18°15' \text{ (no control magnet).}$$

In the case of the small fiber, the scale value is, without the control magnet 1.0γ, say, so that the control magnet must increase the scale value by 1.5γ. From equation (61) $\tau = 8.377$. From (89) $F = 0.03438$. Using these values, we obtain from equation (130)

$$\varphi = 17° \ 12' \text{ (small fiber and control magnet).}$$

In the case of the large fiber if it is assumed that the scale value without the control magnet is 5.5γ, the control magnet must reduce this by 3.0γ. From equation (61) $\tau = 1.523$. From (89) $F = 0.06876$. With these values in (130) we obtain

$$\varphi = 21° \ 9' \text{ (large fiber with control magnet).}$$

Thus the variometer having the large fiber has the greatest stability.

24. **Recording range of the unifilar variometer.**—As the scale value increases with ordinate the unifilar can manifestly record all large values of H; that is, all values of H greater than the base line value. When H is low enough, the magnet reaches a state of instability. We wish to ascertain the value of H corresponding to the critical angle, at which the magnet becomes unstable.

Expressing the equation of equilibrium (82) in terms of φ

$$MH \cos \varphi = h \ (\tau - \varphi) + FM \sin \varphi \tag{132}$$

When $\varphi = 0$,

$$MH_0 = h\tau \tag{133}$$

From (132) by transposition

$$MH \cos \varphi - FM \sin \varphi = h \ (\tau - \varphi)$$

From this equation and (133)

$$\frac{H \cos \varphi - F \sin \varphi}{H_0} = \frac{\tau - \varphi}{\tau}$$

$$H = \frac{H_0 \ (\tau - \varphi)}{\tau \cos \varphi} + F \tan \varphi \tag{134}$$

or

$$= \frac{S_0 \ (\tau - \varphi)}{\cos \varphi} + F \tan \varphi$$

From this equation we infer that, for a particular value of φ, H is smaller in proportion as τ is smaller. Also that the critical value of H is larger on account of the control magnet. If, however, the control magnet is reversed, the critical value of H becomes less.

When the critical angle has been computed from equation (130), the critical value of H is computed from equation (134).

In the use of equations (130) and (134) it should be observed that the value of H in (130) is the value of H derived from (134), so that both equations must be simultaneously satisfied.

The desired accuracy may be attained by a process of continual approximation.

Let us now compute the critical values of H for three variometers, each having the scale values of 2.5γ in the three cases mentioned above. It should be noted, however, that the critical angles have been computed on the assumption that the low or critical value of H is 0.192, which we have heretofore regarded as the base line or station value. To save computations, we shall reverse the problem and determine what the base line or station value must be in order that $H=0.192$ shall be the critical value. For this purpose we may write equation (134)

$$H_0=\frac{H\tau\cos\varphi-F\tau\sin\varphi}{\tau-\varphi} \tag{135}$$

One other point is to be considered. The value of τ computed from equation (61) depends on the value of H_0 which we are seeking. Merely for illustration, however, we shall use τ computed from equation (61), with the value of $H=0.192$. We thus obtain from equation (135)

$H_0=0.20150$, range $=0.20150-0.19200=950\gamma$ (no control magnet)
$H_0=0.20070$, range $=0.20070-0.19200=870\gamma$ (small fiber)
$H_0=0.20360$, range $=0.20360-0.19200=1160\gamma$ (large fiber)

As a matter of fact, the angle φ, and hence the low value of H, is limited by the angular spacing of the regular and reserve mirrors, and by the mechanical construction of the variometer. When ϕ has been determined, the lower limit of H may be computed from equation (134).

During a magnetic storm, when H reaches low values, the scale value may become so small and the motion of the magnet so rapid that the recording spot of light fails to make any impression on the photographic paper. To meet this situation the scale value of the variometer must be so adjusted that at low H the motion of the magnet is slow enough to produce a record.

Solving equation (134) for τ we get,

$$\tau=\frac{H_0\,\varphi}{H_0-H\cos\varphi+F\sin\varphi} \tag{136}$$

When φ has been determined from mechanical considerations, and when the low value of H and the field intensity of the control magnet have been decided upon, τ may be computed from equation (136) and the scale value may be then computed from equation (88).

For a single example, let $\varphi=6°30'$, $H_0\gamma=19200$, and $H\gamma=18400$, and $F=0$ (no control magnet). We obtain

$$s_0=3.54$$

From equation (129) the scale value corresponding to $H\gamma=18400$ is 2.65γ

For small angles, as in this case, it will be sufficient to compute the scale value at low H by the usual scale value formula

$$s = 3.54 + .0035n$$
$$= s_0 + a\ n$$

Here $n = -260$, giving $s = 2.63$.

The lower limit of H which the variometer can record may be determined experimentally by bringing the auxiliary magnet, which is used in the usual scale value deflections, near enough to the variometer to bring the reserve recording spot to its lowest possible limit. The magnetic moment m, of the auxiliary magnet, is known from the deflections of the D variometer. The change in the field intensity at the H magnet is, in gammas,

$$\frac{2m \times 10^5}{r^3} = \Delta H_\gamma$$

The lower limit of H is then $H_0 - \Delta H$.

25. **Comparison of characteristics of the bifilar and unifilar variometers.**—

	Bifilar variometer.	Unifilar variometer.
Sensitiveness	Adjusted by distance between fibers. Requires no control magnet.	Usually requires control magnet.
Scale value	Increased by turning torsion head so as to increase ordinate.	Increased by turning torsion head so as to increase ordinate.
Scale value	Decreases with ordinate.	Increases with ordinate.
a factor	Negative and small	Positive and relatively large. Can be reduced by control magnet and large fiber.
Deflections	Practically symmetrical	Unsymmetrical. Lack of symmetry can be reduced by control magnet and large fiber.
Scale value=0	Only when H=0	At low H at critical angle.
Stability	Stable for all values of H, high and low.	Unstable at low H at critical angle. Stability increased by control magnet and large fiber.
Recording range	All values of H, high and low	All high values of H. No H less than that corresponding to the critical angle. Lower range may be increased by control magnet and large fiber.

IV. CHARACTERISTICS COMMON TO BOTH TYPES OF VARIOMETER.

26. **Temperature coefficient of bifilar variometer.**—Temperature changes can affect the bifilar variometer by changing the distance between the fibers, owing to the expansion of the metallic materials used in the construction of the suspension system. Attempts have been made to compensate a bifilar variometer for temperature by using metals of different coefficients of thermal expansion. Such methods, however, have not proved satisfactory.

The effect of temperature on the bifilar variometer is chiefly due to the change in the magnetic moment of the magnet.

The equation of equilibrium is

$$MH \sin \theta = A \sin (\delta - \theta) \tag{3}$$

Differentiate this with respect to θ and M

$$d\theta[MH \cos \theta + A \cos (\delta - \theta)] = -H \sin\theta dM \tag{137}$$

From equation (4)

$$MH \cos \theta + A \cos (\delta - \theta) = MS \sin \theta \tag{4}$$

From (137) and (4)

$$MSd\theta = -HdM$$

For small changes we may write $Sd\theta = -\Delta H$ and $dM = \Delta M$, and therefore

$$\frac{\Delta H}{H} = \frac{\Delta M}{M} \tag{138}$$

In words, the proportional change in H equals the proportional change in the magnetic moment.

Let

$$M = M_0 (1 - qt) \tag{139}$$

and so

$$\Delta M = -M_0 qt \tag{140}$$

where t is the temperature, and q is the temperature coefficient of the magnet. From (138) and (140)

$$q = \frac{\Delta H}{Ht} \tag{141}$$

Thus the temperature coefficient of the magnet may be derived from the observed apparent change in H, when the variometer is subjected to a change of temperature alone.

27. **Temperature coefficient of unifilar variometer.**—Temperature changes can affect the unifilar variometer by changing the elasticity of the fiber, but this effect is small. Thus the temperature coefficient is chiefly due to the changes in the magnetic moments of the suspended and control magnets. From (82) the equation of equilibrium, when the control magnet is placed in the magnetic prime vertical, is

$$MH \sin \theta = h (\delta - \theta) - FM \cos \theta \tag{142}$$

Under the same conditions $\left(\kappa = \frac{\pi}{2}\right)$ the scale value equation (83) may be written

$$MH \cos \theta + h - FM \sin \theta = MS' \sin \theta \tag{143}$$

In equations (142) and (143)

$$F = \frac{M'}{r^3} \tag{144}$$

Let us adopt the relation

$$M' = cM \tag{145}$$

(143) becomes

$$MH \cos \theta + h - \frac{M^2 c}{r^3} \sin \theta = MS' \sin \theta \tag{146}$$

and (142) becomes

$$MH \sin \theta = h (\delta - \theta) - \frac{c}{r^3} M^2 \cos \theta \tag{147}$$

Differentiate equation (147) with respect to θ and M and collect terms.

$$\left(MH\cos\theta+h-\frac{c}{r^3}M^2\sin\theta\right)d\theta=-dM\left(H\sin\theta+\frac{2c}{r^3}M\cos\theta\right) \quad (148)$$

Substituting from equation (146)

$$MS'\sin\theta\, d\theta=-dM\left(H\sin\theta+\frac{2c}{r^3}M\cos\theta\right) \quad (149)$$

For small changes we may write

$$\begin{aligned} S'd\theta&=\Delta H\\ dM&=\Delta M\end{aligned} \quad (150)$$

and (149) becomes

$$\frac{\Delta H}{H}=-\frac{\Delta M}{M}\left(1+\frac{2cM\cot\theta}{Hr^3}\right) \quad (151)$$

Using equation (140)

$$q=\frac{\Delta H}{Ht\left(1+\frac{2cM\cot\theta}{Hr^3}\right)} \quad (152)$$

When there is no control magnet, $c=o$ and

$$q=\frac{\Delta H}{Ht} \quad (141)$$

It will be observed that the temperature coefficient in equation (152) is affected by the control magnet. However, about the largest value the term $\frac{2cM\cot\theta}{Hr^3}$ could possible take is 0.006. Hence the effect of the control magnet on the temperature coefficient, for practical purposes, may be disregarded.

28. **Base-line drift.**—It is well known that magnets gradually grow weaker in the course of time. To show the effect of the slow loss of the magnetic moment of the magnet, let us for simplicity, drop the control magnet term in equation (149), and divide through by $\Delta t, t$ representing time. We have

$$MS'\frac{d\theta}{dt}=-H\frac{dM}{dt} \quad (153)$$

Now, if p is the coefficient of magnetic loss with time, the magnetic moment of the magnet at any time is

$$M=M_0\,(1-pt) \quad (154)$$

and

$$\frac{dM}{dt}=-M_0p \quad (155)$$

Substituting this into equation (153)

$$MS'\frac{d\theta}{dt}=M_0Hp \quad (156)$$

showing that $\frac{d\theta}{dt}$ is plus when p is plus. Thus, on account of the loss of magnetic strength, the angle continually increases; that is, the recording spot continually drifts downward toward the base line. Consequently there is a gradual increase in the base line values, called the base line drift. Multiplying both sides of equation (153) by Δt

$$MS'\frac{d\theta}{dt}\Delta t = -H\frac{dM}{dt}\Delta t \tag{157}$$

Now

$$S'\frac{d\theta}{dt}\Delta t = \Delta H$$

and

$$\frac{dM}{dt}\Delta t = \Delta M$$

Substituting these in (157) we get

$$\frac{\Delta H}{H} = -\frac{\Delta M}{M} \tag{158}$$

That is, the apparent fractional change in H—i. e., a downward drift of the recording spot—equals the fractional change in magnetic moment:

From (154) and (158)

$$p = \frac{\Delta H}{Ht} \tag{159}$$

Comparing the equations in this section with those in the preceding section we notice that they are identical. In fact, the temperature coefficient and the coefficient of magnetic loss are due to the same physical cause, namely, the weakening of the magnet, and manifest themselves in the same way by an apparent decrease in H.

29. **H variometer compensated for temperature.**—Let us investigate the possibility of compensating an H variometer for temperature by placing a magnet north (or south) of the variometer, and with south end north. The equation of equilibrium is

$$\left(MH - \frac{2MM_c}{r^3}\right)\sin\theta = h(\delta - \theta) \tag{160}$$

and this may be written

$$\frac{\delta - \theta}{\sin\theta} = \frac{1}{h}\left(MH - \frac{2MM_c}{r^3}\right) \tag{160'}$$

Here M_c is the magnetic moment of the compensating magnet. Differentiating with respect to t and M and M_c

$$f(\theta)\frac{d\theta}{dt} = \frac{1}{h}\left(\frac{HdM}{dt} - \frac{2M}{r^3}\frac{dM_c}{dt} - \frac{2M_c}{r^3}\frac{dM}{dt}\right) \tag{161}$$

Now let

$$M = M_0(1 - q_1 t)$$
$$M_c = M_{c0}(1 - q_2 t) \tag{162}$$

q_1 and q_2 being temperature coefficients. From (162)

$$\frac{dM}{dt} = -M_0 q_1$$

$$\frac{dM_c}{dt} = -M_{c0} q_2 \tag{163}$$

By substituting (163) in (161)

$$f(\theta)\frac{d\theta}{dt} = \frac{1}{h}\left[-H M_0 q_1 + \frac{2 M_0 M_{c0}}{r^3}(q_1 + q_2)\right] \tag{164}$$

If the variometer is not to be affected by temperature changes $\frac{d\theta}{dt}$ must be equal to zero. Therefore,

$$\frac{2 M_{c0}}{r^3} = \frac{H q_1}{q_1 + q_2} = F_c \tag{165}$$

And if the temperature coefficients are equal $q_1 = q_2$

$$\frac{2 M_{c0}}{r^3} = F_c = \frac{H}{2} \tag{166}$$

That is, the field intensity of the compensating magnet at the center of the suspended magnet must be one-half the earth's magnetic field. As the temperature coefficients of magnets vary considerably, magnets of the same temperature coefficients might not be easily obtained. However, by adjusting the distance the compensation could be effected by trial.

In the case of our sample unifilar variometer, a compensating magnet of magnetic moment $M_c = 10$ would be placed at a distance $r = 5.93$ cm. north or south of the suspended magnet provided the temperature coefficients of both magnets were equal.

The results above will, of course, hold true for the bifilar variometer.

30. **Design for a uniflar variometer.**—In the equation of equilibrium (160) it will be observed that the effect of the compensating magnet is to reduce the intensity of the earth's magnetic field at the suspended magnet. It is this reduced value that must be used in investigating the scale value, the a factor, the critical angle, stability, and the recording range. Denoting the reduced value by H', the relation of the reduced field to the base line or station value is

$$H_0 = F_c + H' \tag{167}$$

When condition (166) is satisfied

$$H' = \frac{H}{2} \tag{168}$$

Assuming that equation (166) holds true, let us investigate the possibility of designing a unifilar variometer that will possess the following characteristics: An a factor equal to zero, and consequently constant scale value, and symmetrical deflections; compensation for temperature; ability to record a range of 800γ below station or base-line value at an angle $6° 30' = \varphi$. The reduced value of H is here 0.09600, and the reduced value of H at the angle $\varphi = 6° 30'$ is 0.088. Let us try these values in equation (136), assuming there is no control magnet, and use the value of τ thus obtained in equation (61). We shall obtain a scale value of 3.3γ.

Turning to equation (102) we see that by using a scale value $s_0 = 5.0$ (without a control magnet) and by reducing this to 3.5γ (s'_0) by means of a control magnet, the a factor becomes zero. Our problem is solved. Therefore, at a station where $H_\gamma = 19200$, a unifilar variometer so designed (a), will be compensated for temperature by a magnet placed north or south of it, which reduces the earth's field by one-half (b), will have a constant scale value of 3.5γ (c), can record at an angle considerably within the critical angle, a low value H_γ equal to 18400, which is 800γ below the station value, and is as low as is likely to be encountered in the severest magnetic storms.

31. X Y **variometers.**—Referred to XY axes directed toward the geographical north and east, the couple acting on the suspended magnet is evidently

$$C = -MX \sin\theta + MY \cos\theta. \tag{169}$$

If the X and Y intensities vary about a mean value, θ will vary about a mean direction. We may regard the angle θ in this equation as the mean direction of the magnet. The equation shows that the effect of the Y intensity will be small or negligible when θ is nearly $\frac{\pi}{2}$. In the same way the effect of the X intensity will be small or negligible when θ is nearly zero. Since the variations of the X and Y intensities of the earth's magnetic field are small, the magnet will depart little from its mean position, so that the X variometer will be practically unaffected by variations in Y, and the Y variometer will be practically unaffected by variations on X. On the other hand, if the mean position of the magnet does not coincide with the X or Y axis, both X and Y are effective and the recorded variations are to that extent erroneous.

The same reasoning will apply to any other set of rectangular axes. We conclude, then, that a suspended magnet will respond to and correctly record that component intensity only to which it is perpendicular. Two variometers, whose suspended magnets are perpendicular to each other, will resolve the resultant horizontal intensity into two rectangular components. Thus, it will be seen, that the ordinary D and H variometers are but perpendicular kinds of XY variometers.

The characteristics of XY variometers may be investigated in the same way as the case of D and H variometers. Control magnets may be used to increase or decrease the sensitiveness. Scale values may be derived from the observed deflections produced by a known change in that component field intensity which the variometer records.

32. **Scale values of** H **variometer when magnet is in any position.**—It will be of importance to derive expressions for the scale values of an H variometer when the suspended magnet is not perpendicular to the magnetic meridian. For this purpose it will be necessary to refer the suspended magnet to fixed axes.

A little consideration will show that the magnetic meridian, so called, is not a fixed direction like the astronomic meridian. The magnetic meridian, being defined by the declination, varies periodically during the course of a day. We will, therefore, take as our fixed axes the mean direction of the magnetic meridian, and the mean direction of the magnetic prime vertical. We shall denote the northward intensity by N and the eastward intensity by E. The following relations subsist between the H, N, and E intensities, φ being the angle between H and N.

$$\begin{aligned} H^2 &= N^2 + E^2 \\ N &= H \cos \varphi \\ E &= H \sin \varphi \end{aligned} \tag{170}$$

In cases where φ is a small angle, N and H are practically equal in magnitude, but differ slightly in direction, so that in the following discussion N can be replaced by H without appreciable error. It will be sufficient to consider only one type of variometer, namely, the unifilar variometer.

The equation of equilibrium is

$$MN \sin \theta - MF \cos \theta = h\,(\delta - \theta) \tag{171}$$

The N scale value is

$$S_N = -\frac{dN}{d\theta} = N \cot \theta + \frac{h}{M \sin \theta} + E \tag{172}$$

From equation (171)

$$h = \frac{MN \sin \theta - ME \cos \theta}{\delta - \theta} \tag{173}$$

Substituting this into equation (172)

$$S_N = N\left(\cot \theta + \frac{1}{\delta - \theta}\right) + E\left(1 - \frac{\cot \theta}{\delta - \theta}\right) \tag{174}$$

The E scale value is

$$\frac{dE}{d\theta} = S_E = N + \frac{h}{M \cos \theta} + E \tan \theta \tag{175}$$

Substituting h, equation (173)

$$S_E = N\left(1 + \frac{\tan \theta}{\delta - \theta}\right) + E\left(\tan \theta - \frac{1}{\delta - \theta}\right) \tag{176}$$

We can make an interesting application of the E scale value, equation (175). Suppose the magnet is at rest in the magnetic meridian, which implies that $\theta = E = 0$. Then from (175)

$$S_E = H + \frac{h}{M}$$

and this is the force per radian acting on the magnet. To determine h, turn the torsion head 90°. The magnet will be deflected through a small angle f, and the equation of equilibrium for this case is

$$MHf = h\,(90° - f)$$

or

$$h = \frac{MHf}{90° - f} \tag{177}$$

Substituting for h,

$$S_E = H\left(1 + \frac{f}{90° - f}\right) \tag{178}$$

And the couple or torque per radian is

$$MS_E = MH\left(1 + \frac{f}{90° - f}\right)$$

If displaced the magnet will oscillate according to the equation

$$K\theta'' + MH\left(1 + \frac{f}{90° - f}\right)\theta = 0$$

From this equation we derive

$$MH = \frac{\pi^2 K}{T^2\left(1 + \frac{f}{90° - f}\right)} \tag{179}$$

33. The declination variometer as an E intensity variometer.— From equation (170) we have

$$\tan\varphi = \frac{E}{N} \tag{180}$$

Regarding N as constant, we infer that E and φ change together. In other words, it is changes in the E intensity that cause changes in the declination. Thus the declination variometer is, in fact, an E intensity variometer. Let us determine its scale value in the case where φ and, therefore, E are both small. In this case θ in equation (175) becomes identical with φ, and $E \tan\varphi$ is negligible.

As in the preceding section we get

$$S_E = H\left(1 + \frac{f}{\rho - f}\right) \tag{178}$$

or, in gammas,

$$s_E = H_\gamma\left(1 + \frac{f}{\rho - f}\right)\epsilon$$

where the general angle ρ has been written for the particular angle 90°. If the recording box is placed at such a distance as to make 1 millimeter equal to 1 minute of arc.

$$s_E = \frac{H_\gamma}{3437.8}$$

$$= 5.58 \text{ for } H_\gamma = 19200$$

Thus the D scalings could be reduced to E intensities in the same way as the H scalings are reduced to H intensities. Doubtless the use of the D variometer in this way would be of value in some physical investigations.

By means of control magnets the E intensity variometer could be made as sensitive as desired.

34. Maladjustment of the H **variometer.**—We shall now apply the equations for the N and E scale values to the case of an H variometer in which the magnet makes an angle (10° or less) with the magnetic prime vertical. From equation (180)

$$E = N \tan \varphi \tag{180}$$

Substituting this in the scale value, equations (174) (176),

$$S_N = N\left[\left(\cot\theta + \frac{1}{\delta - \theta}\right) + \left(1 - \frac{\cot\theta}{\delta - \theta}\right)\tan\varphi\right]$$

$$S_E = N\left[\left(1 + \frac{\tan\theta}{\delta - \theta}\right) + \left(\tan\theta - \frac{1}{\delta - \theta}\right)\tan\varphi\right]$$

Now φ, the deviation of the declination from the mean, is a periodic angle, and is either very small or zero. These scale values are then, approximately,

$$S_N = H\left(\cot\theta + \frac{1}{\delta - \theta}\right) \tag{181}$$

$$S_E = H\left(1 + \frac{\tan\theta}{\delta - \theta}\right) \tag{182}$$

The angular displacement of the suspended magnet caused by the E intensity is $\frac{E}{S_E}$ and true H is altered to an extent

$$\Delta H = \frac{S_N}{S_E}\ E = \frac{s_N}{s_E}\ E_\gamma \tag{183}$$

The change from true H per 1′ change in declination is, from (183) and (180),

$$\Delta H = \frac{S_N}{S_E} H \tan 1' \tag{184}$$

As an example, let the torsion head of our sample unifilar variometer be turned 6°, so that $\theta = 84°$, and the magnet makes an angle of 6° with the magnetic prime vertical: Then

$$s_N = 1.95 + (0.0035 \times 240) = 2.79,$$

since $360' = 240$ mm. Expressing equation (182) in gammas, and using the value $\delta = 5.867$, we compute $s_E = 26.5$. From (184) we obtain $\Delta H = 0.59\gamma$ per $1'$ change in declination.

When θ is near $\frac{\pi}{2}$ the scale values are

$$S_N = \frac{H}{\delta - \theta} \tag{185}$$

$$S_E = H\,\frac{\tan\theta}{\delta - \theta} \tag{186}$$

H in equation (182) being negligible in comparison with $H\,\frac{\tan\theta}{\delta-\theta}$. From these two equations

$$\tan\theta = \frac{S_E}{S_N} = \frac{s_E}{s_N} \tag{187}$$

That is, the ratio of the E scale value to the H scale value is the tangent of the position angle of the magnet. For a practical method, then, of determining the position of the magnet with respect to the magnetic prime vertical, determine the scale value when the deflection magnet is placed in the magnetic prime vertical on east and west sides of the variometer. The ratio of the scale value thus obtained to the regular H scale value will, in accordance with equation (187), determine the position angle of the magnet.

It is evident that both H and E act concurrently in making the actual curve on the magnetogram. A general idea of the effect of E on H may be obtained by assuming that H can be expressed

$$H = A\cos(\omega t + \alpha),$$

and that the effect of E can be expressed

$$\Delta H = B\cos(\omega t + \beta)$$

where $\omega = \frac{2\pi}{T}$ and α and β are phase angles. The recorded curve will be of the form

$$C = A\cos(\omega t + \alpha) + B\cos(\omega t + \beta)$$

This can be written

$$C = P\cos(\omega t + \lambda)$$

Where

$$P = [(A\cos\alpha + B\cos\beta)^2 + (A\sin\alpha + B\sin\beta)^2]^{1/2}$$

$$\sin\lambda = \frac{A\sin\alpha + B\sin\beta}{P}$$

$$\cos\lambda = \frac{A\cos\alpha + B\cos\beta}{P}$$

$$\tan\lambda = \frac{A\sin\alpha + B\sin\beta}{A\cos\alpha + B\cos\beta}$$

From these equations we should expect that the true H curve and the recorded curve C should differ in amplitude and in phase angle, and consequently in the value and time of a maximum or minimum. For example, take a curve represented by the following

$$C = 25\,\gamma \cos \frac{\pi}{12}\frac{t}{T} - 4\,\gamma \sin \frac{\pi}{12}\frac{t}{T} = H - \Delta H,$$

in which $T = 24$ hours, the unit of time being the hour. When written thus:

$$H - C = 4\gamma \sin \frac{\pi}{12}\frac{t}{T}$$

the equation shows that true H is decreased during the first half of the day and increased during the second half. The phase angles are:

$\alpha = O,\ \beta = \frac{\pi}{2},\ \tan \lambda = \frac{B}{A} = \frac{25}{-4},\ \lambda = -81^\circ = 5.4$ hours. The amplitude is

$$P = (A^2 + B^2)^{1/2} = 25.3\gamma$$

This curve now is

$$C = 25.3\gamma \cos \frac{\pi}{12\,T}(t - 5.4)$$

This is a maximum when t equals 5.4 hours. In this case there is little change in the value of the maximum, but a large change in the time of the maximum (or minimum). If the curve were this:

$$C = 25\,\gamma \cos \frac{\pi t}{12\,T} + 4\,\gamma \cos \frac{\pi t}{12\,T}$$

$\alpha = \beta = \lambda = O,\ P = 29\gamma$. So the curve is

$$C = 29\,\gamma \cos \frac{\pi t}{12\,T}$$

In this case the value of the maximum is considerably greater, but the time of the maximum (or minimum) remains unchanged.

In both cases the true H curve is evidently

$$H = 25\,\gamma \cos \frac{\pi t}{12\,T}$$

Part II.—THEORY OF SUSPENSIONS OF HORIZONTAL INTENSITY VARIOMETERS.

V. BIFILAR SUSPENSION.

35. **The simplified bifilar suspension.**—In the figure the fibers and their points of attachment are projected on a horizontal plane through the center of gravity of the magnet 0. Fiber l_1 is attached to the magnet S and to the support at A. Its projection is $AS = x$, and it is inclined to the vertical at an angle ψ_1. (See Fig. 1.) Fiber l_2 is attached to the magnet at N, and to the support at B. Its projection is $BN = y$, and it is inclined to the vertical at an angle ψ_2.

The horizontal distance between the points of attachment on the support is a, and b is the horizontal distance between the points of attachment to the magnet.

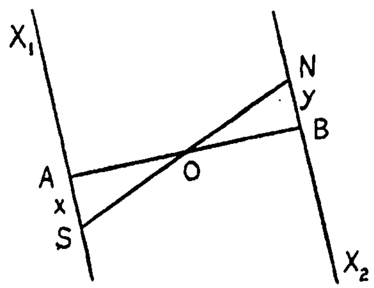

Fig. 1.

In general, when periodic forces are applied at A and N, the magnet will oscillate through an angle φ and its center of gravity will also oscillate round a mean position. When the bifilar suspension is used in a variometer, the center of gravity should remain stationary, since the angular motion only of the magnet is required.

Let us ascertain what relations must subsist in order that 0 shall remain at rest:

Let T_1 and T_2 = tension in fibers,
Z_1 and X_1 = vertical and horizontal forces at S,
Z_2 and X_2 = vertical and horizontal forces at N,
v_1 and v_2 = vertical distance between S and A, and N and B, respectively. Since the moments around any horizontal line through 0 are equal

$$Z_1 \cdot OS = Z_2 \cdot ON \tag{188}$$

$$Z_1 \cdot x = Z_2 \cdot y. \tag{189}$$

Therefore,

$$\frac{OS}{x} = \frac{ON}{y} \tag{190}$$

Hence, from dynamical considerations, the triangle AOS and NOB are similar, and as AS and BN are parallel

$$\begin{aligned} Z_1 &= T_1 \cos \psi_1 \\ X_1 &= T_1 \sin \psi_1 \end{aligned} \tag{191}$$

$$\begin{aligned} Z_2 &= T_2 \cos \psi_2 \\ X_2 &= T_2 \sin \psi_2. \end{aligned} \tag{192}$$

From (191)

$$\frac{X_1}{Z_1}=\tan\psi_1=\frac{x}{v_1}=\frac{x}{l_1\cos\psi_1}. \tag{193}$$

From (192)

$$\frac{X_2}{Z_2}=\tan\psi_2=\frac{y}{v_2}=\frac{y}{l_2\cos\psi_2} \tag{194}$$

If the center of gravity is to remain at rest

$$X_2=X_2. \tag{195}$$

Then from (193) and (194)

$$\frac{Z_1x}{l_1\cos\psi_1}=\frac{Z_2y}{l_2\cos\psi_2}. \tag{196}$$

From (196) and (189)

$$l_1\cos\psi_1=l_2\cos\psi_2. \tag{197}$$

The simplest construction is to make l_1 equal to l_2 and then ψ_1 equals ψ_2. Furthermore, from the second and fourth members of (193) and (194) $x=y$, and, therefore, from (189), $Z_1=Z_2$. From (191) and (192) $T_1=T_2$. From (188) $OS=ON$. Also $Z_1=Z_2=\frac{mg}{2}$, mg being the weight of the magnet. Thus, the simplified bifilar suspension is one whose fibers are equal in length, are equally inclined to the vertical, are under equal tensions, and are equally distant from the center of gravity of the magnet.

36. **Potential energy of the bifilar suspension.**—The potential energy of the bifilar suspension is

$$V=mgz_1, \tag{198}$$

in which $z_1=$ the difference of the vertical distance between the points A and S (or N and B) before and after the magnet is turned. Before the magnet is turned through an angle λ, the vertical distance between the center of gravity of the magnet and the support is l.

After the magnet is turned, the vertical distance is

$$(l^2-x^2)^{\frac{1}{2}}$$

The difference is

$$z_1=l-(l^2-x^2)^{\frac{1}{2}}=l\left[l-\left(l-\frac{x^2}{l^2}\right)^{\frac{1}{2}}\right]=\frac{x^2}{2l}\text{ very approximately.} \tag{199}$$

But

$$x^2=\frac{a^2}{4}+\frac{b^2}{4}-\frac{ab\cos\lambda}{2}. \tag{200}$$

From (199), (200), and (198)

$$V=\text{constant}-mg\,\frac{ab}{4l}\cos\lambda. \tag{201}$$

37. **The suspension couple, bifilar suspension.**—The couple due to the suspension is

$$-\frac{dV}{d\lambda}=-mg\,\frac{ab}{4l}\sin\lambda \tag{202}$$

which tends to decrease the angle λ.

It will be of interest to derive the couple another way. Let C denote the couple then

$$C = X_1 p, \tag{203}$$

where p is the perpendicular distance between x and y.

$$X_1 = \frac{mgx}{2l} \text{ very approximately} \tag{204}$$

as will be seen from equation (193). Also

$$\frac{px}{2} = \frac{ab}{4} \sin \lambda,$$

or

$$p = \frac{ab \sin \lambda}{2x}. \tag{205}$$

Substituting (204) and (205) in (203)

$$C = mg \frac{ab}{4l} \sin \lambda, \tag{202}$$

We still have to make an allowance for the torsion in the individual fibers. If the fibers are free from torsion when a and b are parallel, the total suspension couple acting on the magnet is evidently

$$C = mg \frac{ab}{4l} \sin \lambda + h\lambda. \tag{206}$$

We may, however, safely neglect the term $h\lambda$ in the case of the bifilar suspension.

Referring the magnet and suspension to a vertical plane and an x axis in the magnetic meridian, and denoting the position angle of the magnet by θ and the position angle of the torsion head (a) by δ we shall have

$$\lambda = \delta - \theta \tag{207}$$

Neglecting the term $h\,\lambda$, substituting (207) in (206), and equating the resulting equation to the couple due to the magnetic intensity H, we obtain for the equation of equilibrium

$$MH \sin \theta = mg \frac{ab}{4l} \sin (\delta - \theta) \tag{208}$$

Comparing this equation with equation (3)

$$MH \sin \theta = A \sin (\delta - \theta) \tag{3}$$

we observe that

$$A = mg \frac{ab}{4l} \tag{209}$$

When $\theta = \frac{\pi}{2}$ the scale value equation (4) becomes

$$S_0 = \frac{A \cos \tau}{M} = mg \frac{ab}{4Ml} \cos \tau, \tag{210}$$

From this equation it will be seen how the scale value can be changed by changing the distances between the fibers; that is, by altering a and b.

38. **Scale value of the bifilar variometer by method of weights.**—The fact that the weight, mg, is a factor in A suggests another method of determining the scale value of the bifilar variometer.

Differentiate equation (3) with respect to θ and A,

$$d\theta = \frac{dA \sin(\delta - \theta)}{MH \cos\theta + A \cos(\delta - \theta)} \tag{211}$$

But equation (4) is

$$S = \frac{MH \cos\theta + A\cos(\delta - \theta)}{M \sin\theta} \tag{4}$$

Multiplying equations (211) and (4) member by member

$$Sd\theta = \frac{dA \sin(\delta - \theta)}{M \sin\theta} \tag{212}$$

From the equation of equilibrium

$$\frac{\sin(\delta - \theta)}{\sin\theta} = \frac{MH}{A} \tag{213}$$

And this substituted in (212)

$$Sd\theta = \frac{HdA}{A}$$

If we denote the weight of the magnet and its appurtenances by W, equation (209) will be

$$A = W\frac{ab}{4l}$$

$$dA = dW\frac{ab}{4l} \text{ and } \frac{dA}{A} = \frac{dW}{W}$$

Also

$$d\theta = n\epsilon$$

Hence

$$S = H\frac{dW}{Wn\epsilon}$$

or in gammas

$$s = H_\gamma \frac{dW}{Wn} \tag{214}$$

In practice it will be convenient to make the added weight a suitable fraction of the weight W.

39. **Standard positions of bifilar suspension.**—When the torsion factor h is not neglected the total couple acting on the magnet is

$$C = -MH \sin\theta + A\sin(\delta - \theta) + h(\delta - \theta)$$

Equating this to the kinetic reaction of the magnet, and transposing all terms to the left side, we get for the equation of motion

$$K\ddot{\theta}+MH\sin\theta-A\sin(\delta-\theta)-h(\delta-\theta) \tag{215}$$

As in paragraph 9, if the magnet is displaced from a position of equilibrium θ_0 by a small angle ψ, so that

$$\theta=\theta_0+\psi$$

Equation (215) assumes the form

$$K\ddot{\psi}+[MH\cos\theta_0+A\cos(\delta-\theta_0)+h]\psi=0 \tag{216}$$

and θ_0 is determined by the condition

$$-MH\sin\theta_0+A\sin(\delta-\theta_0)+h(\delta-\theta_0)=0$$

The period of ψ in (216), and consequently the period of θ is

$$T^2=\frac{\pi^2 K}{MH\cos\theta+A\cos(\delta-\theta)+h} \tag{217}$$

where θ now indicates a position of equilibrium. There are three standard positions of the bifilar suspension, namely, magnet in equilibrium, (1), when in the magnetic meridian, (2), when the torsion head is turned so as to bring the magnet into the prime vertical, (3) when the magnet and head are reversed with respect to the magnetic meridian.

In the first position, $\delta=\theta=0$. The period is

$$T^2=\frac{\pi^2 K}{MH+A+h} \tag{218}$$

Here the magnet oscillates under the influence of the magnetic field, and the torsion factor A and h.

In the second position, $\theta=\frac{\pi}{2}$. The period is

$$T^2=\frac{\pi^2 K}{A\cos\tau+h} \tag{219}$$

The magnet oscillates from the effect of the torsion factors only.

In the third position, $\delta=\theta=\pi$

$$T^2=\frac{\pi^2 K}{A+h-MH} \tag{220}$$

Here it will be noticed that the magnetic field tends to displace the magnet from its position, while at the same time, the torsion tends to restore it. As long as the denominator in equation (220) is plus, the position is a stable one, and this is always a possibility. Evidently the magnet may be made as sensitive as desired by adjusting the value of the denominator in (220).

As the third standard position of the bifilar suspension is of some importance in terrestrial magnetism, we shall bestow a little more attention on it. Moreover, the discussion of this particular case leads to a consideration of a type motion of the magnet more general than any we have hitherto treated, namely, the motion of a magnet under given applied forces. At the same time, in this connection, it will be of interest to point out a property common to all damped variometers.

If, for simplicity, we consider that the angle θ is the angle the south end of the magnet makes with the magnetic meridian, and that the magnetic meridian is the equilibrium position, the equation of the free motion of the magnet is:

$$K\ddot{\theta}+(A+h-MH)\,\theta=0$$

As variometer magnets are inclosed in damping boxes, it will be necessary to add a term to express the damping effect. Denote it by r. The equation of damped motion will be

$$K\ddot{\theta}+r\dot{\theta}+(A+h-MH)\,\theta=0 \tag{221}$$

In the case we are considering the applied force is evidently the E intensity, which we will assume may be expressed

$$E=E_0\cos\omega t$$

Where $\omega=\frac{2\pi}{T_E}$ and T_E is the complete period (double oscillation). The equation of motion which we have to discuss is

$$K\ddot{\theta}+r\dot{\theta}+(A+h-MH)\,\theta=-E_0\cos\omega t=-E_0e^{j\omega t}. \tag{221}$$

The complete solution of this equation contains terms of two classes. One class expresses the damped motion of the magnet, which, as we have found in discussing the method of determining scale values by the method of oscillations, soon dies out. The other class of terms expresses the steady motion of the magnet.

Let us try as a solution,

$$\theta=Pe^{j\omega t} \tag{222}$$

where $j=\sqrt{-1}$ and e is the base of the natural system of logarithms. The real part of $Pe^{j\omega t}$ is $P\cos\omega t$, and the imaginary part is $jP\sin\omega t$. When P has been determined, the solution will be $\theta=P\cos\omega t$. Substituting equation (222) in (221), and solving for P

$$P=\frac{-E_0}{A+h-MH-\omega^2K+\omega r}$$

$$=\frac{-E_0}{A+h-MH-\frac{4\pi^2K}{T_E^2}+\frac{2\pi r}{T_E}}$$

$$=\frac{-E_0}{4\pi^2K\left(\frac{1}{T_f^2}-\frac{1}{T_E^2}\right)+\frac{2\pi r}{T_E}} \tag{223}$$

the last form being derived by considering equation (220) which is here

$$T_f^2 = \frac{4\pi^2 K}{A + h - MH}$$

Should the period of the applied force, T_E, become equal to the free period of the magnet, T_f, a condition of resonance would exist in which the amplitude of the applied force is enormously magnified. But T_f is a matter of a few seconds, while T_E, even in the rapid changes during a magnetic storm, is a matter of minutes, so that terms in T_E in equation 223 may be dropped. The solution of equation (221) is now

$$\theta = \frac{-E}{A + h - MH} \cos \omega t \qquad (224)$$

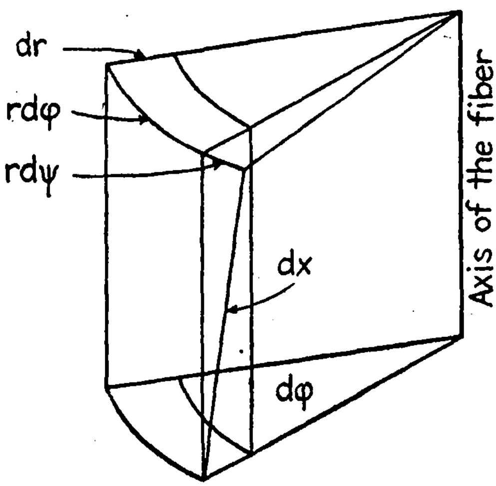

Fig. 2.

We thus conclude that, at least under ordinary conditions, the magnets of variometers follow the changes in the magnetic field to a high degree of fidelity. This is the property alluded to above.

Returning to the third standard position of the bifilar suspension, it is still possible to magnify the E intensity by making the denominator in equation (224) small.

The third standard position of the bifilar suspension is, therefore, a D variometer by which the changes in declination may be magnified to any extent desirable. To determine the scale value of this sensitive type of D variometer, it will be sufficient to compare the recorded changes of the variometer with the changes in declination observed by means of a magnetometer.

VI. UNIFILAR SUSPENSION

40. **Quartz fibers.**—The following physical properties of quartz are taken from the Smithsonian Physical Tables:

Coefficient of cubical thermal expansion: 0.3840×10^{-4}
Coefficient of linear thermal expansion: 0.1280×10^{-4}
Rigidity modulus (selected value): 2.833×10^{11}
Change of rigidity modulus with temperature

$$\mu = \mu_{15}[1 - \alpha(t - 15°)] \qquad \alpha = 0.00012$$

μ_{15} means μ at 15°.

Tensile strength, pounds per square inch, 116000 to 167000 pounds.

41. Coulomb's law.—The important relation in connection with the torsion of fibers of circular section is the following:

$$C=\frac{\pi\mu r^4}{2}\frac{\psi}{l} \tag{225}$$

in which C is the reactive couple, r=radius, μ=modulus of rigidity, l=length, and ψ is the whole torsion in the fiber.

This relation was first given by Coulomb, the French physicist, in 1785, and is sometimes referred to as Coulomb's law. It may be derived as follows:

We may regard the fiber as made up of elementary fibers of cross section $rdrd\varphi$ and length dx, deformed from their normal shape. (See Fig. 2.) The upper and lower faces of the elementary fiber are assumed to be parallel planes perpendicular to the axis of the fiber. The upper face is displaced with reference to the lower face through the distance $rd\psi$. The reactive force F at the upper face, called into play by the distortion of the elementary fiber, is, from the theory of elasticity:

F=area $\times$ modulus of rigidity $\times$ change in angle. The moment of this force is Fr, so that Fr integrated throughout the volume of the fiber will give the reactive couple. The change in angle is $r\dfrac{d\psi}{dx}$ Hence

$$C=\mu\iiiint r^3drd\phi\frac{d\psi}{dx}=\mu\frac{\pi}{2l}r^4\psi \tag{225}$$

The torsional couple varies directly as the rigidity modulus, the amount of torsion in the fiber, and inversely as the length. It should be noticed that the couple varies directly as the *fourth* power of the radius.

This explains why it is so difficult to select a fiber to give a particular scale value.

The factor $\dfrac{\psi}{l}$ is sometimes called the twist; that is, the torsion per unit length. The factor $\mu\dfrac{\pi}{2}r^4$ is called the torsional rigidity. Its value may be found by observing the period of an inertia weight suspended by the fiber.

42. Torsion factor of the unifilar variometer.—The torsional couple of the unifilar suspension is $h\tau$, so that the τ, which has been used in the discussion of the unifilar variometer corresponds to ψ in the expression of Coulomb's law, equation (225). Therefore, we have

$$h\tau=\frac{\pi\mu r^4\tau}{2l}$$

and the torsion factor is

$$h=\frac{\pi\mu r^4}{2l} \tag{226}$$

43. Size of fiber for a given base-line scale value.—From equation (56) when $\theta=\frac{\pi}{2}$, we have

$$MH=h\tau$$

Combining this with equation (226)

$$h=\frac{MH}{\tau}=\frac{\pi\mu r^4}{2l} \tag{226'}$$

Also from equation (60)

$$S_o = \frac{H}{\tau} \tag{60}$$

Eliminating the ratio $\frac{H}{\tau}$ from equation (226′)

$$r^4 = \frac{2\,MS_o l}{\pi \mu} \tag{227}$$

From equations (60) and (61)

$$S_o = \frac{s_o}{\epsilon \times 10^5} \tag{228}$$

Hence the radius of the fiber is given by

$$r^4 = \frac{2\,Ms_o l}{\pi \mu \epsilon \times 10^5} \tag{229}$$

where s_o is the scale value expressed in gammas and all the other quantities are expressed in C. G. S. units.

44. **Fraction of temperature coefficient of unifilar variometer due to temperature changes in fiber.**—Let α = the change of rigidity modulus per degree of temperature, and let β = the coefficient of thermal expansion of quartz. The rigidity modulus may be expressed

$$\mu = \mu_o\ (1 + \alpha t)$$

The radius of the fiber may be expressed

$$r^4 = r^4{}_0(1 + 4\ \beta t)$$

The length of the fiber may be expressed

$$l = l_o\ (1 + \beta t)$$

Hence the torsion factor of the unifilar variometer at any temperature, referred to a standard temperature, which for simplicity is here assumed to be 0°, is

$$\begin{aligned} h &= \frac{\pi \mu r^4}{2\,l} \\ &= \frac{\pi \mu_0 r^4{}_0 (1 + \alpha t)\ (1 + 4\beta t)}{2\ l_0 (1 + \beta t)} \\ &= \frac{\pi \mu_0 r^4{}_0}{2\ l_0} (1 + \alpha t)\ (1 + 3\beta t) \\ &= \frac{\pi \mu_0 r^4{}_0}{2\ l_0} [1 + (\alpha + 3\beta) t] \\ &= h_o [1 + (\alpha + 3\beta) t] \end{aligned} \tag{226}$$

The rate of change of the torsion factor with temperature is

$$\frac{dh}{dt} = h_o\ (\alpha + 3\beta) \tag{230}$$

For the magnetic moment at any temperature

$$M = M_0\ (1-qt)$$

and

$$\frac{dM}{dt} = -M_0 q \tag{231}$$

The equation of equilibrium of the unifilar variometer is

$$MH \sin\theta = h(\delta-\theta) \tag{56}$$

Differentiate this equation regarding M, h, and θ as variables

$$(MH\cos\theta + h)\frac{d\theta}{dt} = -H\sin\theta\frac{dM}{dt} + (\delta-\theta)\frac{dh}{dt}$$

or by means of equation (57)

$$-M\sin\theta\ \frac{dH}{dt} = -H\sin\theta\ \frac{dM}{dt} + (\delta-\theta)\ \frac{dh}{dt}$$

and for the base line position $\theta = \frac{\pi}{2}$

$$-M\frac{dH}{dt} = -H\ \frac{dM}{dt} + \tau\frac{dh}{dt} \tag{232}$$

Substituting equations (230) and 231) in (232)

$$-M\ \frac{dH}{dt} = HM_0 q + h_0\tau\ (\alpha+3\beta) \tag{233}$$

From equation (56), when $\theta = \frac{\pi}{2}$,

$$MH = h_0\tau \tag{234}$$

and equation (233) becomes

$$-M\frac{dh}{dt} = HM_0 q + MH\ (\alpha+3\beta)$$

or very approximately

$$-\frac{dH}{Hdt} = q + \alpha + 3\beta$$

or

$$q' = q\left(1 + \frac{\alpha+3\beta}{q}\right) \tag{235}$$

That is, the temperature coefficient of the variometer q', is $1+\frac{\alpha+3\beta}{q}$ $\times$ the temperature coefficient of the magnet. In the case of our sample unifilar variometer

$$\frac{\alpha+3\beta}{q} = -0.12,\ \text{approximately},$$

and

$$q' = 0.88q.$$

That is, the temperature coefficient of our sample unifilar variometer is some 90 per cent of the temperature coefficient of the magnet.

45. **Useful formulas for reference.**—Angular value of 1 mm. on magnetogram:

$$\epsilon = \tan \epsilon' = \frac{1}{20R} \tag{236}$$

where ϵ is in radians, ϵ' is in minutes of arc, and R is the distance from the manetogram to the face of the movable mirror.

Field intensity of magnet on its axis produced:

$$F = \frac{2M}{r^3} \tag{237}$$

Field intensity of a magnet at a point perpendicular to its axis from its center.

$$F = \frac{M}{r^3} \tag{238}$$

The magnetic moment of a magnet may be experimentally determined by placing it in the magnetic prime vertical east or west of a D variometer.

$$M = \frac{Hr^3}{2} \tan u \tag{239}$$

where u is the observed deflection. For a double deflection in direct and reversed position

$$M = \frac{Hr^3}{4} \tan 2u \tag{240}$$

The magnetic moment may also be determined by placing it in the magnetic meridian north or south of an H variometer whose scale value is known.

$$M = \frac{r^3 u' s}{2 \times 10^5} \tag{241}$$

in which u' is the observed deflection in millimeters. For a double deflection

$$M = \frac{r^3 (2u') s}{4 \times 10^5} \tag{242}$$

General bifilar scale value:

$$s = H_\gamma \epsilon [\cot \theta + \cot (\delta - \theta)] \text{ (gammas)} \tag{5b}$$

Base line bifilar scale value:

$$s_0 = H_\gamma \epsilon \cot \tau \text{ (gammas)} \tag{6b}$$

a ıactor, bifilar scale value:

$$a = H_\gamma \epsilon^2 \left(1 - \frac{1}{\sin^2 \tau}\right) \tag{243}$$

General unifilar scale value:

$$s. = H_\gamma \epsilon \left(\cot \theta + \frac{1}{\delta - \theta}\right) \tag{59}$$

Base line unifilar scale value:

$$s_0 = \frac{H_\gamma \epsilon}{\tau} \tag{61}$$

a factor, unifilar scale value:

$$a = H_\gamma \epsilon^2 \left(1 - \frac{1}{\tau^2}\right) \tag{77}$$

Base-line scale value, unifilar with control magnet:

$$s'_0 = \frac{H_\gamma \epsilon}{\tau} - F_\gamma \epsilon \tag{88}$$

a factor, unfilar scale value, with control magnet:

$$a' = \epsilon^2 \left(H_\gamma + \frac{F_\gamma}{\tau} - \frac{H_\gamma}{\tau^2}\right) = a + \frac{F_\gamma \epsilon^2}{\tau} \tag{97}$$

METHODS OF DETERMINING SCALE VALUES.—(*a*) *Deflections.*—For the same magnet deflecting both the *D* and *H* variometers, at the same distance, we derive from (240) and (242)

$$s = \mathrm{H}_\gamma \frac{\tan 2u}{2\, u'} \tag{244}$$

(*b*) *Oscillations:*

$$s = \epsilon H_\gamma \frac{T_{\mathrm{H}}^2}{T_{\mathrm{d}}^2} \left(1 + \frac{\lambda^2}{\pi^2}\right) \tag{51}$$

(*c*) *Torsion angle:*

$$s = H_\gamma \epsilon \cot \tau \tag{6b}$$

Here the angle τ is measured.

(*d*) *Weight:*

$$s = \frac{H_\gamma d\, W}{u'\, W} \tag{214}$$

Change in scale value due to control magnet parallel to suspended magnet:

$$\Delta s = \pm F_\gamma \epsilon = \pm \frac{M'}{r^3} \epsilon \times 10^5 \tag{89}$$

Change in scale value due to turning torsion head:

$$\Delta s = H_\gamma \epsilon \sin (n_2 - n_1) \epsilon \tag{115}$$

a factor derived from observations: From equations (71) we obtain

$$a = \frac{2s\, (u_2 - u_1)}{u_1^2 + u_2^2} \tag{245}$$

where u_1 is the observed plus deflection, and u_2 is the observed minus deflection, and s is the computed scale value.

If n is the away ordinate, the base line scale value is

$$s_0 = s - an \tag{246}$$

a factor for unifilar with control magnet:

$$a' = \frac{H_\gamma^2 \epsilon^2 - s_0 s_0'}{H_\gamma} \tag{102}$$

Critical angle of instability, unifilar with control magnet:

$$\tan \varphi = \frac{H - F(\tau - \varphi)}{H(\tau - \varphi) + F} \tag{130}$$

τ is obtained from (61).

Critical angle, unifilar with no control magnet:

$$\tan \varphi = \frac{1}{\tau - \varphi} \tag{131}$$

Value of H at critical angle:

$$H = \frac{H_0(\tau - \phi)}{\tau \cos \varphi} \pm F \tan \varphi$$

$$= \frac{S_0\ (\tau - \varphi)}{\cos \varphi} \pm F \tan \varphi \tag{134}$$

Base-line drift follows the relation:

$$\frac{\Delta H}{H} = -\frac{\Delta M}{M} \tag{158}$$

Temperature coefficient:

$$q = \frac{\Delta H}{H \Delta t}$$

Position angle of the suspended magnet:

$$\tan \theta = \frac{S_E}{S_H} \tag{187}$$

Error in recorded H when the suspended magnet does not lie in the prime vertical:

$$\Delta H = \frac{S_H}{S_E} H \tan 1' \tag{184}$$

Suspension couple of bifilar:

$$C = \frac{mg}{4l} ab \sin \lambda \tag{202}$$

Bifilar torsion factor:

$$A = \frac{mg}{4l} ab \tag{209}$$

Coulomb's law:

$$C = \frac{\pi \mu r^4 \psi}{2l} \tag{225}$$

Unifilar torsion factor:

$$h = \frac{\pi \mu r^4}{2l} \tag{226}$$

Size of quartz fiber for given scale value:

$$r^4 = \frac{2\,Mls_0}{\pi \mu \epsilon \times 10^5} \tag{229}$$

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