STUDIES ON THE THERMODYNAMICS OF THE ATMOSPHERE.

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III.—APPLICATION OF THE THERMODYNAMIC FORMULÆ TO THE NONADIABATIC ATMOSPHERE.

THE NONADIABATIC ATMOSPHERE.

In the preceding papers of this series it has been shown that in the latitudes of the temperate zones the atmosphere is not arranged in such a way that the thermal gradients conform to the adiabatic rate of change along the vertical, $-\frac{dT}{dz} = 9.867°$ C. per 1000 meters, but that they depart from that rate, being generally much less. In the tropical zones the few available observations indicate that in the lower strata the temperature gradient exceeds that amount, or is equal to it. Thus O. L. Fassig[1] found the mean of four ascents at Nassau, in June-July, 1904, to be 28.3° C. at the surface and 18.3° C. at 1000 meters, evidently the adiabatic rate. H. Hergesell[2] found for 16 ascents on the Atlantic, in the region between the African coast, the Canaries, and the Azores, the following temperatures:

Height.	T	ΔT	
Meters.	° C.	° C.	
5000	(—10.0)		
		— 8.5	Adiabatic gradient.
4000	— 1.5		
		—10.5	
3000	9.0		
		— 9.0	
2000	18.0		
		— 8.4	
1000	26.4		
		+ 3.4	
0	23.0		

This is an average adiabatic rate from the lower cloud level to 5000 meters, but differs widely from that rate from the surface to 1000 meters. He also reports[3] an adiabatic rate, for the ascensions of 1905, from the surface to 1350 meters, then a zero or even a positive temperature gradient to 3550 meters, above that a rather rapid fall to 13,000 meters, and higher still in the atmosphere a slower rate, indicating an intrusion of warm air.

As the result of my kite work from the U. S. S. *Cæsar*, over the North Atlantic Ocean between Hampton Roads and Gibraltar, during the Spanish Eclipse Expedition, I found the temperatures as follows, for the dates June 24, 26, 28, 29, 30, July 5, and September 22, 1905:

Height.	Mean of 5 ascents.	July 5.	Sept. 22.
Meters.	° C.	° C.	° C.
1000	16.9	7.9	15.6
800	17.1	9.3	17.9
600	17.6	11.1	18.5
400	18.5	13.2	14 6
200	19.6	15.6	17.8
0	22.1	18.0	20.9

These evidently approximate the adiabatic rate on July 5, but depart from it on the other dates, notably on September 22, when the kite ran through a warm stratification, probably blown from the peninsula of Spain over the ocean. These examples show plainly that meteorologists must be prepared to discuss the problems of the circulation of the atmosphere whether the thermal vertical gradients are adiabatic or not, and since our common formulæ are confined to the adiabatic case, it is an important study to learn how they can be practically modified and rendered flexible enough to meet the actually existing conditions.

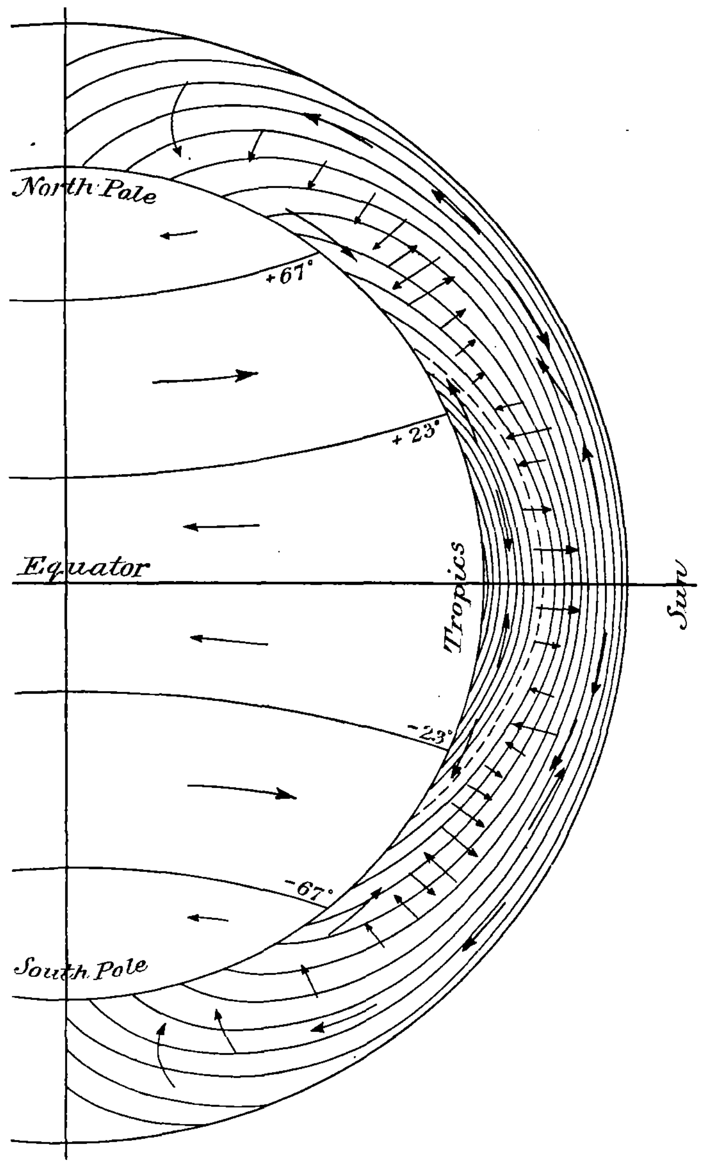


FIG. 11.

I have made an attempt to indicate the probable arrangement of the isothermal surfaces in the earth's atmosphere by means of fig. 11. In the tropical zones the adiabatic rate prevails up to a certain height, as the dotted line, and above that a slower rate. In the temperate zones there is an intrusion of the adiabatic rate into the lower levels and a mixing area, but generally the temperature-fall is less than the adiabatic rate, resulting in a small gradient near the surface and up to 3000 meters, a more rapid fall to 10,000 meters, and again a slower fall due to a second intrusion of warm air from the Tropics. In the polar zones the temperature gradients are probably small, the air being generally cold, and having only small changes from the surface upward. These suggested isothermal lines should be compared with the circulation described in my paper, MONTHLY WEATHER REVIEW, January, 1904, fig. 19, where the results of this intrusion of the types I and II between the temperate and the tropical zones are explained. The arrows are reproduced on fig. 11, where it is seen that three circuits are proposed for each hemisphere; (1)

[1] Kite flying in the Tropics. O. L. Fassig. M. W. R., December, 1903.

[2] Sur les ascensions de cerfs-volant exécutées sur la Méditerranée et sur l'ocean Atlantique . . . ., 1904. H. Hergesell. Note in Comptes Rendus, Jan. 30, 1905.

[3] Die Erforschung der freien Atmosphäre über dem Atlantischen Ocean . . . ., 1905. H. Hergesell. Met. Zeit. November, 1905.

the tropic, circulating anticlockwise; (2) the temperate-tropic, circulating clockwise; and (3) the temperature-polar, circulating feebly anticlockwise for the Northern Hemisphere. In the temperate zones the local cyclonic and anticyclonic systems represent the products of the vertical as well as the horizontal mixing of the currents of air derived by transportation from different latitudes. The excess of heat of the Tropics, producing an adiabatic distribution of temperature in their lower strata, works out poleward at the top and at the bottom by irregular streams, which produce a varying system of temperature gradients in the atmosphere of the temperate zones, standing about midway in value, namely, 5.0° C. per 1000 meters, between that prevailing in the Tropics, 9.87° C. per 1000 meters, and that probably prevailing in the polar zones, as 2.0° to 3.0° C. per 1000 meters. The interchange of heat between the Tropics and the polar zones is by means of these three more or less irregular circuits, which produce primarily the well-known masses of permanent high or low pressure areas standing over the oceans and continents, and secondarily the rapidly migrating cyclonic gyrations of the temperate zones. We shall make an effort to approach our study of this complex circulation by a transformation of the thermodynamic formulæ into forms which will be suitable for computations in the actual atmosphere, as distinguished from an adiabatic but fictitious atmosphere, which has commonly been discussed by meteorologists.

DEVELOPMENT OF THE THERMODYNAMIC FORMULÆ.

In the formulæ derived for discussing the circulation of the atmosphere, it is important that the velocity should be expressed as a function of the temperature in a nonadiabatic atmosphere. It has been generally the custom to treat the velocity as a function of the pressure P, the density ρ, and the gravity g, but it will be equally valid and more valuable to make it a function of the temperature T, the specific heat at a constant pressure C_p, and the gravity g. We must in doing this assume the applicability of two physical laws in the atmosphere. There has been a difficulty in connecting the results obtained by these two methods, which will be pointed out in this paper and their reconciliation will be explained.

I. THE FIRST FORM OF THE BAROMETRIC FORMULA.

The special feature of this formula is that the density ρ is eliminated by the following process: Assume the Boyle-Gay-Lussac law, $P=\rho RT$, and the pressure law, $-dP=\rho g dz$,

(1) Then, $-\frac{dP}{P} RT = g dz.$

Since $R=\frac{P_0}{\rho_0 T_0}$, for the standard conditions, we have,

$$(2)\qquad -\frac{dP}{P}\frac{P_0 T}{\rho_0 T_0}=g dz.$$

By definition $P_0=B_0\rho_m g_0$, and $\frac{dP}{P}=\frac{dB}{B}$, so that,

$$(3)\qquad -\frac{dB}{B}\cdot\frac{B_0\rho_m g_0}{\rho_0 M}\frac{T}{T_0}=g dz, \text{ for common logs.}$$

For the hypsometric formula the gravity g is computed from the standard gravity g_0 by the factors, $(1+\gamma)=(1+0.0026\cos 2\varphi)$ for latitude, and $\left(1+1.25\frac{h}{R}\right)=(1+0.000000196)\,h$ for altitude, since $g_0=g(1+\gamma)\left(1+1.25\frac{h}{R}\right)$.

In integrating for an atmosphere composed of dry and moist air between the heights z_0 and z, the temperature term T, which is variable, is taken as the mean temperature of the air column $z-z_0$, and the moist air is accounted for by the factor, $(1+\beta)=\left(1+0.378\frac{e}{B}\right)$, where e is the vapor tension. Hence, the integral mean temperature is,

$$\frac{1}{z}\int_{z_0}^{z} T=T_m, \text{ and } \frac{T_m}{T_0}=(1+0.367\theta)=(1+\alpha\theta).$$

We must pass from $\frac{P_0}{P}=\frac{B_0}{B}\left(1+1.25\frac{h-h_0}{R}\right)$, to logarithms, $\log\frac{P_0}{P}=(1+.00157)\log\frac{B_0}{B}=(1+\eta)\log\frac{B_0}{B}$, by adding the factor $(1+\eta)$.

Finally, $\frac{B_0\rho_m}{\rho_0 M}=K=18400,$

for $B_0=0.760$ meter, $\rho_m=13595.8$, $\rho_0=1.29305$, $M=0.43429$ in the meter-kilogram system.

Hence, by integration,

$$(4)\qquad -\int_{z_0}^{z}\frac{dB}{B}\cdot\frac{B_0\rho_m g_0}{M\rho_0}\cdot\frac{T\left(1+0.378\frac{e}{B}\right)}{T_0}=\int_{z_0}^{z} g dz, \text{ because,}$$

$$(5)\qquad \log\frac{B_0}{B}\cdot\frac{K g_0}{T_0}T_m\left(1+0.378\frac{e}{B}\right)=g_m(z-z_0).$$

If $\frac{K g_0}{T_0}=K_1$ is computed as a new barometric constant, and

$T_m\left(1+0.378\frac{e}{B}\right)=T_v$, the virtual temperature, then,

$$(6)\qquad K_1 T_v \log\frac{B_0}{B}=g_m(z-z_0) \text{ in mechanical units.}$$

I have computed the logarithmic tables 91, 92, 93 of the International Cloud Report, 1898, in such a form that the dry air temperature term m, the humidity term βm, and the gravity term γm, are kept separate from each other in

$$(7)\qquad \log B_0=\log B+m-m\beta-m\gamma,$$

for the sake of accurate and flexible applications in all possible meteorological computations. Auxiliary tables can be constructed from these primary tables for any desired applications, by way of shortening the work in special cases, such as in numerous reductions to any selected plane, or in computing the pressures from point to point in the atmosphere, using as arguments the temperatures and humidities observed in balloon or kite ascensions. Especially, they can be used to compute the dynamical units of force, or work required to pass from point to point, by simply extracting $(z-z_0)$ from the tables, with the temperature, humidity, and pressure as the arguments and multiplying $(z-z_0)$ by g_m, so that

$$(8)\qquad -\int_{z_0}^{z}\frac{dP}{\rho}=g_m(z-z_0),$$

when there is no circulation or velocity term, $\frac{1}{2}(q^2-q_0^2)$. This result is in conformity with the equation,

$$-dP=\rho g dz,$$

with which we began this discussion.

II. THE SECOND FORM OF THE BAROMETRIC FORMULA IN AN ADIABATIC ATMOSPHERE.

In formula (108a) of my collection in the International Cloud Report the abnormal form for dry air was written:

$$(9)\qquad \frac{P}{P_0}=\left(\frac{T_0-\theta_m h}{T_0}\right)^m, \text{ which is}$$

$$\frac{B}{B_0}=\left(\frac{T_0-\frac{dT}{dh}h}{T_0}\right)^m=\left(\frac{T}{T_0}\right)^m,$$

where $-\frac{dT}{dh}$ is the actual vertical gradient of temperature and the exponent is undetermined. We acquire from the observations a vertical gradient, $-\frac{dT}{dh}$, which generally differs from the adiabatic gradient, $-\frac{dT_a}{dh}$, and seek to determine the proper value of the exponent m.

From my formula (73), in the adiabatic state for $dQ = 0$, we have $0 = C_p dT - RT\frac{dP}{P}$, in mechanical units.

(10) Hence, $\frac{dP}{P} = \frac{C_p}{R}\frac{dT}{T}$, and integrating,

$$(11) \qquad \log\frac{P}{P_0} = \frac{C_p}{R}\log\left(\frac{T}{T_0}\right), \qquad \text{or } \frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{\frac{C_p}{R} = \frac{k}{k-1}}.$$

$$(12) \quad \text{Again, } \frac{C_p}{R} = -g\frac{dz}{dT}\cdot\frac{T_0}{l_0 g_0} = -g\frac{dz}{dT}\cdot T_0\cdot\frac{\rho_0}{P_0}, \qquad \text{and}$$

$$(13) \qquad \frac{C_p}{R} = -\frac{dz}{dT}T_0\frac{g\rho_0}{g_0\rho_m B_0}. \qquad \text{Substituting in (10)}$$

$$(14) \qquad \frac{dB}{B} = -\frac{dz}{dT}\cdot T_0\frac{g}{g_0}\frac{\rho_0}{\rho_m B_0}\cdot\frac{dT}{T}. \qquad \text{Hence,}$$

$$(15) \qquad \int_{z_0}^{z} g\,dz = -\frac{B_0\rho_m g_0}{M\rho_0 T_0}\frac{1}{z}\int_{z_0}^{z} dT\frac{dB}{B}, \qquad \text{for } T = \frac{1}{z}\int_{z_0}^{z} dT,$$

as before. Hence, we see that $\frac{C_p}{R}$ supplies the constants for the barometric constant K *in the adiabatic case only.* These substitutions, (12), (13), can be verified by referring to the formulæ of Table 14. It is well known that the use of the formula $\frac{B}{B_0} = \left(\frac{T}{T_0}\right)^{\frac{k}{k-1}}$ is not applicable in the actual atmosphere, except to give what is called by von Bezold the potential temperature T_0, corresponding with $(B\,.\,T)$ when reduced to the standard pressure B_0.

Making the following substitutions,

$$B_0\rho_m g_0 = P_0, \ \frac{P_0}{\rho_0 T_0} = R, \text{ and } \frac{1}{z}\int_{z_0}^{z} dT = T_m, \text{ we have}$$

$$(16) \quad g\,(z - z_0) = R\,T_m \log\frac{P_0}{P} \text{ in Naperian logarithms, and}$$

$$(17) \qquad \log\frac{P_0}{P} = \frac{g}{R}\frac{z - z_0}{T_m}, \qquad \text{or } \frac{P_0}{P} = e^{\frac{g}{R}\frac{z-z_0}{T_m}}$$

This result can be obtained again by another process.

Assume, $\qquad -dP = +g\rho\,dz$, and $\rho = \frac{P}{RT}$,

Then, $\qquad -\frac{dP}{P} = +\frac{g}{R}\frac{dz}{T}$, and by integrating,

$$(17)_a \qquad \log\frac{P_0}{P} = \frac{g}{R}\int_{z_0}^{z}\frac{dz}{T} = \frac{g}{R}\cdot\frac{z - z_0}{T_m}.$$

III. THE BAROMETRIC FORMULA IN A NONADIABATIC ATMOSPHERE.

In the preceding case it has been assumed that the temperature varies with the height by the adiabatic law, which is, $-\frac{dT_a}{dz} = \frac{g_0}{C_p} = \frac{1000}{P_0} = 0.0098695$ °C., so that the temperatures of the formulæ of section II, of which $\frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{\frac{k}{k-1}}$ is the representative, must have this relation. Now it is known that this formula in the atmosphere does not apply, except in occasional instances, and we, therefore, shall seek a formula of the same type which will admit other temperature gradients, $-\frac{dT}{dz}$, in a quasi-adiabatic atmosphere. It has been assumed that there was no addition or subtraction of heat in the variation of the pressures and temperatures, but as this is only a special case it will be proper to take the general case, where the quantity of heat dQ is added or subtracted, besides that acquired or lost during the expansion and contraction processes. Since in the stratifications of the atmosphere by currents possessing different thermodynamic properties, there is departure from the adiabatic state by the term dQ, we shall resume the full equation for discussion.

Fig. 12 will make our treatment clear.

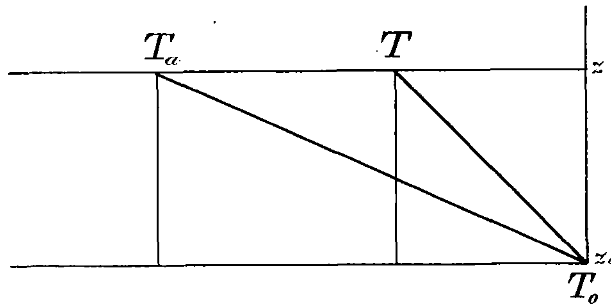


FIG. 12.—The relation of the observed to the adiabatic gradient.

Let T_0 = the initial temperature at the height z_0
T_a = adiabatic temperature at the height z
T = observed temperature at the height z

Then the adiabatic gradient is $a_0 = -\frac{dT_a}{dz} = \frac{T_0 - T_a}{z - z_0}$,

and the observed gradient is, $\quad a = -\frac{dT}{dz} = \frac{T_0 - T}{z - z_0}$.

Let the ratio of these gradients, $n = \frac{dT_a}{dT} = \frac{T_0 - T_a}{T_0 - T}$.

Having regard to the adiabatic thermodynamic equation,

$$(18) \qquad 0 = C_p dT_a - \frac{dP}{\rho}$$

we observe that the thermal mass passes from $(T_0 z_0)$ to $(T_a z)$ by the oblique path marked, T_0 to T_a, in conformity with the formulæ just discussed; it can then be carried from the point T_a to the point T at the same level z by changing the temperature through $(T - T_a)$, and the addition of the heat

$Q = C_p(T - T_a)$. Now we have,

$$(19) \qquad 0 = C_p\,(T_a - T_0) - \int_{z_0}^{z}\frac{dP}{\rho}, \qquad \text{from (18), and adding,}$$

$$(20) \qquad Q = C_p\,(T - T_a), \qquad \text{we obtain,}$$

$$(21) \qquad Q = C_p\,(T - T_0) - \int_{z_0}^{z}\frac{dP}{\rho}, \qquad \text{or in the differential form}$$

$$(22) \qquad dQ = C_p dT - \frac{dP}{\rho}.$$

Since $dT_a = n\,dT$, we have $dQ = C_p(dT - dT_a) = C_p dT - C_p n\,dT$.

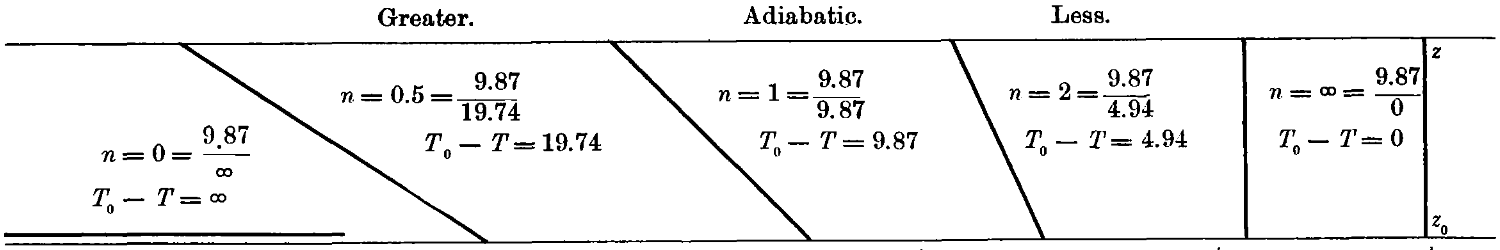
Subtracting this value of dQ from equation (22) we find,

$$(23) \qquad 0 = C_p n\,dT - \frac{dP}{\rho}, \qquad \text{in a quasi-adiabatic form,}$$

which is true in a stratum where n is constant, that is, where the gradient $-\frac{dT}{dz}$ is not changing.

Substituting, $\frac{1}{\rho} = \frac{RT}{P}$, we have,

$$(24) \qquad 0 = n\,C_p\,dT - RT\frac{dP}{P}, \qquad \text{and}$$



$\frac{P}{P_0} = \left(\frac{T}{T+\infty}\right)^{0\frac{k}{k-1}}$	$\frac{P_{0.5}}{P_0} = \left(\frac{T}{T+19.74}\right)^{0.5\frac{k}{k-1}}$	$\frac{P_1}{P_0} = \left(\frac{T}{T+9.87}\right)^{1\frac{k}{k-1}}$	$\frac{P_2}{P_0} = \left(\frac{T}{T+4.94}\right)^{2\frac{k}{k-1}}$	$\frac{P_\infty}{P_0} = \left(\frac{T}{T}\right)^{\infty\frac{k}{k-1}}$
$= \left(\frac{T}{T+\infty}\right)^{0}$	$= \left(\frac{T}{T+19.74}\right)^{1.73}$	$= \left(\frac{T}{T+9.87}\right)^{3.46}$	$= \left(\frac{T}{T+4.94}\right)^{6.92}$	$= \left(\frac{T}{T}\right)^{\infty}$
$= \left(\frac{1}{\infty}\right)^{0} = 0$	$= A_{0.5} < A_1$	$= A_1 < A$	$= A_2 < 1$	$= 1^\infty = 1$
$P = 0 \times P_0 = 0$	$P_{0.5} = A_{0.5} P_0 < P_1$	$P_1 = A_1 P_0 < P_2$	$P_2 = P_0 A_2 < P_0$	$P_\infty = P_0$

FIG. 13.—The variations of the ratio $n = \frac{dT_a}{dT}$.

$$(25) \qquad 0 = \frac{nC_p}{R}\frac{dT}{T} - \frac{dP}{P}, \qquad \text{so that,}$$

$$(26) \qquad \frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{\frac{nC_p}{R}} = \left(\frac{T}{T_0}\right)^{n\frac{k}{k-1}} = \left(\frac{T}{T_0}\right)^{-\frac{g}{R}\frac{dz}{dT}} = \left(\frac{T}{T_0}\right)^{\frac{g}{Ra}}.$$

The last forms are found as follows:

By definition, $-\frac{dT_a}{dz} = a_0 = \frac{g}{C_p}$, and the ratio,

$$n = \frac{dT_a}{dT} = \frac{a_0}{a} = \frac{g}{C_p a} = \frac{g}{C_p}\frac{dz}{dT}. \qquad \text{Since } \frac{k}{k-1} = \frac{C_p}{R},$$

$$(27) \qquad n\frac{k}{k-1} = \frac{g}{C_p a}\frac{k}{k-1} = \frac{g}{Ra} = -\frac{g}{R}\frac{dz}{dT}.$$

Our formula, therefore, differs from the adiabatic formula by the factor n in the exponent with $\frac{k}{k-1}$. This ratio n between the adiabatic and observed gradients, depends upon the amount of heat added or subtracted from an adiabatic atmosphere to produce the given observed atmosphere within the stratum $z—z_0$ where the gradient remains a constant. We can evidently pass from one stratum to an adjoining stratum either continuously by changing n gradually, or discontinuously by changing n abruptly. The ratio n is a new variable to be introduced into the thermodynamic equations in their application to the atmosphere, so that all the standard thermodynamic equations and discussions become available with this simple modification. Such an exposition as was given by M. Margules in his admirable paper, "Über die Energie der Stürme," which is limited to the adiabatic case, may be modified in this way and be made very useful in practical meteorology. It is rarely the case that computations of T_0 to T, from one level to another, z_0 to z, can be made by general dynamic formulæ, but they must usually be observed with balloons and kites.

The ratio, $n = \frac{dT_a}{dT} = \frac{\text{adiabatic gradient}}{\text{observed gradient}}$, can range between the limits $n = 0$ and $n = \infty$; for $n = 1$ the gradient is adiabatic; for $n < 1$ the cooling is more rapid than in the adiabatic gradient, as in summer afternoons when the ground is superheated and cumulus clouds are forming; for $n > 1$ the cooling is less rapid than in the adiabatic gradient, as generally in the temperate and polar zones; the Tropics probably conform to the adiabatic gradient in the lower strata of the atmosphere.

IV. CONSTRUCTION OF THE PRIMARY DIFFERENTIAL EQUATION.

Under the assumption that n is variable we now differentiate the equation with the variables P, T, n,

$$(28) \qquad \frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{n\frac{k}{k-1}}.$$

Passing to logarithms,

$$(29) \qquad \log\frac{P}{P_0} = n\frac{k}{k-1}\log\left(\frac{T}{T_0}\right), \qquad \text{or for one limit,}$$

$$(30) \qquad \log P = n\frac{k}{k-1}\log T. \qquad \text{Differentiate,}$$

$$(31) \qquad \frac{dP}{P} = n\frac{k}{k-1}\frac{dT}{T} + \frac{k}{k-1}\log T dn. \qquad \text{Substitute } \frac{1}{P} = \frac{1}{\rho RT},$$

$$(32) \qquad \frac{dP}{\rho RT} = n\frac{k}{k-1}\frac{dT}{T} + \frac{k}{k-1}\log T dn. \qquad \text{Substitute } R\frac{k}{k-1} = C_p,$$

$$(33) \qquad \frac{dP}{\rho} = nC_p dT + C_p T\log T dn. \qquad \text{In common logs and to } dz,$$

$$(34) \qquad \frac{dP}{\rho dz} = nC_p\frac{dT}{dz} + C_p T\log T\frac{dn}{dz}, \text{ for the vertical direction.}$$

Again, since $nC_p\frac{dT}{dz} = -g$, by this substitution we have,

$$(35) \qquad \frac{dP}{\rho dz} = -g + C_p T\log T\frac{dn}{dz}, \qquad \text{and hence,}$$

$$(36) \qquad dP = -\rho g dz + \rho C_p T\log T dn.$$

We see then that the effect of the change from an adiabatic atmosphere to any other gradient is accomplished by adding the term $\rho C_p T\log T dn$.

If it should happen that besides the strictly mechanical velocities thus indicated there is a further expenditure of heat by radiation, it would be necessary to add the special term, $Q_0 - Q$, making, from (33),

$$(37) \qquad \frac{P}{\rho} - \frac{P_0}{\rho_0} = (Q_0 - Q) + nC_p(T - T_0) + C_p T\log T(n - n_0)'.$$

It is better to say that the full term $C_p\, T\log\, T\,(n - n_0)$ has a radiation part, $(Q - Q_0)$, and a velocity part, $C_p\, T\log\, T\,(n - n_0)'$.

The factor n, due to an addition or subtraction of heat other than by adiabatic expansion and contraction, fully accounts for the presence of a nonadiabatic gradient, through the stratification of the layers of air due to transportation horizontally from one latitude to another, or generally from one place to another; or else through the addition or subtraction of latent heat in the condensation of aqueous vapor to water, or by the vaporization of water to aqueous vapor. In effect, by the practical use of the factor n, we can dispense with the difficult computations which occur in making an allowance for the action of the vapor contents of the atmosphere; or, on the other hand, we can substitute for n its equivalent in terms of such other computations as may be found convenient for particular purposes.

The corresponding formulæ involving P, T, R, ρ, and n, in terms of the temperature T, become,

$$(38)\quad \frac{P}{P_0}=\left(\frac{T}{T_0}\right)^{\frac{nk}{k-1}};\ \log P=\log P_0+n\frac{k}{k-1}(\log T-\log T_0).$$

$$(39)\quad \frac{\rho}{\rho_0}=\left(\frac{T}{T_0}\right)^{\frac{n}{k-1}};\ \log \rho=\log \rho_0+\frac{n}{k-1}(\log T-\log T_0).$$

$$(40)\quad \frac{R}{R_0}=\left(\frac{T}{T_0}\right)^{n-1};\ \log R=\log R_0+(n-1)(\log T-\log T_0).$$

$$(41)\quad \frac{\rho}{\rho_0}=\left(\frac{P}{P_0}\right)^{\frac{1}{k}};\ \log \rho=\log \rho_0+\frac{1}{k}(\log P-\log P_0).$$

It is evident that R is not constant except in the adiabatic system for $n=1$; and that only that density determined through the use of n is generally valuable in the atmosphere.

V. APPLICATION TO THE GENERAL EQUATIONS OF MOTION.

We will now make the connection between this system of equations and the general equations of motion which have been employed in meteorology. From the equations (200) of the Cloud Report, we have, in connection with the differentiations of equation (37) along the axes x, y, z, for the *acceleration*,

$$(42)\quad \begin{cases} -\frac{1}{\rho}\frac{\partial P}{\partial x}=\frac{du}{dt}-\cos\theta\,(2n+\nu)\,v+\frac{uw}{r} \\ =\frac{\partial Q}{\partial x}-C_p n\frac{\partial T}{\partial x}-C_p T\log T\frac{\partial n}{\partial x} \\ -\frac{1}{\rho}\frac{\partial P}{\partial y}=\frac{dv}{dt}+\sin\theta\,(2n+\nu)\,w+\cos\theta\,(2n+\nu)\,u \\ =\frac{\partial Q}{\partial y}-C_p n\frac{\partial T}{\partial y}-C_p T\log T\frac{\partial n}{\partial y} \\ -\frac{1}{\rho}\frac{\partial P}{\partial z}=\frac{dw}{dt}-\sin\theta\,(2n+\nu)\,v-\frac{u^2}{r}+g \\ =\frac{\partial Q}{\partial z}-C_p n\frac{\partial T}{\partial z}-C_p T\log T\frac{\partial n}{\partial z}. \end{cases}$$

Multiplying by dx, dy, dz, the equations for *work* are,

$$(43)\quad \begin{cases} -\frac{\partial P}{\rho}=udu-\cos\theta\,(2n+\nu)\,v\partial x+\frac{uw\partial x}{r} \\ =\partial Q-C_p n\,\partial T-C_p T\log T\partial n \\ -\frac{\partial P}{\rho}=vdv+\sin\theta\,(2n+\nu)\,w\partial y+\cos\theta\,(2n+\nu)\,u\partial y \\ =\partial Q-C_p n\,\partial T-C_p T\log T\partial n \\ -\frac{\partial P}{\rho}=wdw-\sin\theta\,(2n+\nu)\,v\partial z-\frac{uu}{r}\partial z+g\partial z \\ =\partial Q-C_p n\,\partial T-C_p T\log T\partial n. \end{cases}$$

Since, by substituting $v\partial x=u\partial y$, $w\partial x=u\partial z$ and $w\partial y=v\partial z$, the terms in $(2n+\nu)$, the angular velocity of the earth and the atmosphere relative to it, disappear in the summation, they represent a deflecting force at right-angles to the direction of motion at the velocity q, which does not modify the circulation but only the path of motion. The integral, therefore, becomes between two places,

$$(44)\quad \begin{cases} \frac{P_0}{\rho_0}-\frac{P}{\rho}=\frac{1}{2}(q^2-q_0^2)+g(z-z_0) \\ =Q-Q_0-C_p n\,(T-T_0)-C_p T\log T\,(n-n_0). \end{cases}$$

It is noted that the term for the circulation $\frac{1}{2}(q^2-q_0^2)$ must be added to the equations of sections I, II, III, IV, to pass from the static state there considered to the circulating state here computed. Since we have

$$(45)\quad g\,(z-z_0)=-C_p n\,(T-T_0),\text{ it follows that}$$

$$(46)\quad \tfrac{1}{2}(q^2-q_0^2)=-C_p T\log T\,(n-n_0),\text{ for } Q-Q_0=0,$$

so that the circulation can be computed directly in terms of T and $(n-n_0)$. This proves that the energy of circulation is derived from the difference of temperature gradients in neighboring masses of air, where $n-n_0$ is not equal to zero. Moreover, since the integral of gdz around a closed curve is zero, $\int_0 (gdz)\,ds=0$, and we have the remaining,

$$(47)\quad -\int_0\frac{dP}{\rho}.\,ds=\int_0\frac{dq}{dt}\,ds+\int_0(gdz)\,ds=\int qdq.$$

This is the equation employed by Bjerknes in his discussion of the circulation of the atmosphere, and is applicable only in *closed curves*, along all points of which P, ρ, q, or qdq must be known by observations. The difficulty of securing such observed data simultaneously along the circuit at a given time is so great that this special case of the general equation will seldom be serviceable. In ordinary meteorology it is required to integrate between the two points, as in the same horizontal plane, or in a vertical direction. Since the term $\frac{1}{2}(q^2-q_0^2)$ is expressed in mechanical measures and represents work done, then it may be taken as equivalent to $\frac{1}{2}(q^2-q_0^2)=g(z'-z'_0)$, so that

$$(48)\quad \tfrac{1}{2}q^2=g\,z',\text{ and } q^2=2\,g\,z'.$$

The circulation is therefore always equivalent to a falling velocity through the height z', which may be computed.

Furthermore, since $Q-Q_0$ is also given in mechanical units, it may be taken as equivalent to

$$(49)\quad Q-Q_0=g\,(z''-z''_0),\text{ so that}$$

$$(50)\quad Q=gz''$$

and the stored up energy of radiation is equivalent to a vertical work.

It follows from these considerations that we obtain

$$(51)\quad \begin{cases} q\frac{dq}{dx}=\frac{dQ}{dx}-C_p n\frac{dT}{dx}-C_p T\log T\frac{dn}{dx},\text{ in latitude.} \\ q\frac{dq}{dy}=\frac{dQ}{dy}-C_p n\frac{dT}{dy}-C_p T\log T\frac{dn}{dy},\text{ in longitude.} \\ q\frac{dq}{dz}=\frac{dQ}{dz}\qquad\qquad\quad -C_p T\log T\frac{dn}{dz},\text{ in vertical.} \end{cases}$$

Since $P=B\rho_m g_0$, we obtain in a stratum of mean ρ,

$$(52)\quad \begin{cases} (B_0-B)=\frac{\rho}{2\,g_0\rho_m}\left[(q^2-q_0^2)+2\,g(z-z_0)\right] \\ =\frac{\rho}{2\,g_0\rho_m}\left[(Q-Q_0)-C_p n\,(T-T_0)-C_p T\log T\,(n-n_0)\right] \end{cases}$$

It is readily perceived that the introduction of the factor n and the correlation of the pressure, velocity, gravity, radiation, specific heat, temperature, and gradient, in this double equation leads to an innumerable number of special combinations, taken in connection with the equations of thermodynamics. These embrace the first and second laws of thermodynamics, cyclic processes, the entropy S, the inner energy U, the thermodynamic potentials (F. Φ); the adiabatic, isodynamic, isometric, isothermal physical processes; differential relations with pairs of variables; thermodynamic surfaces and lines in gases; the adiabatic, isodynamic, isenergetic, and isopiestic processes with other variables in pairs; the gaseous, liquid, and solid phases; latent and specific heat; mixtures and chemical transformations, chemical dissociation, their solutions, and other relations, involving ionization, electrical and magnetic fields of force. This vast subject is open to meteorological investigation in the atmosphere, and will no doubt eventually lead to important practical results.

VI. FOUR SYSTEMS OF CONSTANTS FOR THE ATMOSPHERE.

In the application of these formulæ to computations of thermodynamic and dynamic problems in the atmosphere, it will be convenient to have for ready reference a table of the most important constants, with their equivalents in the four systems of units likely to be used. Table 14 presents such a compilation of constants in the following systems of mechanical or gravitational units:

1. Meter-kilogram-second-centigrade degrees.
2. Centimeter-gram-second-centigrade degrees.
3. Meter-gram-second-centigrade degrees.
4. Foot-pound-second-Fahrenheit degrees.

There is often so much confusion in discussing meteorological problems arising from the use of now one system, again another system, and even a hybrid system, that it may be a check against errors for those students who conform to the constants here given. The short formulæ in the first column define the quantities with precision, and the numerous transformations possible among them give rise to many combinations such as occur in various mathematical discussions. Indeed, it is surprising to note how large an amount of current meteorology, occurring in treatises and analytical papers, can be readily reduced to these elementary formulæ, and in reading a new presentation of primary principles it is proper to find whether they conform to these elementary theorems or not. We use the symbols:

TABLE 14.—*Mechanical systems of constants for the atmosphere in gravitational units.*

Formulæ.	S.	° C. Meter-kilogram.	Log.	° C. Centimeter-gram.	Log.	° C. Meter-gram.	Log.	° F. Foot-pound.	Log.
$P_0 = g_0 \rho_m B_n$	g_0	9.8060	0.99149	980.60	2.99149	9.8060	0.99149	32.172	1.50748
	ρ_m	13595.8	4.13340	13.5958	1.13340	13.5958	1.13340	846.728	2.92774
	B_n	0.760	9.88081	76.0	1.88081	0.760	9.88081	2.4934	0.39680
	P_0	101323.5	5.00571	1013235.	6.00571	101.3235	2.00571	67923.5	4.83202
$P_0 = g_0 \rho_0 l_0$	g_0	9.8060	0.99149	980.60	2.99149	9.8060	0.99149	32.172	1.50748
	ρ_0	1.29305	0.11162	0.00129305	7.11162	0.00129305	7.11162	0.080529	8.90595
	l_0	7991.04	3.90260	7991.04	5.90260	7991.04	3.90260	26217.3	4.41859
	P_0	101323.5	5.00571	1013235.	6.00571	101.3235	2.00571	67923.5	4.83202
$P_0 = R_0 T_0 \rho_0$	R_0	287.0334	2.45793	2870334.	6.45793	29.2712	1.46644	1716.43	3.23463
$= F \frac{C_p}{g_0}$	T_0	273.	2.43616	273.	2.43616	273.	2.43616	491.4	2.69144
	ρ_0	1.29305	0.11162	0.00129305	7.11162	0.0012935	7.11162	0.080529	8.90595
$= -F \frac{dh}{dT}$	P_0	101323.5	5.00571	101323.5	6.00571	101.3235	2.00571	67923.5	4.83202
$p_0 = {P_0}/{g_0}$	p_0	10332.8	4.01422	1033.28	3.01422	10.3328	1.01422	2111.23	3.32454
$C_p = \frac{k}{k-1} \frac{l_0}{T_0} g_0$	$\frac{k}{k-1}$	3.461545	0.53927	3.461545	0.53927	3.461545	0.53927	3.461545	0.53927
$= \frac{k}{k-1} R$	l_0	7991.04	3.90260	799104.	5.90260	7991.04	3.90260	26217.3	4.41859
	g_0	9.8060	0.99149	980.60	2.99149	9.8060	0.99149	32.172	1.50748
$= \frac{g_0 P_0}{F}$	$\frac{1}{T_0}$	$\frac{1}{273}$	7.56384	$\frac{1}{273}$	7.56384	$\frac{1}{273}$	7.56384	$\frac{1}{491.4}$	7.30856
$= -g_0 \frac{dh}{dT}$	C_p	993.5787	2.99720	9935787.	6.99720	993.5787	2.99720	5941.57	3.77390
$R_0 = \frac{l_0}{T_0} g_0$	l_0	7991.04	3.90260	799104.	5.90260	7991.04	3.90260	26217.3	4.41859
$= \frac{B_n \rho_m}{T_0 \rho_0} g_0$	g_0	9.8060	0.99149	980.60	2.99149	9.8060	0.99149	32.172	1.50748
	$\frac{1}{T_0}$	$\frac{1}{273}$	7.56384	$\frac{1}{273}$	7.56384	$\frac{1}{273}$	7.56384	$\frac{1}{491.4}$	7.30856
$= \frac{P_0}{\rho_0 T_0}$	R_0	287.0334	2.45793	2870334.	6.45793	287.0334	2.45793	1716.43	3.23463
$C_v = C_p - R_0$	C_v	706.5453	2.84914	7065453.	6.84914	706.5453	2.84914	4225.14	3.62584
$k = \frac{C_p}{C_v}$	k	1.4062486	0.14806	1.4062486	0.14806	1.4062486	0.14806	1.4062486	0.14806
	$k-1$	0.4062486	9.60879	0.4062486	9.60879	0.4062486	9.60879	0.4062486	9.60879
	$\frac{k}{k-1}$	3.461545	0.53927	3.461545	0.53927	3.461545	0.53927	3.461545	0.53927
	$\frac{1}{k-1}$	2.461545	0.39121	2.461545	0.39121	2.461545	0.39121	2.461545	0.39121
$-\frac{dT}{dh} = \frac{g_0}{C_p}$	$-\frac{dT}{dh}$	0.0098695	7.99429	0.000098695	5.99429	0.0098695	7.99429	0.0054147	7.73358
$\frac{1}{A_m} = \frac{g_0}{A}$	$\frac{1}{A_m}$	4185.57	3.62175	41855700.	7.62175	4185.57	3.62175	25027.7	4.39842
$A_m = \frac{A}{g_0}$	A_m	0.0002389	6.37829	2.389×10^{-8}	2.37829	0.0002389	6.37829	0.00003995	5.60158
Pr. Th. U.	Θ							3.968	0.59856
	F	1000.	3.00000	100	2.00000	1	0.00000	367.8	2.56560
	A	0.002343	7.36978	2.343×10^{-5}	5.36978	0.002343	7.36978	0.0012855	7.10906
	$\frac{1}{A}$	426.837	2.63022	42683.7	4.63022	426.837	2.63022	777.9	2.89094

P_0 = pressure in units of force, g_0.
ρ_0 = the weight of a given mass of atmosphere, $\rho_m B_n = \rho_0 l_0$.
C_p = the specific heat at constant pressure.
C_v = the specific heat at constant volume.

$$R = C_p - C_v, \qquad k = \frac{C_p}{C_v}$$

$-\frac{dT}{dh}$ = the temperature fall per unit height in adiabatic state.
$\frac{1}{A}$ = the mechanical equivalent of heat, 426.8 and 777.9.
$\frac{g_0}{A}$ = the factor to change mechanical units to heat units.
F = the factor connecting the thermal gradient and P_0.
θ = the number of British thermal units in 1 kilogram-degree.

VII. THE THERMODYNAMIC CONSTANTS FOR THE SUN.

There is much difficulty in passing from the thermodynamic conditions on the earth to the corresponding thermodynamic conditions on the sun. I have already approached this subject from the side of radiation in my "Eclipse Meteorology and Allied Problems," 1902, and from the method of Nipher's Formulæ, in my studies on the "Circulation of the Atmospheres of the Sun and of the Earth," 1904. I shall briefly present the same subject as the immediate development of the fundamental formulæ introduced in this paper. It is not so difficult to produce a self-consistent system of quantities as it is to find one which conforms to the actual physical state of the sun, and I conceive that it is proper to discuss this subject in several ways.

Specific heat.

From the preceding formulæ, we have,

(53) $$-\frac{dT}{dz} = \frac{g_0}{C_p} = \frac{F}{P_0} = \frac{F}{g_0 \rho_m B_n} = \frac{F}{g_0 \rho_0 l_0}.\quad \text{Hence,}$$

(54) $$C_p = \frac{\rho_m B_n}{F}\cdot g_0^2 = \frac{\rho_0 l_0}{F}\cdot g_0^2.$$

Since $\rho_m B_n = \rho_0 l_0$ is a given mass, and F is constant for a given system of units, it follows that C_p is proportional to the square of the gravity. Taking the force of gravity on the sun,

(55) $(g)_{sun} = g_0 \times G = 9.806 \times 28.028 = 274.843$

it follows that the specific heat on the sun is

(56) $(C_p)\ \text{sun} = C_p \times G^2 = 993.5787 \times (28.028)^2 = 780524.$

Adiabatic rate of temperature-fall.

(57) For the earth $-\frac{dT}{dz} = \frac{g_0}{C_p} = 9.8695°$ per 1000 meters.

(58) For the sun $-\left(\frac{dT}{dz}\right)_{sun} = \frac{g_0 G}{C_p G^2} = \frac{9.8695°}{28.028} = 0.32862°.$

Mechanical equivalent of heat.

(59) From $-\frac{dT}{dz} = \frac{g_0}{C_p}$ for the earth, we have on the sun,

(60) $$-\frac{dT}{G dz} = \frac{g_0 G}{C_p G^2}.\quad \text{Hence, by integration,}$$

(61) $$-C_p G^2 \int dT = g_0 G \int G dz, \text{ or,}$$

(62) $$-C_p G^2 (T - T_0) = g_0 G (z - z_0).$$

If the change of temperature is 1° then,

(63) $$-C_p G^2 = g_0 G (z - z_0) G,$$

is the mechanical equivalent of heat, and is obtained by the fall of a mass through the height $(z - z_0) G$ under the force of gravity $g_0 G$. Whereas on the earth,

(64) $$\frac{1}{A_m} = 4185.57 = 426.8 \times 9.8060, \text{ we have}$$

(65) $$\left(\frac{1}{A_m}\right)_{sun} = 4185.57 \times (28.028)^2 = 3288046.$$

Boyle-Gay-Lussac Law.

From the formulæ of Table 14, we have,

(66) $$g_0 l_0 = \frac{P_0}{\rho_0} = RT_0 = C_p T_0 \frac{k-1}{k} = -g_0 \frac{dh}{dT} T_0 \frac{k-1}{k},$$

on the earth, and we infer that we shall have on the sun,

(67) $$g_0 G \cdot l_0 G^2 = \frac{P_0 G^3}{\rho_0} = RG^2 . T_0 G = C_p G^2 . T_0 G$$
$$= -g_0 G . \frac{dh}{dT} G . T_0 G \frac{k-1}{k}.\quad \text{Hence,}$$

(68) $l_0 G^2 = 7991.04 \times (28.028)^2.$
(69) $P_0 G^3 = 101323.5 \times (28.028)^3.$
(70) $RG^2 = 287.0334 \times (28.028)^2.$
(71) $T_0 G = 273° \times 28.028 = 7652°,$

if $\frac{k-1}{k}$ is retained a constant in both cases.

If the atmosphere of the sun is composed of some other material than ρ_0 of the earth's atmosphere, then the proper modification of the preceding quantities can be readily computed from terrestrial data.

Specific heat at constant volume.

For the earth, $C_v = C_p - R$, and hence, for the sun,

(72) $$C_v G^2 = C_p G^2 - RG^2$$

(73) $$(C_v)_{sun} = C_v G^2 = 706.5453 \times (28.028)^2 = 555040.$$

(74) Finally, $k = \frac{C_p . G^2}{C_v . G^2} = 1.4062486$, as a check.

This system throws the entire emphasis upon a change of gravity depending upon the mass of the central body, rather than upon the change of physical conditions implied in altering the ratio of the specific heats k. Since the temperature of the photosphere may in this way be taken as about 7652°, and the temperature gradient —0.32862° per 1000 meters, it follows that the effective temperature of radiation as determined by bolometer measures, 6100°, will be reached at the height of 4418 kilometers, or 2745 miles above the surface of the photosphere. This change of 1552° may be sufficient to meet the requirements of the spectroscopic observations in regard to the absorption and reversal of the spectrum lines. The gradient, —0.32862° per 1000 meters, is 28.028 times greater than that obtained by my other methods, the difference arising from the different distribution of the gravity factor G, which seems to be fully accounted for in these formulæ.

GERMAN AERIAL RESEARCH STATION.

According to Science (April 6, 1906, p. 559), the German Government has decided to establish a meteorological station on Lake Constance, near Friedrichshafen. It will cost $15,000, the states of Bavaria, Wurttemberg, Baden, and Alsace-Lorraine joining in the expense. Extensive study of the atmosphere will be made daily by means of kites flown from specially constructed boats on the lake. Similar kite and balloon stations already exist in northern Germany, at Lindenberg and Hamburg, and plans are being made to erect still another station in the northeastern part of the Empire.

A NEW DEPARTURE IN FORECASTING.

The following statement has been sent by the Chief of Bureau in reply to a recent letter, requesting some details regarding the "new departure in forecasting weather conditions a month in advance:"

Beyond the statement made by me in New York, in March, that the Weather Bureau believes that it is in possession of a sound scientific basis on which to make forecasts for a considerable period in advance, nothing will be announced in regard to the matter for several months to come.