STUDIES ON THE THERMODYNAMICS OF THE ATMOSPHERE.

By Prof. FRANK H. BIGELOW.

V.—THE HORIZONTAL CONVECTION IN CYCLONES AND ANTICYCLONES.[1]

SOME OF THE DIFFICULTIES IN THIS PROBLEM.

If one wishes to follow the exact process occurring in the natural circulation of the atmosphere, then the next step in the orderly development of the analysis of the problem of the structure of cyclones and anticyclones is exceedingly difficult, and some time must elapse before meteorologists will be able to complete the solution in a rigorous manner. This may be explained by resuming our study of the interchange of energy in the nonadiabatic circulation between high and low areas.[2] Equations (44) and (52), so far as they relate to the circulation in a horizontal plane xy, in the integrated form give the following:

$$C_p n_0 (T-T_0) + C_p\ T_0 \log T_0 (n-n_0) = (Q-Q_0) - \tfrac{1}{2} (q^2-q_0^2).$$

Since there is to be an interchange of energy between the cold area, whose center will be marked C, and the warm area whose center is W, the following notation will be employed:

n_0, the gradient ratio[3]
T_0, the temperature
q_0, the vector velocity
Q_0, the heat energy
} in the cold area, C.

n, the gradient ratio[3]
T, the temperature
q, the vector velocity
Q, the heat energy
} in the warm area, W.

The C and W areas lie between the centers of high and low pressure, marked H and L, respectively, in the order from west to east, as follows:

H (high); C (cold); L (low); W (warm);

as illustrated in the diagrams of papers No. I, II, III, and IV of this series. (MONTHLY WEATHER REVIEW, 1906, January, February, March, and June, respectively.)

(A) One problem is to show the relations between the thermodynamic centers C and W, and the hydrodynamic centers H and L in the moving atmosphere. It will not be proper to make model circulations by erecting chambers around given masses, and then removing certain internal partitions. This process really evades the entire problem to be solved, and substitutes some ideal or experimental system in place of that occurring in the atmosphere.

(B) Another problem is concerned with the gradient factors (n_0 and n) and the temperatures (T_0 and T), and may be stated in the following form. Since the gradients of temperature are changing from point to point in the vertical and in the horizontal directions in a very complex fashion, it seems impracticable to assign temperature functions in advance of the actual observations, and therefore analytic formulas of sufficient flexibility to express the entire existing conditions are impossible. If a simple function of the temperature is adopted, it is certain that this functon will not be applicable to the cyclonic structure taken as a whole, and hence it is very hard to derive the pressures from the temperatures by the simple quasi-adiabatic formulas.

(C) Furthermore, the most troublesome problem of all, in the present state of meteorology, is to show what is the relation between the velocity terms (q_0 and q) and the heat terms (Q_0 and Q). The cyclonic circulation constitutes an effort to bring back to equilibrium the energy-difference represented in the cold and warm areas, and this is done by setting up an extensive series of internal vortices, graduated in size from the large storm areas, down thru tornadoes or secondaries to the minute whirls that are not accessible to any instrumental records. In this interchange of heat between the warm and cold masses, a portion of the energy is absorbed in maintaining the velocity of the masses of air, a second portion goes into radiation, and a third part into equalizing the temperatures. The velocity of the wind in a cyclone does not measure the true velocities (q_0 and q), since the latter include the total internal circulation as well as the flow of the main stream; but there seems to be no way to separate these parts from one another. In a word the total energy is given by the terms

$$C_p n_0 (T-T_0) + C_p\ T_0 \log T_0 (n-n_0),$$

but I can as yet discover no method of distributing the respective portions of this total among the equivalent terms,

$$(Q-Q_0) - \tfrac{1}{2} (q^2-q_0^2) + \text{radiation}.$$

Until all these difficulties have been overcome it will be possible to make only tentative and incomplete discussions of the great problem involved in analytic meteorology.

(D) Finally, the general question as to the reason why the observed gradients of temperature differ from the adiabatic gradient is closely bound up with the distribution of the available energy between the q and Q terms. If a mass of air is moved from one level to another, as from 5000 meters to 4000 meters, in an adiabatic atmosphere, the pressure and the temperature change according to the adiabatic law; in a nonadiabatic atmosphere, the change of temperature does not correspond with the pressure, but a divergence exists depending on the proportion represented by the difference of the ratios $n - n_0$. If in a nonadiabatic atmosphere there is vertical displacement of an air mass, the interchange of energy is partly as heat and partly as velocity, and at the moment a mass moving adiabatically in the midst of a nonadiabatic mass arrives at such a displacement, $z - z_0$, as to be appreciable in respect to $n - n_0$, there is set up a small local interchange of energy between these masses in the form of a minor gyration of some sort. There is thus a continual tendency to balance these two expenditures of energy, the one against the other, in the most economical way, and the resultant temperature and circulation represents the outcome of this physical process. (See fig. 19.)

If instead of one rising current of warm air, $A\ C$, which becomes overcooled, and one current of cold air, $E\ W$, which becomes overheated by adiabatic expansion and contraction with the change of level, there are several such rising and falling masses in a series stretching from west to east, the interchange of heat becomes more complicated. Thus the cold mass C will be found between two masses of warm air W, and the warm mass between two cold masses on the same horizontal level. In this case each warm mass W will divide and seek $C\ C$ on either side of it; the cold mass will also divide and seek $W\ W$ on either side. Since these small horizontal currents can not flow together from opposite directions to the center because a congestion of mass would occur, the motion is transformed into an inflowing helix with vertical component upward for a low pressure center L, and a counterflowing helix with downward vertical component with a high pressure H at the center of the vortex. This process is the cause of the minor whirls in the atmosphere, and contributes something to the formation of cyclones and anticyclones. In the latter case the warm and cold masses are not produced by vertical adiabatic changes, but by transportation of horizontal currents from great distances. The same tendency to divide the warm mass in the northern quadrants between the low and high pressure centers and to curl the cold mass into two branches in the southern quadrants of the high and low pressure areas, has been already found in the observations of the stream lines and the distribution of the temperatures. The tendency to divide and

[1] This paper logically follows No. IV, in the Review for June, 1906, but its publication has been delayed.—EDITOR.

[2] See Monthly Weather Review, March, 1906, page 114.

[3] See Monthly Weather Review, March, 1906, page 113.

curl about the respective branches is common to all mixing masses of different temperatures.

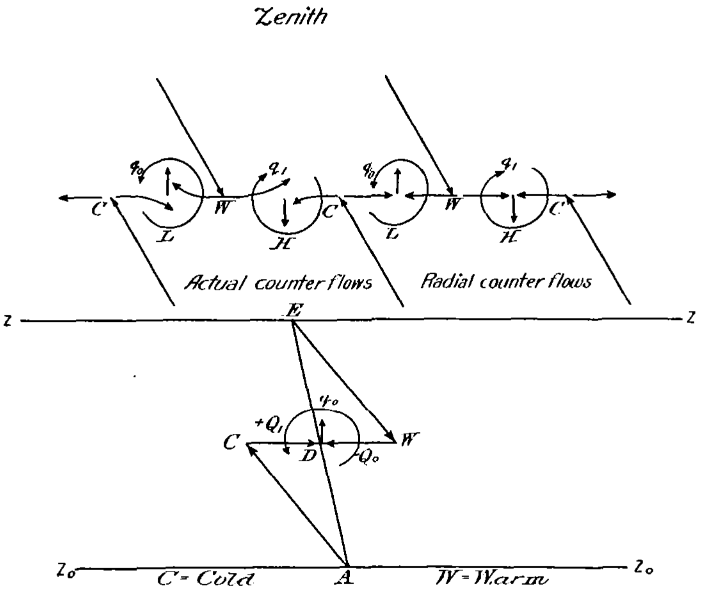

FIG. 19.—Scheme of the transformation of adiabatic gradients into observed temperature gradients thru the heat terms $(Q - Q_0)$ and velocity terms $\left[(q^2 - q_0^2)\right]$.

AE = Observed nonadiabatic gradient.
AC = Adiabatic gradient for warm rising air.
EW = Adiabatic gradient for cold descending air.
CD = Quantity of heat $+ Q_1$ to be added to restore the equilibrium at the height $z - z_0$.
WD = Quantity of heat $- Q_0$ to be lost in restoring the equilibrium.
At the level CDW other amounts of heat, $+ \Delta Q_1$, $- \Delta Q_0$, are expended in setting up a velocity q_0 which is converted into a vortex with a vertical component.

If we find an adiabatic rate of temperature fall in the Tropics such as 10.0° C. per 1000 meters, but one of 5.0° C. in the temperate zones, and of only 2.0° C. in the polar zones, then this distribution between the Tropics and the polar zones is maintained by circulation and heat interchange. The streams of warm air in the lower strata, 0 to 3000 meters, and in the upper strata, 10,000 to 14,000 meters, on moving from the Tropics to cooler latitudes, gradually lose heat by expending the energy thru a series of minor and major gyrations which are set up. These streams near the surface tend by their rising to higher levels, as they approach the polar zones, to stratify the warmer air higher up in proportion to their departure from the Tropics, and thus to lessen the temperature fall from the surface; likewise, above the 10,000-meter level the same phenomenon occurs. Similarly, the cold polar currents flowing toward the equator tend to sink to lower levels, and this diminishes the temperature gradient in the middle latitudes. These two systems of currents can not traverse the space between the Tropics and the polar zones without encountering one another, and interacting upon each other, in the cyclones and anticyclones, and the general effect of the entire process is to maintain a gradient of temperature which differs from the adiabatic rate. The divergence between the actual and the adiabatic rate is very different from place to place, as shown by the observations. There is an incessant turmoil of adjustment at all levels, and in all latitudes, whose outcome is the wind, clouds, rain, and temperature actually prevailing. As above stated, it seems to be impossible to treat this physical complex as an analytic unit in the present state of meteorology, and hence I shall confine my discussion to a series of more or less detached studies, which yet tend to elucidate the general problem.

THE HORIZONTAL CIRCULATION.

In Tables 29 and 30,[4] under the columns $z - z_0$, are given the vertical distances thru which the cold masses must fall and the warm masses rise, in order to attain an equilibrium on their respective levels. Thus, for the maximum cold masses in the east quadrant of the high area and the west quadrant of the low area, and for the maximum warm masses in the west quadrant of the high area and the east quadrant of the low area, we find the displacements in the winter, respectively, as follows:

TABLE 41.—*Vertical displacement, $z - z_0$, from equilibrium.*

Height in meters.	High east.	Low west.	Mean.	High west.	Low east.	Mean.
10000	+325	+454	+390	− 325	−487	−406
9000......	+366	+440	+403	− 352	−440	−396
8000......	+425	+467	+446	− 425	−382	−404
7000......	+498	+498	+498	− 569	−356	−463
6000	+592	+563	+578	− 740	−326	−533
5000......	+748	+650	+699	− 926	−325	−626
4000	+793	+646	+720	−1033	−333	−683
3000......	+854	+726	+790	−1024	−299	−662
2000......	+984	+777	+881	−1036	−311	−674
1000......	+836	+593	+715	− 890	−351	−621
0......	+511	+414	+463	− 487	−268	−378

The sign (+) means that the mass is too high by the given number of meters for thermodynamic equilibrium, and the sign (−) that the mass is too low. The cold masses can fall thru $z - z_0$ meters and the warm masses can rise thru $z - z_0$ meters on their respective levels, under the given conditions.

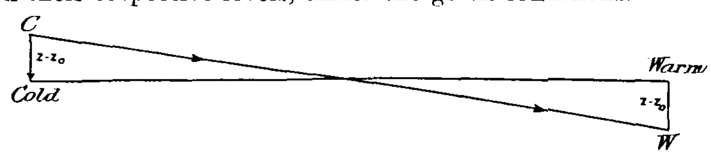

FIG. 20.—The conversion of vertical falls into horizontal circulation.

Thus at the 4000-meter level the cold mass can fall about 720 meters and the warm mass can rise 683 meters to bring about thermal equilibrium, *when there is no horizontal circulation.* If the cold air could sink to the level of the warm mass thermally, it would have a potential fall of 1403 meters, supposing this warm mass to remain unchanged in position and energy. The tendency is then for the cold mass in seeking the *lowest thermal level* not to fall vertically, but in the main to move almost horizontally down a gradient defined by CW. Assuming that the distance between the maxima C and W averages as in the ordinary cyclone about 1000 kilometers, or 1,000,000 meters we have a possible gradient,

$$G = \tan^{-1}\left(\frac{1403}{1,000,000}\right) = \tan^{-1}(0.001403) = 0° \ 5' \ 3''.$$

As this large gradient would give very rapid horizontal motions there is too much power to be expended in this simple manner. The warm mass is really rising and the cold mass falling simultaneously, not vertically but toward each other in the manner indicated by the diagrams, figs. 9, 10.[5] In these, and from other descriptions of the prevailing circulation found in the reports of the Weather Bureau, we infer that in all of the levels the cold current flows southeastward toward the warm area, while the warm current flows northwestward toward the cold area.

This flow is *not directly toward the respective centers* of the warm and cold waves, making the currents meet along an axis,

[4] See Monthly Weather Review for June, 1906, pp. 267–271.
[5] See Monthly Weather Review, February, 1906, pp. 77–78 and lithograph plate at the end.

because this would produce a congestion of the density and make the flow impossible. The system of internal reactions in the circulating fluid, in combination with the deflecting force due to the earth's rotation, will cause the stream lines to flow about the center up to a certain limited amount of congestion on the outer circles. It is evident that a compromise or resultant between these opposite tendencies must be brought about, and then the stream lines will approximate to spirals converging toward the center in the cyclone, but diverging in the anticyclone. In order to avoid the congestion, a vortex motion is thus established with an ascending component over all areas contained within the closed isobars of the cyclone, but descending in the anticyclone. The conflict of this localized circulation with the general circulation, the continuous absorption of the former by the latter, produces the entire observed cyclone system. Quite similar reasoning accounts for the downward component in the anticyclone, which is generated and fed from the other portions of the cold and warm areas, since it has been shown that both of these masses divide into two branches and are absorbed in consecutive high and low pressure areas.

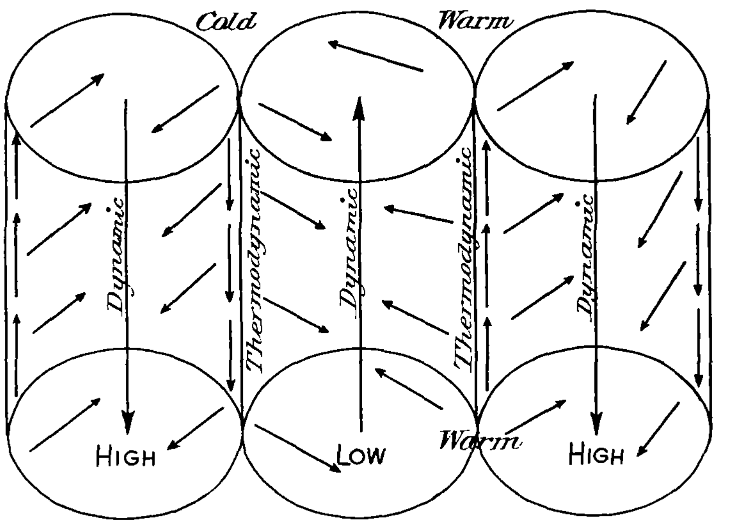

FIG. 21.—Scheme of the horizontal circulation in cyclones and anticyclones.

In the low area, in the strata from the surface to about 4000 meters, to the southward of the center, the cold mass tends to underrun the warm mass, while to the northward of the center in the strata above 4000 meters, the warm mass tends to overflow the cold mass. On the other hand, in the high pressure area, similar conditions exist tho the sectors or quadrants are inverted in their order. The cold air near the surface separates or divides into two branches, which tend to underrun the warm areas on either side, and in the high levels the warm air divides into two branches which tend to overflow the adjacent cold masses on either side.

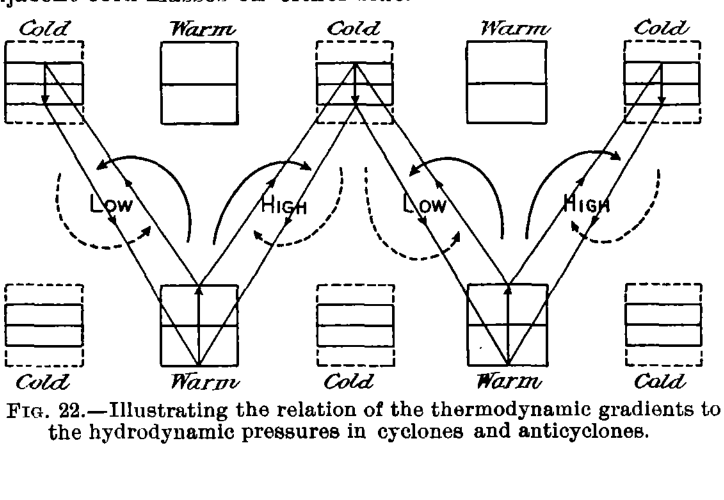

FIG. 22.—Illustrating the relation of the thermodynamic gradients to the hydrodynamic pressures in cyclones and anticyclones.

In the warm areas the isobars are farther apart than in the cold areas, and by the ordinary rules the circulations are in the directions indicated. The warm mass divides into two branches which overflow the cold masses to the north, while the cold mass divides into two branches which underrun the warm masses to the south. The outcome is to produce more stable equilibrium by superposing air of less potential density upon air of greater potential density. At the same time there is an interchange of heat and a manifestation of dynamic energy in the form of large and small vortices on the horizontal planes with dynamic components in the vertical directions. In this process there are involved: (1) an interchange of heat; (2) a more stable equilibrium, since gravity has pulled the air of great potential density downward, while that of lower potential density is pushed up; (3) an amount of kinetic energy corresponding to the movements of the air masses from one level surface to another; (4) important horizontal motions with minor vortex motions whose kinetic energy represents a large fraction of that mentioned in the preceding item.

THE HORIZONTAL PRESSURE GRADIENTS.

In order that the reason for this overflow of warm masses upon cold masses in the upper strata, with underflow of cold masses beneath warm masses in the lower strata may be evident, we need only compute the pressures B in the several strata of the warm and cold masses, respectively, from the surface up to 10,000 meters. Combine the temperatures given in Table 21[6] thus: Take the mean of the temperatures of the east sector of the high area and the west sector of the low area for the mean temperature in the cold mass, and the mean of the temperatures of the west sector of the high area and the east sector of the low area for the mean temperature of the warm mass, on each of the 1000-meter levels. The result will be found in Table 43, Section II, and is transferred to the first column of the cold and warm masses in Table 42, and marked t.

The mean t of the successive strata gives the mean temperature of the air column, $\theta = \frac{t_z + t_{z-1}}{2}$, in the second column. This is the argument for m in Table 91, International Cloud Report, and we may assume that the observed t is the virtual temperature, and that it includes the dry air and the vapor contents as they occur. With H the height and θ as arguments, the value of m is extracted. It is now necessary to assume some value of the pressure B at the surface in warm and cold areas, independent of any variation due to the circulation in the high and low areas, and I have taken two pressures, 10 millimeters different, as fairly representing known surface pressures under the prescribed conditions. Thus, for 770 millimeters in the cold mass, we shall have 760 millimeters in the warm mass, as the barometric pressure at the surface. Adopt these values, take log B, 2.88649 in the cold area, 2.88081 in the warm area, add successively the m on the several levels, and then take the corresponding B_c, B_w. Comparing B_c with B_w it is seen that the cold area pressure is greater than the warm area pressure up to 4000 meters, and that the warm area pressure is greater than the cold area pressure above that level. Hence, cold air flows to warm areas below, while warm air flows to cold areas above 4000 meters, conforming to well-recognized principles.

We can compute the vertical distance thru which 1 millimeter of air extends in the several levels. Take the difference between the pressures in the successive 1000-meter levels, $B - B_0$, the second difference, $\Delta (B - B_0)$, showing the variation with the height, then divide 1000 by $B - B_0$ for Δz, the required height in meters thru which 1 millimeter of air, that is the weight of air measured by 1 millimeter of mercury, extends. It changes from 11 meters near the surface to 31 meters near the 10,000-meter level, and shows the spaces that exist in a vertical direction between successive isobaric surfaces.

[6] Page 268, Monthly Weather Review, June, 1906.

Since the tendency of gravity is to make these spaces equal in the same stratum, a circulation is set up to bring this about; this is the flowing of the air which, thereupon, builds up the observed cyclones and anticyclones in combination with the other forces, inertia, expansion and contraction, deflection, centrifugal, friction, and internal vortical motion. This complex network of forces can be reduced to a rigid analytic discussion only with the greatest difficulty, even without the term involving the interchange of heat energy into velocity, and it seems nearly useless to attempt it until further experimental knowledge of this process in the free air has been obtained by a careful discussion of the temperature conditions observed in balloon and kite ascensions.

TABLE 42.—*Computation of the pressure B in the cold and warm maxima on each 1000-meter level.*

Height in meters.	In cold masses. t	θ	m	B_c	B_c	In warm masses. t	θ	m	B_w	B_w
	°C.	°C.		log.	mm.	°C.	°C.		log.	mm.
10000...	−56.6			2.28423	192.41	−51.7			2.29445	196.99
		−53.6	6796				−48.4	6595		
9000...	−50.5			2.35219	225.01	−45.0			2.36040	229.30
		−47.2	6561				−41.4	6397		
8000...	−43.8			2.41780	261.70	−37.8			2.42437	265.69
		−40.5	6371				−34.1	6200		
7000...	−37.1			2.48151	303.05	−30.4			2.48637	306.46
		−33.9	6195				−26.8	6016		
6000 ..	−30.6			2.54346	349.51	−23.1			2.54653	351.99
		−27.5	6033				−19.7	5849		
5000...	−24.3			2.60379	401.60	−16.2			2.60502	402.74
		−21.3	5885				−13.4	5705		
4000..	−18.2			2.66264	459.86	−10.6			2.66207	459.27
		−15.5	5752				− 8.3	5595		
3000...	−12.7			2.72016	525.00	− 5.9			2.71802	522.42
		−10.2	5636				− 3.7	5499		
2000...	− 7.6			2.77652	597.75	− 1.6			2.77301	592.94
		− 5.5	5536				0.0	5425		
1000...	− 3.3			2.83188	679.02	+ 1.7			2.82726	671.83
		− 0.8	5461				+ 3.5	5355		
0...	+ 1.7			2.88649	770.00	+ 5.2			2.88081	760.00

Vertical distance for 1 mm. of pressure between strata of different temperature.

Height.	$B-B_0$	$\Delta(B-B_0)$	$(B-B_0)$	Δ_z	Δ_z	$B-B_0$	$\Delta(B-B_0)$	$(B-B_0)$	Δ_z	Δ_z
	mm.	mm.	log.	log.	mm.	mm.	mm.	log.	log.	mm.
10000...										
	32.60		1.51322	1.48678	30.68	32.31		1.50934	1.49066	30.95
9000...		4.09					4.08			
	36.69		1.56455	1.43545	27.25	36.39		1.56098	1.43902	27.48
8000 ..		4.66					4.38			
	41.35		1.61648	1.38352	24.18	40.77		1.61034	1.38966	24.53
7000...		5.11					4.76			
	46.46		1.66708	1.33291	21.52	45.53		1.65830	1.34170	21.96
6000...		5.63					5.22			
	52.09		1.71675	1.28325	19.20	50.75		1.70544	1.29456	19.70
5000...		6.17					5.78			
	58.26		1.76537	1.23463	17.16	56.53		1.75228	1.24772	17.69
4000...		7.08					6.62			
	65.14		1.81385	1.18615	15.35	63.15		1.80037	1.19963	15.84
3000...		7 61					7.37			
	72.75		1.86183	1.13817	13.75	70.52		1.84831	1.15169	14.18
2000...		8.52					8.37			
	81.27		1.90993	1.09007	12.30	78.89		1.89702	1.10298	12.68
1000...		9.71					9.28			
	90.98		1.95895	1.04105	10.99	88.17		1.94532	1.05468	11.34
0...										

THE HORIZONTAL INTERCHANGE OF HEAT ENERGY.

We can secure some idea of the process involved in the interchange of the heat energy on the horizontal surfaces by a computation of the formula:

Term I Term II

$$C_p\, n_0\, (T_1 - T) + C_p\, T_0 \log T_0\, (n_1 - n) = (Q_1 - Q) - \tfrac{1}{2}\, (q_1^2 - q^2).$$

The necessary data are collected in Table 43, and they are gathered in the same way as described for the temperatures, by combining the sectors of cold and of warm masses, respectively. The mean value of the gradient ratio n is found by extracting n from Tables 25 and 26, and taking the means, n for cold areas and n_1 for warm areas. Then the difference, $n_1 - n$, and the mean, $n_0 = \frac{1}{2}\,(n + n_1)$, are taken out for use in the formula. We adopt the notation (n, t, q, Q) for the cold mass, (n_1, t_1, q_1, Q_1) for the warm mass, and (n_0, t_0, q_0, Q_0) the mean values of the cold and warm masses when required.

TABLE 43.

I.—*Mean values of the gradient ratio n in the cold and warm maxima.*

Height in meters	Ratio. n High east.	Low west.	Mean cold.	Ratio. n_1 High west.	Low east.	Mean warm.	n_1-n W—C.	Mean n_0
10000......	1.778	1.659	1.718	1.535	1.547	1.541	−.177	1.630
9000......	1.537	1.500	1.518	1.319	1.458	1.389	−.129	1.454
8000......	1.495	1.443	1.469	1.246	1.447	1.347	−.122	1.408
7000......	1.523	1.447	1.485	1.234	1.471	1.353	−.132	1.419
6000......	1.567	1.498	1.533	1.272	1.518	1.395	−.138	1.464
5000	1.623	1.562	1.593	1.430	1.629	1.530	−.063	1.562
4000......	1.725	1.690	1.708	1.974	1.766	1.870	+.162	1.789
3000......	1.894	1.876	1.885	2.443	1.974	2.209	+.324	2.047
2000......	2.285	2.150	2.218	3.056	2.518	2.787	+.569	2.503
1000......	2.179	2.213	2.196	3.439	2.611	3.025	+.829	2.611
000......	1.769	2.065	1.917	3.290	2.367	2.829	+.912	2.373

II.—*Mean values of the temperature T in the cold and warm maxima.*

Height in meters.	Temperature. t High east.	Low west.	Mean cold.	Temperature. t_1 High west.	Low east.	Mean warm.	t_1-t W-C.	Mean T_0 $t_0+273°$	Log. T_0
	°C.	°C.	°C.	°C.	°C.	°C.	°C.	Abs.	
10000......	−56.2	−57.0	−56.6	−52.2	−51.2	−51.7	+4.9	218.8	2.34005
9000	−50.2	−50.7	−50.5	−45.3	−44.7	−45.0	+5.5	225.2	2.35257
8000......	−43.6	−43.9	−43.8	−37.6	−37.9	−37.8	+6.0	232.2	2.36586
7000......	−37.1	−37.1	−37.1	−29.6	−31.1	−30.4	+6.7	239.2	2.37876
6000	−30.7	−30.5	−30.6	−21.7	−24.5	−23.1	+7.5	246.1	2.39111
5000......	−24.6	−24.0	−24.3	−14.3	−18.0	−16.2	+8.1	152.7	2.40261
4000......	−18.6	−17.8	−18.2	− 8.7	−12.5	−10.6	+7.6	258.6	2.41263
3000......	−13.0	−12.4	−12.7	− 4.2	− 7.6	− 5.9	+6.8	263.7	2.42111
2000......	− 8.0	− 7.2	− 7.6	− 0.2	− 3.0	− 1.6	+6.0	268.4	2.42878
1000......	− 3.7	− 2.8	− 3.3	+ 2.7	+ 0.7	+ 1.7	+5.0	272.2	2.43489
000......	+ 1.5	+ 1.9	+ 1.7	+ 5.6	+ 4.7	+ 5.2	+3.5	276.5	2.44170

III.—*Mean values of the velocity term in the cold and warm maxima.*

Height in meters.	Velocity. $\frac{1}{2}(q_1^2-q^2)$ High west.	Low east.	Mean warm.	Velocity. $\frac{1}{2}(q_1^2-q^2)$ High east.	Low west.	Mean cold.	Average.
10000	+54	+175	+115	−127	− 95	−111	113
9000....	+56	+216	+136	−122	−122	−122	129
8000......	+72	+228	+150	−101	−132	−117	134
7000......	+72	+220	+146	− 91	−121	−106	126
6000......	+65	+160	+113	− 83	− 83	− 83	98
5000......	+88	+ 88	+ 88	− 74	− 50	− 62	75
4000......	+72	+ 72	+ 72	− 56	− 32	− 44	58
3000......	+60	+ 86	+ 73	− 58	− 14	− 36	55
2000	+10	+ 52	+ 31	− 28	− 28	− 28	30
1000......	+ 3	+ 19	+ 11	− 12	− 25	− 19	15
000......	+ 8	+ 18	+ 13	− 8	− 8	− 8	11

These data are given in Section I of Table 43; the temperature data in Section II of that table are taken from Tables 21 and 22; T_0 and log T_0 are computed; finally, $\frac{1}{2}\,(q_1^2 - q^2)$ are taken from Tables 33 and 34. Since the velocity energy is a small term in comparison with $(Q_1 - Q)$, there is no need to be particular about the exact velocities, and approximate values are sufficient. In order to learn the relation between the values of the ratio n, n_1 in cold and warm areas in the several strata, they are plotted in fig. 23. It is seen that the curves cross each other between the 4000 and the 5000-meter level, showing that there is a reversal of the physical process at that elevation, as warming below and cooling above, so that the cold mass is warming below and the warm mass is cooling above in conformity with the preceding statements. Since the adiabatic gradient is $-9.87°$ C. per 1000 meters, and $a = \frac{a_0}{n}$, we find the gradients corresponding with n at the several levels by using the lower horizontal argument in the diagram.

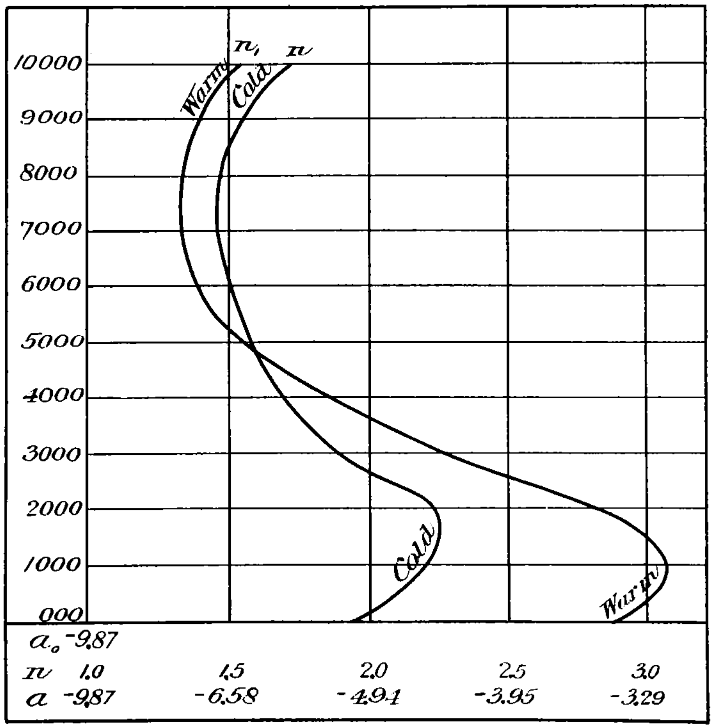

FIG. 23.—Mean values of the gradient ratio, n, at the cold and warm maxima.

The computation of the terms $\mathrm{I} = C_p\, n_0\, (T_1 - T)$ and $\mathrm{II} = C_p T_0 \log T_0\, (n_1 - n)$ gives the results that are found in Table 44, for the several 1000-meter levels. Term I is positive for all levels, and term II reverses the sign at about the 5000-meter level. The sum I + II is reduced to calories by the factor $A_m = 0.0002389$ in Table 14.[7] In the last column of Table 44, a mean value of $\frac{1}{2}\,(q_1^2 - q_0^2)$ is added as computed by Section III, Table 43. A comparison of columns 4 and 6 shows how small the velocity term is in comparison with the heat term. An unknown (R) is added in the formula to represent the waste of energy in passing thru friction into motion. It stands between the energy and velocity terms, but can not be evaluated, and it is presupposed in the unexprest function that connects heat with motion. In the same way there is the unknown radiation term, J, wherein some heat energy is wasted so far as the motion of the atmosphere is concerned. The function uniting $(Q_1 - Q) - \frac{1}{2}(q_1^2 - q_0^2) + (R) + (J)$ being undetermined, it is very difficult to make satisfactory progress in this direction, and the problem must wait for further developments. Reviewing columns I + II in calories, which is the heat energy available from the temperature distribution, it is seen that it is positive and diminishes up to the 5000-meter level, above which it is small and negative. Comparing this column with Tables 37 and 38 it is observed that the vertical heat potentiality is about the same as the horizontal capacity for motion. If a kilogram of air is moved as noted by the conditions of the problem, this amount of heat must be interchanged. In the actual atmosphere this transfer is not so simple, and hence only a portion of the Q-energy is actually produced. How much less is really generated depends upon the efficiency of the thermodynamic engine in the practical physical operations of the air.

[7] See Monthly Weather Review, March, 1906, page 115.

TABLE 44.—*Values of the terms in the formula.*

$$\overset{\mathrm{I}}{C_p\, n_0\, (T_1 - T)} + \overset{\mathrm{II}}{C_p\, T_0 \log T_0\, (n_1 - n)}$$
$$= (Q_1 - Q) - \tfrac{1}{2}\,(q_1^2 - q^2) + R + J.$$

Energy terms in the horizontal convection.

Height in meters.	I	II	I+II	I+II in calories.	$\frac{1}{2}\,(q_1^2 - q^2)$
10000......	7936	—90042	—82106	— 19.6	113
9000	7946	—67905	—59959	— 14.3	129
8000......	8394	—66587	—58193	— 13.9	134
7000......	9443	—74624	—65181	— 15.6	126
6000......	10909	— 80684	—69775	— 16.7	98
5000......	12571	—38004	—25433	— 6.1	75
4000......	13509	100427	113936	+ 27.2	58
3000......	13830	205529	219359	+ 52.4	55
2000......	14921	368533	383454	+ 91.6	30
1000......	12971	545913	558884	+133.5	15
0......	8252	611771	620023	+148.1	11

SOME CASES OF RESTRICTED CONDITIONS.

In order to approach this intricate problem by a mathematical analysis, it will be desirable to study some simpler cases, or models, wherein the conditions are limited by ideal restrictions. These consist in placing two masses of air in adjoining chambers, or in one chamber with a movable partition, whereby two fixed masses under given conditions when set into communication react upon each other. Dr. M. Margules has made several such studies in his paper, Über die Energie der Stürme, and for the sake of profiting by this excellent work, I have prepared a brief synopsis of the results as *modified by myself to meet nonadiabatic conditions.* It is proposed to give the assumed data and the resulting formula, omitting the algebraic reductions, and to urge that the student should not fail to read that paper. In order to preserve the notation of my formula, the following table of equivalents will be useful:

	Margules.	*Bigelow.*
External kinetic energy	K to	$(K) = \frac{1}{2}\int \rho\, q^2\, dv = \frac{1}{2}\, m\, q^2.$
External potential energy	P to	$V = \int \rho\, (-g\, r + \frac{1}{2}\, w_0^2\, \varpi^2)\, dv.$
Internal kinetic energy Internal potential energy	I to	$U = \left\{ \begin{matrix} H_m\ (\text{molecules}) + H_a\ (\text{atoms}) \\ J_m\ (\text{molecules}) + J_a\ (\text{atoms}) \end{matrix} \right\} = C_v \int T \rho\, dv.$
Quantity of heat		$Q = \int dt \int \frac{dQ}{dt}\, \rho\, dv.$
Work of expansion	A to	$-\mathrm{W} = -\int dt \int \frac{p}{\rho}\, \frac{d\rho}{dt}\, dv = \int dt \int p\, v\, \frac{d\frac{1}{v}}{dt}\, dv.$

	Margules.		*Bigelow.*
Potential energy + centrifugal force	W	to	$V_1 = -g r + \frac{1}{2} w_0^2 \varpi^2$.
Friction			$(R) = -\int d t \int R q \cos (R q) \rho d v$.
Velocity	c, V	to	q
Volume	k	to	v
Density	μ	to	ρ
Ratio of specific heats	γ	to	$k = \frac{C_p}{C_v}$.
Adiabatic constant	$\frac{1}{x}$	to	$\frac{k}{k-1} = \frac{C_p}{R} = \frac{g_0}{R a_0}$.
Height	c	to	h
Surface	O	to	S
Entropy temperature	θ	to	T_0
Potential temperature	τ	to	T_0
Drive temperature	ϑ	to	T_0

GENERAL THERMODYNAMIC EQUATIONS.

(1) Conservation of energy.

$$\begin{cases} \partial (K) + \partial V - \partial W + (R) = 0. \\ Q = \partial U + \partial W + (R) = \partial (K) + \partial V + \partial U + (R). \\ Q = [\partial (K) + \partial V] \text{ external} + [\partial H + \partial J] \text{ internal} = \partial W + \partial U. \end{cases}$$

(2) Variation of heat.

$$\begin{array}{lll} & \text{External work.} & \text{Internal heat.} \\ d Q = & R T \frac{d v}{v} & + \frac{1}{A} C_v d T. \\ d Q = & -R T \frac{d p}{p} & + \frac{1}{A} C_p d T \\ d Q = & p d v & + \frac{v d p + p d v}{A R}. \end{array} \quad \text{in mechanical units.}$$

(3) External potential energy.

$$V = \int_0^\infty g z \rho d z = \int_0^h g z \rho d z + g z M_h. \qquad p_h = g M_h = g \rho h.$$

$$V = \quad = -\int_p^{p_h} z d p + Z p_h = + \int_0^h p d z - z p_h + Z p_h.$$

$$V = \int_0^h p d z + (Z - z) p_h = R \int T d m + \text{const.}$$

(4) Internal energy. $U = C_v \int T d m + \text{const.}$

$$(U + V) = (C_v + R) \int T d m + \text{ const.} = C_p \int T d m + \text{const.}$$

(5) Transformation of energy.

$$\begin{cases} -\partial (U + V) = (U + V)_a (\text{initial}) - (U + V)_e (\text{final}) = C_p \int (T - T^1) d m. \\ \partial (K) + (R) = \frac{1}{2} M q^2 = C_p \int (T - T^1) d m = C_p (T - T^1) M. \end{cases}$$

(6) Entropy variations.

$$\begin{cases} S - S_0 = \int \frac{d Q}{T} = C_v \log \frac{T}{T_0} + R \log \frac{v}{v_0}. \\ S - S_0 = \int \frac{d Q}{T} = C_p \log \frac{T}{T_0} - R \log \frac{p}{p_0}. \\ \frac{\partial S}{\partial z} = \frac{1}{T} \frac{\partial Q}{\partial z} = \frac{C_p}{T} \frac{\partial T}{\partial z} - \frac{R}{p} \frac{\partial p}{\partial z}. \end{cases}$$

(7) Potential temperature.

$$T_0 = T \left(\frac{P_0}{P} \right)^{\frac{k-1}{k}}.$$

(8) In linear vertical changes.

$$\int_0^h T d m = \frac{1}{g} \frac{1}{1 + \frac{k-1}{n k}} (p_0 T_0 - p T).$$

(9) Auxiliary equations.

$$\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{\frac{n}{k-1}} = \left(\frac{P}{P_0}\right)^{\frac{1}{k}} = \frac{v_0}{v}.$$

$$\frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{\frac{nk}{k-1}} = \left(\frac{\rho}{\rho_0}\right)^{k} = \left(\frac{v_0}{v}\right)^{k}.$$

Adiabatic. Observed.

$$\frac{k}{k-1} = \frac{C_p}{R} = \frac{g}{Ra_0}. \qquad \frac{nk}{k-1} = \frac{nC_p}{R} = \frac{g}{Ra}. \qquad a = \frac{a_0}{n} = \frac{g}{nC_p}.$$

$$\frac{1}{\rho} = \frac{1}{P} R\,T. \qquad \frac{1}{\rho}\frac{dP}{dz} = -\,g. \qquad \frac{1}{P}\frac{dP}{dz} = -\frac{1}{RT}\,g.$$

CASE I. CHANGE OF POSITION OF THE LAYERS IN A COLUMN OF AIR.

In consequence of the general and local circulations of the atmosphere, a certain gradient $a = \frac{a_0}{n}$ prevails at a given locality in a column above the earth's surface. This requires an amount of heat Q_0 and a temperature T_0 at each level z_0 to maintain the stratum in equilibrium. If the heat energy changes to Q for any reason or the temperature is altered to T there must follow a change in elevation to z to restore the equilibrium. The equation of equilibrium,

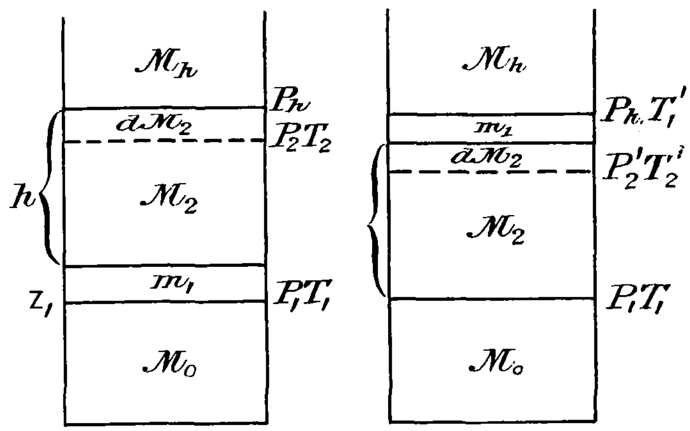

FIG. 24 A. Initial. Final.

$$(10)\quad \tfrac{1}{2}(q^2-q_0^2) = (Q-Q_0) - C_p n\,(T-T_0) - C_p T_0 \log T_0\,(n-n_0) - g\,(z-z_0),$$

is available for the computation of the motion due to stratifications in the column. In order to take a simple case we assume that each air mass retains its own heat energy or $Q = Q_0$, and that the gradient is the same thruout the column or $n = n_0$. Hence when starting from rest or $q = 0$, the equation becomes for the unit mass.

$$(11)\qquad \tfrac{1}{2}\,q^2 = -C_p\,n\,(T-T_0) - g\,(z-z_0).$$

This must be applied to each mass moved, so that finally

$$(12)\qquad \tfrac{1}{2}\,m\,q^2 = \sum\Big[-C_p\,n\,(T-T_0) - g\,(z-z_0)\Big]\,m.$$

Let the column be separated from the surrounding air by walls and consist of four parts. M_0 is a lower section not affected by the transfer; the next layer m_1, under pressure P_1 and temperature T_1, is not in equilibrium, so that the stratified layer m_1 must rise if T_1 is too warm and fall if T_1 is too cold for its elevation z_1. If it rises thru a height $h = z_2 - z_1$, and by expanding cools to a given temperature T_1^1, the pressure P_1 will become P_h and be in equilibrium; the section M_2 of thickness h falls a certain distance and changes its temperature; for the upper differential layer $d\,M_2$ the initial values $P_2\,T_2$, become P_2^1, T_2^1, and the function must be integrated thruout the mass M_2; the temperature of the mass M_h is not affected by the mutual transfer of $m_1\ M_2$, but rises or falls like a piston in the chamber, while its lower surface maintains the pressure P_h. Hence, we have the conditions,

Layer.	Initial.	Final.	Pressure.
$d\,M$	$P_2\ T_2$	$P_2^1\ T_2^1$	$P_2^1 = P_2 + g\,m_1$
m_1	$P_1\ T_1$	$P_h\ T_1^1$	$P_1 = P_h + g\,M_2$

$$(13)\quad T_2^1 = T_2\left(\frac{P_2^1}{P_2}\right)^{\frac{k-1}{nk}} = T_2\left(1 + \frac{g\,m_1}{P_2}\right)^{\frac{k-1}{nk}} = T_2\left(1 + \frac{k-1}{n\,k}\frac{g\,m_1}{P_2}\right)$$

$$= T_2 + T_2\,\frac{R}{n\,C_p}\,\frac{g\,m_1}{P_2}.$$

$$(14)\quad T_1^1 = T_1\left(\frac{P_h}{P_1}\right)^{\frac{k-1}{nk}}.$$

Substituting in the equation,

$$(15)\quad \text{Kinetic energy} = C_p\Big[\int (T_1 - T_1^1)\,d\,m_1 + \int (T_2 - T_2^1)\,d\,M_2\Big].$$

$$(16)\qquad \tfrac{1}{2}\,m_1\,q^2 = C_p\left[T_1 - T_1\left(\frac{P_h}{P_1}\right)^{\frac{k-1}{nk}}.\right]m_1 - g\,\frac{h}{n}\,m_1,$$

since

$$(17)\qquad \frac{RT_2}{P_2}\int\frac{d\,M_2}{n} = \int\frac{d\,M_2}{n\rho_2} = \int\frac{d\,z}{n} = \frac{h}{n}.$$

The gravity terms in these equations disappear, because the mechanical work in each case, $g\,h\,M_1$ and $g\,(Z_2 - Z^1_2)\,M_2$ (where Z_2 is the height of the center of gravity of M_2) is of the same amount and oppositely directed. Every expansion or contraction of air masses begins on an adiabatic gradient, and hence the formulas must be founded on that basis. But minor interchanges of energy as heat Q and velocity $\frac{1}{2}q^2$ almost immediately begin in the mixing process, so that the theoretical conditions soon suffer modifications which it is quite impracticable to follow out.

CASE II. THE TEMPERATURE IS A CONTINUOUS FUNCTION OF THE HEIGHT, $T_2 = T_1 - a\,h$.

It is important to eliminate the pressures from the formula and express the function in terms of g, h, T, and the gradients. Several forms of the function for the temperature distribution may be employed to represent the atmosphere, but it is only occasionally that these formulas can be used to replace the actual pressure and temperature observations at different levels. For the observed gradient we have

$$(18)\quad \text{Observed gradient.}\ \Big\}\ \frac{P_h}{P_1} = \left(\frac{T_2}{T_1}\right)^{g/Ra} = \left(\frac{T_1 - ah}{T_1}\right)^{g/Ra}.$$

Hence,

$$(19)\quad \left.\begin{array}{r}\text{Adiabatic}\\ \text{gradient.}\end{array}\right\} \left(\frac{P_h}{P_1}\right)^{\frac{k-1}{k}} = \left(1 - \frac{ah}{T_1}\right)^{g/_{Ra}\frac{k-1}{k}} = \left(1-\frac{ah}{T_1}\right)^{g/C_p a}$$

Then,

$$(20)\quad C_p m_1 (T_1 - T_1^1) = C_p m_1 \left(T_1 - T_1 + \frac{gh}{C_p} - \tfrac{1}{2}\frac{g^2h^2}{C_p^2 T_1} + \frac{\tfrac{1}{2}gh^2a}{C_pT_1}\right).$$

Finally,

$$(21)\quad \tfrac{1}{2} m_1 q^2 = g h m_1 - \tfrac{1}{2}\frac{g h^2}{C_p T_1}\cdot\frac{g}{C_p} + \tfrac{1}{2}\frac{g h^2}{C_p T_1}\cdot a - g h m_1$$

$$= \tfrac{1}{2}\frac{g h^2}{T_1} m_1 \left(a - \frac{g}{C_p}\right),$$

$$= \tfrac{1}{2}\frac{g h^2}{T_1} m_1 \left(\frac{a_0}{n} - a_0\right).$$

The mass m_1 is driven from its position with a velocity-energy inversely proportional to the temperature, so that warm air has less driving power than cold air. The drive depends upon the departure-ratio n and vanishes when $n=1$, that is, for an adiabatic expansion in an adiabatic gradient. When $a > a_0$ the mass m_1 is in unstable equilibrium—is too cold for its position and tends to fall. Example, for $n=0.5$, $a=19.74 > a_0 = 9.87$. When $a < a_0$ the mass m_1 is in stable equilibrium. Example, for $n=2$, $a=4.94 < a_0 = 9.87$ It is not possible to drive the small mass m_1 thru any great height h in the atmosphere, because the differential energy in the expanding mass sets up minor whirls which tend to interchange the Q-energy by mechanical effects and internal friction.

The result is to change the gradient from a_0 to $a = \frac{a_0}{n}$. If the displacement of the mass m_1 takes place in the medium of gradient a then the drive may be exprest by terms of the form,

$$(22)\quad \tfrac{1}{2} g \frac{h^2}{T_1} m_1 \left(\frac{a_0}{n_1} - \frac{a_0}{n}\right) = \tfrac{1}{2}\frac{gh^2}{T_1} a_0 \left(\frac{n - n_1}{n n_1}\right),$$

where n_1 is the effective ratio of the moving mass m_1 and a that of the prevailing general gradient.

CASE III. FOR LOCAL CHANGES BETWEEN TWO ADJACENT STRATA OF DIFFERENT TEMPERATURES, WHERE ON THE BOUNDARY THE PRESSURE $P = P_1^1 = P_2^1$, AND THE TEMPERATURE IS DISCONTINUOUS.

Take the following conditions:

Layer.	Initial.	Final.	Pressure.	Temperature.
m_2	$P_2\, T_2$	$P_2^1\, T_2^1$	$P_2^1 = P_2 + g m_1$	$T_2^1 = T_2 \left(\frac{P_2 + g m_1}{P_2}\right)^{\frac{k-1}{nk}}$
m_1	$P_1\, T_1$	$P_1^1\, T_1^1$	$P_1^1 = P_1 - g m_2$	$T_1^1 = T_1 \left(\frac{P_1 - g m_2}{P_1}\right)^{\frac{k-1}{nk}}$

The equation of equilibrium becomes, for $P^1_1 = P^1_2 = P$,

$$(23)\quad \begin{array}{l}\text{Kinetic}\\ \text{energy}\end{array} = C_p\Big[m_1(T_1 - T^1_1) + m_2(T_2 - T^1_2)\Big],$$

$$= C_p\Big[m_1\Big(T_1 - T_1 + T_1 \frac{R}{n\,C_p}\frac{g\,m_2}{P}\Big) + m_2\Big(T_2 - T_2 - T_2\frac{R}{n\,C_p}\frac{g\,m_1}{P}\Big)\Big],$$

$$= m_1 m_2 \frac{R g}{n P}(T_1 - T_2),$$

$$= m_1 m_2 \frac{g}{n}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right).$$

Since $\frac{R\,T_1}{P_1} = \frac{1}{\rho_1}$ and $\frac{R\,T_2}{P_2} = \frac{1}{\rho_2}$, therefore

$$(24)\quad \tfrac{1}{2} M q^2 = m_1 m_2 \frac{g}{n}\frac{\rho_2 - \rho_1}{\rho_1 \rho_2}.$$

The kinetic energy inducing an interchange is proportional to the difference of the densities and inversely proportional to the product of the densities. Hence, if strata of different densities are flowing over one another in the general circulation which is temporarily stratified, these two strata tend to mix by interpenetration according to this law.

AUXILIARY THEOREM. EVALUATION OF $\int T\,dm$ IN LINEAR VERTICAL TEMPERATURE CHANGES.

$$(25)\qquad \text{Assume } T = T_0 - a z,\qquad P = P_0\left(\frac{T}{T_0}\right)^{g/_{Ra}},\qquad \int T\,dm = \int T\rho\,dz.$$

$$\int_0^z T\rho\,dz = \frac{1}{R}\int_0^z P\,dz = \frac{1}{R}\int_0^z P_0\left(\frac{T}{T_0}\right)^{g/_{Ra}} dz = \frac{1}{R}\int_0^z P_0\, T^{g/_{Ra}}\, T_0^{-g/_{Ra}}\, dz$$

Change the limits of integration from z to T.

$$(26)\qquad T = T_0 - a z,\quad dT = -a\,dz,\quad -\frac{1}{a}\,dT = dz,\quad \int_0^z T^x\,dz = -\frac{1}{a}\int_{T^0}^{T} T^x\,dT.$$

$$(27)\qquad \int_0^z T\rho\,dz = \frac{1}{Ra} P_0 T_0^{-g/_{Ra}} \int_T^{T_0} T^{g/_{Ra}}\,dT = \frac{1}{Ra} P_0 T_0^{-g/_{Ra}} \left[\frac{1}{1+g/Ra} T^{g/_{Ra}+1}\right]_T^{T_0}$$

$$= \frac{1}{g + Ra} P_0 T_0^{-g/_{Ra}} \left(T_0^{g/_{Ra}+1} - T^{g/_{Ra}+1}\right) = \frac{1}{g + Ra}(P_0 T_0 - P\,T).$$

For any gradient other than the adiabatic we have,

$$(28)\qquad \int_0^z T\rho\,dz = \frac{1}{g}\,\frac{1}{1 + \frac{k-1}{nk}}\,(P_0 T_0 - P\,T).$$

CASE IV. THE OVERTURN OF DEEP STRATA IN THE COLUMN.

Let the pressures, temperatures, and heights be arranged in the initial and final states as indicated in the diagrams (fig. 24 B). The greatest entropy in 1 is less than the least in 2, so that the cold mass 1 will fall beneath the warm mass 2. The heights of the masses will change as well as the pressures and temperatures.

Assume P_0, T_{02}, h_2, T_{i1}, h_1, as known in the initial state.

Pressures.	Temperatures.
(29) $P_i = P_0 \left(\frac{T_{i2}}{T_{02}}\right)^{\frac{nk}{k-1}}$.	$T_{i2} = T_{02} - \frac{g h_2}{n C_p}$.
(30) $P_h = P_i \left(\frac{T_{h1}}{T_{i1}}\right)^{\frac{nk}{k-1}}$.	$T_{h1} = T_{i1} - \frac{g h_1}{n C_p}$

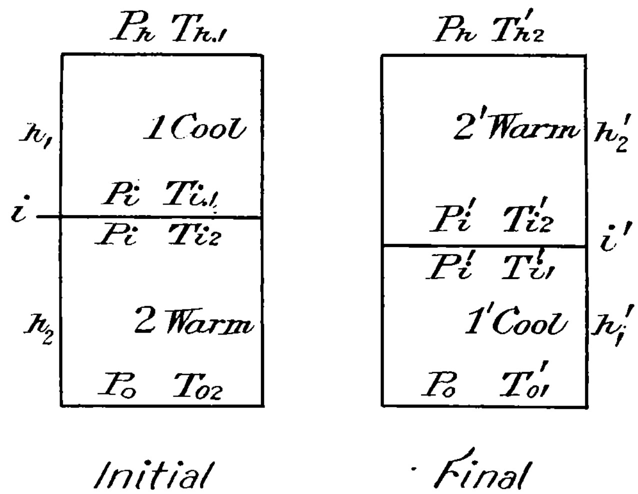

FIG. 24 B.

Substitute in $C_p\left(\int T\,dm - \int T_1\,dm_1\right)$ successively.

(31) Initial, $(V+U)_a = C_p\int T\,dm = \frac{C_p}{g}\frac{1}{1+\frac{k-1}{nk}}(P_0 T_{02} - P_i T_{i2} + P_i T_{i1} - P_h T_{h1}) + \text{const.}$

(32) Final, $(V+U)_e = \frac{C_p}{g}\frac{1}{1+\frac{k-1}{nk}}(P_0 T_{01}^1 - P_i^1 T_{i1}^1 + P_i^1 T_{i2}^1 - P_h T_{h2}^1) + \text{const.}$

(33) Kinetic energy $= (V+U)_a - (V+U)_e = \frac{1}{2} M q^2 = \frac{1}{2} \frac{P_0 - P_h}{g} q^2$.

(34) Heights, $h_1^1 = \frac{n C_p}{g}(T_{01}^1 - T_{i1}^1)$, $h_2^1 = \frac{n C_p}{g}(T_{i2}^1 - T_{h2}^1)$.

(35) Approximate solution of Case IV. $\frac{1}{2} q^2 = \frac{g}{n}\frac{h_1 h_2 (T_{i2} - T_{i1})}{h_1 T_{i2} + h_2 T_{i1}}$.

CASE V. TRANSFORMATION OF TWO MASSES OF DIFFERENT TEMPERATURES ON THE SAME LEVEL INTO A STATE OF EQUILIBRIUM.

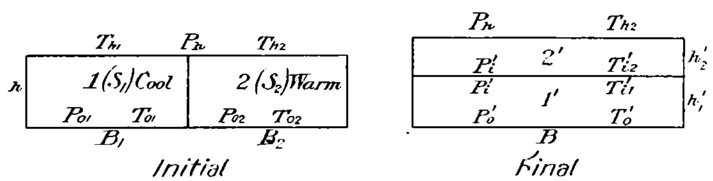

FIG. 24, C.

Given as data at the height h, T_{h_1}, T_{h_2} P_h, the areas B_1, B_2, the entropy $S_1 < S_2$. Hence by the formulas,

(36) $P_{01} = P_h\left(\frac{T_{01}}{T_{h_1}}\right)^{\frac{nk}{k-1}} = P_h\left(1 + \frac{g h}{n C_p T_{h_1}}\right)^{\frac{nk}{k-1}}$. $T_{01} = T_{h_1}\left(1 + \frac{g h}{n C_p T_{h_1}}\right)$.

(37) $P_{02} = P_h\left(\frac{T_{02}}{T_{h_2}}\right)^{\frac{nk}{k-1}} = P_h\left(1 + \frac{g h}{n C_p T_{h_2}}\right)^{\frac{nk}{k-1}}$. $T_{02} = T_{h_2}\left(1 + \frac{g h}{n C_p T_{h_2}}\right)$.

(38) Initial. $(V+U)_a = C_p \frac{1}{g}\frac{1}{1+\frac{k-1}{nk}}\frac{B}{2}(P_{01} T_{01} - P_h T_{h_1} + P_{02} T_{02} - P_h T_{h_2}) + \text{const.}$

(39) $P^1_i = P_h + \frac{1}{2}(P_{02} - P_h)$. $P^1_0 = P_h + \frac{1}{2}(P_{02} - P_h) + \frac{1}{2}(P_{01} - P_h)$.

(40) Final. $(V+U)_e = C_p \frac{1}{g}\frac{1}{1+\frac{k-1}{nk}} B (P^1_0 T^1_0 - P^1_i T^1_{i1} + P^1_i T^1_{i2} - P_h T_{h2}) + \text{const.}$

(41) Kinetic energy. $\frac{1}{2} M q^2 = (V+U)_a - (V+U)_e$.

(42) Mass and heights. $M = \frac{B}{g}(P^1_0 - P_h)$. $h^1_1 = \frac{C_p}{g}(T^1_0 - T^1_{i1})$. $h^1_2 = \frac{C_p}{g}(T^1_{12} - T_{h2})$.

(43) Approximate solution for Case V. Take $\tau = \frac{T_2 - T_1}{T}$. $T^2 = T_1 T_2$. $M = B P_h \frac{h}{R T} = B \rho h$ (approximate).

(44) $$\tfrac{1}{2} M q^2 = \tfrac{1}{2} M \frac{B_1 B_2}{B^2} g h \tau.$$

CASE VI. CONTINUOUS HORIZONTAL TEMPERATURE DISTRIBUTION WITH ADIABATIC VERTICAL GRADIENT.

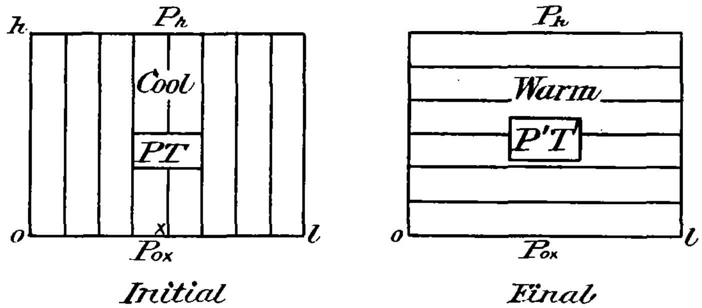

FIG. 24 D.

(45) Assume $T = f(x) - \frac{g}{C_p} z.$

(46) $$P^1 = P_h + \frac{1}{l}\int_{l-x}^{l} (P_{0x} - P_h)\, dx = P - \left[P - P_h - \frac{1}{l}\int_{l-x}^{l} (P_{0x} - P_h)\, dx\right].$$

(47) $$T - T^1 = T - T\left(\frac{P^1}{P}\right)^{\frac{k-1}{k}} = \frac{k-1}{k}\left(T - \frac{P_h}{R\rho} - \frac{1}{l R \rho}\int_{l-x}^{l} (P_{0x} - P_h)\right) d x.$$

(48) $$T_x \frac{P_{0x} - P_h}{g} = T_x \int_0^h \rho\, d z = \int_0^h T \rho\, d z.$$

(49) $$\int_0^h (T - T^1) \rho\, d z = \frac{k-1}{k}\,\frac{1}{g}\, P_h T_0 \left(\frac{g h}{R T_0}\right)^2 \left(\frac{1}{2} - \frac{x}{l} + \frac{\tau}{2}\frac{x}{l} - \frac{\tau}{2}\frac{x^2}{l^2}\right).$$

(50) $$\tfrac{1}{2} M q^2 = C_p \int_0^h (T - T^1)\, d m = l P_h \frac{g h}{R T_0}\, h\, \frac{\tau}{12}.$$

(51) $$q = \sqrt{\frac{g h \tau}{6}}.$$

CASE VII. POSITION OF LAYERS OF EQUAL ENTROPY WHEN THE PRESSURE AT A GIVEN LEVEL IS CONSTANT AND THE TEMPERATURE AT THIS LEVEL IS A FUNCTION OF THE HORIZONTAL DISTANCE AND A LINEAR FUNCTION OF THE HEIGHT.

Let the gradient ratio which distinguishes one stratification of the air from another having a different temperature gradient be n.

(52) $P = P_h \left(\frac{T}{T_h}\right)^{\frac{n C_p}{R}}$. $T = T_h + \frac{g}{n C_p}(h - z).$

(53) The curves. $F(x z) = n \log T_h - (n-1) \log T =$ const.

(54) Angle of curves. $\tan \alpha = \frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z} = \frac{n}{n-1}\left(\frac{h-z}{T_h} + \frac{C_p}{g}\right)\frac{\partial T_h}{\partial x}.$

CASE VIII. FINAL CONDITION OF TWO AIR MASSES UNDER CONSTANT PRESSURE WITH GIVEN INITIAL LINEAR VERTICAL TEMPERATURE FALL.

On removing the partition the layers 1 and 2 spread out, change their heights, and there is a mixed stratum between them.

(55) Temperatures $$\begin{cases} T_1 = T_{h_1} + \dfrac{g}{n_1 C_p}(h - z). \\ T_2 = T_{h_2} + \dfrac{g}{n_2 C_p}(h - z). \end{cases}$$

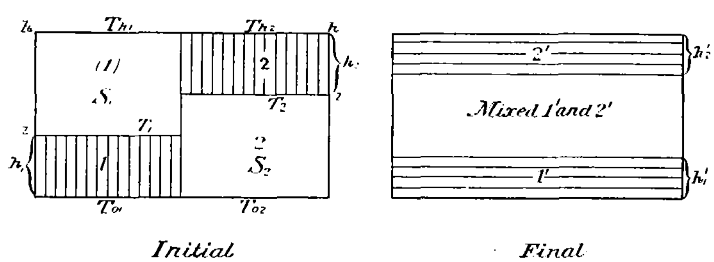

FIG. 24 E.

(56) Entropy $$\begin{cases} S_1 = C_p\,[n_1 \log T_{h_1} - (n_1 - 1) \log T_1] + \text{const.} \\ S_2 = C_p\,(n_2 \log T_{h_2} - (n_2 - 1) \log T_2) + \text{const.} \end{cases}$$

(57) $$\log \frac{T_{02}}{T_1} = \log \frac{T_2}{T_{h_1}} = \frac{n}{n-1} \log \frac{T_{h_2}}{T_{h_1}}.$$

(58) Heights $$\begin{cases} h_1 = \dfrac{n C_p}{g}(T_{01} - T_1). \\ h_2 = \dfrac{n C_p}{g}(T_2 - T_{h_2}). \end{cases}$$

If the vertical temperature fall of the masses 1 and 2 is smaller than in adiabatic equilibrium, then the entropy increases with the height, and it can happen that in the colder mass (1) the entropy at the height h_1 will be as great as in the warmer mass (2) at the ground. The higher layers in (1) form a series with an entropy equal to the layers in (2) up to the height $h - h_1$. In the final state the under part of (1) will spread out on the ground, above it will be layers which are mixtures of (1) and (2), and farther up will lie the masses of (2) which initially were between $(h - h_2)$ and h. On the boundaries of the three layers the temperature transition is continuous.

It will be convenient to approach the dynamic equations of motion in cyclonic vortices thru a study of the Cottage City waterspout of August 19, 1896. It should be recognized that in ordinary cyclones the vortices are not perfect and it is only rarely and in highly developed storms that anything like pure vortex motion is attained. The waterspout, therefore, offers a good example of vortex motion in the atmosphere with which to test the above equations. I may remark that the theory first advanced in my International Cloud Report, 1898, for the generation of cyclones and anticyclones in the general circulation seems to be practically confirmed by these studies based upon actual observations.

VILLARD'S THEORY OF THE AURORA.

By WM. R. BLAIR, Assistant Physicist. Dated Mount Weather, Va., January 18, 1907.

In his "Essai de Théorie de l'Aurore Boréale",[1] M. P. Villard desires especially to account for the movements of the aurora and the various forms in which it appears. He assumes that the auroral light is due to the motion of cathode rays under the influence of the earth's magnetic field, and he argues that these rays are of terrestrial origin. The auroral arch, auroral draperies, and dance of the rays, as usually defined, are the peculiarities to be explained.

The earth's magnetic field is conceived to be similar to that existing between the poles of a Ruhmkorff electro-magnet (the coils being in line with each other). Using such a magnet and the theory, already developed, of how a cathode particle moves in a magnetic field, experiments were devised and carried out for the reproduction of the auroral phenomena on a small scale, in an evacuated bulb. Electrodes were sealed in the bulb; the negative electrode was especially devised for projecting into the field of the magnet, in a suitable direction, a small bundle of cathode rays. Photographs of these reproductions were obtained.

The first three of the following figures and their descriptions serve as a review of the effects of a magnetic field on the motion of projected cathode particles, the fourth, as a basis for the explanation of the forms and movements of the aurora.

Fig. 1 represents the earth's magnetic field. A A' is the magnetic axis, N and S the poles. This field is such that the distribution of magnetic force in a plane thru B B' and perpendicular to A A' is symmetric with respect to the point at which the plane cuts the axis.

Fig. 2 shows the path followed by a cathode particle projected vertically into the earth's field in this equatorial plane, i. e., at right-angles to the line of force. The curve traced is an epitrochoid.

Fig. 3 illustrates the motion of an electron in a uniform magnetic field. Its path is a helix lying lengthwise in the direction of the field. In this case the electron entered the field in a direction other than at right-angles to the lines of force.

The more general case in which the magnetic field is not uniform, but, like that of the earth, has converging lines of force, can not be readily represented by means of a diagram. It will be explained by the use of figs. 2 and 3. The electron is projected into the magnetic field at an angle to the equatorial plane, and consequently its path is a combination of the helical and epitrochoidal paths with this additional feature. In the increasing field the successive spires of the helix, according to Villard, decrease in diameter and in forward

[1] Annales de Chimie et de Physique, September, 1906.

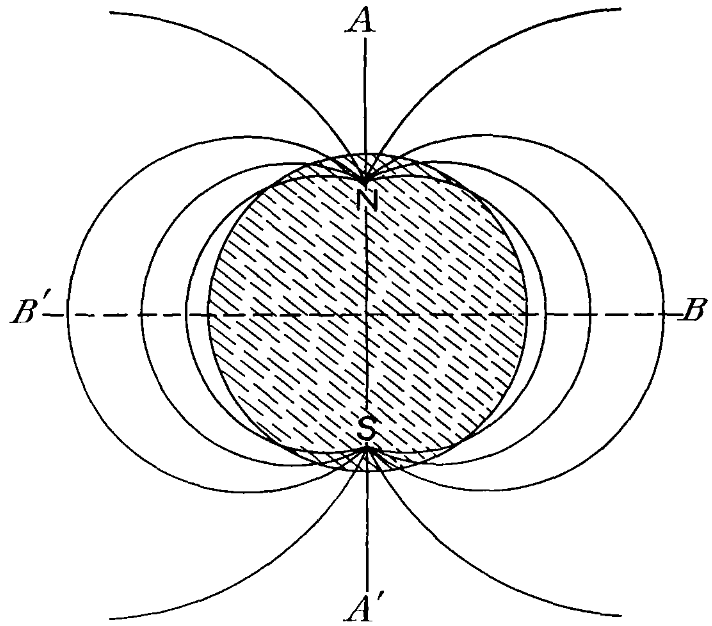

FIG. 1.—The earth's magnetic field.

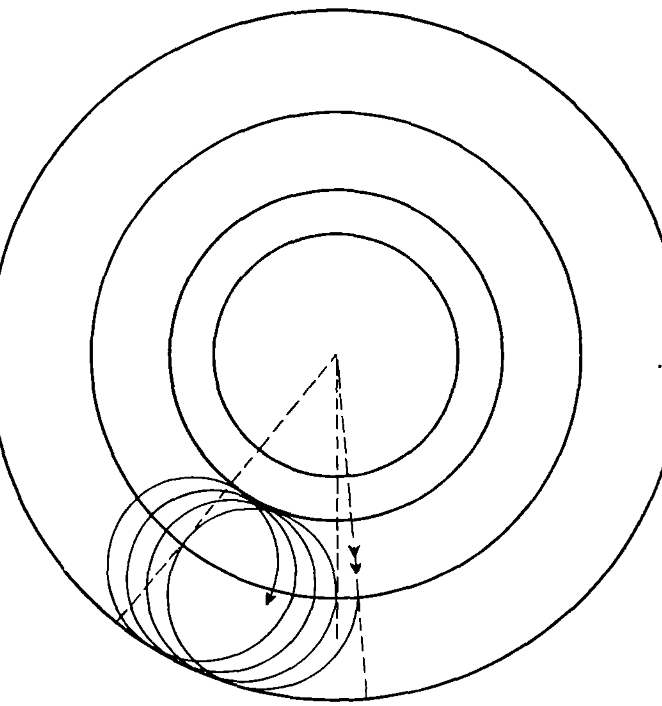

FIG. 2.—The path followed by a cathode particle.

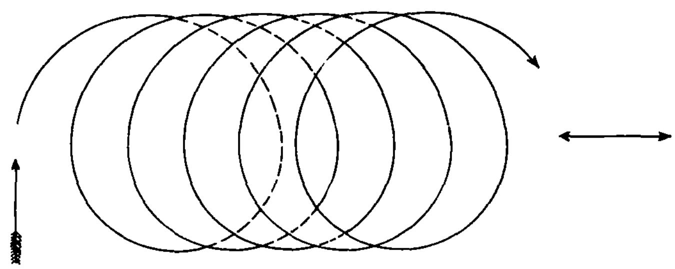

FIG. 3.—The motion of an electron in a uniform magnetic field.