islands offshore, would afford excellent tests of Seemann's theory. Such observations could be very easily made and the work is certainly most attractive.

Although in considering land and sea breezes, we are dealing with one of the less important meteorological phenomena, it is well, in closing this review, to call attention to the correlation of these breezes with other winds of more importance but of similar origin. The warming and cooling of the land which gives rise to the on- and offshore breezes we have been studying, is a diurnal affair depending on the warming by the sun's heat by day and the cooling by radiation and conduction by night. The changes of temperature and of pressure to which these breezes are due, are not, however, restricted to the coastal region, as the breezes themselves are, but extend all over the land surface. Now what we see as a diurnal phenomenon in the case of the land and sea breezes, we see as a seasonal phenomenon in the increase of pressure over the continents in winter and the decrease of pressure in summer, whereby they become alternately areas of high and low pressure and their wind circulation changes accordingly. A winter continent, therefore, has outflowing winds, and a summer continent has inflowing winds, and these, which may be called continental winds, combine with the larger class of terrestrial winds to form the general winds of the earth. The class of seasonal, or continental winds is simply a larger example of the smaller class of land and sea breezes. In the former case the continuance of the temperature and pressure conditions is for some months at a time; in the latter it is for a few hours only. In the former case, therefore, a general continental circulation of the winds can be established; in the latter there is only time for the establishment of a local and incomplete circulation.

NOTE.—ADDED JULY 24, 1893.

Since preparing the above review, the writer has received a copy of Dr. Otto Krümmel's "Geophysikalische Beobachtungen der Plankton-Expedition" (Kiel & Leipzig, 1893), in which are presented the results derived from the meteorological observations made by Dr. Krümmel during the scientific exploring voyage of the Plankton Expedition in the Atlantic Ocean during July–November, 1889. Dr. Krümmel calls attention to a point in connection with sea breezes which is worthy of note here, and of careful observation in any future investigation of this class of winds. He noticed that during the time the vessel *National* was anchored off Para the sea breeze was most marked during a flood tide. This fact the author finds referred to in several previous accounts of land and sea breezes. In Staff Commander James Penn's "Sailing Directions for the West Coasts of France, Spain and Portugal" (London, 1867, p. 273) it is stated that at Cadiz—

> The sea breezes vary from west to north-northwest and are generally strongest at the full and change of the moon, when they not unfrequently blow during the whole night. They set in most commonly with the flood * * *.

Further, in the "Annalen der Hydrographie," 1887, p. 164, the captain of the German cruiser *Habicht* states that at Kamerun the sea breeze is the strongest when the flood tide comes in the afternoon. The explanation of this fact is found in the mechanical raising, by the rising tide, of the mass of air lying over the water near the shore, thus causing stronger gradients aloft and consequently a more active circulation.

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GRAPHICAL INTEGRATION OF FUNCTIONS OF A COMPLEX VARIABLE WITH APPLICATIONS.

By S. DOUGLAS KILLAM, Ph. D.

[Dated, University of Alberta, November 5, 1913.]

Many problems in mathematics can be solved more simply by graphical than by analytical methods; especially is this true when the problem is presented graphically, and we wish a graphical representation of the solution. The object of this present paper is to give some of the results of graphical integration of functions of a complex variable as obtained by the author and published in part in his Dissertation (1).

Functions of a complex variable arise in a great many problems of mechanics and physics. In these the graphical method of solution is of great advantage, and gives results in which the errors of graphic methods are so small that they may be disregarded.

As an introduction to the integration of functions of a complex variable I wish to give a short and accurate method of graphically integrating functions of a real variable (2), since graphical integration of functions of a complex variable can be reduced to repeated successive graphical integrations of functions of a real variable.

In this paper I shall only consider the mathematical side of the problem and give one or two examples worked out in detail, so that the physicist or student of applied mechanics can easily apply my method to other problems.

1. GRAPHICAL INTEGRATION OF FUNCTIONS OF A REAL VARIABLE.

A function $f(x)$ of the real variable x can be represented graphically in the x,y plane. On the x axis (fig. 1) we take a number of points x_1, x_2, x_3, $\cdots$; through them draw the ordinates cutting the curve $f(x)$ in the points a, A, B, C, $\cdots$. We now draw M_1c parallel to the x axis so that the area of aM_1b equals the area of bcA. In the same way we draw de; fg; hi; $\cdots$ and produce them to cut the y axis in the points M_1, M_2, M_3, M_4, $\cdots$. Take a point P on the x axis so that Px_1 shall equal unity; and join P to these points M_1, M_2, M_3 $\cdots$. Now the area under the curve $f(x)$ equals the area under the "step-curve" $aM_1bcdefg \cdots$. We obtain the integral of the broken line by starting from x_1 and draw lines parallel to the lines PM_1, PM_2, $\cdots$ meeting the ordinates drawn through the points x_2, x_3, x_4, $\cdots$ at the points A',B',C', $\cdots$. This broken line $A'B'C' \cdots$ gives the values of $\int_0^x f(x)dx$ for $x=x_1$; $x=x_2$; $\cdots$. In order to obtain the graph of the curve for all values of x we draw a smooth curve through the points $A'B'C' \cdots$. If our points x_1, x_2, x_3, $\cdots$ are drawn close enough together the broken line $A'B'C'D' \cdots$ closely approximates the curve $\int f(x)dx$. The Integraph (a machine for mechanically integrating curves) may be used for integrating the graph of the curve $\int f(x)dx$; but I have found by careful work with both methods that the graphical method of integration is shorter and more accurate. With reasonable care and the use of a large scale the results obtained will be exact enough for the solution of all problems arising in applied mathematics.

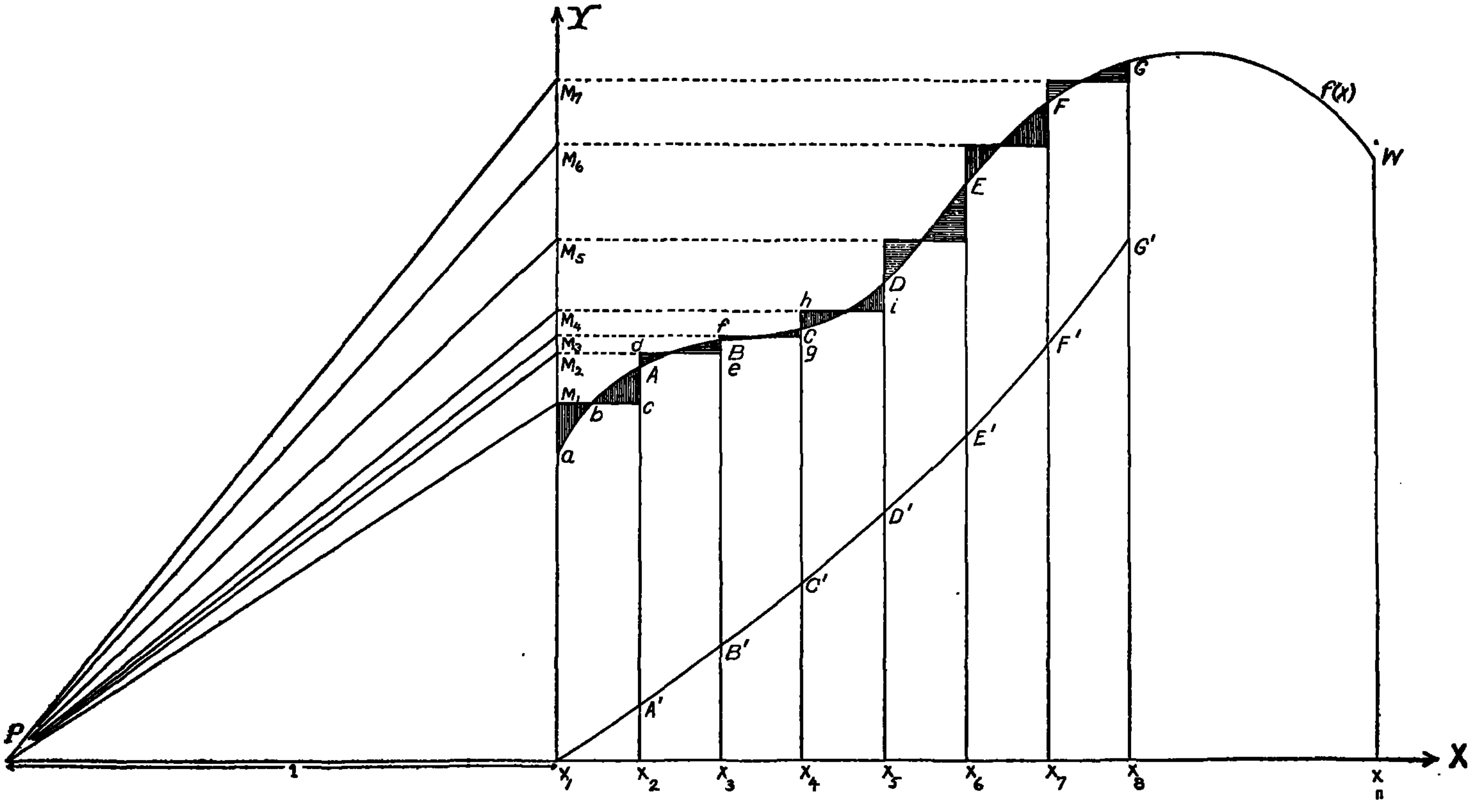

FIG. 1. Graphical representation of $f(x)$ in the x,y plane.

2. GRAPHICAL INTEGRATION OF FUNCTIONS OF A COMPLEX VARIABLE.

A function $w=f(z)$ of the complex variable $z=x+iy=re^{i\phi}$ is given graphically by the conformal representation of the z plane on the w plane. We can choose the net of curves in the z plane in any way we like; but in order to have a control over our analytical work we choose a system of orthogonal curves

$$r=r_n(n=1,2,3, \cdots)$$

and

$$\phi=\phi_n(n=1,2,3, \cdots)$$

where r_n and ϕ_n are such values that the z plane is covered by a net of small squares as in figure 2.

In accordance with the laws of conformability, our function $f(z)$ will be represented in the w plane by a system of orthogonal curves which cover the plane with a net of small squares (see fig. 3).

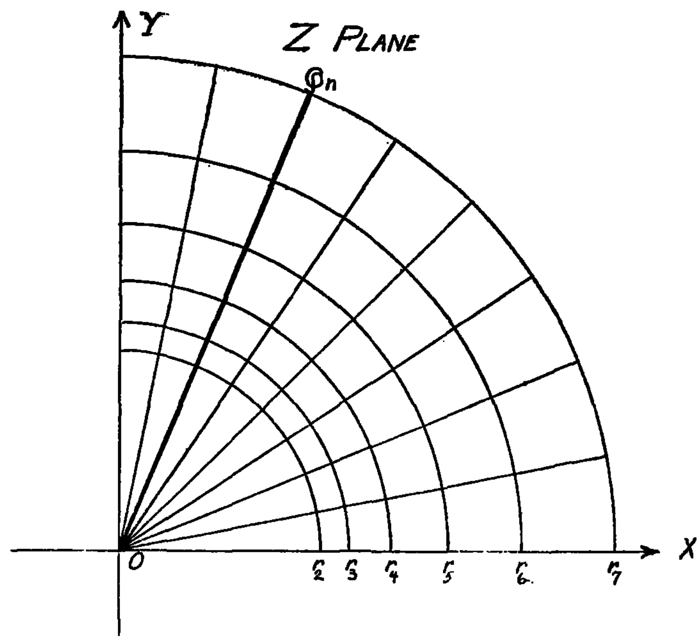

FIG. 2. A system of orthogonal curves in the z plane.

We now seek a graphical representation of the function

$$Z=\int_0^z f(z)dz$$

in the Z plane; i. e., we seek the curves $r=r_n$ and $\phi=\phi_n$

in our Z plane. If we integrate $f(z)$ from $z=0$ to $z=r_n e^{i\phi_n}$ along the curve or path $\phi=\phi_n$ we have

$$Z=X+iY=\int_0^{z_n} f(z)dz$$
$$=\int_0^{r_n}\Big[u(r,\phi_n)+iv(r,\phi_n)\Big]d(re^{i\phi_n})$$
$$=e^{i\phi_n}\Big[\int_0^{r_n}u(r,\phi_n)dr+i\int_0^{r_n}v(r,\phi_n)dr\Big] \tag{1}$$

where

$$f(z)=u(r,\phi_n)+iv(r,\phi_n).$$

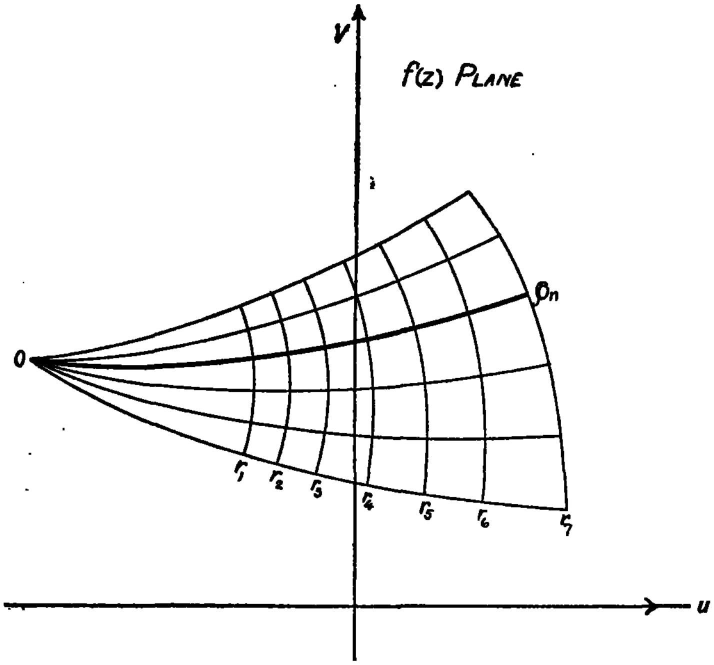

FIG. 3.

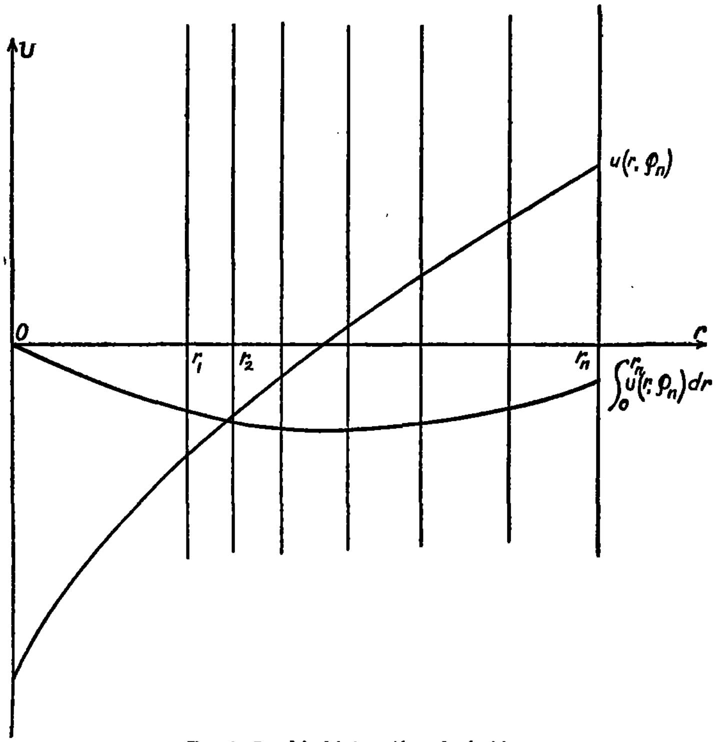

FIG. 4. Graphical integration of $u(r,\phi_2)$.

Now $u(r,\phi_n)$ is a function of the real variable r and can be represented graphically in a u,r plane. Similarly $v(r,\phi_n)$ can be represented in a v,r plane. We now integrate graphically by the method of §1 to obtain the values of

$$\xi=\int_0^{r_n} udv$$

and

$$\eta=\int_0^{r_n} vdr\ (n=0,1,2,3,\cdots)$$

(see figs. 4 and 5).

In order to draw the graphs of the curves

$$u(r,\phi_n) \text{ and } v(r,\phi_n)$$

we obtain the values of u and v which correspond to

$$r_n(n=1,2,3,\ .\ .\ .)$$

from figure 3.

In our

$$Z=\int_0^z f(z)dz \text{ plane}$$

we draw the axes $\int udr$ and $\int vdr$ so that the angle between the X axis and the $\int udr$ axis equals ϕ_n (see fig. 6).

From equation (1) we see that the factor $e^{i\phi_n}$ means that the X axis must rotate through an angle of ϕ_n in order to coincide with the ξ axis. In the ξ,η plane we mark the points $r_1, r_2, r_3, \cdots$ with the coordinates ξ and η. We get these values from the graphical integration of the functions $u(r,\phi_n)$ and $v(r,\phi_n)$ (see figs. 4 and 5). They can be transferred over to our new Z plane by measurement; or, by having our ξ,η plane drawn on transparent paper, we can mark off the points $r_1,r_2,r_3,\cdots$ without measuring the values of ξ and η. This latter method eliminates a small error of measurement.

Through the points $r_1, r_2, \cdots$ we draw a smooth curve which is the required graphical representation of the curve $\phi_n=\phi_n$ in the Z plane.

In the same way we obtain the curves $\phi=\phi_1$; $\phi=\phi_2$; .... Through the points r_n on each curve $\phi=\phi_n$ we

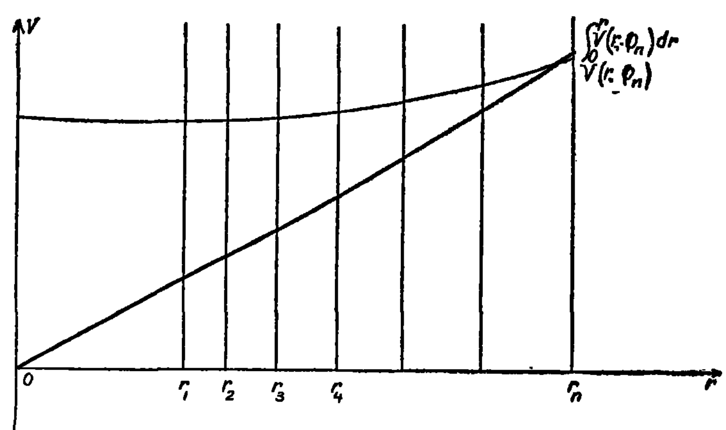

FIG. 5. Graphical integration of $v(r,\phi_2)$.

draw a smooth curve and obtain a net of small squares covering the Z plane which is the graphical representation of the function

$$Z=\int_0^z f(z)dz.$$

3. GRAPHICAL INTEGRATION OF $f(z) = (1-z^4)^{-\frac{1}{2}}$.

Let $f(z)$ in the preceding section be $(1-z^4)^{-\frac{1}{2}}$ then we have the special problem of the graphical integration of $(1-z^4)^{-\frac{1}{2}}$.

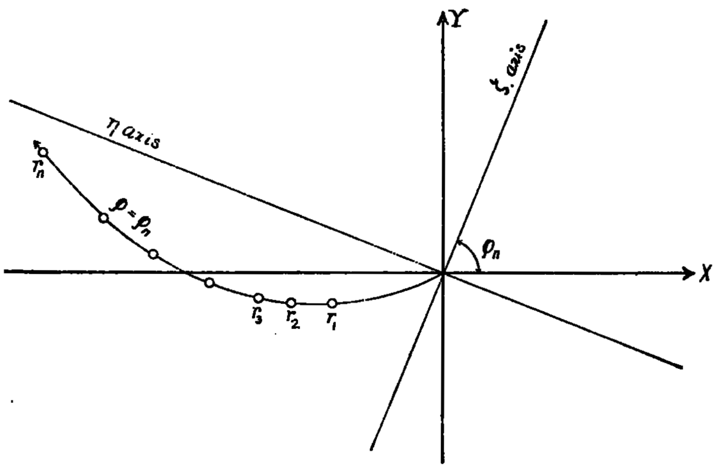

FIG. 6.

We represent

$$z = x + iy = re^{i\phi} \text{ in the } z \text{ plane}$$

by the curves $r = r_n (n = 0,1,2, \cdots)$ and $\phi = \phi_n$. (Figure 7). These curves are transformed by the function

$$f(z) = (1-z^4)^{-\frac{1}{2}}$$

into the curves $r = r_n$ and $\phi = \phi_n$ $(n = 1,2,3, \cdots)$ in the $f(z)$ plane (see figure 8).

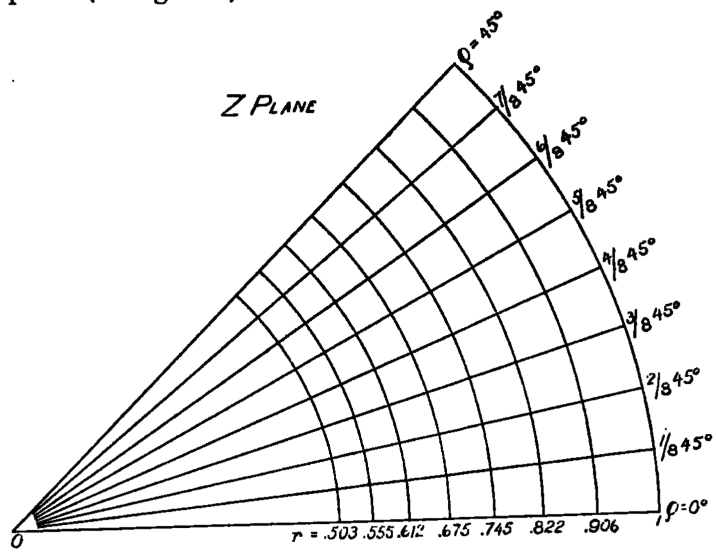

FIG. 7.

Now in order to graphically represent the function

$$Z = \int_0^z f(z)dz$$

we integrate at first along the line $\phi = 0$. Now, of course z is a real variable. If we put $z = \sin t$, we have for $\phi = 0$

$$\int_0^1 \frac{dz}{\sqrt{1-z^4}} = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1+\sin^2 t}} = \int_0^{\frac{\pi}{2}} f(t)dt.$$

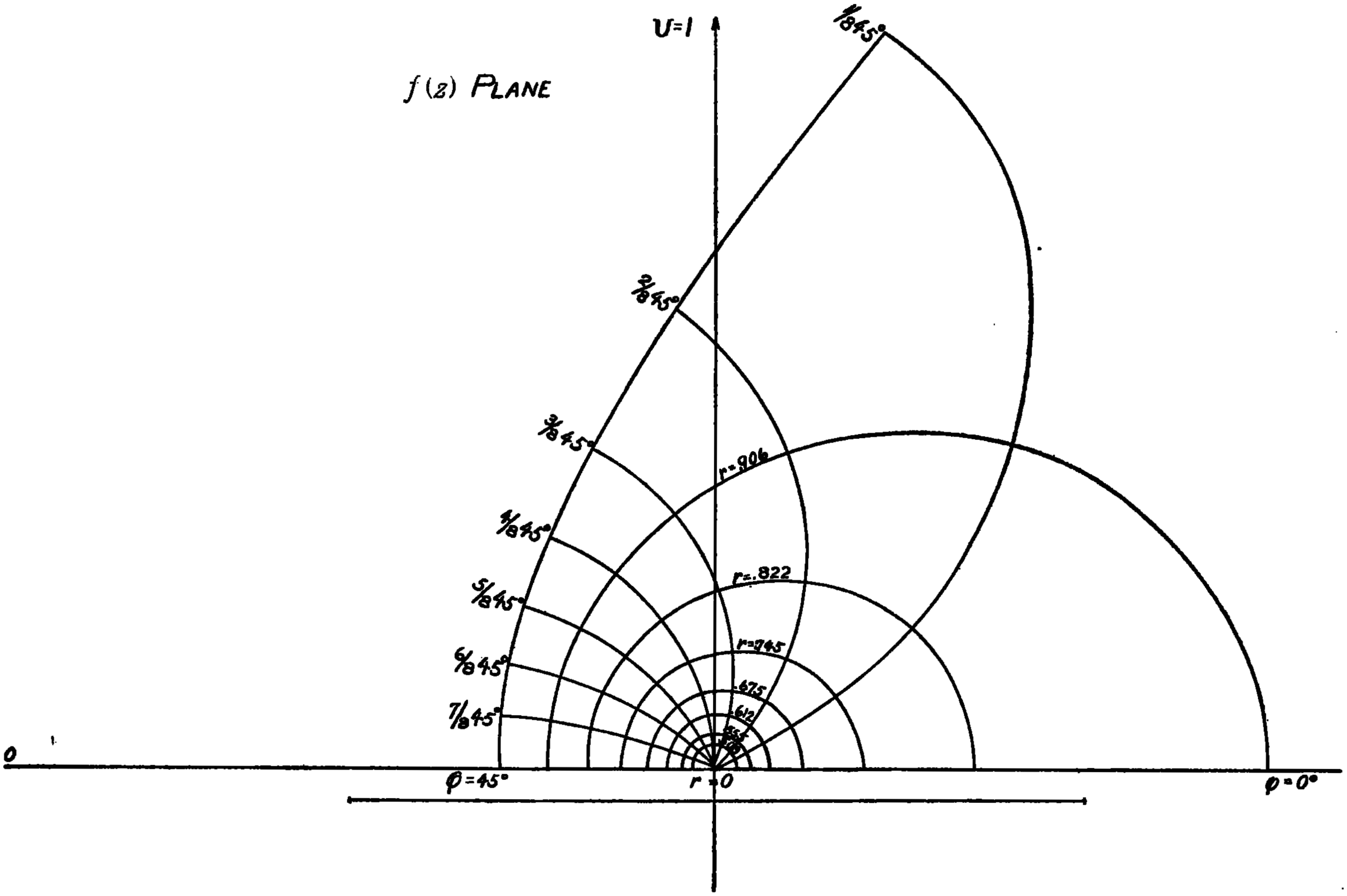

FIG. 8. Curves $r = r_n$ and $\phi = \phi_n$ in the $f(z)$ plane.

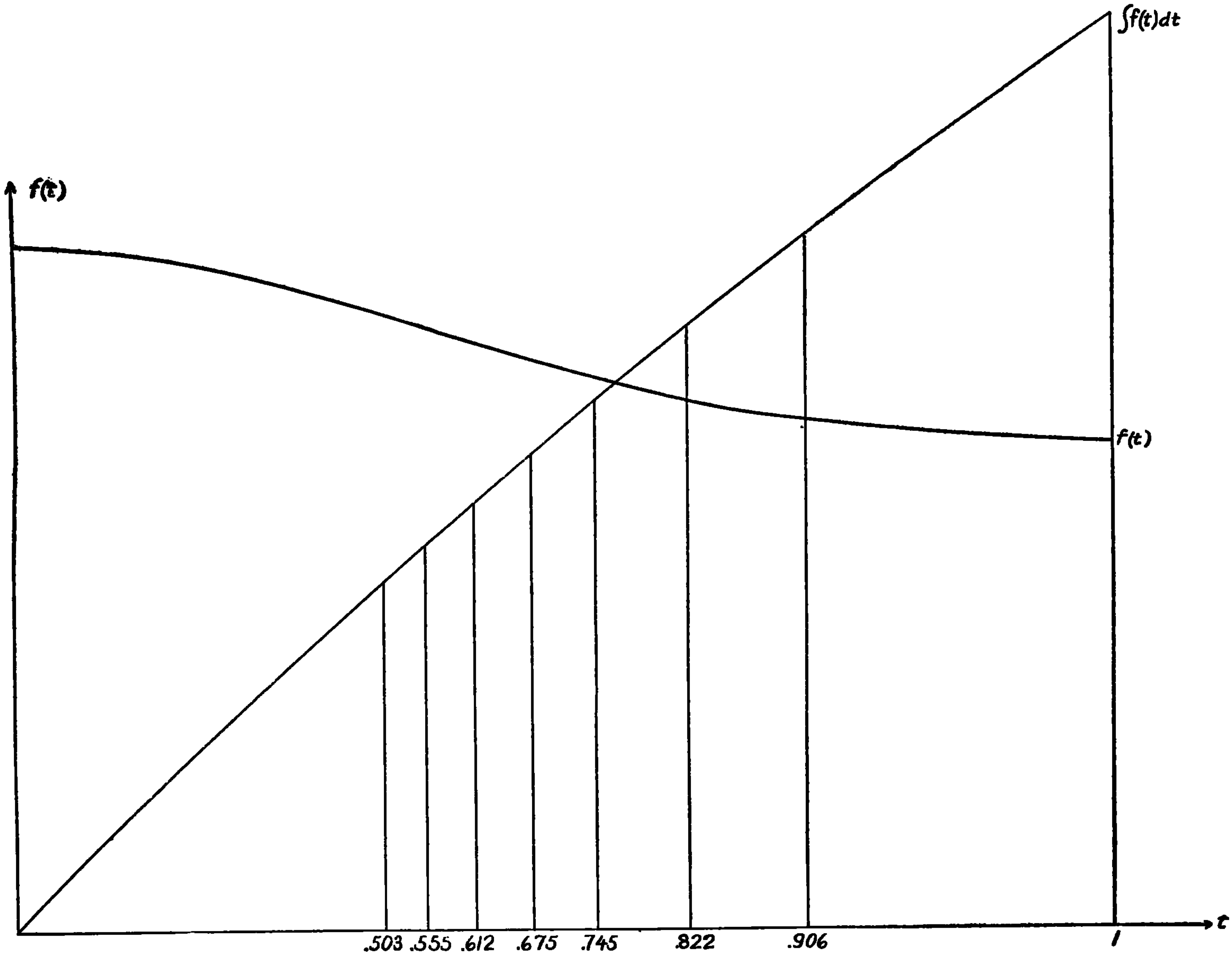

FIG. 9. Graphic representation of $f(t)$ in a $f(t),t$ plane.

We represent the function $f(t)$ graphically in a $f(t),t$ plane; integrate graphically and get the values of $\int_0^z \frac{dz}{\sqrt{1-z^4}}$ for all real values of z between $z=0$ and $z=1$ (see figure 9).

Next we integrate along the lines

$$\phi=\tfrac{1}{8}\,45°;\ \phi=\tfrac{2}{8}\,45°;\cdots$$

and get as explained in §2 the curves

$$\phi=\tfrac{1}{8}\,45°;\ \phi=\tfrac{2}{8}\,45°;\cdots$$

in our Z plane. The graphical integration is carried out

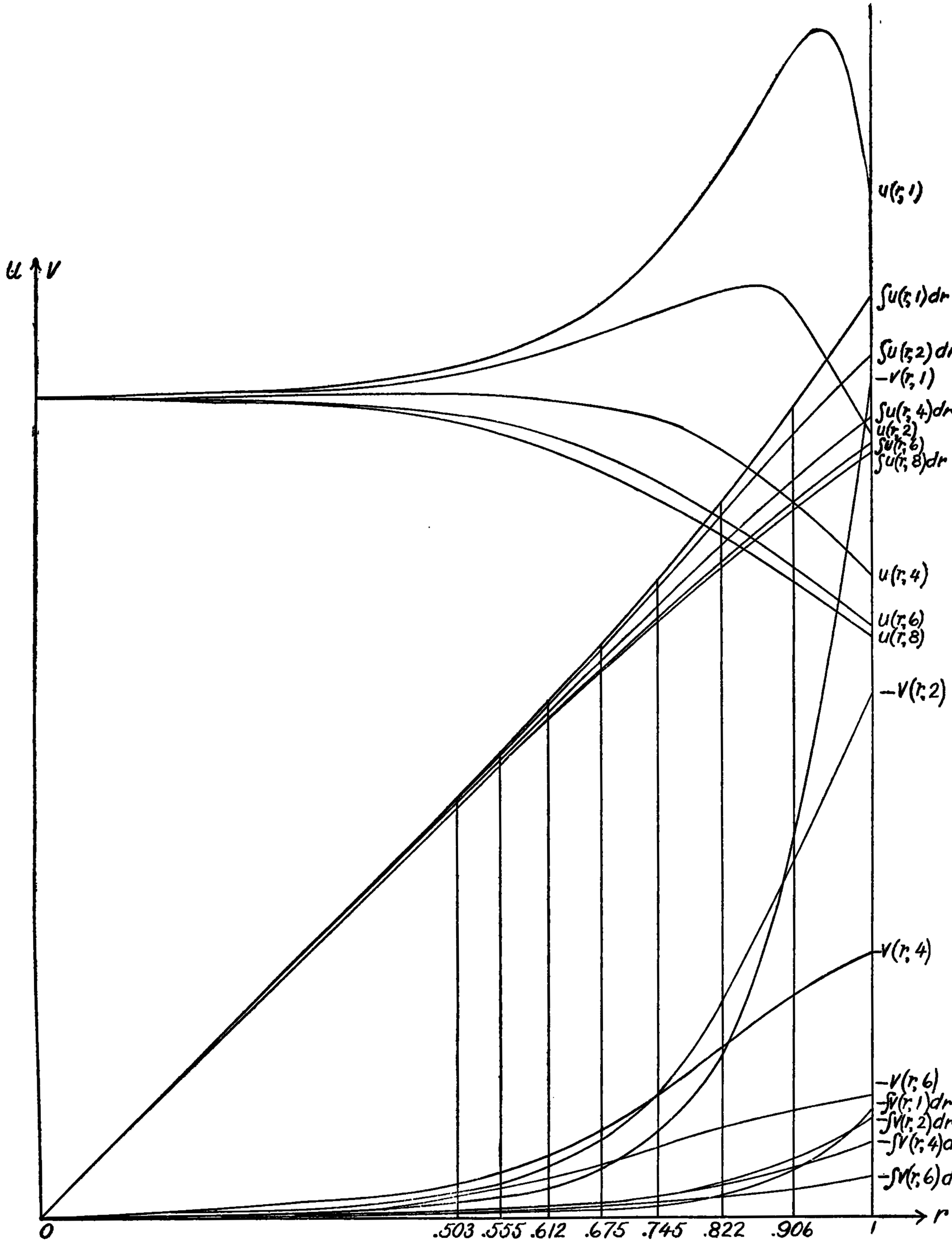

FIG. 10. Graphical integration along $\phi=\frac{1}{8}\,45°$, $\phi=\frac{2}{8}\,45°$, etc.

as in figure 10 and the results shown graphically in figure 11.

We now have in the Z plane the graphical representation of our function Z for all values of z in the first octant

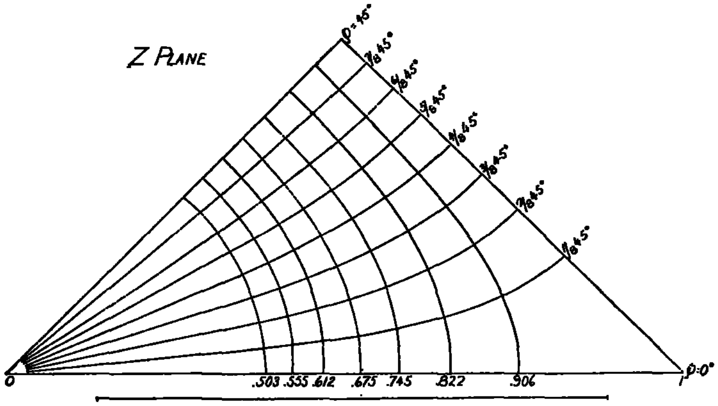

FIG. 11. Result of graphical integration of $f(t)$ in a $f(t),t$ plane.

of the circle with radius unity. The function is continuous everywhere even at the point $z=1$ which is a branch

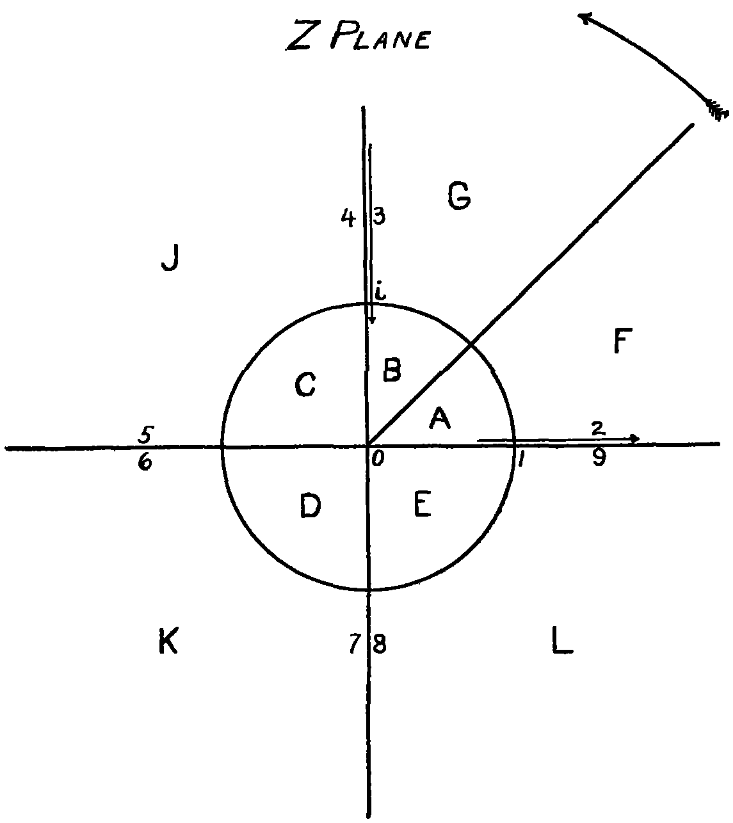

FIG. 12. Illustrating "rebattement" for the function in the second octant.

point of the first order. We obtain the representation of our function for the second octant by simply rebatting [from the French geometrical term "rebattement" "The rotation of a plane about its intersection with a plane of projection, until it coincides with that plane"] around the line $\phi=45°$. In the same way we obtain the representation in the 3d, 4th, . . . 8th octant. (See figure 12). The areas A, B, C, D, E, correspond to the areas A_1, B_1, C_1, D_1, E_1. The points ± 1 and $\pm i$ are branch points of the first order. If we let $z=z'^{-1}$ we have

$$\int_0^\infty \frac{dz}{\sqrt{1-z^4}} = i\int_0^1 \frac{dz'}{\sqrt{1-z'^4}}$$

that is, we only need to rebatt our figure around the line $r=1$ to obtain the graphical representation of Z for the whole z plane.

The area inside the circle with radius unity is transformed into the area inside the square *abcd*. (Figure 13.)

The perimeter of the circle with radius unity is transformed into the perimeter of the square *abcd*.

The area outside the circle with radius unity is transformed into the finite areas E_1, G_1, J_1, K_1, and L_1.

In a similar way we can integrate any functions of the complex variable z, and obtain a graphical representation of Z for all values of z.

This method is especially valuable in integrating elliptic integrals of the first, second, and third kind. The same

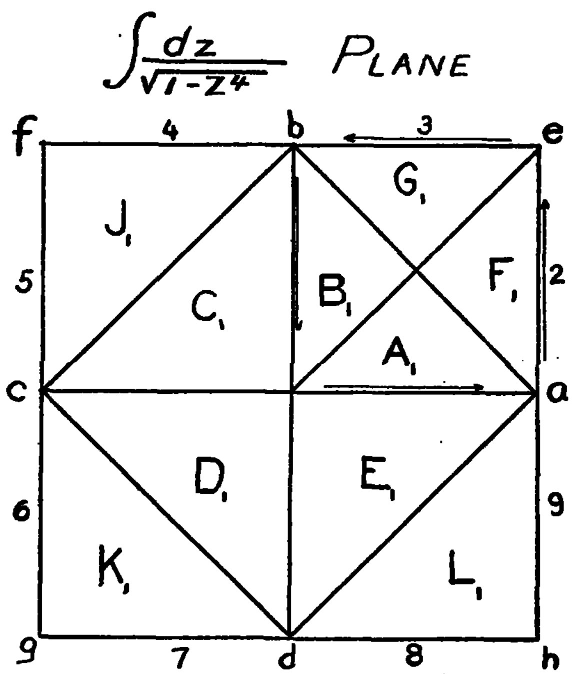

FIG. 13.

methods can be used for solving all problems involving graphical integration of functions of a complex variable that arise in mechanics and applied mathematics.

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