with the gases of external space. Similarly, little progress has been made toward a determination since the time of Laplace of the total mass of the atmosphere. If the earth did not rotate and if the distribution of the atmosphere were adiabatic, its height would be limited to about 17 miles and the total mass would be a little more than one-millionth part of the entire mass of the earth. But the earth rotates and while the distribution near the surface of the oceans is approximately adiabatic it is probable that this law does not hold at any great height. Hence, if we suppose the mass distribution to be such as conceived by Laplace, the total mass must be much larger than the lower limit just assigned. Here, then, are two capital problems available for research by aid of the more recently acquired resources of knowledge concerning the constitution of gases.

Recent researches, and especially those of Bjerknes in his Dynamic Meteorology, have gone far toward a rational treatment of the kinetic properties of the atmosphere, and we may confidently entertain the hope that rapid progress will be practicable in the near future.

NOTE ON A MIRAGE AT SEA.

[Dated July 3, 1919.]

Ship's position [U. S. S. *Radnor*] 40° 26′ N., longitude 64° W., apparent time of ship, 4 p. m. Light easterly winds, smooth sea, barometer 29.94 inches; my personal barometer, 30.00 inches; air, wet bulb, 59°; air, dry bulb, 63°; temperature of sea at surface, 53°; overcast with cumulus clouds. Strong mirages noted all around. Four other ships were in sight at the time. These vessels appeared at times to be steaming along at the top of a hugh wall of ice; at other times the bodies of the ship seemed to rise out of the water at least twice their height. Horizon had all the appearance of a long, rugged coast line.—*M. S. Harloe, Lieut. Commander, U. S. N. R. F.*

THE SUN'S INFLUENCE ON THE DIURNAL VARIATION OF THE ATMOSPHERIC POTENTIAL-GRADIENT.[1]

By W. F. G. SWANN.

Professor of Physics, University of Minnesota.

[Dated Minneapolis, Minn., July 10, 1919.]

SYNOPSIS.—The paper presents a view as to a possible origin of part of the diurnal variation of the atmospheric potential-gradient.

Various phenomena in cosmical physics lend support to the view that the upper atmosphere is so highly conducting as compared with the air near the earth's surface that, for electrostatic considerations, it may be looked upon as perfectly conducting. Thus, for example, we may look upon all points on a sphere in the upper atmosphere, concentric with the earth, as being at the same potential.

If the conductivity of the atmosphere depended only upon the altitude, and were independent of the position on the earth's surface with respect to the sun, the surfaces of equal conductivity would be spheres concentric with the earth, and the conduction current-density and potential-gradient would be independent of the position on the earth's surface. If the sun emits an ionizing radiation, however, we may expect the surfaces of equal conductivity to be dented inward toward the earth when the sun is at the zenith. Thus, the total resistance of a column of air of unit cross section, extending from the earth's surface to a given altitude in the upper atmosphere, would be least when the sun was at the zenith. It would result that the atmospheric conduction current-density would be greatest when the sun was at the zenith; and, if the conductivity of the air at the surface of the earth were the same at all places, we should have a higher potential-gradient on the sunlit portions of the earth than on those remote from the sun.

The ideas contained in the above outline are illustrated in the second portion of the paper by an example which is worked out mathematically. The bearing of these considerations upon the effect of an eclipse on the potential-gradient is also discussed.

Various phenomena in cosmical physics lend support to the view that the upper atmosphere is highly conducting as compared with the air near the earth. Thus, for example, Schuster has developed a theory of the diurnal variations of terrestrial magnetism which invokes, among its requirements, a conductivity about 3×10^{11} times as great as that at the earth's surface. Such a value appears by no means unreasonable when viewed in the light of our beliefs as to the processes at work in the upper atmosphere. The sun's ultra-violet light alone, although insufficient to account for a conductivity of the order of magnitude required by Schuster's theory is able, as the writer has shown,[2] to account for a magnitude one thousandth of this amount, i. e., a conductivity so high that a column of air extending one-fourth of the way around the earth in the upper atmosphere would offer no more resistance than would a column of surface air of equal cross section, but only 3 cms. long.

If the upper atmosphere has a conductivity approximating even to that which the ultra-violet light is capable of accounting for in the sunlit regions, it may, for most electrostatic purposes, be considered as a perfect conductor.

If two concentric spheres be maintained at a difference of potential, the field at the surface of the inner sphere will, from symmetry, be the same at all points of the surface. If a dent be made in the outer sphere, the distance between the spheres at this point will be decreased, and, since each sphere is at the same potential all over, the field at the surface of the inner sphere, under the dent, will be stronger than that elsewhere. Or if, instead of making a dent in the outer sphere we fill the space between the spheres with a medium which is slightly conducting, and in which the conductivity is the same at all points except that at one place it increases from its normal value as we go from the inner to the outer sphere, then the current density will be greater at this place than elsewhere, and the potential-gradient at the surface of the inner sphere will be greater here than elsewhere.

The above crude illustration suggests that if the upper atmosphere is in a highly conducting state, but the conductivity mounts more rapidly with altitude on the sunlit side than on the side more remote from the sun, the potential-gradient should be higher in the former case than in the latter. Here, then, would be an influence playing a part in the determination of the diurnal variation of the potential-gradient, and of such a type as to predict a maximum of the potential-gradient by day and a minimum by night.

On a view of this kind the air-earth conduction current-density at the earth's surface should go through a diurnal variation of the same kind as the potential-gradient, unless the surface conductivity also varies throughout the day. Actual observations bring out the fact that, as a general rule, the conductivity at the earth's surface varies in the opposite sense to the potential-gradient, and in such a way as to maintain the product of the two (the air-earth current-density) more nearly independent of the time of day than either of its constituent factors. The general nature of the phenomena is such as to suggest that the quantity fundamentally determined is the air-

[1] Read before the May meeting of the Mathematical Association of America.
[2] Terr. Mag., v. 21, pp. 1-8, 1916.

earth current-density and that the diurnal variation of the potential-gradient is the direct outcome of the automatic adjustment of the electric field of the atmosphere to maintain constancy in the air-earth current-density while the conductivity suffers a diurnal variation. However, it is by no means universally true that the conduction current-density remains even approximately constant. Thus, for example, if we take the observations for potential-gradient and ionic-content obtained over the ocean on the fourth cruise of the yacht *Carnegie*,[1] by Messrs. H. F. Johnston, B. Jones, I. A. Luke, and S. J. Mauchly, using the apparatus designed by the writer, and if we assume the air-earth current-density to be proportional to the product of the ionic-content and potential-gradient, we obtain a curve such as figure 1, showing a maximum at 9 a. m. and a smaller maximum at midnight.

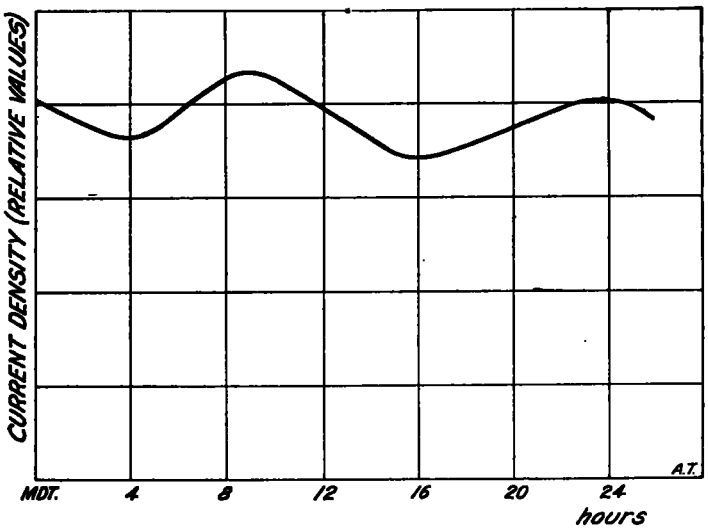

FIG. 1.

It thus appears, at any rate, worth while to consider in rather greater detail possibilities such as that suggested at the beginning of this article for explaining at any rate a portion of the diurnal variation of the potential-gradient otherwise than as a mere secondary effect resulting from a diurnal variation of the conductivity in the immediate vicinity of the earth, and to this end we shall work out an illustrative example.

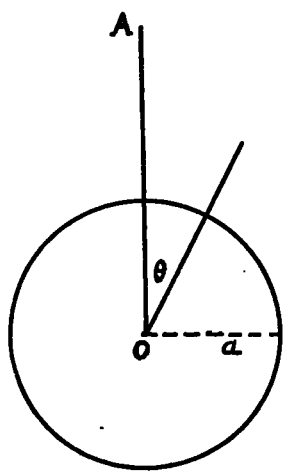

FIG. 2.

The problem to be considered may be specified as follows: Let OA (fig. 2) represent the direction of the sun, the circle of radius a representing the earth. We may assign a conductivity increasing with altitude to a very high value according to some prescribed law of such a kind as to correspond to a more rapid increase in the sunlit portions of the atmosphere than elsewhere, and we may calculate the distribution of potential-gradient in the atmosphere, and in particular over the earth's surface. Or we may state the problem to be solved in the converse manner: Assign a variation of potential-gradient over the earth's surface and determine a conductivity which shall increase to a very high value with altitude, and shall have such a distribution as shall correspond to the assigned distribution of potential-gradient over the earth's surface.

If σ is the conductivity, the equation of continuity takes the form:

$$\frac{\partial}{\partial r}\left(\sigma r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\ \frac{\partial}{\partial\theta}\left(\sigma\sin\theta\frac{\partial V}{\partial\theta}\right) \tag{1}$$

A limited class of solutions, but one which will fit our purpose, is one in which the current density is derivable from a potential ψ, so that

$$\frac{\partial\psi}{\partial r}=\sigma\frac{\partial V}{\partial r},\qquad \frac{\partial\psi}{\partial\theta}=\sigma\frac{\partial V}{\partial\theta} \tag{2}$$

Under these conditions, equation (1) becomes:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{\sin\theta}\ \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0 \tag{3}$$

which is La Place's equation.

Equations (2) lead immediately to:

$$d\psi=\sigma dV \tag{4}$$

Moreover, since ψ and V are perfect differentials, equations (2) lead to

$$\frac{\partial V}{\partial r}\cdot\frac{\partial\sigma}{\partial\theta}-\frac{\partial V}{\partial\theta}\cdot\frac{\partial\sigma}{\partial r}=0 \tag{5}$$

Solving these equations by La Grange's method, the subsidiary equations become

$$\frac{dr}{\left(\frac{\partial\sigma}{\partial\theta}\right)}=\frac{-d\theta}{\left(\frac{\partial\sigma}{\partial r}\right)}=\frac{dV}{0} \tag{6}$$

The solution of the first of these two is $(\sigma-\text{const.})=0$, and the second equation requires that $z=\text{const.}$ Hence, the solution of (5) is $V=\phi\ (\sigma)$, where ϕ is an arbitrary function.

Hence, from (4), ψ is a function of σ. Thus both σ and V are functions of ψ.

Thus, if ψ be chosen as a solution of La Place's equation, and σ be taken as a function of ψ, the corresponding expression for V will be given by:

$$\frac{dV}{d\psi}=\frac{1}{\sigma} \tag{7}$$

The restriction that the currents shall be derivable from a potential ψ thus limits us to problems where σ is constant when V is constant, and where σ is expressible as a function of ψ, which is itself a solution of La Place's equation.

Let us now suppose that $V=0$ when $r=a$, i. e., $\psi=$ const. when $r=a$, since V is a function of ψ. This will automatically necessitate the condition $\sigma=$constant (say σ_0) when $r=a$, since σ is a function of ψ. Let us further suppose that, when $r=a$, $\left(\frac{\partial V}{\partial r}\right)$ is some assigned function of θ which we shall imagine expanded in a series of Legendre coefficients in the form:

$$\left(\frac{\partial V}{\partial r}\right)_{r=a}=A_0+A_1P_1(\mu)+A_2P_2(\mu)+\ \cdot\ \cdot\ \cdot\ \text{etc.} \tag{8}$$

where $\mu=\cos\theta$.

From (2) we have,

$$\left(\frac{\partial\psi}{\partial r}\right)_{r=a}=\sigma_0A_0+\sigma_0A_1P_1(\mu)+\sigma_0A_2P_2(\mu)+\ \cdot\ \cdot\ \cdot\ \text{etc.} \tag{9}$$

[1] Publication No. 175 (Vol. III) of the Carnegie Institution of Washington, p. 417.

But, since ψ is a solution of La Place's equation, it must be of the form:

$$\psi=\frac{a_0}{r}+\left(a_1r+\frac{1}{r^2}\right)P_1(\mu)+\cdots\left(a_nr^n+\frac{b_n}{r^{n+1}}\right)P_n(\mu)\ \cdots$$

and

$$\left(\frac{\partial\psi}{\partial r}\right)_{r=a}=-\frac{a_0}{a^2}+\left(a_1-\frac{2b_1}{a^3}\right)P_1(\mu)+\cdots \left[na_na^{n-1}-\frac{(n+1)b_n}{a^{n+2}}\right]P_n(\mu) \tag{10}$$

Hence, from (9) and (10),

$$na_n-\frac{(n+1)b_n}{a^{2n+1}}=\frac{\sigma_0A_n}{a^{n-1}} \tag{11}$$

Again, since $\psi=$const. when $r=a$, we have:

$$a_n+\frac{b_n}{a^{2n+1}}=0 \tag{12}$$

(11) and (12) serve to determine a_n and b_n in terms of A_n, and we finally have

$$\psi=-\frac{\sigma_0A_0a^2}{r}+\frac{\sigma_0A_1}{3}\left(r-\frac{a^3}{r^2}\right)P_1(\mu)+\cdots +\frac{\sigma_0A_n}{2n+1}\left[\frac{r^n}{a^{n-1}}-\frac{a^{n+2}}{r^{n+1}}\right]P_n(\mu)+\cdots \tag{13}$$

If σ be taken as any function of ψ, and V be determined from (7), these values of σ and V will be consistent with the distribution of potential-gradient given by (8).

As an example, let us suppose that, when $r=a$,

$$\left(\frac{\partial V}{\partial r}\right)_{r=a}=X_0+X_0\alpha(1+\cos\theta) \tag{14}$$

where α is a constant.

This is of the form (8), where $X_0(1+\alpha)=A_0$, and $X_0\alpha=A_1$, so that

$$\frac{\psi}{a}=-\sigma_0X_0(1+\alpha)\frac{a}{r}+\frac{\sigma_0X_0\alpha}{3}\left(\frac{r}{a}-\frac{a^2}{r^2}\right)\cos\theta+\cdots \tag{15}$$

Now choose σ of the form:

$$\sigma=\sigma_0e^{\beta\left[1-\left\{\frac{a}{r}-\frac{\alpha}{3(1+\alpha)}\left(\frac{r}{a}-\frac{a^2}{r^2}\right)\cos\theta\right\}\right]} \tag{16}$$

where β is a constant.

This value of σ is a function of ψ, and reduces to σ_0 when $r=a$. From (2) and (16) we have:

$$\frac{\partial V}{\partial r}=X_0\left[(1+\alpha)\frac{a^2}{r^2}+\frac{\alpha}{3}\left(1+\frac{2a^3}{r^3}\right)\cos\theta\right]e^{-\beta\left[1-\left\{\frac{a}{r}-\frac{\alpha}{3(1+\alpha)}\left(\frac{r}{a}-\frac{a^2}{r^2}\right)\cos\theta\right\}\right]} \tag{17}$$

If, for example, $\alpha=0.25$, we see from (14) that the value of σ given by (16) corresponds to a condition where the value of the potential-gradient with the sun at the zenith is 1.5 times the value with the sun at the antipodes. Further, if, with this value of α, we choose β so that the conductivity at an altitude of 10 kilometers, with the sun at the zenith, is 30 times the corresponding value at the earth's surface,[1] we find $\beta=1817$. Of course, (17) shows that the potential-gradient at this altitude would attain only about one thirtieth of its value at the earth's surface, as experiment shows to be the case. This latter conclusion, which is only a bi-product of the present problem, would of course have followed on much more elementary considerations than those here presented.

Writing

$$z=1-\left[\frac{a}{r}-\frac{\alpha}{3(1+\alpha)}\left(\frac{r}{a}-\frac{a^2}{r^2}\right)\cos\theta\right]$$

we see that

$$\frac{dz}{dr}=\frac{a}{r^2}+\frac{\alpha}{3(1+\alpha)}\left(\frac{1}{a}+\frac{2a^2}{r^3}\right)\cos\theta$$

so that z increases for all values of r for $\theta<\frac{\pi}{2}$, and even for $\theta=\pi$, z increases with r for values of r/a less than that given by

$$\left(\frac{a}{r}\right)^2=\frac{\alpha}{3(1+\alpha)}\left[1+\frac{2a^3}{r^3}\right]$$

For $\alpha=0.25$ as cited above, the value of r/a greater than unity which satisfies this equation is approximately 3.82, giving $z=0.99$ approximately. With the value 1817 for β, which was chosen above so as to make the conductivity at an altitude of 10 km. thirty times that at the surface, we should have, for the minimum value of σ at $r=3.82a$,

$$\sigma=\sigma_0\times10^{781}$$

so that expression (16) indicates a practically perfect conductivity at $r=3.82a$. Further, the expression indicates a more rapid increase of conductivity with altitude when $\theta=$zero than for any other value of θ, and is thus of a type to represent qualitatively the influence of a radiation from the sun. The constant α may be taken as indicating the intensity of the sun's effect. Absence of any effect due to the sun would be symbolized by a zero value for α, in which case (16) would indicate equal rates of increase of σ and equal rates of decrease of $\frac{\partial V}{\partial r}$ for all values of θ. Moreover, (14) would, for this case, indicate constancy of the potential-gradient over the earth's surface.

It may be observed that the form which we have chosen for σ in this illustrative example, leads to a diminution with altitude when r is sufficiently great and $\cos\theta$ is negative. This fact is not fundamental to our problem, however; for if, in place of (16) for σ, any other function be chosen which is sensibly the same as (16) for $r<3.82a$, then, for a given total charge on the earth's surface, the distribution of $\left(\frac{\partial V}{\partial r}\right)$ in the region $r<3.82a$ will be sensibly equal to (17), as may be seen from the following argument, which is very similar to the usual proof of the uniqueness theorem for solutions of La Place's equation, and the equation of continuity.

[1] The results of A. Wigand (Terr. Mag., v. 19, pp. 93-101, 1914) indicate a value 37×10^{-4} E. S. U. for δ at 9,000 meters. For σ at the earth's surface, values ranging all the way from 0.4 to 3.0 E. S. U. are quoted.

Suppose there be two solutions V and V_1. Then, from Green's theorem,

$$\int\int\int (V-V_1)\left[\frac{\partial}{\partial x}\left(\frac{\sigma\partial(V-V_1)}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\sigma\partial(V-V_1)}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\sigma\partial(V-V_1)}{\delta z}\right)\right]dxdydz=-\int\int\int\sigma\left[\left\{\frac{\partial(V-V_1)}{\delta x}\right\}^2+\left\{\frac{\partial(V-V_1)}{\partial y}\right\}^2+\left\{\frac{\partial(V-V_1)}{\partial z}\right\}^2\right]dxdydz+\int\int(V-V_1)\left[\sigma l\frac{\partial(V-V_1)}{\partial x}+\sigma m\frac{\partial(V-V_1)}{\partial y}+\sigma n\frac{\partial(V-V_1)}{\partial z}\right]ds \quad (18)$$

The left-hand side is zero on account of the equation of continuity. The surface integral is taken over the surface composed of $r=a$ and $r=3.82a$. Over $r=a$, V and V_1 are both constants. Over $r=3.82a$, V and V_1 are sensibly constant, in view of the enormous conductivity there. Hence, over each surface, the integral is proportional to the corresponding value of

$$\int\int\left[\sigma l\frac{\partial(V-V_1)}{\partial x}+\sigma m\frac{\partial(V-V_1)}{\partial y}+\sigma n\frac{\partial(V-V_1)}{\partial z}\right]ds \quad (19)$$

This integral is zero: For, over $r=a$, σ is constant, and the integral is proportional to the difference between the total charges on the earth's surface for the two solutions, which difference is zero by the conditions of the problem. The integral (19) over the surface $r=a$ represents the difference of the total currents from the earth for the two solutions, so that this difference is zero; and, as a result, the difference between the total currents through $r=3.82a$, for the two solutions, must be zero. It follows from this last conclusion that the value of (19) over $r=3.82a$ is zero. Hence, the total surface integral in (18) is zero.

We thus conclude that the remaining triple integral in (18) is zero, and this necessitates that $V-V_1$ shall be, at most, a constant, since the triple integral is a sum of squares. Again, the constant must be zero, since $V=V_1$ over $r=a$. Hence, within $r=3.82a$, the two solutions V and V_1 are sensibly the same.

From the physical standpoint the significance of the above demonstration exists in the fact that the highly conducting layer acts as an electric shield, which prevents external agencies from doing more than produce a constant potential within. This, again, is of importance in showing us that the distribution of potential-gradient over the sphere $r=a$ has really resulted from the variable conductivity distribution assumed below the sphere $r=3.82a$, and is not a mere result of the conductivity distribution, and its concomitant charge distribution which our function (16) happens to predict for the extreme outer regions of the atmosphere ($r>3.82a$).

An interesting field of extension of the principles outlined in this article is to be found in the solar eclipse. We should expect that the sudden removal of the sun's rays from any part of the upper atmosphere would have the effect of decreasing the potential-gradient at the corresponding point on the earth's surface. This is in line with such experiments as have been made on the variation of the potential-gradient during the eclipse. Reverting to the crude illustration occurring at the beginning of this article, we may liken the removal of the sun's rays to the production of a dent on the inside of the outer conducting sphere figuring in the illustration.

RADIO DIRECTION CHANGES AND VARIATIONS OF AUDIBILITY.

By CARL KINSLEY and ALBERT SOBEY.

[Presented at the September meeting of the Institute of Radio Engineers, New York, N. Y., and published in their proceedings.]

SYNOPSIS.—The war afforded very good opportunity for experimenting with devices for discovering the source of wireless waves. This paper discusses experiments made with the radiogoniometer and with the audibility meter with a view to determining a possible relation between direction of source and audibility.

It is at once evident that variations of audibility and direction are in a large measure due to the changes in the uniformity of the transmitting medium. This is discussed to determine whether or not the observed discontinuities are explainable upon this basis. It is suggested that much is dependent upon the ionization of the atmosphere by sunlight and the consequent absorption of electrical energy, which would account for the greater radius of transmission at night; the existence of the reflecting Heaviside layer, at an altitude of about 80 km.; and the nighttime irregular character of the ionization at the base of the stratosphere, at about 10 km., forming variable reflecting surfaces for waves.—*C. L. M.*

INTRODUCTION.

The development of methods for the determination of the point from which radio signals originate was actively carried on during 1918. The radiogoniometer measurements played an increasingly important rôle during the fall of last year. It seemed probable that the so-called radio compass might be called on to an extraordinary extent to assist in the projected operations of the spring of 1919, when an extensive use of aircraft would make necessary an exact method of navigation for cloudy weather both by day and night.

There had been no opportunity, however, for an examination of the degree to which reliance could be placed upon that method of determining directions when the origin of the radio signals was known. During the winter and spring of 1919 we were fortunately able to carry on an extended series of measurements under the peculiarly favorable circumstances of widely separated stations operated by well-trained observers.

DISCUSSION OF PROBLEM.

It has been found that radio messages can be read at much greater distances than the previously existing theories of the transmission of electromagnetic waves had indicated as possible. As soon as an appreciable arc of the earth's surface had been spanned it became evident that the radio waves must bend around the earth. MacDonald [1] was one of the first to show that the waves might be refracted sufficiently. This theory criticised, corrected, and finally approved, has been discussed by Rayleigh and Poincaré,[2] also by Poincaré in a series of articles, [3] and by Love.[4] Other equations were developed by Nicholson in a series of papers,[5] and by Rybegynski.[6] Sommerfeldt [7] had introduced a correction term by considering a surface wave at the discontinuity in the specific inductive capacity which occurs at the earth's surface.

All of the comparatively simple theories differed materially in the assumptions made so that the constants differed widely. In one particular, however, they agreed,

[1] Proc. Royal Soc., vol. 71, p. 251, 1903. Proc. Royal Soc., vol. c. 72, p. 59, 1904. Trans. Royal Soc., vol. c. 210, A, p. 113, 1910.
[2] Proc. Royal Soc., vol. 72, A, 1903.
[3] La Lumière Électrique, vol. 42, 1908.
[4] Trans. Royal Soc., vol. 215 A, p. 105 and 125, 1915.
[5] Phil. Mag., vol. 19, 1910.
[6] Ann. der Physik, vol. 41, 1913.
[7] Ann. der Physik, vol. 28, 1909.