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MATHEMATICAL THEORY OF THE GRAPHICAL EVALUATION OF METEOROGRAPH SOUNDINGS BY MEANS OF THE STÜVE (LINDENBERG) ADIABATIC CHART

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INTRODUCTION

The use of airplanes by meteorological services to obtain pressure, temperature, and humidity values in the free air necessitated the adoption of rapid methods for the evaluation of the records, in order that the data might be communicated as promptly as possible to designated centers for use in air-mass analysis and daily forecasting. Previously, when kites, and captive and sounding balloons were the principal means of making such soundings, it was customary in some services to use a more or less laborious arithmetical method, based on the Laplace hypsometric equation, for computing the heights of the successive levels attained by the recording instrument. This was unsatisfactory because of the time consumed in the computations; and several semigraphical or graphical methods of obtaining essentially the same results were devised. One such method is that of V. Bjerknes (**1**), which necessitates the construction of a diagram with virtual temperature [1] plotted against the logarithm of the barometric pressure, and the use of four auxiliary tables in the computation of heights. A more straightforward graphical method is that of Stüve (**2**), which is an adaptation of the older Hertz-Neuhoff (**3, 4**) adiabatic diagram, with at least one feature (graphical humidity correction) apparently adopted from the Bjerknes virtual-temperature diagram. The resulting diagram (fig. 1), commonly known as the adiabatic chart, also embodies several new and valuable features, and facilitates greatly the evaluation of aerological soundings with an accuracy comparable with that of the observational data. It therefore was adopted by the United States Weather Bureau and other services several years ago.

It is our purpose in this paper: (1), to describe the groundwork coordinate system that forms the adiabatic chart; (2), to show how an airplane meteorological sounding, excepting the determination of altitude, is evaluated through its use; (3), to develop the mathematical theory that underlies the graphical computation of altitude by means of the chart; and (4), to indicate the various graphical methods of different degrees of approximation which may be used on the chart for determining the altitudes of the successive levels of a sounding.

I. GROUNDWORK OF THE ADIABATIC CHART

1. *Temperature and pressure.*—The network of vertical and horizontal lines on the chart (fig. 1) represents a rectangular Cartesian coordinate system, with a uniform scale of temperature for the abscissas, and a logarithmic scale of barometric pressure for the ordinates. The vertical lines, separating unit distances on the abscissa scale, are drawn for each whole degree of temperature in degrees centigrade, with values ordinarily ranging from +35° C. on the right to −45° C. on the left. The horizontal lines extending entirely across the chart and cutting the logarithmic ordinate scale are drawn for every 10 millibars [2] of pressure, with values ordinarily ranging from 1,050 mb at the bottom to 400 mb at the top. The indicated ranges of pressure and temperature suffice for airplane ascents made to elevations of 17,000 feet in continental United States.

In the actual construction of the coordinate system of the adiabatic chart, it is customary in the United States to use a space of 4.5 mm to represent an interval of 1° C. It is also customary to construct the horizontal pressure lines by measuring from the line for 1,000 mb as a datum and drawing the remainder of the lines at distances therefrom in accordance with the relationship $75 \log_{10}\left(\frac{1{,}000}{P}\right)$ cm, where P is pressure in mb. The number 75 is here merely an arbitrary modulus. A smaller scale which is convenient for the construction of larger charts used in the evaluation of sounding balloon observations is two-thirds of that just stated, viz, 3.0 mm per 1° C., and 50 for the modulus of the pressure scale.

On this coordinate system, free-air temperatures obtained from an aerological sounding may be plotted against the corresponding barometric pressures.

2. *Potential temperature.*—The family of curves that run in the general direction from the upper left hand corner of the chart to the lower right hand corner are curves of constant (or equal) potential temperature [3] called "adiabats". The potential temperature, θ in ° K., of air containing an average amount of water vapor (disregarding effects of possible condensation of water vapor), for any given temperature and pressure is computed according to the equation $\theta = T\left(\frac{1000}{P}\right)^{0.288}$, where T is in degrees absolute (Kelvin) and P is in mb and the exponent is the value of $\left(\frac{R}{JC_p}\right)$ in which J is the mechanical equivalent of heat, and R and C_p are respectively the gas constant and specific heat at constant pressure for the humid air. The figures associated with

[1] For the definition of this term see the next section of this article (p. 125).

[2] 1 millibar (mb) represents a pressure of 1,000 dynes per square centimeter, or 1 mb $\simeq \left\{\begin{array}{l}0.75006 \text{ mm} \\ 0.0295299 \text{ inch}\end{array}\right\}$ of mercury under standard conditions of temperature and gravity.

[3] The potential temperature of absolutely dry air may be defined as the temperature it would assume if brought adiabatically, i. e., without gain of heat from the environment or loss thereto, from its given initial conditions of temperature and pressure to a final constant reference pressure, usually (and here) chosen to be 1,000 mb.

the adiabats and running nearly diagonally in the direction from the lower left to the upper right hand corners of the chart represent the potential temperatures corresponding to the given adiabats. It is obvious that an adiabat may be constructed by selecting for θ an arbitrary value, and solving for the values of T corresponding to a series of values assumed for P, or alternatively one might solve for the values of P corresponding to a series of values assumed for T.

Since θ is merely a function of T and P, the potential temperature corresponding to any given temperature and pressure may be immediately read from the family of curves under consideration.

3. *Relative humidity.*—There is no special scale printed on the chart for plotting relative humidity against barometric pressure; however, it is customary to select some space to the left of the chart, and regard a horizontal interval corresponding to 10° of temperature as representing an interval from 0 to 100 percent of relative humidity. Accordingly, free-air relative humidities may be plotted against barometric pressures (ordinates) for corresponding levels of a sounding.

4. *Vapor pressure.*—The partial pressure of aqueous vapor in the atmosphere, or vapor pressure, is computed from the relative humidities and temperatures observed in the sounding according to the following fundamental relation:

$$e = r \cdot e_s(t)$$

where e = vapor pressure; r = relative humidity (expressed as a decimal); $e_s(t)$ = partial pressure of saturated aqueous vapor (*over water*, where the hair hygrometer is used) at the given temperature of the air, t. Values of $e_s(t)$ are given in the Smithsonian Meteorological Tables and other standard works. The vapor pressure, obtained in the manner just shown, may be plotted (on a horizontal scale) against barometric pressure (on the vertical scale) in the allotted space on the farthest right-hand side of the chart. The scale for this purpose, indicated over the

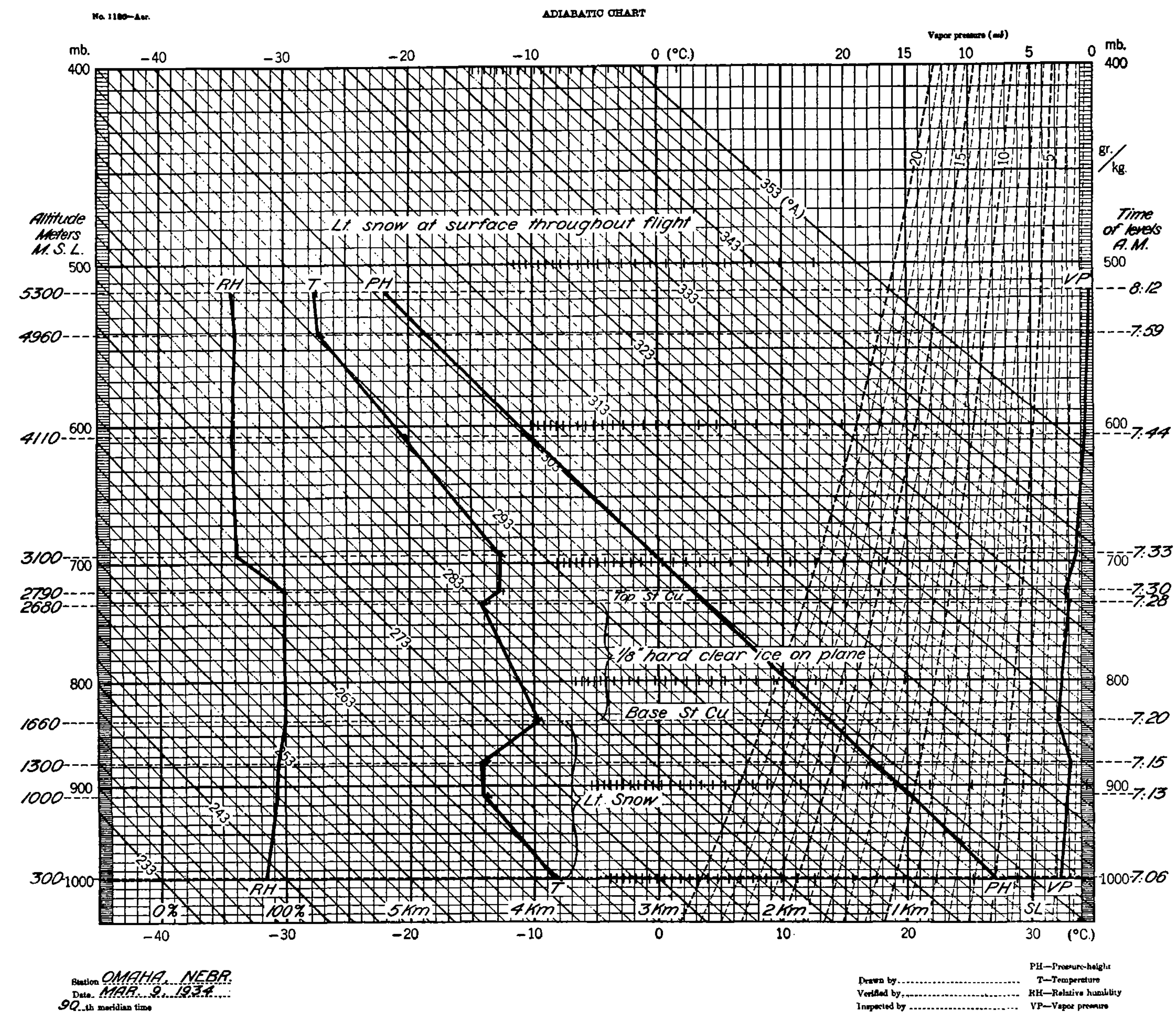

Figure 1.—Adiabatic chart showing representation of meteorograph sounding made at Omaha, Nebr., March 9, 1934.

upper right-hand corner of the chart, is such that a horizontal distance corresponding to 1° of temperature represents a vapor pressure of 1 mb. The zero of this scale coincides with the farthest right-hand margin of the chart, and *increasing values* of vapor pressure are represented *to the left of this.*

5. *Specific humidity.*—The family of curves printed in the form of dashed lines on the right-hand side of the chart represent curves of constant (or equal) specific humidity.[4] The specific humidity of a space that contains the normal proportion of ordinary atmospheric constituents plus water vapor is computed from the equation

$$s=623\,\frac{e}{p-0.377e}\ \mathrm{g/kg}$$

where s=specific humidity in grams of water vapor per kg of moist air; e=vapor pressure; p=barometric pressure. The two arguments necessary for this equation, viz. vapor pressure and barometric pressure, are respectively represented by the horizontal vapor pressure scale on the extreme right side of the chart referred to in the last paragraph, and by the vertical logarithmic scale of barometric pressure forming the ordinates of the chart. The curves corresponding to specific humidities of 5, 10, 15, and 20 g/kg are indicated by figures associated with the appropriate curves. These figures are near the upper portions of the dashed curves (see fig. 1). These curves enable one to ascertain the specific humidities that correspond to given values of vapor pressure and barometric pressure.

6. *Virtual temperature.*—The virtual temperature [5] on the absolute scale, which is used as an auxiliary in computing altitudes by means of the chart, is given by the equation

$$T_v=\frac{T}{1-0.377\frac{e}{p}}$$

where T is the temperature of the air in the absolute scale (degrees Kelvin), $T=(273.2+t)$, t in ° C. The difference between the virtual temperature and the actual temperature of the air is given by

$$(T_v-T)=T\left(\frac{1}{1-0.377\frac{e}{p}}-1\right)$$

If the space under consideration were saturated, the vapor pressure, e, in the above equations would take on a limiting value $e_s(T)$, that is, the saturation vapor pressure which is a function of the air temperature alone. Hence the virtual temperature would take on a corresponding limiting value which we will call the saturation virtual temperature represented by the symbol T_{sv}. Then obviously

$$(T_{sv}-T)=T\left(\frac{1}{1-0.377\frac{e_s(T)}{P}}-1\right)$$

Thus $(T_{sv}-T)$ is a function of T and P alone. The magnitudes of the differences $(T_{sv}-T)$ are indicated on the chart by the horizontal distances between the short vertical dashes shown on the horizontal lines corresponding to the multiples of 100 mb. of pressure. The value of T upon which is based each of the differences so indicated is that corresponding to the center of the respective interval.

The last two equations given above may be simplified by performing the indicated divisions on the right-hand sides and neglecting terms containing second or higher powers of $0.377\frac{e}{p}$ or $0.377\,\frac{e_s(T)}{p}$. This is permissible since the latter rarely exceeds 0.02, and thus squares and higher powers are negligible in comparison with the first power. Therefore

$$(T_v-T)=T\left(0.377\frac{e}{p}\right),\ \text{approximately;}$$

and

$$(T_{sv}-T)=T\left(0.377\frac{e_s(T)}{p}\right),\ \text{approximately.}$$

Since $e=r\cdot e_s(T)$, (see section 4, where r=relative humidity expressed as a fraction), we have

$$(T_v-T)=r\cdot(T_{sv}-T),\ \text{approximately.}$$

Thus

$$T_v=T+(T_v-T)=T+r\cdot(T_{sv}-T).$$

Hence if for a given pressure we wish to determine the horizontal position which corresponds to the virtual temperature under the given conditions of actual temperature and relative humidity, we merely find the position that corresponds to the actual temperature and make a displacement *to the right* equal to the fractional part r of the value $(T_{sv}-T)$. The latter is indicated by the horizontal distance between the nearest short vertical dashes on the given (horizontal) pressure line. For example, if the relative humidity is 75 percent, the virtual temperature is to the right of the actual temperature by an amount equal to ¾ $(T_{sv}-T)$; etc.

For pressures intermediate between the multiples of 100 mb. upon the lines for which the short vertical dashes representing values of $(T_{sv}-T)$ are printed, it is obviously necessary to interpolate values of $(T_{sv}-T)$ vertically from the printed dashes to obtain values appropriate to the given pressures.

7. *Height.*—A special scale for indicating heights is not printed on the chart; however, as will be proved later, the temperature scale may be made to serve this purpose. A broken line, called the pressure-height curve (indicated by *PH* in fig. 1), is constructed according to a procedure to be described in a later section (IV), and by its aid is obtained the height that corresponds to any pressure observed in the sounding. This is accomplished by first going (vertically) along the pressure scale until one locates on the *PH* curve a point at the pressure for which one desires to find the corresponding height. Then as the theory developed in section III shows, the height (in meters) at which this pressure exists *above the station elevation* is obtained when the constant 102 is multiplied by the *horizontal projection* (*measured in degrees centigrade*) of the *PH* curve in passing from its lower extremity to the given point. To facilitate the computations of height, it is customary to enter near the foot of the chart a height scale, employing a temperature interval of 10° C. to represent a height interval of 1,000 m. However, as stated above, the projections of the *PH* curve must be multiplied by 102 to give heights in meters, so that a

[4] The specific humidity of a given space may be defined as the ratio of the mass of water vapor contained in the space to the total mass of moist air (including water vapor or other gases) in the same space. The unit customarily used to represent specific humidity is: Grams of water vapor per kilogram of moist air, i. e. g/kg, which is a unit 1/1000th as large as that given by the definition.

[5] The virtual temperature of (moist) air at a given barometric pressure is the temperature at which dry air would possess the same density as the given (moist) air when at the same barometric pressure.

10° C. projection really represents 1,020 meters, or 1° C. represents 102 meters. Heights are customarily represented *above sea level* by drawing the *PH* curve so that it is displaced horizontally with respect to the zero of the height scale by an amount (in ° C.) equal to the height (in meters) of the surface station above sea level divided by 102, and then regarding zero height as sea level. In this manner, readings on the height scale, when increased by 2 percent, give true heights above sea level directly.

If the true heights above sea level corresponding to given barometric pressures are known for a number of levels in the free air (as for example when such data are received in daily airplane weather reports), a *true PH* curve may be directly constructed from the data. This curve does not require the 2-percent correction referred to above. (Section IV of this paper may be consulted for further details regarding this type of *PH* curve.)

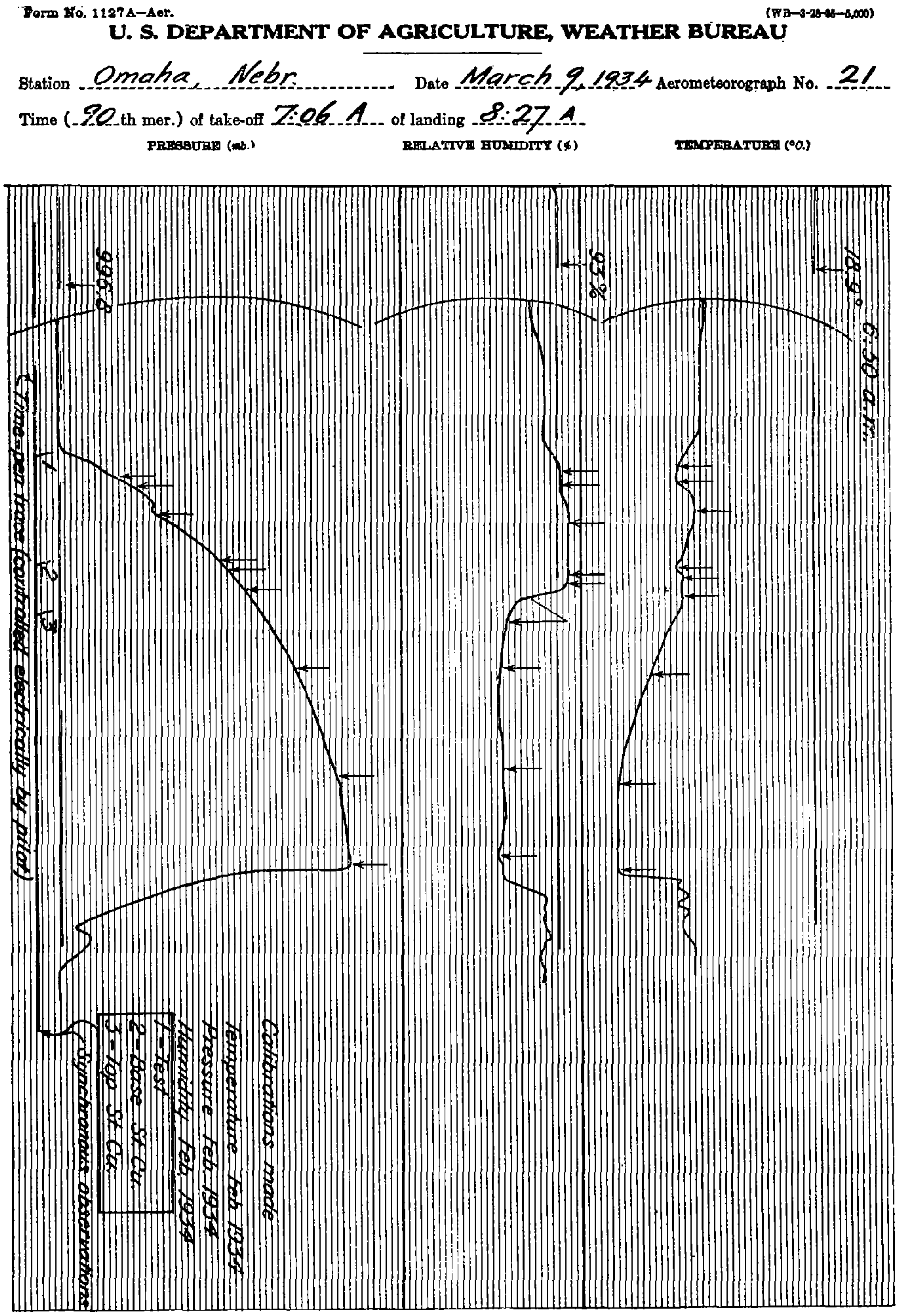

FIGURE 2.—Meteorogram obtained in the sounding made at Omaha, Nebr., March 9, 1934. Arrows indicate points representing "significant levels."

II. PROCEDURE FOLLOWED IN EVALUATING A SOUNDING, EXCEPT FOR DETERMINATION OF ALTITUDES

Figure 2 shows a meteorogram on which are inscribed simultaneous records of the temperature, relative humidity, and barometric pressure in the free air as obtained by means of an aerometeorograph which is illus-

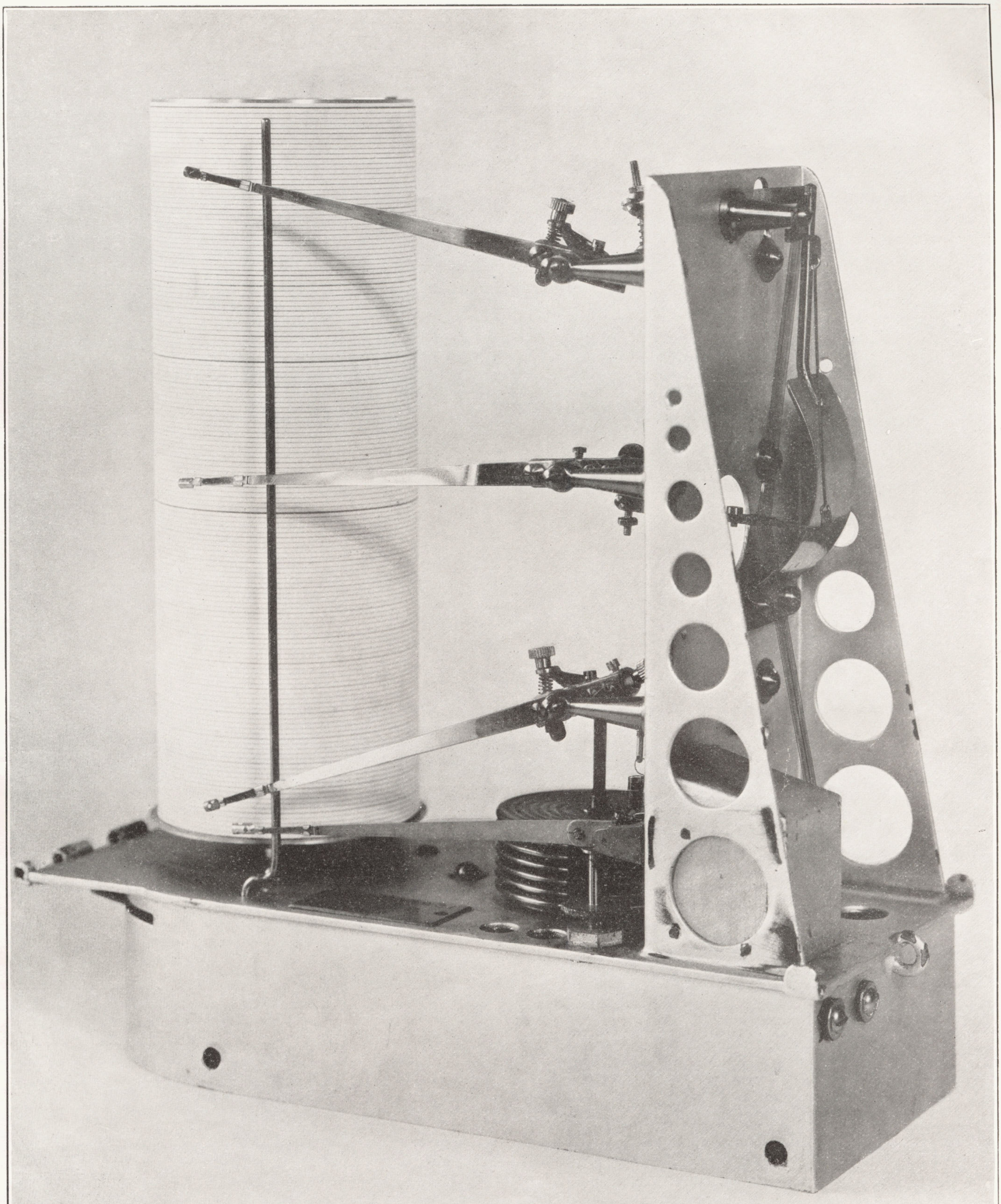

FIGURE 3.—Side view of the Friez type aerometeorograph, with protective cover removed, the instrument carried on United States Weather Bureau observation airplanes to measure and record automatically the temperature, relative humidity and barometric pressure of the free air. On the left is the cylindrical drum which rotates by clockwork and carries a ruled sheet, the meteorogram, on which the three upper pens trace records of the three data in question. Near the lower center is the sylphon element, an evacuated box of thin metal which serves in the capacity of an aneroid barometer. Near the right center, mounted on the upright support, is the curved bimetallic element for measuring the temperature; and beside it, to the left and extending above and below, may be seen a strand of human hairs whose change in length with change in relative humidity permits the measurement of the moisture in the atmosphere. The lowest pen, actuated by a small electro-magnet controlled by a button switch in the pilot's cockpit, is contacted with the meteorogram when the pilot wishes to note the time of any observation during flight.

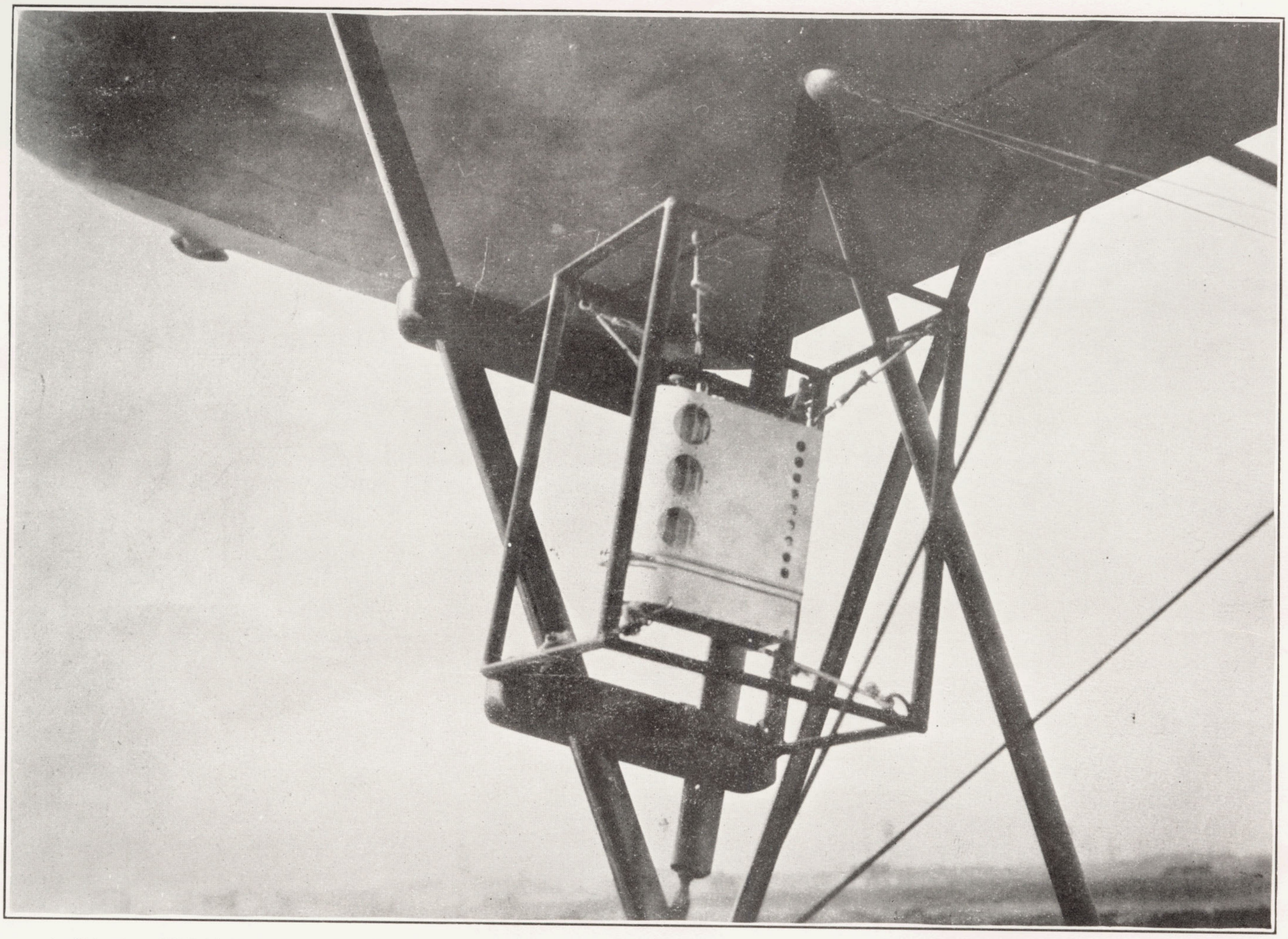

FIGURE 4.—Side view of Friez type aerometeorograph, with protective cover fastened into place, mounted between the wings of a biplane in a skeleton rectangular frame of metal which is rigidly attached to the sides of the struts. The connections between the corners of the aerometeorograph and the frame are of rubber shock-absorbing cord to reduce vibration of the instrument. The holes on the right of the cover are for ventilating the temperature and humidity measuring elements. The three mica windows near the left permit a view of the pens with the cover closed. The leading edge (to the left) is stream-lined. Dimensions: Length from leading edge to rear =8¾ inches, width =4 inches, height =10 inches, approximately.

trated in figure 3. Figure 4 illustrates the latter instrument mounted on a biplane in position for a sounding.

The following paragraph describes briefly the procedure employed by the Weather Bureau in evaluating the continuous record of free-air conditions made by the instrument in flight after the record is placed at the disposal of the ground observer following the completion of the airplane ascent.

After the completion of a sounding, points are marked off on the meteorogram traces to indicate significant levels, that is, levels where marked changes in the vertical temperature or relative humidity gradients have occurred. Synchronous points on the three traces are determined by taking equal distances along the horizontal time scale measured from the arcs which mark the known time when the pens are finally set to give a continuous record (called second pens-down). The vertical displacements of these synchronous points are measured from the respective base lines of the three traces. (The base lines are horizontal portions of the traces made just prior to the sounding while the aerometeorograph is subjected to known constant conditions of pressure, temperature, and relative humidity in a well-ventilated highly humid box.) The displacements of the given base lines from the base lines employed during the calibration of the instrument are determined by referring the known base-line conditions of p, t, and r to the calibration tables (or curves) for the given instrument and finding the corresponding ordinate displacements. These respective base-line displacements from the calibration positions are added to the appropriate displacements of the synchronous points from the respective base lines for the given meteorogram, and the total ordinate displacements thus obtained when referred to the calibration tables (or curves) give the corresponding values of free-air pressure, temperature, and relative humidity.

[All pressure readings of the instrument are corrected for the effect of temperature upon the pressure recording element in accordance with the well-known equation (5)

$$\delta p = -\Delta t(A + \alpha p)$$

where δp=error in the recorded pressure; Δt=(temperature of instrument at the time error is to be determined) *minus* (temperature of the instrument at the time of the pressure calibration), algebraically; p=pressure indicated by the instrument at the time the error (δp) is to be determined; A and α are constants. $\alpha = -0.00013$, very closely, for the type of instrument pictured in figure 3.

In this type of instrument, when the pressure element is properly calibrated, the value of A for a given instrument should lie between 0.05 and 0.10, preferably at about 0.078. It is not desirable to use a pressure element which has a value of A greater than 0.30.]

The columns headed "Observed values" in table 1 exhibit the free-air barometric pressures, temperatures, and relative humidities obtained from the meteorogram illustrated in figure 2, in the manner outlined in the above paragraph.

The vapor pressures corresponding to the observed values of temperature and relative humidity are next computed by means of the equation

$$e = r . e_s(t)$$

which has already been discussed (I:4). The results of these computations are also shown in table 1.

TABLE 1.—*Evaluation of ascent of Aerometeorograph no. 24, Mar. 9, 1934, 90th meridian time, Omaha, Nebr.*

Significant levels	Observed values				Computed values	Graphically obtained values from chart			
	Time (a. m.)	Barometric pressure	Temperature	Relative humidity	Vapor pressure	Altitude above sea level	Potential temperature	Specific humidity	Virtual temperature −273
		Mb	*°C.*	*Percent*	*Mb*	*Meters*	*°K.*	*Gr./Kg.*	*°C.*
Surface	7:06	998.3	−8.5	84	2.71	300	264.6	1.7	−8.2
1	7:13	910.2	−14.1	94	1.94	1,000	266.0	1.3	−13.9
2	7:15	877.3	−14.3	94	1.91	1,301	268.6	1.4	−14.1
3	7:20	835.2	−9.7	100	2.94	1,657	277.3	2.2	−9.4
4	7:28	732.4	−14.2	100	2.04	2,683	283.1	1.7	−13.9
5	7:30	722.3	−12.8	100	2.29	2,790	285.8	2.0	−12.5
6	7:33	692.6	−12.8	61	1.40	3,096	289.2	1.3	−12.6
7	7:44	606.5	−20.4	58	.70	4,111	291.7	.7	−20.3
8	7:59	540.8	−27.3	60	.39	4,957	293.3	.4	−27.2
9	8:12	515.3	−27.7	57	.36	5,304	296.9	.4	−27.6

PILOT'S NOTES.—Maximum altitude of 17,400 feet reached at 8:12 a. m. Entered base of St Cu clouds at 5,000 feet at 7:20 a. m. (contact no. 2 on meteorogram). Peak of St Cu clouds at 8,800 feet at 7:28 a. m. (contact no. 3 on meteorogram). These were 10/10 overcast. There were 3/10 St clouds at time of take-off but was unable to get contact on base or peak of these. They were estimated to be at 1,500 and 3,000 feet, respectively. Hard, clear ice of about ⅛ inch thickness was accumulated. Altitudes in feet above surface by altimeter.

ADDITIONAL NOTE.—⅛ inch hard, clear ice formed on all exposed parts of plane. Times when plane entered and left clouds in which ice formed: Entered, 7:20 a. m.; left, 7:28 a. m. Times of beginning and ending of precipitation encountered during flight: Snow, 7:06 a. m. (take-off) to 7:20 a. m. Snowing from surface to base of St Cu clouds. Light snow observed at surface throughout flight by weather observer at the airport.

The steps necessary in plotting the results of the sounding on the adiabatic chart are as follows (see fig. 1):

(1) *Isobaric lines.*—First, sets of points are located on the vertical logarithmic pressure scales, one on each side of the chart, to correspond with the respective barometric pressures for the several significant levels. The sets of points on each side of the chart are connected by straight horizontal lines (called isobaric lines).

(2) *Temperature-pressure curve.*—On each of these horizontal (isobaric) lines there is indicated by a dot the temperature prevailing at the corresponding level. These points are connected by straight lines. The letter T is placed at the top and bottom of the broken line thus obtained to identify it as the temperature-pressure curve.

(3) *Humidity-pressure curve.*—A convenient 10° interval to the left of the chart is selected for the humidity scale; then, below the isobaric line that represents the surface barometric pressure there are written the limiting values 0 percent and 100 percent, respectively, at the left and right extremities of this interval. On this horizontal scale an interval of 1° C. thus represents 10 percent of relative humidity. As in the case of temperature, there is indicated by a point on the isobaric line for each significant level the corresponding relative humidity,

and all the points thus plotted are connected by straight lines. The top and bottom of the resulting broken line is labeled with the letters $R\ H$ for identification purposes.

(4) *Vapor pressure-pressure curve.*—Using the horizontal vapor-pressure scale indicated near the upper right-hand corner of the chart, the vapor pressures for the several levels are indicated by placing a point on each of the respective isobaric lines. Straight lines are drawn connecting the points thus obtained, and the broken line so formed is labeled with the letters $V\ P$ at top and bottom.

(5) *Virtual temperatures.*—On each of the isobaric lines drawn, there is indicated by a point the virtual temperature for the level in question. This is accomplished graphically by applying the proper virtual-temperature correction (T_v-T), which is always positive, to the temperature for the same level. To obtain (T_v-T), one goes vertically on the chart from the temperature at the given pressure level to the nearest horizontal lines, above and below, representing pressures which are multiples of 100 mb. Then the values $(T_{sv}-T)$ there indicated by the distances between the short vertical dashes are subdivided graphically to obtain the proportion thereof equal to the relative humidity at the given level. This gives values (T_v-T) for the standard 100 mb isobaric lines, and a slight interpolation is then necessary to reduce to the correction applicable at the isobaric line for the given level. This correction is indicated by a dot (fig. 1) placed at the proper distance to the right of the point representing the temperature for the significant level under consideration. The temperatures corresponding to the positions of these dots consequently represent the virtual temperatures for the levels. The column in table 1 headed "Virtual temperature−273" indicates the values obtained from the results of the sounding illustrated in figure 1.

(6) *Potential temperature.*—The potential temperature for any level, if desired, may be found from the adiabatic chart by noting the value assigned to the adiabat passing through the intersection of the temperature-pressure curve (T) with the isobaric line for the level in question. The data in the column of table 1 headed "Potential temperature" shows the values thus found from figure 1.

(7) *Specific humidity.*—The specific humidity for any level, if desired, may be found from the adiabatic chart by noting the value assigned to the curve of constant specific humidity which passes through the intersection of the vapor pressure-pressure curve, $V\ P$, with the isobaric line for the given level. The column of table 1 headed "Specific humidity, g/kg" indicates the values thus obtained from figure 1.

III. MATHEMATICAL THEORY UNDERLYING THE GRAPHICAL DETERMINATION OF HEIGHTS OF LEVELS IN THE FREE AIR BY MEANS OF THE ADIABATIC CHART

Symbols employed

ρ=density of moist air (grams/cm^3).
P=total barometric pressure of moist air (dynes/cm^2)*.
e=vapor pressure, i. e. partial pressure of aqueous vapor of moist air (dynes/cm^2)*.
T=absolute temperature of moist air (°K.).
T'=virtual temperature of moist air (°K.).
ρ_0=density of pure dry air under standard conditions of pressure and temperature (grams/cm^3).
P_0=standard pressure (dynes/cm^2).
T_0=standard temperature (°K.).
R=gas constant for 1 gram of pure dry air, $\frac{\text{dyne.cm.}}{\text{gram.°K.}}$
g=acceleration due to gravity at level of moist air (cm./sec.^{2}).
h=height of moist air (cm).
J=mechanical equivalent of heat, ergs/cal.
C_p=specific heat of dry air at constant pressure, cal./gram. °K.

*Pressure units used are dynes/cm^2 *up to* equation (5), but mb. beginning with equation (5) and continuing.

From the hydrostatic equation, we have, in the free air where the density of moist air is ρ and the gravity acceleration g,

$$dP=-\rho g\cdot dh, \tag{1}$$

in which dP denotes vertical change in barometric pressure and dh change in height.

From the characteristic equation for perfect gases, and from the known ratio between the density of water vapor and that of pure dry air, viz. 0.623 under average conditions, we may express the density of moist air in terms of measurable quantities. Thus

$$\rho=\frac{1}{R}\,\frac{P}{T}\left(1-0.377\frac{e}{P}\right),\ \text{where} \tag{2}$$

$$\frac{1}{R}=\rho_0\frac{T_0}{P_0} \tag{3}$$

from which $R=2.8686\times10^6\ \frac{\text{dynes}\cdot\text{cm.}}{\text{grams}\cdot°\text{K.}}$, where we take the standard values

ρ_0=density of pure dry air under standard conditions P_0 and T_0, viz. 0.001293 gm./cm.^{3}.

P_0=1 atmosphere (=760 mm. of mercury)=1013250 dynes/cm^2.

T_0=freezing point of water=273.18° K. **(6)**

Substituting (2) in (1), we obtain

$$\frac{dP}{P}=-\frac{g}{R}\cdot\frac{1}{T}\left(1-0.377\frac{e}{P}\right)dh, \tag{4}$$

which is the differential form of the hypsometric equation.

We now proceed to show how the terms in equation 4 may be evaluated by means of the adiabatic chart, and the equation thus solved graphically.

Consider an adiabatic chart formed of a rectangular coordinate system with absolute temperatures, T, as abscissas and $\log_e\left(\frac{1000\ mb.}{P_{mb.}}\right)$ as ordinates (where P *from hereon* is understood to be measured in millibars,[6] i. e., 1,000 dynes/cm^2, instead of dynes/cm^2), upon which coordinate system there is constructed the family of curves of constant potential temperature given by the equation

$$\theta=T\ \left(\frac{1000}{P}\right)^{\frac{R}{JC_p}}, \tag{5}$$

where J=mechanical equivalent of heat; C_p=specific heat of dry air at constant pressure; R=gas constant for 1 gram of pure dry air.

[6] This is permissible since P hereafter occurs only in the form of ratios $\frac{dP}{P}$ or $\frac{e}{P}$, so that if the same units, viz, mb., are used in both numerators and denominators, the values of the ratios remain unchanged whether expressed in dynes/cm^2 or mb.

The exponent of $\left(\frac{1000}{P}\right)$ is later denoted by k. (See page 131.)

Taking natural logarithms[7] of both members of this equation, we get

$$\log \theta=\log T+\left(\frac{R}{JC_p}\right)\log\left(\frac{1000}{P}\right), \tag{6}$$

whence

$$\theta=\epsilon^{\log T+\left(\frac{R}{JC_p}\right)\log\left(\frac{1000}{P}\right)}, \tag{7}$$

where ϵ=base of Napierian logarithms.

From (7) it is evident that θ is a scalar point function of the coordinates of the adiabatic chart, T and $\log\left(\frac{1000}{P}\right)$, that is, with each point on the chart specified by a value T and a value $\log\left(\frac{1000}{P}\right)$, there is associated a value of the scalar quantity θ that depends only upon the coordinates of the point. It is therefore permissible to find the rate of change of θ with distance along any direction in the field of the scalar point function θ, viz, the adiabatic chart.

Let s represent distance measured on the adiabatic chart; then taking the directional derivative of θ in the direction of the tangent to any one member of the family of curves θ=constant (adiabats), defined by either of the last three equations, we get from (7), after simplification, since $\theta\neq 0$,

$$\left(\frac{d\theta}{ds}\right)_\theta=0=\frac{1}{T}\left(\frac{dT}{ds}\right)_\theta+\left(\frac{R}{JC_p}\right)\left(\frac{d\log\left(\frac{1000}{P}\right)}{ds}\right)_\theta \tag{8}$$

where the notation $\left(\frac{d}{ds}\right)_\theta$ represents differentiation with respect to s in such a direction as to keep θ constant. We note that by taking the directional derivative along the tangent to an adiabat, we are choosing a direction in which θ remains constant, hence the left member of equation (8) becomes zero as indicated.

From (8) we get

$$\frac{1}{T}=-\left(\frac{R}{JC_p}\right)\frac{\left(\frac{d\log\left(\frac{1000}{P}\right)}{ds}\right)_\theta}{\left(\frac{dT}{ds}\right)_\theta} \tag{9}$$

Equation (9) obviously reduces to

$$\frac{1}{T}=-\left(\frac{R}{JC_p}\right)\left(\frac{d\log\left(\frac{1000}{P}\right)}{dT}\right)_\theta \tag{10}$$

where the expression

$$\left(\frac{d\log\left(\frac{1000}{P}\right)}{dT}\right)_\theta$$

just introduced represents the slope of an adiabat in the coordinate system T, $\log\left(\frac{1000}{P}\right)$. Hence (10) states that the reciprocal of the absolute temperature at any point on the adiabatic chart is proportional to the slope of the tangent to the adiabat through that point. A deduction we may draw from (10) is that the slopes of the adiabats at all points along a given isotherm (vertical line on the adiabatic chart) are identical.

Introducing the following simplifying notation, let

$$y\equiv\log\left(\frac{1000}{P}\right),\text{ and} \tag{11}$$

$$x\equiv T;\text{ then equation (10) obviously may be rewritten} \tag{12}$$

$$\frac{1}{T}=-\left(\frac{R}{JC_p}\right)\left(\frac{dy}{dx}\right)_\theta,\text{ where} \tag{13}$$

$\left(\frac{dy}{dx}\right)$ represents the slope of the adiabat on the chart at the coordinates T, $\log\left(\frac{1000}{P}\right)$, i. e., (x, y).

We have thus arrived at a graphical means of expressing the reciprocal of T, which is necessary for the graphical solution of equation (4). There remains the factor $\left(1-0.377\frac{e}{P}\right)$ to take into account. This may be done as follows: From the considerations in the paragraph immediately following equation (10), we note that the slope of an adiabat at a point on the chart depends on the value of the absolute temperature corresponding to that point, but not on the value of $\log\left(\frac{1000}{P}\right)$. Hence for each temperature, T, there is a corresponding slope of the adiabats which is independent of the pressure. Now, consider a point on the adiabatic chart with coordinates T', $\log\left(\frac{1000}{P}\right)$, where, in general, $T'\neq T$. From (13) we get

$$\frac{1}{T'}=-\left(\frac{R}{JC_p}\right)\left(\frac{dy}{dx}\right)'_\theta \tag{14}$$

where the prime associated with the symbol on the right denotes that the pertinent temperature is T' instead of T. From (13) and (14), we get

$$\left(\frac{dy}{dx}\right)'_\theta=\left(\frac{dy}{dx}\right)_\theta\cdot\frac{T}{T'} \tag{15}$$

Let T' have the particular value

$$T'\equiv\frac{T}{\left(1-0.377\frac{e}{P}\right)},\text{ which is the expression for virtual temperature; then (15) becomes} \tag{16}$$

$$\left(\frac{dy}{dx}\right)'_\theta=\left(\frac{dy}{dx}\right)_\theta\left(1-0.377\frac{e}{P}\right). \tag{17}$$

Now multiplying both members of (13) by the factor $\left(1-0.377\frac{e}{P}\right)$, and substituting (17) in the equation thus obtained, we get

$$\frac{\left(1-0.377\frac{e}{P}\right)}{T}=-\left(\frac{R}{JC_p}\right)\left(\frac{dy}{dx}\right)'_\theta \tag{18}$$

We note that

$$\frac{dP}{P}=d\log P=-d\log\left(\frac{1000}{P}\right)=-dy; \tag{19}$$

[7] Logarithms hereafter designated by "log" represent Napierian logarithms. Logarithms to the base 10 are designated by "$\log_{10}$".

hence on substituting equations (18) and (19) in (4) and simplifying, we obtain the result

$$dh = -\left(\frac{JC_p}{g}\right)\frac{dy}{\left(\frac{dy}{dx}\right)'_\theta} \tag{20}$$

From geometrical considerations, we see that $\left(\frac{dy}{dx}\right)'_\theta$ is the trigonometric tangent of the angle between the adiabat and the isobar (horizontal line) at the point where the coordinates on the adiabatic chart are T', log $\left(\frac{1000}{P}\right)$; and dy is the vertical distance between two neighboring isobars at the same coordinates. Therefore, the ratio of the latter to the former represents the horizontal projection of that portion of the geometric tangent to the adiabat which is bounded by the two neighboring isobars. That is,

$$dh = -\left(\frac{JC_p}{g}\right)(dx)'_\theta = -\left(\frac{JC_p}{g}\right)(dT)'_\theta \tag{21}$$

where $(dx)'_\theta \equiv (dT)'_\theta$ is the projection, on the axis of abscissas (T axis), of the line-segment tangent to the adiabat at T', log $\left(\frac{1000}{P}\right)$ and delimited by the isobars

$$\left[\log\left(\frac{1000}{P}\right) - \tfrac{1}{2} d \log\left(\frac{1000}{P}\right)\right]$$

and

$$\left[\log\left(\frac{1000}{P}\right) + \tfrac{1}{2} d \log\left(\frac{1000}{P}\right)\right]$$

Therefore, with the neglect of the small vertical variation in gravity, equation 21 tells us that on the adiabatic chart, *horizontal projections* of small segments of adiabats give a relative (graphical) measure of the differences in height that correspond to given differences in pressure when the virtual temperatures for the intervals of pressure in question are known. Then integrating equation (20) between the limits of height (h_2, h_1) and the corresponding limits of pressure (P_2, P_1), we obtain

$$\int_{h_1}^{h_2} dh = -\left(\frac{JC_p}{g}\right)\int_{P_1}^{P_2}\frac{dy}{\left(\frac{dy}{dx}\right)'_\theta} \tag{22}$$

If the virtual temperature, T', is constant throughout a given finite interval of pressure, then the slope of the adiabat corresponding to this value of T' is constant, that is $\left(\frac{dy}{dx}\right)'_\theta$ =constant; and the integration in equation (22) may be performed since $dy \equiv d \log \left(\frac{1000}{P}\right)$ is integrable. Hence in the special case where T'=constant, we obtain from equation (22) the result

$$(h_2 - h_1) = \left(\frac{JC_p}{g}\right)\frac{(y_1 - y_2)}{\left(\frac{dy}{dx}\right)'_\theta} \tag{23}$$

where

$$y_1 = \log\left(\frac{1000}{P_1}\right) \text{ and } y_2 = \log\left(\frac{1000}{P_2}\right)$$

If T' varies in value throughout a given finite interval of pressure, the segments of the adiabats corresponding to the various values of T' must vary in slope; and in general we cannot rigorously perform the integration indicated in equation (22). However, through *small* finite intervals of pressure in the free atmosphere where T' is variable, we may as an approximation adopt a constant value of the slope $\left(\frac{dy}{dx}\right)'_\theta$ which corresponds to the mean value of T' for the interval. That is, we may assume that the integration will be sufficiently exact if we employ a single value of slope that represents the *mean* slope of the segments of the adiabats corresponding to the variable values of T' throughout the interval of pressure. Where the value of T' varies linearly with log $\left(\frac{1000}{P}\right)$, as is customarily assumed to be the case in aerological work for each successive layer of air (usually) marked off by discontinuities in vertical temperature gradient at top and bottom, the mean value of T' is obviously to be found at the center of the virtual temperature-pressure curve in the interval of log $\left(\frac{1000}{P}\right)$ for any given layer. Therefore the *mean* slope of the segments of the adiabats that correspond to the variable values of T' in the interval is to be found by determining the center point of the virtual temperature-pressure curve for the interval and taking the slope of the *adiabat passing through, and at, this center point*. Hence if $\left(\frac{dy}{dx}\right)'^{\text{(mean)}}_\theta \equiv m_1$ denotes the slope of the adiabat at the temperature which corresponds to the *mean* virtual temperature (i. e., center) of the interval of log $\left(\frac{1000}{P}\right)$ in question, we obtain from (22) the result

$$(h_2 - h_1) = \left(\frac{JC_p}{g}\right)\frac{(y_1 - y_2)}{m_1} \tag{24}$$

which is more general than equation (23).

The variables y_1 and y_2 obviously represent ordinates of isobars on the adiabatic chart, and m_1 represents a slope. Suppose we construct a straight line segment that has slope m_1 and terminates in the isobars y_1 and y_2; then since the equation $(y_1 - y_2) = m_1(x_1 - x_2)$, where x_1 and x_2 are abscissas corresponding to the ordinates y_1 and y_2 respectively, is the rectangular equation of a straight line, we recognize the expression $(y_1 - y_2)/m_1$ on the right of equation (24) as the value $(x_1 - x_2)$ of the projection, on the axis of abscissas, of the straight line segment from the point (x_1, y_1) to the point (x_2, y_2). The abscissas x_1, x_2 are temperatures on the chart hence if $T_1 \equiv x_1$ and $T_2 \equiv x_2$, equation (24) becomes

$$(h_2 - h_1) = \left(\frac{JC_p}{g}\right)\cdot(T_1 - T_2). \tag{25}$$

Therefore, if we construct a straight line segment on the adiabatic chart, of slope $m_1 \equiv \left(\frac{dy}{dx}\right)'^{\text{(mean)}}_\theta$ as defined above, and terminate the segment in the isobars $y_1 \equiv \log\left(\frac{1000}{P_1}\right)$ and $y_2 \equiv \log\left(\frac{1000}{P_1}\right)$ for the given layer, then the horizontal projection of the line segment expressed in ° K. (or ° C.) gives a value which, when multiplied by the constant factor $\left(\frac{JC_p}{g}\right)$ is the difference in height between the top and bottom of the layer. Since both m_1 and (y_1, y_2) may be constructed graphically on the adiabatic chart, we are thus provided with a graphical means of computing thicknesses of layers of air in the free atmosphere, provided the necessary observational data are at hand.

If it is required to compute the elevation of the upper limit of a layer superimposed upon several others, then equation (24) provides the means; for if $h_1, h_2, h_3, \ldots h_n$ are the elevations of the respective boundary surfaces separating the successive layers (h_1 being the bottom of the lowermost layer, and h_n the top of the uppermost layer), $y_1, y_2, y_3, \ldots y_n$ are the corresponding ordinates on the adiabatic chart as derived from the observed barometric pressures, and $m_1, m_2, m_3, \ldots m_{(n-1)}$ are the appropriate slopes for the respective layers (subscripts indicating the identification numbers of the lower boundaries), then for each layer we obtain an equation similar to equation (24) and by addition of these $(n-1)$ equations we obtain the result

$$(26)\quad (h_n-h_1)=\left(\frac{JC_p}{g}\right)\cdot\left[\frac{(y_1-y_2)}{m_1}+\frac{(y_2-y_3)}{m_2}+\ldots+\frac{(y_{n-1}-y_n)}{m_{n-1}}\right]$$

If h_1 is the elevation of the surface of the ground above sea level, then the elevation h_n above sea level is obviously

$$(27)\quad h_n=\left(\frac{JC_p}{g}\right)\left[\frac{g}{JC_p}\cdot h_1+\frac{(y_1-y_2)}{m_1}+\frac{(y_2-y_3)}{m_2}+\ldots+\frac{(y_{n-1}-y_n)}{m_{n-1}}\right]$$

Equation (27) tells us that we may graphically determine the elevation above sea level of any point in the free air, in terms of the total horizontal projection on the adiabatic chart of the appropriate straight line segments for the successive layers of air, provided we add an allowance for the elevation of the ground equal to $\left(\frac{g}{JC_p}\cdot h_1\right)$; (see the curve designated *PH* in fig. 1).

It is now necessary to evaluate the term $\left(\frac{JC_p}{g}\right)$. When the value of C_p refers to perfectly dry air we recognize this as the expression for the reciprocal of the dry adiabatic lapse rate (see, for example, Humphreys' Physics of the Air, second edition, 1929, p. 28). To compute this term on the basis of the latest accepted values of the several constants involved, we shall adopt the following values:

$J=4.1852 \times 10^7$ ergs per 15° calorie (given by Birge (**6**)); $C_p=0.2405$ cal.$_{15°}$/gram . ° K. at one atmosphere pressure and 0° C. for dry air free of CO_2 (given by Holborn, Scheel, and Henning (**7**)); this should be essentially the same as C_p for atmospheric dry air containing 3 parts of CO_2 per 10,000 parts of air, since the value of C_p for CO_2 is 0.197 at 0° C.

The value of C_p given by Moody (**8**) (viz, 0.2412 cal.$_{20°}$/g.°K. at one atmosphere pressure and 20° C.) when reduced to the standard conditions employed above, becomes 0.24057, in essential agreement with that adopted herein.

The values of gravity acceleration (g) herein adopted are based on the United States Coast and Geodetic Survery formula (**9**) for the variation of g with latitude at sea level, and Helmert's formula (**10**) for the variation of g with height above sea level. Since most airplane weather observation ascents attain an elevation of about 5,000 meters, we shall employ values of g pertinent to the mean height of 2,500 m. Thus we obtain:

g (at $\varphi=30°$, $h=2500$ m.)$=978.556$ cm./sec^2.
g (at $\varphi=40°$, $h=2500$ m.)$=979.400$ " "
g (at $\varphi=50°$, $h=2500$ m.)$=980.299$ " "
g (standard gravity) $=980.665$ " "

where φ=latitude, and h=height above sea level.

Let k denote the exponent of $\left(\frac{1000}{P}\right)$ in eq. (5) [also the constant coefficient in eq. (18)], then (compare equations 4, 18, 19, and 20) from the above values of the adopted constants we find

$$\frac{1}{k}\cdot\left(\frac{R}{g}\right)=\frac{1}{\left(\frac{R}{JC_p}\right)}\cdot\left(\frac{R}{g}\right)=\left(\frac{JC_p}{g}\right)$$

$=10286$ cm./°K. for $\varphi=30°$, $h=2500$ m.
$=10277$ " " $\varphi=40°$, "
$=10268$ " " $\varphi=50°$, "
$=10264$ " " standard gravity.

Converting these to round numbers expressed in meters per ° K., we finally obtain 103 meters per ° K. for the value of $\left(\frac{JC_p}{g}\right)$. Hence applying this to equations (23)-(27), we arrive at the conclusion that *if the adiabats are constructed* on the adiabatic chart *in accordance with the constants here adopted*, a horizontal projection of 1° C. by a straight line segment of proper slope corresponds to a difference of elevation of 103 meters.

If the adiabats are not constructed on the adiabatic chart in accordance with the values of the constants adopted above, then the factor 103 just discussed is not applicable in general. Thus if we evaluate the exponent $k\left[=\left(\frac{R}{JC_p}\right)\right]$ in equation (5) on the basis of the adopted constants, we obtain $k=0.2850$ instead of 0.288 as usually employed; the latter value is commonly assumed to be the one appropriate to average values of humidity. Assuming that $k=0.288$, then for $\phi=40°$, $h=2500$ m.,

$$\frac{1}{k}\cdot\left(\frac{R}{g}\right)=\frac{1}{0.288}\cdot\frac{2.8686\times10^6}{979.400}=10170\text{cm./°}K$$

and for standard gravity 10157 cm./°K.

Therefore, *when the adiabats are constructed on the basis of 0.288 as the exponent, which is the case in figure 1, we must employ a factor of 102 meters per °K.* in determining heights graphically by means of the adiabatic chart.

NOTE.—One may inquire why, if the actual adiabatic chart (fig. 1) has its ordinate system constructed on the basis of the relation $y_{10}\equiv75\log_{10}\left(\frac{1000}{P}\right)$, is it permissible to consider the theoretical chart which we have discussed in this section with an ordinate system constructed on the basis of the relation $y_e \equiv \log_e\left(\frac{1000}{P}\right)$. Since $\log_{10}\left(\frac{1000}{P}\right)=M\log_e\left(\frac{1000}{P}\right)$, where M is the modulus of common logarithms, 0.43429+, $y_{10}=(75\,M)\log_e\left(\frac{1000}{P}\right)=(75\,M)\,y_e$. Then $dy_{10}=(75\,M)\,dy_e$, and if $x\equiv T$ as before, $\left(\frac{dy_{10}}{dx}\right)'_\theta=(75\,M)\left(\frac{dy_e}{dx}\right)'_\theta$. By division, we have $\frac{dy_{10}}{\left(\frac{dy_{10}}{dx}\right)'_\theta}=\frac{dy_e}{\left(\frac{dy_e}{dx}\right)'_\theta}$; but equation (20) is equivalent to

$$dh=-\left(\frac{JC_p}{g}\right)\frac{dy_e}{\left(\frac{dy_e}{dx}\right)'_\theta},$$

hence by virtue of the equality just previously found we may rewrite this equation in the form

$$dh=-\left(\frac{JC_v}{g}\right)\frac{dy_{10}}{\left(\frac{dy_{10}}{dx}\right)_\theta'}.$$

Therefore, all the discussion regarding the graphical computation of height is as valid in the case of the actual adiabatic chart as in the case of the theoretical adiabatic chart.

IV. VARIOUS METHODS FOR THE CONSTRUCTION OF THE PRESSURE-HEIGHT (PH) CURVE ON THE ADIABATIC CHART, AND GRAPHICAL COMPUTATION OF HEIGHTS OF FREE-AIR LEVELS

From equation (27) it is obvious that we may graphically determine the elevation above sea level corresponding to any pressure in the free air by adding the horizontal projections (on the axis of abscissas= temperature) of several straight line segments of appropriate slope, one for each successive stratum of air, assuming an allowance is made for the elevation of the ground above sea level. We recall that the straight line segments in question must terminate in the isobars (ordinates) for the upper and lower limits of each respective stratum, and that they must each have an appropriate slope corresponding to the slope of the tangent to the adiabat at the center of the "virtual temperature-pressure" curve for the stratum (which is assumed to be linear). Under these circumstances, a horizontal projection of any segment equal to 1° C. corresponds to a difference in elevation of 102 meters.

To facilitate the addition of the horizontal projections of the straight-line segments required by equation (27), instead of constructing each segment so that its center coincides with the center of the virtual temperature-pressure curve for its particular stratum, which requires the addition of separate horizontal projections, we may displace each one horizontally, maintaining the same slope, until they join at their extremities and form a continuous (broken-line) curve. The continuous curve thus formed is called the pressure-height (*PH*) curve, shown in figure 1.

To facilitate the determination of the magnitude of the total horizontal projection of any portion of the *PH* curve from its origin (lower terminus) to any point on the curve, it is most convenient to place the origin so that the projections in degrees Centigrade may be read off at a glance. Since each degree Centigrade of projection corresponds to 102 meters elevation, it is convenient to regard the temperature scale as a sort of height scale, whereby the projection in degrees increased by 2 percent gives the elevation in hundreds of meters. Thus a 10° horizontal projection corresponds to 1,020 meters, or 1 km plus 20 meters.

The following instructions briefly indicate a procedure for the construction of the *PH* curve on the adiabatic chart, and the determination of the elevations that correspond to the pressures at the various levels attained in an aerological ascent (see fig. 1):

(1) Immediately below the isobar that corresponds to the barometric pressure at the surface, write in the height-scale values: 0 km (or S. L., representing "sea level"), 1 km, 2 km, 3 km, 4 km, 5 km, 6 km, etc., increasing values extending to the left, one value beneath successive vertical lines for consecutive printed 10° intervals of temperature. The origin (i. e., S. L. position) of this scale is placed somewhere to the right of the chart at any suitable arbitrary point located on a vertical line representing a 10° multiple of temperature. (This scale is in error by 2 percent as mentioned above, but serves to facilitate the exact computation of elevations.)

(2) Place a dot on the surface pressure isobar with reference to the height scale (constructed as just indicated) at the apparent height equal to ($h_1/1.02$) (see first term in brackets in equation (27)), where h_1 is the height of the station above sea level, more particularly the height at which the isobar in question represents the existing barometric pressure. (This should be the same height as that at which the station temperature reading was made—usually the height of the instrument shelter). For example, if both the surface barometric pressure and temperature refer to the height 300 meters above sea level, the dot should be placed at the position on the horizontal height scale corresponding to the height 300/1.02 or 294 meters. This dot is the origin of the *PH* curve.

(3) To construct the segment of the *PH* curve for the first stratum of air above the station, first refer to the temperature-pressure, *T*, curve for this stratum and mark by dots the virtual temperatures corresponding to the top and bottom, respectively, of the stratum. Imagine a line through these two dots, the "virtual temperature-pressure" curve, and mark a dot at the *center* of this imaginary line. (This dot represents the *mean* virtual-temperature for the stratum.) Find the adiabat passing through this (central) dot, or if an adiabat does not happen to pass through the dot, place some identifying mark such as a small cross on a point located vertically above the dot and lying on an adiabat. Place a draftsman's triangle at the dot or cross in question and orient it until one edge thereof is tangent to the adiabat at the point marked by the dot or cross. (This gives the slope of the straight line segment corresponding to the mean virtual temperature for the stratum.) Place a straightedge, such as a ruler, against one of the edges of the triangle other than that which is tangent to the adiabat, and slide the triangle by parallel displacement until the tangential edge passes through the origin of the *PH* curve, which is the dot marked in the manner indicated in paragraph (2) above. Draw a straight line segment along the tangential edge, starting it in the dot just referred to and ending it in the isobar for the top of the stratum. This is the first segment of the *PH* curve.

NOTE.—Instead of employing a draftsman's triangle and a straightedge as indicated above, a set of parallel rules may be used to advantage, especially in making the parallel displacement of slope.

(4) The segments of the *PH* curve for the other successive strata of air may be obtained in a manner similar to that just described for the first stratum; in each case the *initial point for any segment is the terminal point of the segment for the preceding lower stratum.* Thus a succession of segments may be constructed to form a continuous curve, marked by the letters *PH* in figure 1.

(5) To determine the elevations corresponding to the several isobars on the chart which represent the significant levels attained in the sounding, from each point where the *PH* curve intersects an isobar project an imaginary vertical line down on the height scale and read the value there indicated. *Increase each reading by 2 percent of itself*, thus finally obtaining the actual elevations in meters above sea level. The results found in this manner for the aerological sounding illustrated in figure 1 are entered in table 1 in the column headed "Altitude

above sea level, meters." These results are also entered in figure 1 in the left-hand margin just above each respective isobar.

NOTE.—Once the *PH* curve is constructed, the pressure, temperature, or other data corresponding to any particular elevation may be obtained. Thus:

(6) To determine the pressure corresponding to any given elevation, the *PH* curve must be employed in a manner the inverse of that just described. First divide the given elevation by the factor 1.02, then imagine a point on the horizontal height scale at this *diminished* elevation (i. e., $h/1.02$) and move vertically on the chart from this point until the *PH* curve is intersected. The required pressure may be determined by reading the pressure scale on the side of the chart at the isobar passing through the point of intersection in question.

(7) The values of the temperature, virtual-temperature, relative humidity, vapor pressure, potential temperature, or specific humidity corresponding to any given elevation may be determined by finding the isobar corresponding to that elevation as outlined in paragraph (6) and then noting the values of the desired elements at the intersections of the appropriate curves with the isobar in question.

It may be seen from equation (27) that if the *slope* of each segment of the *PH* curve constructed in the manner indicated in the preceding paragraphs *be decreased in the proportion (1/1.02)*, then the horizontal projections of the resulting straight line segments will fulfill the condition that 1° C. of projection corresponds to 100 meters difference in elevation instead of 102 meters. Hence, if we can modify the slopes in this proportion, the horizontal height scale referred to in paragraph (1) will give *actual* heights above sea level directly without a correction of 2 percent being necessary. This assumes, of course, that the origin of the new *PH* curve is placed at the position on the scale corresponding to height h_1 instead of $(h_1/1.02)$ as before. The slopes of the segments may easily be *decreased* in the required proportion if the horizontal projection of each segment of the original *PH* curve which we have described be *increased* in the proportion (1.02/1). This is most easily accomplished by (1) constructing a *PH* curve in the manner described in paragraphs (1)–(4) or preferably by placing a dot on each isobar where the original *PH* curve would intersect the isobar; (2) reading the horizontal projections of these dots on the height scale and increasing these readings by 2 percent; (3) placing new dots on the respective isobars at positions vertically above the height scale corresponding to the readings increased by 2 percent just referred to; and (4) constructing a new *PH* curve formed of straight line segments connecting the new dots. This new *PH* curve has the advantage that elevations corresponding to any pressure attained in the sounding may be read directly on the height scale. Conversely, pressures corresponding to any elevation may be determined directly by noting the pressures on the side of the chart at the level of the isobar that passes through the point in the *PH* curve vertically above the point on the height scale where the given elevation is marked.

Instead of making the slope of each segment of the *PH* curve parallel to the *tangent* to the appropriate adiabat at the point where it passes through the center of the virtual-temperature-pressure curve (or vertically above or below that central point), one might employ the approximation of making the slope of each segment respectively parallel to the *chord* of the portion of the appropriate adiabat bounded by the isobars for the given stratum. This may easily be accomplished with the aid of a draftsman's compass. Briefly, the instructions for this procedure are as follows:

(*a*) Beginning with the stratum between the ground and the first significant level, set the point of a compass on the intersection of the proper printed adiabat and the surface pressure isobar. With the compass thus centered, mark a point on the surface pressure isobar at the place on the horizontal height scale (see paragraphs nos. 1 and 2 above) that corresponds to the station elevation divided by the factor 1.02 (i. e., $h_1/1.02$). With this same setting of the compass, place one compass point on the intersection of the proper adiabat with the isobar for the first level. With the compass thus set, mark a point on this isobar with the other compass point extended toward the left of the chart. The point thus finally obtained on the isobar for the first level and the point designated by $h_1/1.02$ on the horizontal height scale marked on the surface pressure isobar, represent the upper and lower termini respectively of the first segment of the original *PH* curve discussed in paragraph number (3) above.

(*b*) Next, adjust the compass to the distance between the point last obtained and the intersection of the proper printed adiabat for the second stratum of air with the isobar for the first level. With this setting, place one point of the compass on the intersection of the adiabat just referred to with the isobar for the second significant level, and mark a point on this isobar with the other compass point extended toward the left of the chart. This latter point represents the upper terminus of the segment of the original *PH* curve for the second stratum, while the point last obtained in the preceding paragraph represents the lower terminus of this segment.

In a similar manner the points for all the segments of the *PH* curve may be obtained. If desired, these points may be displaced to the left in the increased proportion of 2 percent on the horizontal height scale, thus giving a new set of points through which a *PH* curve may be drawn that gives actual elevations above sea level directly without correction.

The objection to the method just outlined, wherein chords of adiabats instead of tangents are used to obtain the segments of the *PH* curve, is that it is tantamount to assuming that the vertical virtual-temperature gradient of each stratum is always equal to the dry adiabatic gradient of approximately -1 ° C. per 100 meters, whereas actually the vertical virtual-temperature gradients may have a wide range from positive values (inversions) to super-adiabatic values (algebraically less than -1 ° C./100 m). This may be seen from the following considerations. A section of an adiabat subtended by a chord must have identically the same horizontal projection as the chord. Therefore, since the difference in elevation between two levels is determined merely by the horizontal projection of the portion of the *PH* curve lying between the respective isobars for the two levels, the chord may be replaced by the portion of the adiabat which it subtends when constructing the portion of the *PH* curve lying between the isobars in question. That is, the portion of the *PH* curve for the given stratum may be made curvilinear instead of linear and yet give the same difference in elevation between the top and bottom of the stratum so long as the same horizontal projection obtains in both cases. From the interpretation of the symbol

$$\left(\frac{d\log\left(\frac{1000}{P}\right)}{dT}\right)'_\theta,$$

(see discussion following equations 10, 14, and 20, sec. III), which controls the construction of segments of the *PH* curve, and from equation (18), we note that a *given slope* of a tangent to an adiabat at a point must correspond to an *equivalent virtual temperature* represented by an isotherm (*vertical line*) passing through the point of tangency. Also, the virtual temperature corresponding to a given point on the *PH* curve must be represented as being at the same pressure that is applicable at the point in question, i. e., the isobar (*horizontal line*) passing through the point must intersect the virtual temperature-pressure curve for the stratum at a point having the appropriate virtual-temperature value. Thus, if we chose a point *A* on the portion of the adiabat which forms the *PH* curve for the stratum, there must in general, be a point *B* on the virtual temperature-pressure curve for the same stratum which lies both on a vertical line passing through *A* and on a horizontal line passing through *A* also. The only way in which both these things may occur is for *B* to coincide with *A*. If we apply this argument to every point on the curvilinear section of the *PH* curve, we conclude that the virtual temperature-pressure curve must coincide with the portion of the adiabat in question. Therefore, by constructing the *PH* curve out of chords of adiabats, we do the equivalent of assuming that the vertical virtual-temperature gradient is equal to the dry adiabatic gradient. The only justification for the procedure under consideration would appear to be the practical one that tangents to adiabats cannot be obtained with great accuracy in ordinary work, and that the errors involved through the use of chords are of the same order of magnitude as those involved through the determination of tangents.

We have shown in the preceding paragraph that the *PH* curve for a stratum wherein the vertical virtual-temperature gradient is equal to the dry adiabatic gradient should rigorously be curved and not straight, and should have a curvature and form identical with that of the portion of the adiabat along which the virtual temperature-pressure curve for the stratum lies. However, when the vertical virtual-temperature gradient is zero, that is when the virtual temperature remains constant with decreasing pressure throughout a stratum, equation (18) indicates that the slope of the adiabats at every point along the virtual temperature-pressure curve must remain constant. Therefore, when conditions in a stratum are isothermal with respect to virtual temperature, the *PH* curve for the stratum should rigorously be a straight line. On the other hand, when the vertical virtual-temperature gradient is such that the virtual

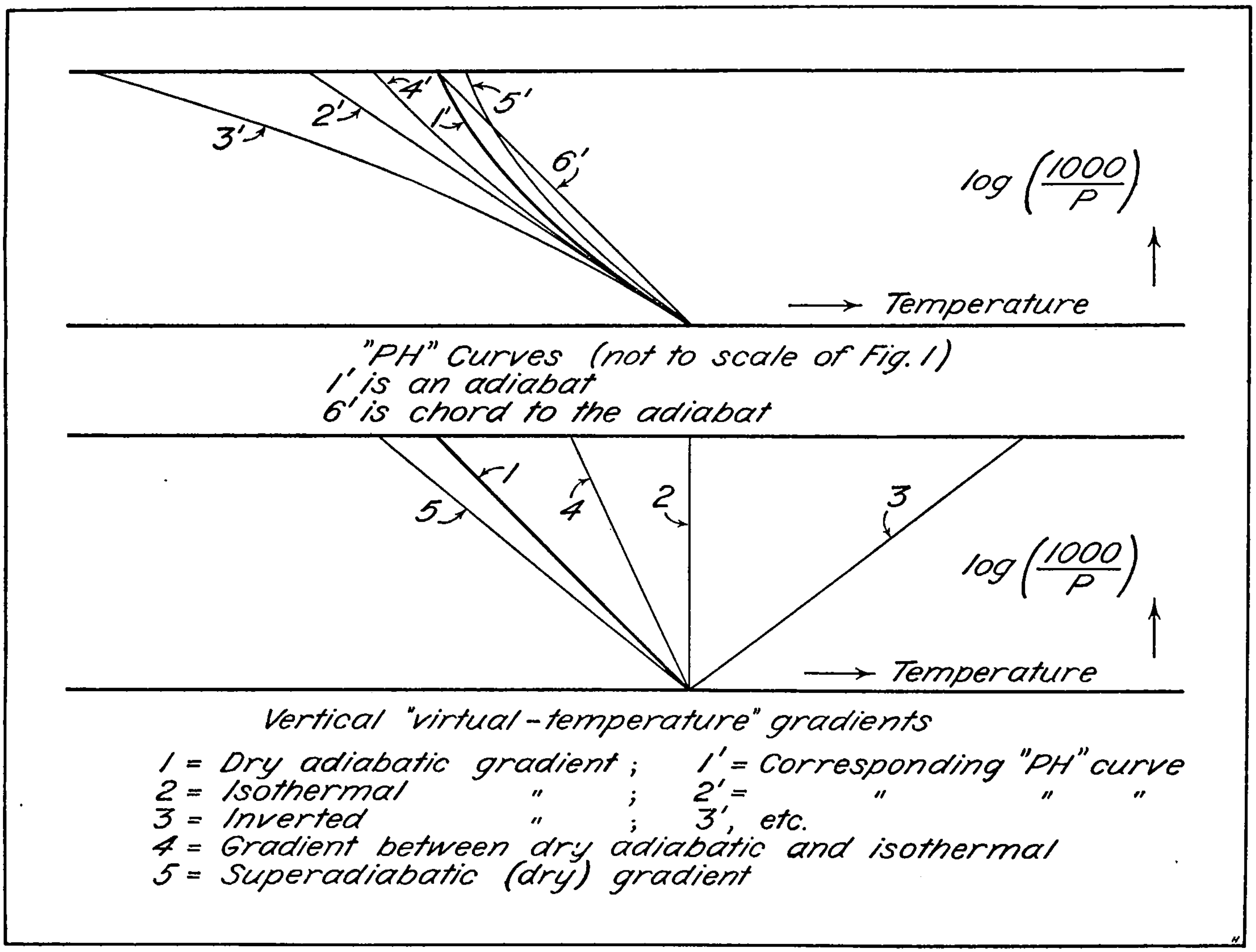

FIGURE 5.—Examples of appropriate *PH* curves (greatly exaggerated) for various vertical virtual-temperature gradients.

temperature increases with decreasing pressure, or in other words when an inversion of virtual temperature prevails in a stratum, the *PH* curve for the stratum should again be curved. Reference to the adiabatic chart portrayed in figure 1 will show that the adiabats are concave *upwards*, and that tangents to the adiabats tend to approach verticality at lower temperatures, and horizontality at higher temperatures. Therefore it follows that the appropriate *PH* curve for a stratum that has an inversion of virtual temperature should rigorously be concave *downwards*, i. e., opposite in curvature to the *PH* curve for a stratum with a dry adiabatic vertical virtual-temperature gradient.

By virtue of the facts presented in the preceding two paragraphs, it is obvious that a linear *PH* curve rigorously has a sort of median position between the appropriate *PH* curves for dry adiabatic and inverted vertical virtual-temperature gradients. This may be best seen from figure 5, which shows the three types of *PH* curves in question, *with greatly exaggerated curvature* to illustrate their respective differences; the *mean virtual temperature for the stratum* is different for each different vertical virtual-temperature gradient. Since the virtual temperature at the bottom of the given stratum shown in figure 5 is fixed, the slope of the *PH* curve at the corresponding pressure level is the same no matter what the vertical virtual-temperature gradient is in the stratum immediately above. Therefore the three *PH* curves in question all start with the same slope, viz, that corresponding to the slope of the adiabats at the virtual temperature of the bottom of the stratum. This must be the slope of the linear *PH* curve which would obtain for isothermal conditions of virtual temperature. Then as a consequence of the upward concave character of the curve appropriate for dry adiabatic conditions, and of the downward concave character of the curve appropriate for inverted conditions of virtual temperature, the two curves in question must deviate from the straight line, the former to the right and the latter to the left. The differences between the horizontal projections of the respective *PH* curves shown in figure 5 are largely accounted for by the different mean virtual temperatures for the stratum that correspond to the different virtual temperature-pressure curves shown in the lower part of the figure. If the mean virtual temperature were identical for each given virtual temperature-pressure curve, then the horizontal projections would be the same in each case.

The discussion in the last paragraph suggests that a very nearly rigorous *PH* curve may be constructed by employing templates of various curvatures depending upon the mean virtual temperatures and upon the vertical virtual-temperature gradients of the particular strata. For example, one can prepare sets of templates for say three different mean virtual temperatures. Each set may consist of templates for the following vertical virtual-temperature gradients: (1) Dry adiabatic, (2) half-way between dry adiabatic and isothermal, (3) isothermal (use straight edge), (4) moderate inversion, and (5) strong inversion. Possibly (4) and (5) may be replaced by one suitable intermediate template. The templates may be made of some transparent material, such as celluloid, similar in form to that of a triangle with one side having a curvature appropriate to a given mean virtual temperature and a given vertical virtual-temperature gradient. This triangle may conveniently be slid to and fro with one straight edge parallel to the isobars, while the curved edge retains the proper direction relative thereto. The range of vertical virtual-temperature gradients to which the given template pertains may be graphically indicated by two fine dark lines of proper slope inscribed on the transparent material. These refinements are of course not justified in many cases; under such circumstances the methods previously outlined for the construction of the *PH* curve may be employed.

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FLOODS IN THE SACRAMENTO VALLEY DURING APRIL 1935

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[Weather Bureau, Sacramento, Calif., May 1935]

On April 6–7 a low-pressure area of marked intensity and wide extent, whose center moved inland from the ocean along the California-Oregon boundary, caused heavy rains generally throughout northern California, culminating in torrential downpours in several localities of the Sacramento basin. This was the primary cause of the flood in question. However, there were two other contributing factors of importance.

First, there was an unusually heavy snow cover in the mountain area at moderate elevations; and the rapid run-off that occurred over the American, Feather-Yuba, and upper Sacramento drainage areas was augmented by melting snow from rains that extended well up into the mountain snow fields. The material source of snow water was in a belt from about 4,000 to 5,000 feet, where the snow was less compact. The winter and early spring