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ON THE RELATION BETWEEN RAINFALL AND STREAM FLOW

By Richmond T. Zoch

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INTRODUCTION

In the act of Congress transferring the meteorological work of the Signal Office to the Weather Bureau of the Department of Agriculture, approved October 1, 1890, the duties of the service are thus summarized (1):

> The Chief of the Weather Bureau, under the direction of the Secretary of Agriculture, shall have charge of forecasting the weather; the issue of storm warnings; the display of weather and flood signals for the benefit of agriculture, commerce, and navigation; the gaging and reporting of rivers; the maintenance and operation of seacoast telegraph lines and the collection and transmission of marine intelligence for the benefit of commerce and navigation; the reporting of temperature and rainfall conditions for the cotton interests; the display of frost, cold wave, and other signals; the distribution of meteorological information in the interest of agriculture and commerce; and the taking of such meteorological observations as may be necessary to establish and record the climatic conditions of the United States, or as are essential for the proper execution of the foregoing duties.

To carry out the provision, "the display of * * * flood signals for the benefit of agriculture, commerce, and navigation; the gaging and reporting of rivers" the Weather Bureau has established some sixty-odd river district centers. These centers supervise the work of about 900 substations, and issue forecasts and warnings of river stages to the public.

It is conservative to state that $3,000,000 worth of property is saved annually as a result of the Weather Bureau's flood warnings.[1]

At the present time, empirical methods solely are used by the Weather Bureau in forecasting river stages and issuing flood warnings. In large rivers where the day-to-day changes are gradual these methods are very efficient, and it is doubtful whether any other methods based on theory, however elaborate, will add very much to them. As an example of the precision of the present methods, the following is quoted from Talman: "Thus in the flood of 1903 the exact time when the crest would reach New Orleans was correctly foretold 28 days in advance and the prediction of the height of the crest was only 5 inches in error." (2)

In small streams, however, where the height of the water surface may change several feet in a few hours the present empirical methods are not so efficient. Contrast the precision of the New Orleans forecast just mentioned with a forecast for a station on a small river made recently, when a stage exceeding any that had occurred there for 15 years was predicted when, as a matter of fact, the river rose only 1.3 feet, a bare one-fifth of the rise expected.[2]

The reason for the great accuracy in predictions for large streams is the fact that for them the forecasts need not be made until the crest has passed some upstream gage; and the further fact that when an upstream gage reaches a certain stage then a gage farther downstream will, at a so much later time, reach a corresponding stage, and this relation can be worked out quite closely. In the small streams, either there may be no gage farther upstream or, if there is one, the time of flood crest travel is so short that it is not feasible to wait for the upstream gage reading; in each case, therefore, flood forecasts must be based on the rainfall. In the large streams, about the only things that cause a flood forecaster to be appreciably in error are the breaking of a levee or dam, or some other similar engineering or flood-protection work, and the erratic movements of ice. In a small stream where the flood forecasts are based on rainfall, not only are there these items to contend with but also the dryness and other conditions of the soil on which the rain falls; possible unequal distribution of rainfall over the watershed; the presence of a snow cover, if any; the effect of evaporation and transpiration; and, of course, the basic factors, i. e., the depth and the rate of the rainfall.

Evidently, then, a study of the underlying principles of the relation between rainfall and consequent run-off and stream flow ought to improve the efficiency of the Weather Bureau's flood warnings in the smaller streams. Accordingly a series of articles, of which this is the first, has been prepared on this subject.

It is well to bear in mind the limitations of any system of flood forecasting. Near the headwaters of a stream, flood crests are reached very soon after the flood-producing rain stops. Indeed, by the time a rainfall observer in the upper portion of the drainage basin has telegraphed the rainfall to a river district center, and the center in turn has tabulated such reports from the several stations in a watershed, determined the forecast, and issued it to the public, the flood crest may already have passed. Thus any system of flood forecasting based upon rainfall, however exact, is of no practical use near the headwaters of streams. Again, for the large streams the simple empirical methods are to be preferred to any complex method based on theory, when the use of the latter effects only a very slight gain in the accuracy of the forecasts. Hence the theory developed in this series of articles

[1] An excellent description of the Weather Bureau's river and flood service is given in ch. X of "Meteorology" by W. I. Milham, New York, 1912.
Further explanations are contained in a pamphlet of the U. S. Department of Agriculture entitled "The Weather Bureau", Miscellaneous Publication No. 114.
These references despite the fact that in some ways they are not up to date, give a good summary of the river and flood service.

[2] It also is interesting to know that about 3 weeks later, rains not quite so heavy occurred above this station, and on account of the previous experience no flood warning was issued. However, the river reached a crest 0.5 foot over the flood stage and caused slight damage.
In another instance, on a different river, when the gage reading was 13 feet a forecast was issued for a crest stage between 20 and 21 feet; however, the river fell.

is practically applicable only to that reach of a river that is comparatively near the headwaters, but not too near.

Yet even for these portions of the rivers of the United States it is believed that it is possible to improve greatly the accuracy of the Weather Bureau's flood warnings, and thus bring about the saving of much property annually.

The articles in this series are in three groups. The first group deals with the development of the mathematical theory of the relation between rainfall, run-off, and stream flow. The second applies this theory to some of the rivers of the United States, and shows how it is possible to forecast floods accurately from rainfall. The third group shows how it is feasible to construct nomograms from which it is possible to read off resulting river stages from given initial conditions of rain, evaporation, and soil capacity. It might be well to point out here that the purpose of the articles in the third group is to develop a ready method of applying the theory; for the practical value of any theory would be vitiated if from 1 to 6 hours of tedious computation were required, after the rainfall observer's reports were received, before a flood warning could be issued.

There will be seven articles in the first group. This, the first, treats the simple case in which the drainage area is rectangular; evaporation and transpiration are neglected; the rate of rainfall, the velocity of the water in the stream, and the condition (dryness) of the soil are constant; and where no snow cover is present. The second article will deal with irregularly shaped drainage areas, the third with evaporation (transpiration is a special case of evaporation), the fourth with varying rates of rainfall, the fifth with varying conditions of the soil, the sixth with varying velocity of water in the stream, and finally in the seventh article, the last of the first group, various combinations of the above factors will be treated.[3]

Each article consists of two parts. The first part tells in words what is accomplished mathematically, by symbols, in the second part. Those readers who are interested in flood forecasting from rainfall, or in the relation between rainfall, run-off, and stream flow, but who do not care to follow the mathematical formulas, need not read the second part of any article. However, those who intend to follow the mathematics should also read the first part of each article. The two parts of some of the articles are subdivided into sections.

PART I

SECTION 1.—THE DISCHARGE FROM A SMALL AREA

When rain falls upon the ground, part of it soaks into (or remains upon) the soil, and part runs off. Consider now that part which runs off. The volume of water running off a unit area per unit of time at any given time is here termed the *rate of run-off*. The volume of water which runs off a unit area in a given interval of time is termed the *run-off*. The run-off is a volume per unit area; the rate of run-off is volume per unit area per unit time. Mathematically, the rate of run-off is the first derivative of the run-off with respect to time.

Unfortunately, in the existing literature on hydrology the term "run-off" is used in both the above senses. It is even more confusing since the term "run-off" is also used for the *volume of run-off* from a given area.

Next consider the rain falling upon the ground. The depth of rain water, as a horizontal sheet, which falls upon the ground in a given interval of time is termed the *rainfall*. The rainfall is a length; the Weather Bureau has always measured rainfall in inches, and in these papers that practice is followed. The depth of rain water falling upon the ground per unit of time at any given time is termed the *rate of rainfall*. The rate of rainfall is depth per unit of time, and is here expressed in inches per hour. Mathematically, the rate of rainfall is the first derivative of the rainfall with respect to time. The volume of rain water which falls upon a given area in a given interval of time is termed the *volume of rainfall*. The volume of rainfall is length times area (i. e., volume), and is here expressed in mile-inches in preference to acre-feet—the latter term being common in literature on hydrology. The mile-inch is here defined as the volume of water which will cover 1 square mile to the depth of 1 inch. The mile-inch is the most convenient unit to use for the volume of rainfall since drainage areas are most commonly expressed in square miles and the rainfall in inches.

Consider further that part which runs off. The run-off, as above defined, will be expressed in inches; this is in keeping with present usage in hydrology. The rate of run-off is expressed in inches per hour rather than in cubic feet per second per square mile. It is necessary to introduce two more terms when dealing with areas other than a unit area. The volume of water which runs off a given area in a given interval of time will be termed the *volume of run-off*. The volume of run-off is length times area, and (as in the case of the volume of rainfall) will be expressed in mile-inches. The volume of water running off a given area per unit of time at any given time will be termed the *volume of rate of run-off*. The volume of rate of run-off is length per unit of time times area, and will be expressed in mile-inches per hour in preference to cubic feet per second.

Consider now the water flowing in a stream. The quantity of water flowing past a given cross section of a stream per unit of time at a given time is termed the *discharge*. The quantity of water which flows past a given cross section of a stream in a given interval of time is termed the *volume of discharge*. The volume of discharge is volume; the discharge is volume per unit time. Mathematically, the discharge is the first derivative of the volume of discharge with respect to time. Discharge is expressed in the same units as the volume of rate of run-off, i. e., in mile-inches per hour. Volume of discharge is expressed in the same units as the volume of run-off, i. e., in mile-inches.

As in the case of the term "run-off", it is unfortunate that the term "discharge" has been used in both of the above senses in the literature on hydrology.

If we confine our attention to a small parcel of ground which is drained by a single outlet, we may neglect the time required for the water that runs off to flow from where the rain falls to the outlet; in this case the *volume of run-off* from the whole small parcel of ground is synonymous with the *volume of discharge* at the outlet during any interval of time; and also the *volume of rate of run-off* is synonymous with the *discharge*.

In section 1 of part II, two equations are developed from fundamental principles. Equation (1) expresses the volume of rate of run-off as a function of time while the rain is falling. Equation (2) expresses the volume of rate of run-off as a function of time after the rain stops. In equation (2) evaporation is ignored. Evaporation will be considered in a later article.

In section 1, part II, there is also a discussion of a certain constant of proportionality introduced in the development of equations (1) and (2).

[3] Unforeseen circumstances may necessitate changing the order of some of these articles.

It is important to distinguish carefully between the terms *discharge* and *volume of rate of run-off*. When we are dealing with the volume of rate of run-off then equations (1) and (2) are true regardless of the *size and shape* of the drainage area. When we are dealing with discharge, equations (1) and (2) will be true only when the drainage area is small. The distinction between discharge and volume of rate of run-off will be more clearly brought out in section 2.

For convenience, and as a summary, all of the terms defined in this section, together with the units in which they are expressed, are here tabulated.

Term	Symbol	Units in which expressed
Rate of run-off	z	Inches per hour.
Run-off	$\int_0^t zdt$	Inches.
Rainfall	R	Do.
Rate of rainfall	$\frac{dR}{dt}=r$	Inches per hour.
Volume of rainfall	$AR=\int_0^t Ardt$	Mile-inches.
Volume of run-off	$A\int_0^t zdt$	Do.
Volume of rate of run-off	$Az=Z$	Mile-inches per hour.
Discharge	y	Do.
Volume of discharge	$\int_0^t ydt$	Mile-inches.

SECTION 2.—THE DISCHARGE FROM A RECTANGLE

The primary concern of these articles is to develop a scheme whereby the height of flood crests can be predicted from the rainfall during a storm and attendant modifying factors. The height of a flood crest is a quantity which is easily observed and measured. Now, the discharge of a stream at a given cross-section may be regarded as a function of two things—one, the gage height of the stream at this cross-section; the other, whether the stream is rising or falling, i. e., the rate of change in that height, for it is well known that the discharge of a stream for a given gage height is greater when the stream is rising than for the same gage height when the stream is falling. Therefore, at the time of a crest the discharge of a stream at a given cross-section is a function of the height only, for then the stream is neither rising nor falling, i. e., the rate of change in height is zero. We may also say that the flood crest height is a function of the maximum discharge. Thus if the maximum discharge could be predicted, then the flood crest height could be predicted, and vice versa; assuming, of course, that the functional relation between the maximum discharge and flood crest height is known.

The functional relation between maximum discharge and flood crest height will be taken up in the second group of papers in the series; for the present we shall be concerned with establishing the relation between the rainfall during a storm (and other attendant factors) and the maximum discharge.

The relation between the rainfall during a storm (and accompanying factors) and the consequent volume of rate of run-off is readily obtained, as the reader may observe later; but the relation between the rainfall and the discharge, in general, is exceedingly involved. Therefore, in order to make this relation at all tractable, simple assumptions are made. What actually happens in Nature is replaced, at first, by a reasonable, workable ideal. Thus in section 2 of part II the relation between rainfall and discharge is worked out on the assumptions that: (1) there is no evaporation, (2) the rainfall and also the rate of rainfall are constant for a given storm, (3) the drainage area is rectangular, (4) the velocity of the water in the stream is constant, (5) the condition of the soil in the drainage area is uniform, (6) there is no snow cover on the soil. Now such ideal conditions never occur in Nature; and in later papers the relation between rainfall and discharge will be obtained when the above restrictions are removed one by one. Here again, in order to make this problem tractable it is necessary to confine the treatment to relatively simple cases as the assumptions enumerated above are removed.

For a large drainage area, the time required for the water that runs off to flow from where the rain falls to the outlet is appreciable, and, naturally, varies with the distance from the outlet. For this reason the volume of run-off and the volume of discharge during a given interval of time are by no means synonymous. Neither is the volume of rate of run-off, in general, identical with the discharge. However, after a prolonged period without any rain over a given drainage area, or in other words, when a steady state has been reached, the rate of run-off becomes constant (that is to say, the rate of run-off changes but little with respect to time) and therefore the volume of rate of run-off becomes equal to the discharge. Moreover, if the interval of time is taken to begin at one steady state and end at another steady state, then the volume of run-off equals the volume of discharge.[4]

In section 2, part II, equation (7) gives the time of the flood crest (maximum discharge) in terms of the duration of the rain and other constants. Equation (8) gives the maximum discharge in terms of the rate of rainfall, the time of maximum discharge, the duration of the rain, and other constants.

PART II

FOREWORD

In preparing a paper which contains tedious mathematical developments, the writer is confronted with the question of how fully these developments should be given: If each is given in full, the cost of publication is unduly increased, and moreover the reader is likely to receive the false impression that the paper is very complicated. On the other hand if the developments are condensed too much, by the omission of intermediate steps or by inadequate explanations, the reader may be unable to follow the paper even though he has a good grasp of the mathematics used. In view of the fact that the prediction of flood crests is a very practical problem, it has been deemed advisable to go to much pains in order to make the mathematical developments clear. Accordingly the following procedure is used in this paper: In deriving equation (1), the symbolic expression for each intermediate step is given, as well as a verbal explanation. For the remaining equations, the symbols for the more simple steps are omitted. It is believed that all readers who can follow the development of equation (1) will be able to supply all the omitted steps in the rest of the paper from the explanations given.

The present paper involves no mathematics beyond elementary calculus.

SECTION 1

Equations are here developed for the relation between rainfall and the rate of run-off, on the assumption that the rate of run-off at any given time is proportional to the

[4] It is worth noting that for an interval beginning with a steady state but ending with an unsteady state, the volume of run-off equals the sum of the volume of discharge and the volume of water in the stream at the time that the interval ends.

rainfall remaining with the soil at that time. It is believed that this assumption, which is the basis for all the equations developed in this series of papers, is a very close approximation to, if not precisely, what occurs in Nature.

Let t be the time, and let $t=0$ when the rain begins; let R be the rainfall that has fallen up to time t, and put $dR/dt=r$. Consider now the case when r is constant.

Let A be the area of a parcel of ground. Now the volume of rainfall remaining with A at any time, measured from the beginning of the rain, is a function of that time; that is to say, the volume of rainfall which has fallen up to time t, less the volume of water which to then has run off, is the volume remaining and is a function of the time t; in symbols:

$$\int_0^t Ardt-\int_0^t Zdt=\phi(t),$$

where Z is the volume of rate of run-off, and $\phi(t)$ is the volume remaining.

The fundamental assumption is: $cZ=\phi(t)$, where c is an unknown constant of proportionality. Therefore we can write $cZ=\int_0^t Ardt-\int_0^t Zdt$, and by differentiation we get $cdZ=Ardt-Zdt$, then solving for dt, $\frac{cdZ}{Ar-Z}=dt$, whence by integration $t=-c\log(Ar-Z)-c\log K$; dividing by $-c$ and combining terms, we have $-\frac{t}{c}=\log K(Ar-Z)$. It follows from the definition of a logarithm that this last equation can be written

$$e^{-\frac{t}{c}}=K(Ar-Z),$$

where e is the base of natural logarithms.

To evaluate the constant of integration K, set $Z=0$ when $t=0$; i. e., we assume that when the rain begins there is no water flowing off. Then

$1=KAr$ or $K=\frac{1}{Ar}$, and on substituting this value of K we have

$$e^{-\frac{t}{c}}=1-\frac{Z}{Ar};$$

hence, on solving for Z,

$$Z=Ar\left(1-e^{-\frac{t}{c}}\right). \tag{1}$$

As $t\rightarrow\infty$, then $Z\rightarrow Ar$, which means that if the rain be prolonged without limit a state will be reached when there is just as much water flowing off the area as there is rain falling upon it. This is exactly as would be expected, because when the soil is completely saturated with water a steady state will be reached soon thereafter, when all the water which falls as rain must run off as surface water.

However, it might be argued that when the soil becomes completely saturated with water a greater volume of rain will remain upon the soil than previously soaked into the soil, and that equation (1) will not represent the behavior of the volume of rate of run-off after the soil is completely saturated. To investigate this, consider a lake or a pond. Assume that no water flows into the lake from the surrounding land. Rain falling upon such a lake will behave precisely as rain falling upon saturated level ground. Assume that the lake has one outlet, and that this outlet is sufficiently wide so that the discharge increases linearly with increase of the depth of the water flowing in the outlet. At the beginning of the rain we assume the lake to be just full, that is, there is no water flowing in the outlet. Under these conditions the heaviest rains of record would not cause the depth of water flowing in the outlet to reach a large value. Therefore the assumption that the discharge increases linearly with increase of depth of water flowing in the outlet is certainly valid.

Let h be the depth of water in the lake (and also in the outlet), measured from the height of the surface of the lake when the rain begins, at the time t. Let y be the discharge and let $y=\bar{c}h$ where $\bar{c}$ is a constant. Also let A be the area of the lake.

Now the volume of rainfall which has fallen up to time t, less the volume of water which to then has flowed away, is the volume remaining upon the lake; or in symbols:

$$\int_0^t Ardt-\int^t \bar{c}hdt=Ah.$$

Then by differentiating we get:

$Ardt-\bar{c}hdt=Adh$; solving for dt and integrating, $t=-\frac{A}{\bar{c}}\log(Ar-\bar{c}h)-\frac{A}{\bar{c}}\log K$. Now, multiplying by $-\frac{\bar{c}}{A}$, and combining terms, it follows from the definition of a logarithm that we can write:

$$e^{-\frac{\bar{c}}{A}t}=K(Ar-\bar{c}h).$$

As stated in words above, when $t=0$, then $h=0$, whence $K=\frac{1}{Ar}$. Therefore on substituting this value of K and solving for h, we have:

$$h=\frac{Ar}{\bar{c}}\left(1-e^{-\frac{\bar{c}}{A}t}\right).$$

Multiplying by c and recalling that $\bar{c}h=y$, we can write

$$y=Ar\left(1-e^{-\frac{\bar{c}}{A}t}\right).$$

This last equation is of exactly the same form as equation (1) and was derived not upon the assumption that the rate of run-off at any given time is proportional to the rainfall remaining upon the lake at that time, but upon the assumption that the discharge is proportional to the depth of water in the outlet. Under the stated conditions the depth of water in the outlet is never very large, and the assumption that the discharge is proportional to the depth of water in the outlet is then certainly valid.

From the above discussion of rain falling upon a lake, it will be clear to the reader that equation (1), which is based on the assumption that the rate of run-off at any given time is proportional to the rainfall remaining with the soil at that time, is rigorously true when the soil is completely saturated with water. However, equation (1) may not represent exactly the volume of rate of run-off when the soil is not completely saturated, because certain factors, e. g., capillary action and osmotic pressure (in the roots of plants), have been ignored. For the primary purpose of these articles it is believed that equation (1) is sufficiently accurate, certainly so as a first approximation, whether all the rain soaks into the soil, all remains upon the soil, or part soaks into the soil and part remains upon the soil.

The constant c, equation (1), depends upon the nature and condition of the soil. It does not vary during any particular rain but does vary from rain to rain, depending upon how much moisture the soil contains at the beginning of the rain and whether the ground is freshly tilled or fallow, covered with vegetation or bare, the kind of vegetation if covered, and whether or not the ground is frozen. Moreover, the constant c will change from area to area, depending upon the type of soil.

The constant c may be considered as consisting of two parts, say c' and c''; one, c', due to the fact that the soil is pervious and hence water soaks *into* the soil; and the other, c'', owing to the fact that not all the rain which does not soak into the soil flows off instantaneously, but a part remains *upon* the soil. When c is thus divided, the second part c'' not only remains constant during any particular rain but also from rain to rain for a given small area. The part c'' depends of course upon the precipitous nature of the land, being close to zero for steep slopes. Over generally level land, especially if many sloughs or hollows are present, c'' may have a rather high value. Neither c' nor c'' varies greatly from area to area so long as all the territory in question is of the same geological formation.

Equation (1) expresses the volume of rate of run-off as a function of the time while the rain continues. It may be termed the equation of rise. When the rain stops, the rate of run-off immediately decreases; equation (1) then no longer applies, but an equation of fall which will now be developed.

Let Z_0 be the volume of rate of run-off from the area A at the time the rain stops. Let $t'=t-t_0$ where t_0 is the duration of the rain. (Also, since $t=0$ when the rain begins, t_0 is the time that the rain stops.) Moreover t_0 is a constant; also when $t=t_0$ then $t'=0$ and $Z=Z_0$.

As in the equation of rise, it is assumed that at any time the rate of run-off is proportional to the rainfall remaining with the soil. That is to say, after the rain stops, the volume of rate of run-off at any time t is proportional to the volume which fell as rain, less the volume which ran off while it was raining, less the volume which has run off during the interval from the time the rain stopped to the time t. Now the volume which fell as rain is a constant, viz. AR, and the volume which ran off while it was raining is also a constant, say $F(t_0)$; whence we can write the previous sentence in symbols, thus: $cZ=AR-F(t_0)-\int_{t_0}^{t} Zdt$. (It may be well to point out that in this last equation the constant c has the same value for a given piece of ground as it had at the beginning of the rain, as in equation (1); and that $F(t_0)$ is the integral of equation (1) between the limits 0 and t_0.) Then, since the first two terms on the right-hand side of this last equation are constants, we have by differentiating: $cdZ=-Zdt$. Clearly, $dt=dt'$. Therefore $cdZ=-Zdt'$; and on dividing by cZ and integrating, $-\frac{t'}{c}=\log KZ$, whence from the definition of a logarithm we have:

$$KZ=e^{-\frac{t'}{c}}.$$

To evaluate the constant of integration K, put $Z=Z_0$ when $t'=0$; then $KZ_0=1$ or $K=\frac{1}{Z_0}$, and

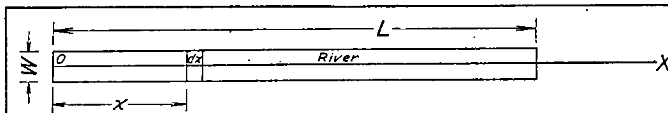
$$Z=Z_0e^{-\frac{t'}{c}}, \tag{2}$$

which is the equation of fall.

SECTION 2

As stated previously, equations (1) and (2) when regarded as expressing the volume of rate of run-off will be correct regardless of the size and shape of the drainage area. When regarded as expressing the discharge, equations (1) and (2) are correct only when the drainage area is small.

Consider now a large drainage area that is rectangular. Let W be the width and L be the length of the rectangle. (See fig. 1.) Let the gaging station be at the origin, and let the river coincide with the X-axis. Let x be the distance of each small (infinitesimal) element of area above the outlet (gaging station). Let v be the velocity of the moving surface water, and consider v to be constant. Let y be the discharge at time t at the gaging station, and let $t=0$ when the rain begins. Assume that the river is dry when the rain begins. Each infinitesimal area above the gaging station may be considered as contributing an infinitesimal portion to the discharge y at the gaging station. At time t each infinitesimal area contributes to y not its discharge at time t but its discharge at the time t diminished by the time required for its water to flow to



The gaging station is at the origin "O"
The river coincides with the X-axis
Distances to the right of O are positive
The river flows to the left

Figure 1

the gage. The time required for the water from an infinitesimal strip, Wdx, to flow to the gage is x/v, where x is the distance of the infinitesimal strip above the gaging station. Hence, it follows from equation (1) that, at time t, each infinitesimal area contributes a discharge of $Wr\left[1-e^{-\frac{1}{c}\left(t-\frac{x}{v}\right)}\right]dx$ to the discharge y at the gaging station. Now, y is the sum of the discharges from all of the infinitesimal areas above the gage and is therefore the integral of the above expression. If the interval from the beginning of the rain to the time t is so small that the water from the upper portions of the drainage area has not had time to reach the gage, the upper limit of the integral is not the length of the drainage area L but is that value of x such that $t=\frac{x}{v}$, i. e., it is that value of x such that the water from distance x has had just sufficient time to reach the gage. Hence, integrating the above expression from 0 to tv,

$$y=Wrv\left[t-c+ce^{-\frac{t}{c}}\right]. \tag{3}$$

Equation (3) expresses y as a function of t. When $t=0$ then $y=0$, and when $t=\frac{L}{v}$, then

$$y=Wr\left[L-cv+cve^{-\frac{L}{cv}}\right]. \tag{3a}$$

Equation (3) holds only on the range $0 \leqq t \leqq \frac{L}{v}$, with the further restriction that $t \leqq t_0$. It should be noted that t is not restricted to this range for any mathematical reasons, but because of the physical nature of the problem. During the interval $0 \leqq t \leqq \frac{L}{v}$ the discharge at the gaging station, y, is increasing from two causes: first, the soil in the drainage area is approaching the limit of its capacity for water, hence a greater and greater portion of the rain falling upon the drainage area is flowing off as free water; secondly, the area which contributes to y is increasing. As soon as the whole drainage area above the gage is contributing to y, then y increases from the first cause only.

Suppose $t \geqq \frac{L}{v}$ and also $t \leqq t_0$. Then integrating the same expression as before, from 0 to L now,

$$y = Wr\left[L - cve^{-\frac{t}{c}}\left(e^{\frac{L}{cv}} - 1\right)\right]. \tag{4}$$

Equation (4) expresses y as a function of t and holds on the range $\frac{L}{v} \leqq t \leqq \infty$ with the restriction that $t \leqq t_0$. In other words, as long as it continues to rain, equation (4) applies. When $t = \frac{L}{v}$ equation (4) reduces to equation (3a), and when $t = t_0$ equation (4) becomes:

$$y = Wr\left[L - cve^{-\frac{t_0}{c}}\left(e^{\frac{L}{cv}} - 1\right)\right]. \tag{4a}$$

Equations (3) and (4) are equations of rise. An equation of fall will now be derived. Suppose that $t > (t_0 + \frac{L}{v})$. Then it follows from equation (2), by reasoning similar to the above, that

$$y = \int_0^L Wr\left[1 - e^{-\frac{t_0}{c}}\right]e^{-\frac{1}{c}\left(t' - \frac{x}{v}\right)}\,dx$$

in which expression the product $Wr\left[1 - e^{-\frac{t_0}{c}}\right]dx$ is the discharge from the infinitesimal strip Wdx at the instant that the rain stops. Whence, by carrying out the indicated integration,

$$y = Wr\left[1 - e^{-\frac{t_0}{c}}\right]cv\left[e^{-\frac{1}{c}\left(t' - \frac{L}{v}\right)} - e^{-\frac{t'}{c}}\right].$$

Then on putting $t' = t - t_0$:

$$y = Wrcv\left[e^{-\frac{1}{c}\left(t - t_0 - \frac{L}{v}\right)} - e^{-\frac{1}{c}(t - t_0)} - e^{-\frac{1}{c}\left(t - \frac{L}{v}\right)} + e^{-\frac{t}{c}}\right]. \tag{5}$$

Equation (5) holds on the range $t_0 + \frac{L}{v} \leqq t \leqq \infty$. When $t = t_0 + \frac{L}{v}$ equation (5) becomes:

$$y = Wrcv\left[1 - e^{-\frac{L}{cv}} - e^{-\frac{t_0}{c}} + e^{-\frac{1}{c}\left(t_0 + \frac{L}{v}\right)}\right]. \tag{5a}$$

Also when $t = \infty$ then $y = 0$; that is to say, when t is taken sufficiently large the discharge ceases.

An equation for the range $t_0 \leqq t \leqq \left(t_0 + \frac{L}{v}\right)$ will now be obtained. $\left(\text{It is supposed that } t_0 > \frac{L}{v}\right)$. If $\left(t - \frac{x}{v}\right) < t_0$ the contribution of the area Wdx to the discharge at the gaging station is

$$dy = Wr\left[1 - e^{-\frac{1}{c}\left(t - \frac{x}{v}\right)}\right]dx,$$

while if $\left(t - \frac{x}{v}\right) > t_0$ then the contribution of the area Wdx is

$$dy = Wr\left[1 - e^{-\frac{t_0}{c}}\right]e^{-\frac{1}{c}\left(t - t_0 - \frac{x}{v}\right)}dx.$$

Consider the time t, and select x_0 as the value of x such that $t - \frac{x}{v} = t_0$, i. e., $x_0 = (t - t_0)v$.

Then:

$$\begin{aligned} y &= \int_{x_0}^{L} Wr\left[1 - e^{-\frac{1}{c}\left(t - \frac{x}{v}\right)}\right]dx + \int_0^{x_0} Wr\left[1 - e^{-\frac{t_0}{c}}\right]e^{-\frac{1}{c}\left(t - t_0 - \frac{x}{v}\right)}dx \\ &= Wr\Big[L - x_0 + cv\Big\{e^{-\frac{1}{c}\left(t - \frac{x_0}{v}\right)} - e^{-\frac{1}{c}\left(t - \frac{L}{v}\right)} \\ &\quad + \left(1 - e^{-\frac{t_0}{c}}\right)\left(e^{-\frac{1}{c}\left(t - t_0 - \frac{x_0}{v}\right)} - e^{-\frac{1}{c}(t - t_0)}\right)\Big\}\Big]. \end{aligned}$$

But since $x_0 = (t - t_0)\,v$ and $t - \frac{x}{v} = t_0$ the above reduces to:

$$y = Wr\left[L - (t - t_0)v + cv\left\{1 + e^{-\frac{t}{c}} - e^{-\frac{1}{c}\left(t - \frac{L}{v}\right)} - e^{-\frac{1}{c}(t - t_0)}\right\}\right]. \tag{6}$$

Equation (6) holds on the range $t_0 \leqq t \leqq t_0 + \frac{L}{v}$. When $t = t_0$ equation (6) reduces to equation (4a), and when $t = t_0 + \frac{L}{v}$ equation (6) reduces to equation (5a). Equation (6) may be considered as a transition between equation (4) and equation (5). In equation (4) the contributions from all of the infinitesimal areas are increasing; in equation (5) the contributions from all of the infinitesimal areas are decreasing. Between these two cases lies equation (6), where the contributions from part of the infinitesimal areas are increasing while the contributions from the remainder are decreasing.

The time of the maximum discharge, i. e., the time of the flood crest, may be found by obtaining the first derivative of equation (6) with respect to t and equating it to zero. Thus:

$$\frac{dy}{dt} = Wr\left[-v + v\left\{-e^{-\frac{t}{c}} + e^{-\frac{1}{c}\left(t - \frac{L}{v}\right)} + e^{-\frac{1}{c}(t - t_0)}\right\}\right].$$

Designate the time of the crest stage by t_c; then:

$$-1 - e^{-\frac{t_c}{c}} + e^{-\frac{1}{c}\left(t_c - \frac{L}{v}\right)} + e^{-\frac{1}{c}(t_c - t_0)} = 0;$$

multiplying by $e^{\frac{t_c}{c}}$, and transposing,

$$e^{\frac{t_c}{c}} = e^{\frac{L}{cv}} + e^{\frac{t_0}{c}} - 1,$$

whence

$$t_c = c \log\left(e^{\frac{L}{cv}} + e^{\frac{t_0}{c}} - 1\right). \tag{7}$$

Setting $t=t_c$ in equation (6), and noting that this equation can be written

$$y=Wr\left[L-(t-t_o)v+cve^{-\frac{t}{c}}\left\{e^{\frac{t}{c}}+1-e^{\frac{L}{cv}}-e^{\frac{t_0}{c}}\right\}\right],$$

it follows that the maximum discharge, y_c, is given by the equation:

$$y_c=Wr[L-(t_c-t_o)v]. \tag{8}$$

Equations (7) and (8) were derived from equation (6), and in turn equation (6) was developed on the assumption that $t_0>\frac{L}{v}$. The question arises whether equations (7) and (8) hold when $t_0<\frac{L}{v}$. Suppose that $t_0<\frac{L}{v}$. Then equation (3) will hold only on the range $0\leqq t\leqq t_0$ because as soon as the rain stops (when $t=t_0$) the contributions from the portions of the drainage area near the gage at once begin to decrease, and if $t_0<\frac{L}{v}$ the contributions from the upper portions of the drainage area have not yet reached the gage so that equation (6) does not apply either. When $t>t_0$ the contributions from the lower part of the drainage area are decreasing, and the discharge at the gaging station is given by:

$$y=\int_{x_0}^{tv}Wr\left[1-e^{-\frac{1}{c}\left(t-\frac{x}{v}\right)}\right]dx+\int_0^{x_0}Wr\left[1-e^{-\frac{t_0}{c}}\right]e^{-\frac{1}{c}\left(t'-\frac{x}{v}\right)}dx,$$

where x_0 is that value of x such that $(t-t_0)v=x_0$, and where $t'=t-t_0$. Whence:

$$y=Wr\left[tv-cv-x_0+cve^{-\frac{1}{c}\left(t-\frac{x_0}{v}\right)}\right.$$
$$\left.+\left(1-e^{-\frac{t_0}{c}}\right)\left(cve^{-\frac{1}{c}\left(t-t_0-\frac{x_0}{v}\right)}-cve^{-\frac{1}{c}(t-t_0)}\right)\right],$$

and on putting $x_0=(t-t_0)\,v$,

$$y=Wrv\left[t-c-(t-t_0)+c\left\{e^{-\frac{t}{c}}+\left(1-e^{-\frac{t_0}{c}}\right)\left(1-e^{-\frac{1}{c}(t-t_0)}\right)\right\}\right],$$

or

$$y=Wrv\left[t_0+c\left\{e^{-\frac{t}{c}}-e^{-\frac{1}{c}(t-t_0)}\right\}\right]. \tag{9}$$

Equation (9) holds only on the range $t_0\leqq t\leqq\frac{L}{v}$. When $t=t_0$ equation (9) becomes:

$$y=Wrv\left[t_0+c\left\{e^{-\frac{t_0}{c}}-1\right\}\right]; \tag{9a}$$

and when $t=\frac{L}{v}$ equation (9) takes the form:

$$y=Wrv\left[t_0+c\left\{e^{-\frac{L}{cv}}-e^{-\frac{1}{c}\left(\frac{L}{v}-t_0\right)}\right\}\right]. \tag{9b}$$

Equation (9) does not have a maximum within the range for which it holds, because its derivative is

$$Wrve^{-\frac{t}{c}}\left\{e^{\frac{t_0}{c}}-1\right\},$$

and on putting this derivative equal to zero we obtain $t=\infty$, which is beyond the range for which equation (9) holds.

Now when $t=t_0$ equation (3) reduces to equation (9a), and also when $t=\frac{L}{v}$ equation (6) reduces to equation (9b). Thus equation (9) is a transition between equations (3) and (6); and when $t_0<\frac{L}{v}$ and $t>\frac{L}{v}$ the discharge is given by equation (6). As equation (9) has no maximum it follows therefore, whether $t_0\lesseqgtr\frac{L}{v}$, that equation (7) gives the time of the crest, and equation (8) gives the maximum discharge. In other words, regardless of how short or long the rain lasts, equations (7) and (8) apply.

A method of showing that equations (3), (4), (5), and (6) are correct is to integrate them between the limits of the respective ranges for which they apply, and ascertain that the sum of the four integrals thus obtained is equal to the volume of rainfall which occurs over the drainage area.

In carrying out the above procedure we first obtain the volume of discharge for the period from $t=0$ to $t=\frac{L}{v}$ by integrating the right-hand side of equation (3) with respect to t, thus:

$$\int_0^{\frac{L}{v}}Wrv\left[t-c+ce^{-\frac{t}{c}}\right]dt=$$
$$Wrv\left[\frac{L^2}{2v^2}-\frac{cL}{v}-c^2e^{-\frac{L}{cv}}+c^2\right]. \tag{A}$$

In a similar manner we integrate the right-hand side of equation (4) with respect to t between the limits $\frac{L}{v}$ and t_o, for the volume of discharge from the time $\frac{L}{v}$ to the time t_o, and find this volume of discharge to be

$$Wr\left[Lt_0+c^2ve^{-\frac{1}{c}\left(t_0-\frac{L}{v}\right)}-c^2ve^{-\frac{t_0}{c}}-\frac{L^2}{v}-c^2v+c^2ve^{-\frac{L}{cv}}\right]. \tag{B}$$

Likewise for the period from $t=t_0$ to $t=t_0+\frac{L}{v}$, the integration of the right-hand side of equation (6) with respect to t between these limits gives

$$Wr\left[\frac{L^2}{2v}+cL-c^2ve^{-\frac{1}{c}\left(t_0+\frac{L}{v}\right)}+c^2ve^{-\frac{L}{cv}}-c^2v-c^2ve^{-\frac{1}{c}\left(t_0-\frac{L}{v}\right)}\right.$$
$$\left.+2c^2ve^{-\frac{t_0}{c}}\right]. \tag{C}$$

Finally, integrating the right-hand side of equation (5) for the period $t=t_0+\frac{L}{v}$ to $t=\infty$ (at the time $t=\infty$ the discharge at the gaging station has receded to its value at the time that the rain began, viz., $y=0$) we get:

$$Wrc^2v\left[1-e^{-\frac{L}{cv}}-e^{-\frac{t_0}{c}}+e^{-\frac{1}{c}\left(t_0+\frac{L}{v}\right)}\right]. \tag{D}$$

The sum of expressions (A), (B), (C), and (D) is $WLrt_0$ and is the volume of discharge during the interval between the time $t=0$ (when the rain began) and the time $t=\infty$ (actually when the river again becomes dry). It will be noted that the volume of discharge is the area of the basin times the rate at which the rain falls times the duration of the rain, and is the volume of rainfall. This result is exactly what should be expected when evaporation is neglected.

Expressions (A), (B), (C), and (D) result when it is assumed that $t_0>\frac{L}{v}$. In a similar manner we can obtain other expressions from equations (3), (9), (6), and (5), and show that when $t_0 < \frac{L}{v}$ the volume of discharge equals $WLrt_0$.

In the above development it was assumed that the river was dry when the rain began. Actually this condition seldom occurs in nature. However, it will be clear to the reader that this assumption was made for simplicity, and was not at all essential for the above development. If the river is not dry at the time the rain begins, then the discharge at time t is given by the sum of the right-hand side of one of equations (3), (4), (5), (6), or (9) (the one whose range includes t) and the discharge when the rain begins. Obviously, it is necessary to assume here that the river is at a steady state when the rain begins.

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THE SNOW SURVEY AS AN INDEX TO SUMMER PRECIPITATION [1]

By O. W. Monson

[Montana State College Agricultural Experiment Station, Bozeman, June 1934]

The successful prediction of rainfall, whether it be a single storm or the accumulation for the entire season, involves a knowledge of where the rainstorms originate and the paths they follow. The exact origin of the rain that falls at a given place cannot be definitely traced, but it is the opinion of reliable authorities that as we advance inland from the ocean the percentage of the moisture in the air which originates directly from the ocean becomes smaller.

The following is quoted from "Forests and Water in the Light of Scientific Investigation," by Raphael Zon:

> The precipitation over the land does not depend solely on the amount of water brought as vapor by the prevailing winds from the ocean. * * * The moisture-laden currents soon lose the moisture which they obtain directly from the ocean, but in moving farther into the interior absorb the evaporation from the land. Hence, the farther from the ocean the greater is the proportion which evaporation from the land forms of the air moisture.

Adolph Meyer says:

> It is a common misconception that almost all of the rain which falls on the land comes from moisture evaporated from the ocean. As a matter of fact, the greater portion of the rain which falls in the United States is water reprecipitated after having fallen as rain (or snow) and having evaporated from the land area. (Elements of Hydrology.)

According to these authorities, much of the water which falls as rain at inland points has been evaporated from the residual of previous precipitation not accounted for by run-off or deep percolation. Therefore, the amount of precipitation that occurs at a given place should depend to a great extent upon the moisture conditions on the lands over which the prevailing winds at that point blow. Moisture is picked up from lakes, reservoirs, streams, snow fields, and from swamps and other moist lands. How much is contributed by each should depend, among other things, upon its relative extent.

On the theory that conditions which affect the extent of one of these sources will affect all in about the same proportion, a preseason measurement of the extent of one or more of the above-named sources of moisture should be an index to the amount of summer rainfall at various places located in the path of the moisture-bearing winds.

To test this theory the writer made comparisons between the water content of the snow cover on the watershed of Swiftcurrent Creek of the St. Mary River drainage basin measured early in May and the amount of rain occurring during April, May, June, July, and August at Havre, Geraldine, and other places located eastward from the snow fields. See figure 1.

The water content in inches of the snow cover in the Swiftcurrent cirque of the St. Mary River Basin was plotted for the 12-year period of record from 1922 to 1933, inclusive. The summer rainfall in inches for the several places mentioned was plotted for the same period, and the water content curve was then compared with each rainfall curve to discover if any correlation existed.

A marked similarity was observed in the fluctuations when the water-content curve was compared with the rainfall curves for Havre and Geraldine, as shown in figure 2. The rainfall record at Havre and Geraldine during April, May, June, July, and August and the water content of the snow cover in the Swiftcurrent basin measured on May 1 of each year are given in table 1.

Correlation coefficients calculated between the water content of the snow in the Swiftcurrent basin and the rainfall during April 1 to August 31 at Havre and at Geraldine give values of 0.72 for Havre and 0.71 for Geraldine, which is a high degree of correlation. This apparent relation between the water content of snow in the Swiftcurrent basin and the summer rainfall at Havre is expressed by the equation $R = 0.177W + 4.74$, where R equals inches of rainfall and W is the water content of the snow cover in inches. The equation representing the best fit line for Geraldine is $R = 0.155W + 4.05$.

The reliability of this correlation is limited by the paucity of data available over a short period of record, 12 years. To be conclusive, a much longer period should be studied. Snow surveys are a comparatively recent innovation, but their value is rapidly being recognized.

By means of these two equations the summer rainfall at Havre and Geraldine was calculated for the 12-year period from the water-content measurements in the Swiftcurrent basin and the estimated amount compared with the actual record as in figures 3 and 4. The similarity of the curves is remarkable. It may be noted that the slope of the estimated curves, that is, up or down, showing increase or decrease as compared with the previous year, is correct 11 out of 12 years for Geraldine and 10 out of 12 years for Havre, and that a forecast of "above normal" or "below normal" would have been correct in 10 out of the 12 years for each place.

But this correlation does not necessarily prove a direct causal relationship between the snow cover on the St. Mary River watershed and the summer rainfall at Havre and Geraldine. Perhaps they are associated as kindred effects of a third factor, or perhaps they show similar variations because affected by other similar though distinct underlying influences. This, however, does not detract from the practical value of the apparent relationship.

[1] Contribution from Montana State College, Agricultural Experiment Station. Paper No. 40, Journal Series.