THE GEOMETRICAL THEORY OF HALOS—III

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REFLECTION

The calculation of the image produced by external reflection is easily accomplished as indicated in figure 6.[1] Internal reflection (which occurs whenever an internal ray meets an interface at an angle with the normal greater than $\gamma = \text{arc} \sin \frac{1}{\mu}$, $\mu > 1$) conforms to the same law, but may be associated with refraction at some other point in the complete path of the ray; when the ray is incident

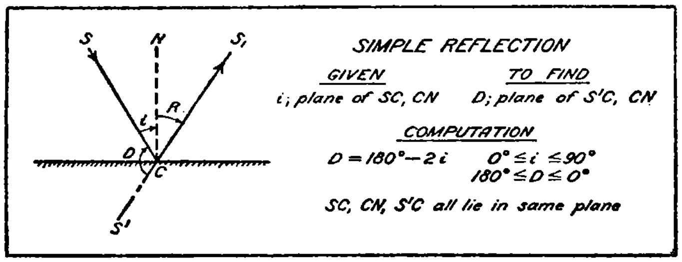

FIGURE 6.—*Simple Reflection:* S, luminous source; SC, incident ray; CN, normal to interface at point of incidence; *i*, angle of incidence; CS_1, reflected ray; R, angle of reflection; S′, virtual image; D, deviation.

on, and also emerges from, a face parallel to that from which it is internally reflected, the emergent ray is parallel to the course it would have taken if merely reflected at the point of incidence.

The geometrical relations on a sphere of indefinitely great radius, figure 7, readily lead to appropriate formulae for computation, Formulae II. It is a corollary of the Law of Reflection, that the source and the image lie at equal distances from the geometric pole of every great circle normal to the reflecting plane (any such pole is on the great circle PP). Furthermore, the projections of the incident and the reflected rays on any plane through the normal to the reflecting surface make equal angles with the normal, just as if they were actual rays. These principles constitute Bravais' Laws of Reflection. The deviation (angular displacement of the image from the true position of an indefinitely distant source) is away from the normal N.

In the case of prismatic refraction accompanied by an internal reflection from a principal plane, figure 8, the projection of the ray on the principal plane is not altered by the reflection; and hence the positions of the images produced with and without such a reflection are symmetric with respect to the principal plane. The calculation of the image, figure 9, is therefore accomplished with Formulae I except that the altitude relative to the principal plane is $-h$ and

$$(6^*) \quad \cos D = \cos^2 h(1+\cos D') - 1 \qquad (\text{I}^*)$$

$$(7^*) \quad \sin A = \frac{\cos h \sin D'}{\sin D}$$

by the Law of Cosines and the Law of Sines, respectively. The deviation is toward V and away from P′.

Prismatic refraction accompanied by an internal reflection from any arbitrarily given plane, figure 10, requires the more complicated Formulae III; see figure 11. When the reflecting plane coincides with a principal plane, $\beta = 0°$ and the formulae reduce to the previous ones; when the reflecting plane coincides with a lateral face, $\beta = 90°$.

Multiple internal reflection will, of course, take place if the once-reflected ray meets the next interface at too great an angle with the normal.

The positions of the images formed by light from the sun or the moon incident on a particular face of an ice crystal of specified form in a given orientation may now be calculated by superposing the appropriate figures on the celestial sphere, with S in the position of the luminary: The preceding formulae give the position of an image relative to the source; and from this, the position relative to the horizon is readily found for any given altitude of the sun or moon, by solving the spherical triangles involved. The formulae required will be given in the next section.

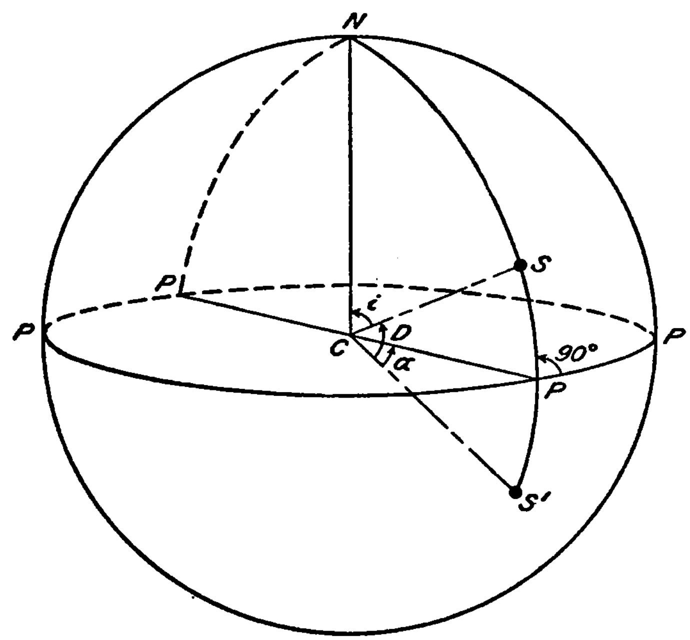

FIGURE 7.—*Calculation of the Image Produced by Simple Reflection:* See Formulae II, PPP, reflecting plane; CN, normal at point of incidence; N, pole of reflecting plane, on same side of plane as source S; S′, image; D, deviation; C, observer. The deviation, D, is away from N.

Formulae II

CALCULATION OF IMAGE PRODUCED BY SIMPLE REFLECTION

Given	*To Find*
Angle of incidence, i	Coordinates of S':
Coordinates of S relative to reflecting plane:	Relative to reflecting plane:
Altitude $90°-i$	Altitude a
Relative Azimuth 0°	Relative Azimuth 0°
	Relative to S
	Deviation, D
	Position Angle 180°

Computation

$$(1) \quad D = 180° - 2i \qquad 0° \leq i \leq 90°$$

$$(2) \quad a = -\frac{D}{2} \qquad -90° \leq a \leq 0°$$

[See figure 7.]

[1] The figures are numbered consecutively with those in paper II.

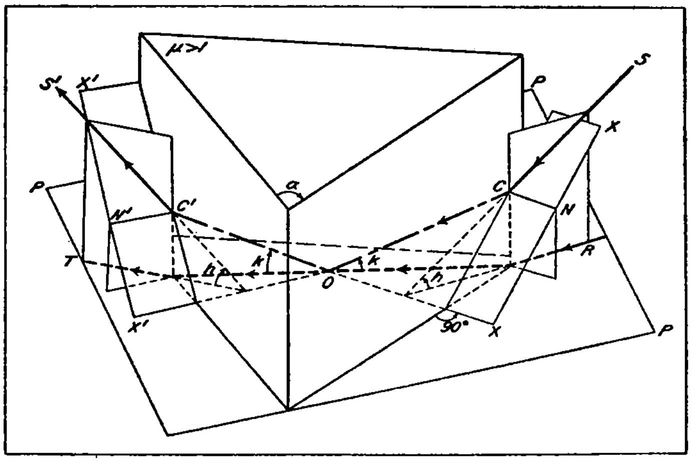

FIGURE 8.—*Internal Reflection from a Principal Plane:* SCOC'S', path of ray; PP, principal plane; CN, C'N', normals; O, point of internal reflection; XX, plane of refraction at incidence; X'X', plane of refraction at emergence.

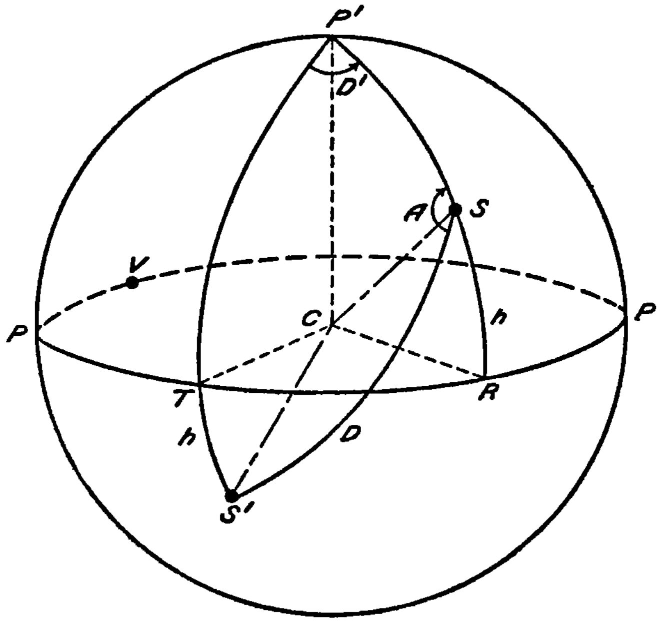

FIGURE 9.—*Calculation of the Image Produced by Internal Reflection from a Principal Plane:* See Formulae I *. PPV, principal plane; P' pole of principal plane on same side of plane as S; V, vertex of refracting angle; S, source; S', image; D, deviation; A, position angle of image; *h*, inclination of incident ray to principal plane; R, T, projections of source and of image on principal plane; D', deviation of projection of ray on principal plane; C, observer. Deviation is always toward V and away from P'.

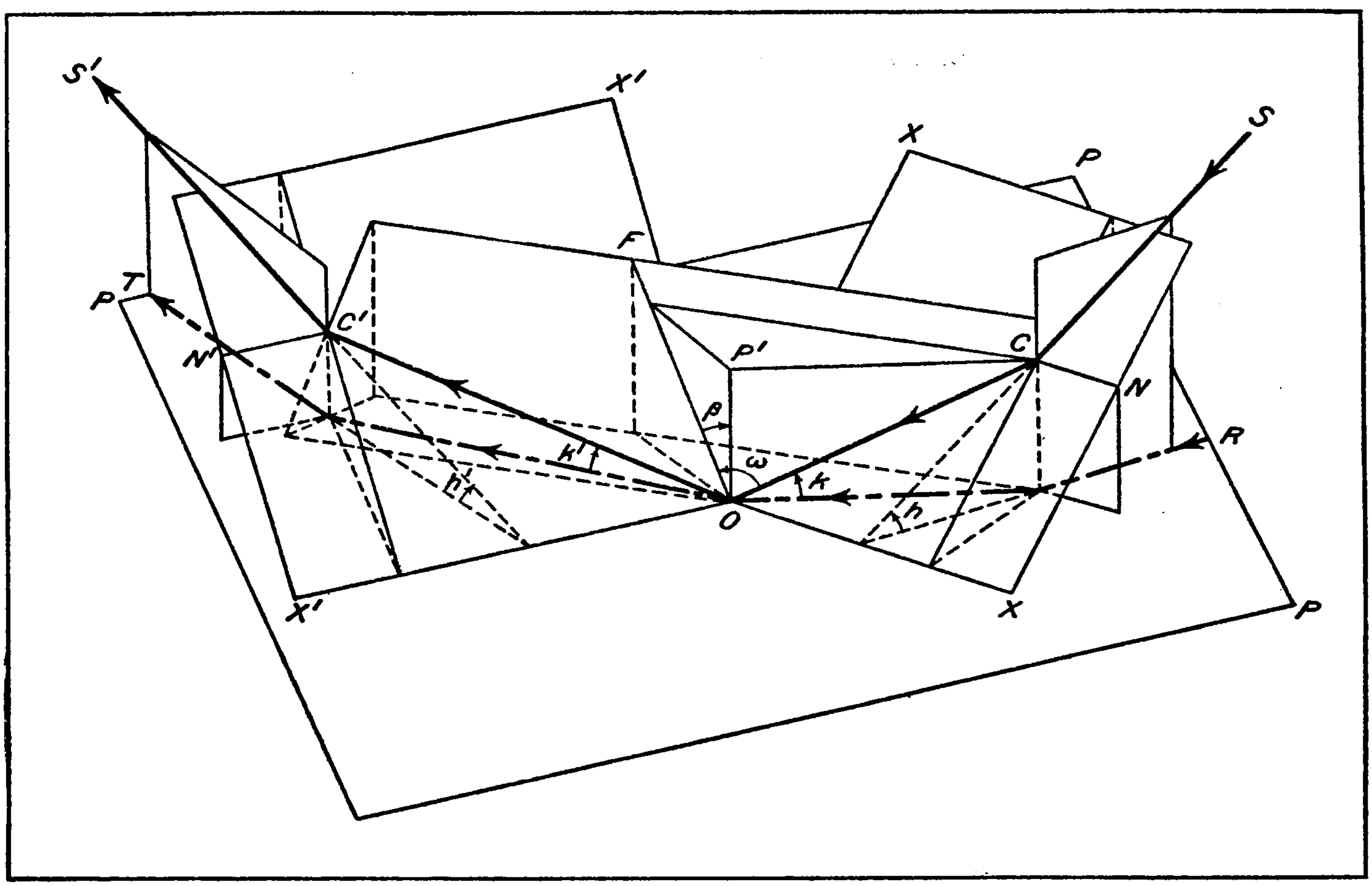

FIGURE 10.—*Prismatic Refraction accompanied by Internal Reflection:* SCOC'S', path of ray; PP, principal plane of refracting angle; OF, normal to reflecting plane; XX, plane of refraction at incidence; FC'OC, plane of incident and reflected rays; X'X', plane of refraction at emergence; CN, C'N', normals to faces of incidence and emergence, respectively.

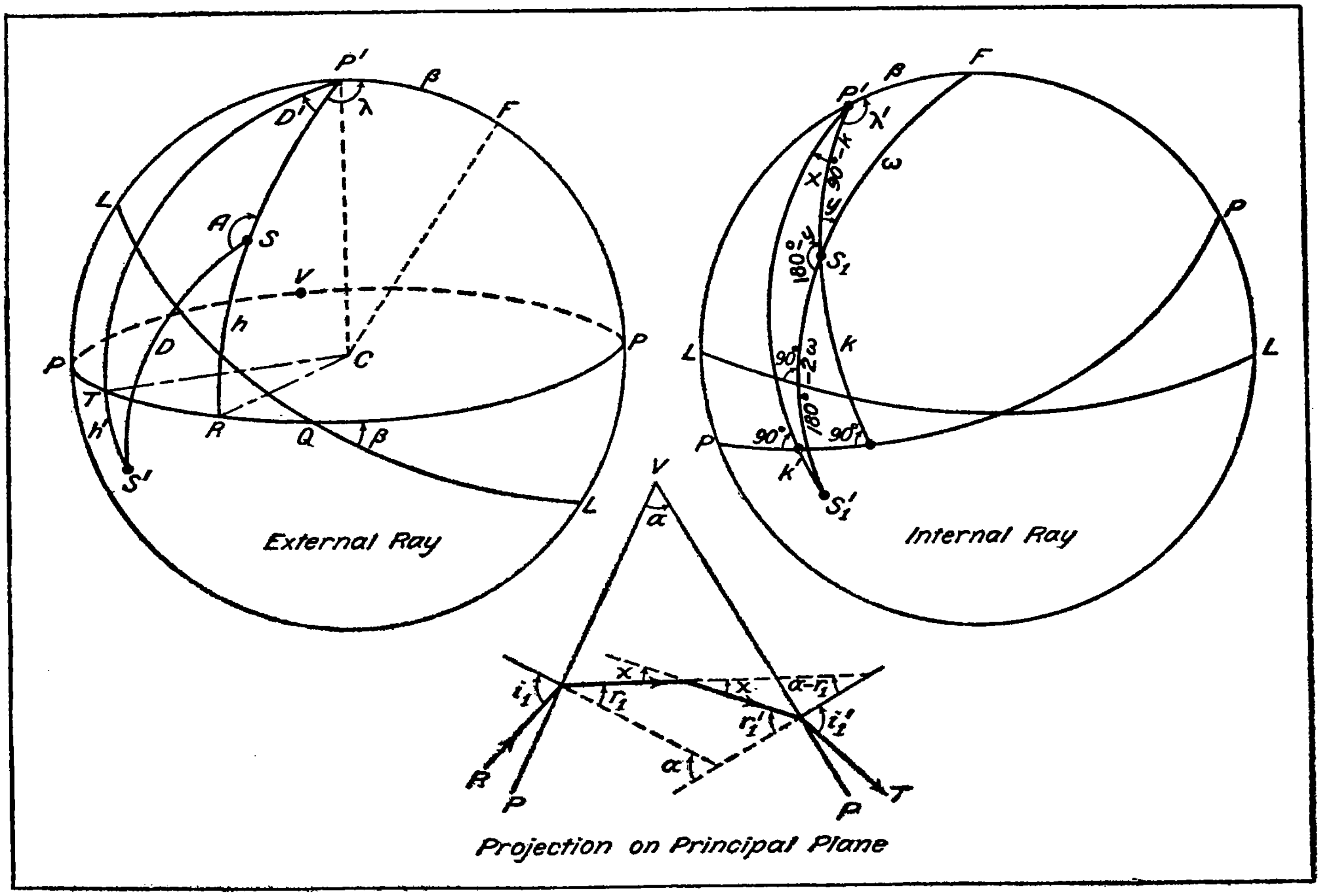

FIGURE 11.—*Calculation of Image Produced by Refraction and Internal Reflection:* See Formulae III. PP, principal plane, with pole P′; LL, reflecting plane, with pole F; S, source; S′, image; R, T, projections of source and image on principal plane; D′ deviation of projection of ray on principal plane; D, deviation; A, position angle. [C, observer at center of sphere is omitted from the diagram; S_1C, S'_1C would correspond to CO, C′O, respectively, in figure 10.]

Formulae III

CALCULATION OF IMAGE PRODUCED BY PRISMATIC REFRACTION COMBINED WITH INTERNAL REFLECTION

Given

μ, α; h; β, λ; i_1

Coordinates of S:
- Relative to Principal Plane
  - Altitude h
  - Relative Azimuth 0°

To Find

Coordinates of S'
- Relative to Principal Plane
  - Altitude $-h'$
  - Relative Azimuth $\pm D'$
- Relative to S
  - Deviation D
  - Position Angle $\pm A$

Computation

(1) $\sin h=\mu \sin k$ } Table 3

(2) $\lambda'=\lambda+i_1-r_1$

(3) $\cos \omega=\sin k \cos \beta+\cos k \sin \beta \cos \lambda'$

(4) $\sin y=\dfrac{\sin \beta \sin \lambda'}{\sin \omega}$

(5) $\sin k'=\sin k \cos 2\omega+\cos k \sin 2\omega \cos y$

(6) $\sin h'=\mu \sin k'$

(7) $\sin x=\dfrac{\sin 2\omega \sin y}{\cos k'}$

(8) $r_1'=\alpha+x-r_1$ (Internal reflection occurs if $r_1'>\gamma'$ *for* h')

(9) $D'=i_1+i_1'-\alpha$ Table 3 with $h=h'$

(10) $\cos D=\cos h \cos h' \cos D'-\sin h \sin h'$

(11) $\sin A=\dfrac{\cos h' \sin D'}{\sin D}$

[See figure 11. The trigonometrical relations on the sphere follow from the Law of Cosines and the Law of Sines.]