Observations of this type—made intermittently during the night and at one or only a few stations—can be misleading. Fitful breezes may import warmer or colder air to the vicinity of the instrument shortly before an observation is taken. Stations may be poorly selected in spite of efforts to find representative or extreme conditions. The "cold air drainage" may actually have its source in none of the locations where measurements are taken. More stations and more nearly continuous observations are desirable. Nevertheless, certain tentative conclusions are suggested by the data.

Group I presents a condition not uncommon on cloudy and windy nights—the surface of the basin soil cooling to a lower temperature than the air overlying it. Group II shows relations on an inversion night—the coldest air in the basin having a temperature lower than that of the basin soil and of the hill-slope soil and air. The air on the hill slope is the thin layer of cold air near the soil which on this particular night was 10° F. colder than the air 5 feet above the ground. The difference in the basin between these two heights is much less, being on this night 4° F.

Readings in a mountain canyon are given in group IV. The drainage of cold air down the canyon was easily detected, this air being colder than the soil of the canyon. This air, however, was not cold enough to reach the basin gravitationally unless it underwent considerably more cooling during its descent. Observations on other nights at lower elevations (group VI) did not reveal any stream of air sufficiently cold to reach the basin gravitationally. If this stream of air at high elevations does cool sufficiently on its journey to descend to the basin, it must have overcome the adiabatic warming of approximately 5.5° F. per 1,000 feet of descent.

The greatest observed difference between nocturnal soil and air temperatures (with the former the colder) are given in group V, the station being a high, isolated peak. Snow on the peak was considerably colder than both air and soil; the soil was frozen to a depth of several inches. Bare rock nearby, however, was warmer than the air. On the same peak, when there was no snow present, as on the night of January 20, 1936, the air became ½° F. colder than the soil; but on the night of May 16, 1936, the soil became 3° F. colder than the air.

At midday in the shade of a cliff at 4,000 feet elevation, moist soil has been observed to be 12.5° F. colder than the air. Beneath the surface this soil was frozen, perhaps the result of low temperatures the previous week. However, snow and frozen soil on the mountains are rare during the season of greatest basin-inversions.

THE GEOMETRICAL THEORY OF HALOS—IV

By EDGAR W. WOOLARD

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PART 2. CALCULATION OF POSITIONS OF IMAGES ON THE CELESTIAL SPHERE

From the general Formulae I, II, or III (or the appropriate special case of some one of these) the position of any image, *relative to the source*, produced by prismatic refraction, simple reflection, or prismatic refraction in combination with an internal reflection, may always readily be computed. These formulae will now be applied to derive formulae for the position *relative to the horizon* when the source is the sun or the moon at a given altitude; the derivation is essentially a transformation from spherical coordinates relative to the principal plane, or the reflecting plane, to the familiar horizon coordinate system of astronomy (with azimuths measured from the vertical circle through the sun).

IMAGES PRODUCED BY PRISMATIC REFRACTION WITHOUT INTERNAL REFLECTION

The derivation of the desired formulae may be accomplished by superimposing figures 5, 7, 11 in the proper orientation on the celestial sphere.[1] Figure 5 is the one required in the case of prismatic refraction without internal reflection.

Images produced by a dihedral angle with vertical refracting edge (principal plane horizontal).—Place figure 5 on the celestial sphere as in figure 12, with P' at the zenith Z, the great circle PRP on the horizon EME, and S in the position of the sun or the moon at altitude H above the horizon.

Then evidently $h=H$; and to compute the altitude H', the azimuth ζ from the solar vertical, the deviation D, and the position angle A' (measured from above the luminary) of the image S', at any given value of H, we have immediately from Formulae I the equations that comprise Formulae A. (It is of course unnecessary to make any actual use of equation (1) in the calculation if table 3 is used.)

FORMULAE A

CALCULATION OF THE IMAGE PRODUCED BY A DIHEDRAL REFRACTING ANGLE WITH VERTICAL REFRACTING EDGE

Parameter: H, $0° \leq H \leq \text{arc cos}\left\{\sqrt{\mu^2-1}\tan\frac{\alpha}{2}\right\}$

Argument: i_1

Calculation of D, A', H', ζ:

(1) $\mu' = \sqrt{\frac{\mu^2-\sin^2 H}{1-\sin^2 H}}$ Table 2

(2) $\sin r_1 = \frac{\sin i_1}{\mu'}$,

$\text{arc sin}\left\{\sin\alpha\sqrt{\mu'^2-1}-\cos\alpha\right\} \leq i_1 \leq 90°$ Table 3

(3) $r_1' = \alpha - r_1$

(4) $\sin i_1' = \mu' \sin r_1'$ Table 3

(5) $D' = i_1 + i_1' - \alpha$

$$(6)\begin{cases} D = 2\,\text{arc sin}\left\{\sin\frac{1}{2}D'\cos H\right\},\ D<D' \\ \qquad D_o' = 2\,\text{arc sin}\left\{\mu'\sin\frac{\alpha}{2}\right\}-\alpha \\ \qquad D_m' = 180° - \left\{\alpha + \text{arc cos}\left[\mu'\sin\left(\alpha - \text{arc sin}\frac{1}{\mu'}\right)\right]\right\} \\ A' = \text{arc cot}\left\{\tan\frac{1}{2}D'\sin H\right\} \end{cases}$$

$$(7)\begin{cases} H' = H \\ \zeta = D' \end{cases}$$

[See figure 12. These formulae are obtained by putting $h=H$ in Formulae I.]

Images produced by a dihedral angle with horizontal refracting edge (principal plane vertical.)—To compute the image produced, at any altitude of the sun or moon, when

[1] These figures appear in Papers II, III. The figures in the present paper are numbered consecutively with those in the preceding two papers.

the refracting edge of an angle is in any azimuth and the faces of the angle at any inclination to the horizontal, superpose figure 5 on the celestial sphere as shown in figure 13. The orientation of the face of incidence is most conveniently specified by the inclination of the normal, as measured by the angle t ($<180°$) between the horizontal and the normal, reckoned from the point on

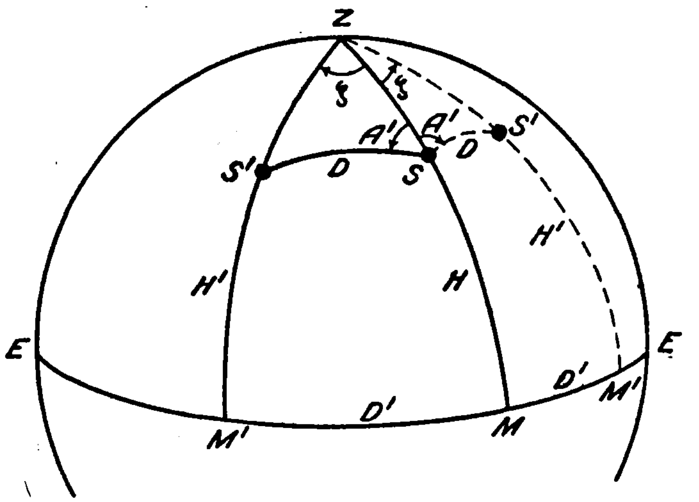

FIGURE 12. *Calculation of the image produced by a dihedral refracting angle with vertical refracting edge.* See Formulae *A*. *Z*, zenith; *EME*, horizon; *S*, luminary, at altitude *H*; *S'*, image; *D*, deviation; *A'*, position angle; *H'* altitude of image; ζ, azimuth of image. (*Cf.* fig. 5.)

the horizon below R, and taken positive or negative according as the normal is directed above or below the horizon.

The numerical value of i_1 is always the arc between N and R (see figures 4 and 5), and hence is the difference between t and δ, the altitude of R; but this difference must be taken with the proper algebraic sign: The range of

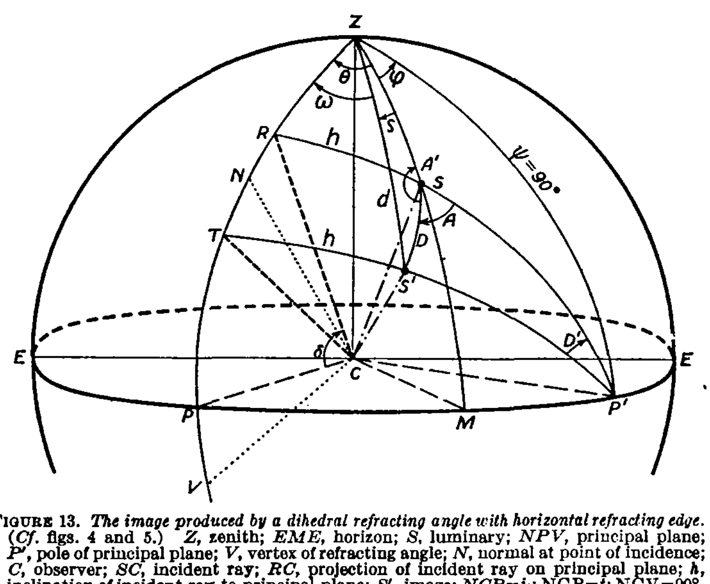

FIGURE 13. *The image produced by a dihedral refracting angle with horizontal refracting edge.* (*Cf.* figs. 4 and 5.) *Z*, zenith; *EME*, horizon; *S*, luminary; *NPV*, principal plane; *P'*, pole of principal plane; *V*, vertex of refracting angle; *N*, normal at point of incidence; *C*, observer; *SC*, incident ray; *RC*, projection of incident ray on principal plane; *h*, inclination of incident ray to principal plane; *S'*, image; *NCR*=i_1; NCP=t; NCV=90°. D' is always toward V.

transmission extends at most (see fig. 2) from a point between V and N to the point 90° beyond N. Take (fig. 14) a refracting angle with the vertex V and the normal N in the positions V_1, N_1, respectively, so that $i_1=90°$; and let the angle rotate clockwise. Until N reaches E, $t<0<\delta$ and $i_1>0$, whence $i_1=\delta-t$; while N passes from E to R, $0<t<\delta$ and $i_1>0$, whence $i_1=\delta-t$; after N passes R, $0<\delta<t$ and $i_1<0$, whence as long as R lies within the range of transmission $i_1=\delta-t$. When N has passed 90° beyond R, to the position N_2, the vertex lies at R and light then ceases to fall on the face under consideration. Meanwhile, light has become incident on the other face of the angle, when its normal N' was in the position N'_2; as soon as R comes within the range of transmission, $t<0<\delta$ and $i_1<0$, whence $i_1=t-\delta$ until N' reaches E; while N' passes

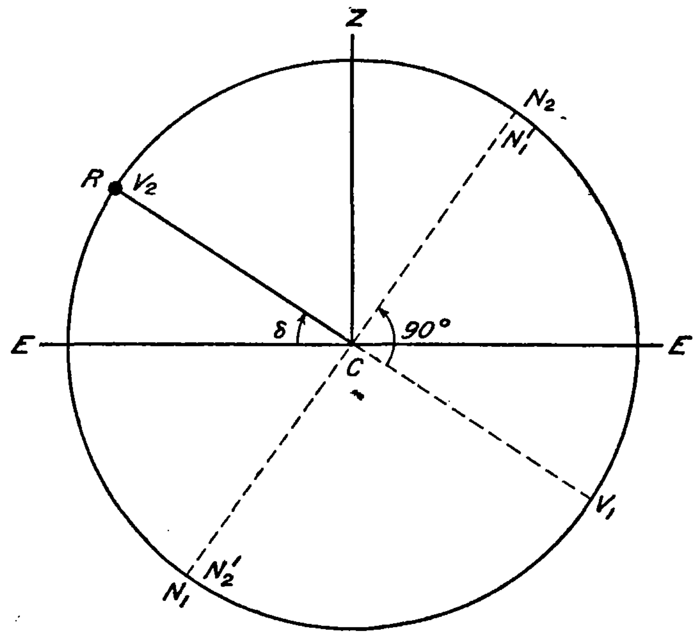

FIGURE 14. Determination of i_1 when refracting edge is horizontal.

from E to R, $0<t<\delta$ and $i_1<0$, whence $i_1=t-\delta$; after N' passes R, $0<\delta<t$ and $i_1>0$, whence $i_1=t-\delta$ until N' has passed 90° beyond R to N'_1. As the angle continues to turn, N, V again come into positions N_1, V_1, respectively, and the light once more becomes incident on the first face. That is, with the place of V given (90° in one of the two directions from N), we have for i_1:

	V, R on opposite sides of Z	V, R on same side of Z
Normal pointing up, t positive	$i_1=t-\delta$	$i_1=\delta-t$
Normal pointing down, t negative	$i_1=\delta-t$	$i_1=t-\delta$

The deviation D' is always toward V.

The position of the image is given by Formulae B; see figure 15.

FORMULAE B

CALCULATION OF THE IMAGE PRODUCED BY A DIHEDRAL REFRACTING ANGLE WITH HORIZONTAL REFRACTING EDGE

Parameters: H, $0°\leq H\leq 90°$; position of V; t (positive when normal is directed upward, negative when downward).

Argument: h (or, if desired, θ; then $\sin h=\sin\theta\cos H$; $0°\leq\theta\leq 90°$).

Calculation of D, A', $H'>0°$; ζ:

(1) $\sin\delta=\sin H\,\mathrm{sec}\,h$ $0°\leq h\leq(90°-H)$ or until total reflection occurs

(2) $\begin{cases} t>0; \ V,\ R \text{ same side } Z \\ t<0; \ V,\ R \text{ opposite sides } Z \end{cases}$

$$\sin r_1=\frac{1}{\mu'}\sin(\delta-t)$$

Tables 2, 3

$$\begin{cases} t>0;\ V,\ R \text{ opposite sides } Z \\ t<0;\ V,\ R \text{ same side } Z \end{cases}$$

$$\sin r_1 = \frac{1}{\mu'} \sin (t-\delta)$$

(3) $r'_1 = \alpha - r_1$

(4) $\sin i'_1 = \mu' \sin r'_1$ Table 3

(5) $D' = i_1 + i'_1 - \alpha$ from R toward V

(6) $\sin H' = \cos h \sin \begin{cases} (\delta + D'),\ D' \text{ upward} \\ (\delta - D'),\ D' \text{ downward} \end{cases}$ $0° \leq (\delta \pm D') \leq 180°$

(7) $\sin \omega = \sin h \sec H'$

(8) $D' + \delta > 90°$

$\zeta = 180° - (\omega + \theta)$

$D' + \delta < 90°$

$\zeta = \begin{cases} \omega - \theta,\ D' \text{ upward} \\ \theta - \omega,\ D' \text{ downward} \end{cases}$

(9) $\sin \frac{D}{2} = \sin \frac{D'}{2} \cos h,\ D < D'$

(10) $\cos ZSP' = -\tan H \tan h$ $90° \leq ZSP' \leq 180°$

(11) $\cot A = \tan \frac{D'}{2} \sin h$

(12) $A' = \begin{cases} ZSP' - A,\ D' \text{ upward}, \\ 360° - (ZSP' + A),\ D' \text{ downward, measured in same direction from solar vertical as } \zeta \end{cases}$

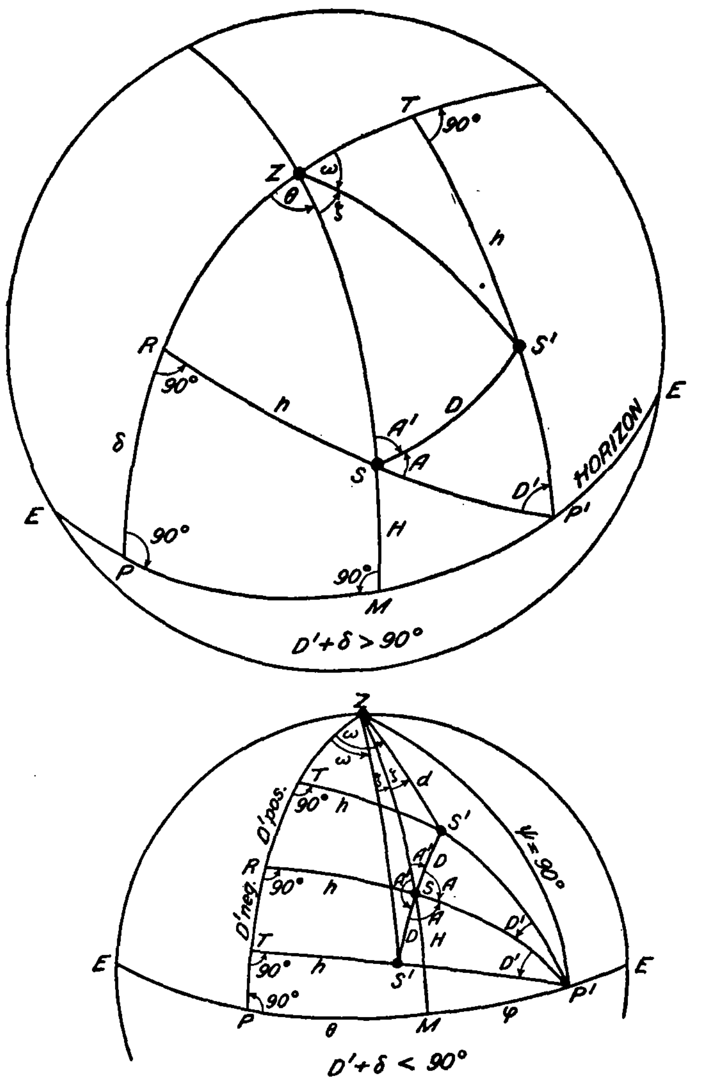

FIGURE 15. *Calculation of the image produced by a dihedral refracting angle with horizontal refracting edge.* See Formulae B. *Z*, zenith; *EME*, horizon; *S*, luminary; *S'*, image; *ZRP*, principal plane; *D*, deviation; *A'*, position angle; zenith distance of image, $d=90°-H'$; ζ, azimuth of image. $RP=\delta$; $\theta+\phi=90°=PP'$. *D'* is always toward *V*. (*Cf.* figs. 5, 13.)

[See figure 15. Formula (1) follows from the Law of Cosines applied to triangle *SZR;* (6) and (7) from the Law of Cosines and Law of Sines, respectively, in triangle *S'ZT;* (10) from Law of Cosines in *ZSP'*; the remaining formulae follow directly from either Formulae I or the geometry of figure 15.]

Images produced by a dihedral angle with the refracting edge in any position.—The superposition of figure 5 on the celestial sphere in an arbitrary orientation gives figure 16, from which Formulae C for the calculation of the image follow.

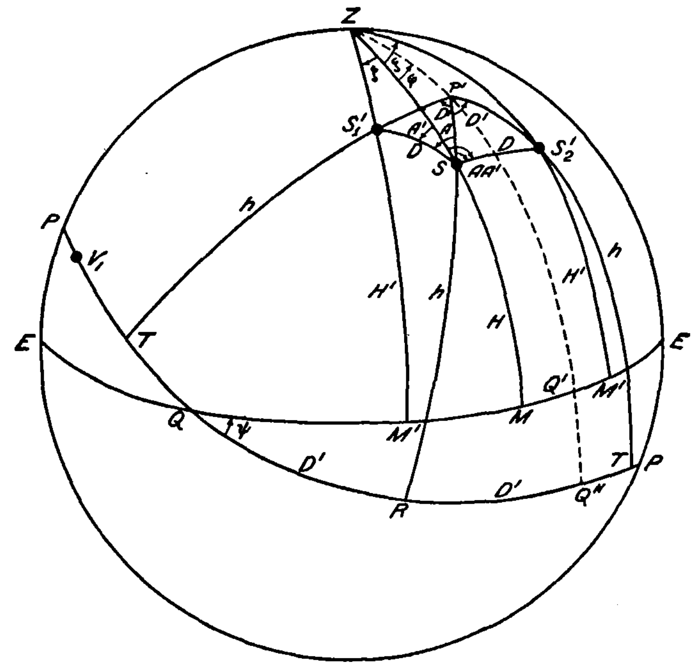

FIGURE 16. *Calculation of the image produced by a dihedral refracting angle with refracting edge in any position.* See Formulae *C*. *Z*, zenith; *EME*, horizon; *S*, luminary; *PRP*, principal plane; *P'*, pole of principal plane, specified by ψ, ϕ; S'_1, S'_2, images; *D*, deviation; ζ, azimuth of image. $ZP'=\psi$; $MQ=90°-\phi$; $QQ''=QQ'=90°$; *D'* is always toward *V*.

When $\psi=0°$, these formulae reduce to Formulae A; and when $\psi=90°$, they reduce to Formulae B.

FORMULAE C

CALCULATION OF THE IMAGE PRODUCED BY A DIHEDRAL REFRACTING ANGLE WITH REFRACTING EDGE IN ANY POSITION

Parameters: H, $0° \leq H \leq 90°$; position of V

Arguments: ψ, ϕ

Calculation of D, A', H', ζ:

(1) $\tan m = \cos \phi \tan \psi$ $0° \leq \phi \leq 90°,\ 0° \leq \psi \leq 90°$

(2) $n = 90° - (H+m)$

(3) $\sin h = \dfrac{\cos \psi \cos n}{\cos m}$

(4) $\tan ZSP' = \pm \dfrac{\tan \phi \sin m}{\sin n},\ H+m \leq 90°$

(5) Compute D', D, A from Formulae I

(6) $A' = A \pm ZSP'$

(7) $\tan m' = \cos A' \tan D,\ n' = 90° - (H+m')$

(8) $\sin H' = \dfrac{\cos D \cos n'}{\cos m'}$

(9) $\tan \zeta = \pm \dfrac{\tan A' \sin m'}{\sin n'},\ H+m' \leq 90°$

[See figure 16. Formulae (1) to (4) constitute the solution of the triangle *ZP'S*, and (7) to (9), the solution of triangle *ZSS'*.]