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A SIMULTANEOUS LAGRANGIAN-EULERIAN TURBULENCE EXPERIMENT [1]

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ABSTRACT

Simultaneous measurements of turbulent vertical velocity fluctuations at a height of 300 ft. measured by means of a fixed anemometer, small floating balloons, and airplane gust equipment at Brookhaven are presented. The resulting Eulerian (fixed anemometer) turbulence energy spectra are similar to the Lagrangian (balloon) spectra but with a displacement toward higher frequencies.

1. THE NATURE OF THE PROBLEM

The term *Eulerian,* in turbulence study, implies consideration of velocity fluctuations at a point or points fixed in space, as measured for example by one or more fixed anemometers or wind vanes. The term *Lagrangian* implies study of the fluctuations of individual fluid parcels; these are very difficult to measure. In a sense, these two points of view of the turbulence phenomenon are closely related to the two most palpable physical manifestations of turbulence, first the irregular or *random* nature of the turbulent fluctuations and second the remarkable ability of fluid in a state of turbulence to disperse properties. Such problems of turbulence in the atmosphere as the effect of gusts on structures (towers, buildings, airframes) lead to the Eulerian kind of analysis, whereas the Lagrangian form arises naturally in the highly important field of atmospheric diffusion.

The irregular nature of turbulence early suggested, in fact imposed, a statistical rather than a deterministic approach to its study. Certain important quantities have emerged, with the development of this theory, that serve to characterize turbulence, chiefly the *energy spectrum function,* and the closely related *autocorrelation function.* This paper reports an attempt to make measurements of both Lagrangian and Eulerian turbulence fluctuations simultaneously. This problem owes its importance largely to the fact that experimenters usually measure the Eulerian-time form of the turbulence spectrum or correlation, which all fixed measuring devices of turbulence record; but theoretical developments as well as practical applications inevitably require the Eulerian-space, or the Lagrangian form. Consequently, both experimental verification of theoretical developments and the application of the theories to many practical problems require some consideration of the Eulerian and Lagrangian interrelationships.

2. OBSERVATIONAL TECHNIQUES

A comparatively wide variety of instruments is available for direct Eulerian-time observation of the natural wind at low elevations, the only requirement being that the device should respond to fluctuations in both the vertical and horizontal directions as rapidly as is required by the frequencies of fluctuations that are to be studied. Hot wire and thermopile anemometers [6], and bidirectional wind vanes of several types [6], [13], have all been used, as well as more unusual devices like the anemoclinometer [12], and tethered balloons [7]. Presumably any of these,

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if available in quantities of at least two, could be used for Eulerian-space observations. Few such analyses have been reported; the observations of Shiotani [24], with hot wire anemometers arranged in the vertical, are one example. Sakagami has recently, in two fascinating papers [22], [23], reported on some spatial wind fluctuation measurements of very small-scale turbulence, obtained by using an array of tiny wind vanes, each only a centimeter long. Airplane gusts measurements, for example those reported by Bunker and McCasland [2] or Press and Mazelsky [20], provide a kind of Eulerian-space information. Although the sampling time of such measurements is finite, it is short enough that a relatively large space is sampled quite rapidly. Such measurements represent a fairly close approach to true Eulerian-space observations.

The method invariably employed in Lagrangian observations of atmospheric turbulence has been to introduce some tracer into the air and follow its motion. The list of tracers that have been employed (most of them were first tried by L. F. Richardson [21]) is, as Edinger [4] remarked, entertaining, including as it does Edinger's soap bubbles, Miller's Kleenex lint experiments [15], and the work done by Badgley [1] using dandelion seed. Balloons of various sizes have of course been used, also; and some work has been done by following smoke puffs, using a camera obscura [3]. The difficulty is to find a tracer that has negligible mass, zero terminal velocity or net buoyancy, is small enough to represent, faithfully, the smallest significant scale of air motions, and that can be traced by some reasonably convenient system. Certain of the above tracers satisfy one or more of these criteria, but no one satisfies all.

Considering this great diversity of measuring systems, it is clear that some care must be exercised in intercomparing various results. Different devices possess entirely different response properties, and size alone introduces a strong selective effect upon the fluctuations that can be measured. Bunker's PBY airplane is certainly indifferent to a range of small turbulent fluctuations that, say, Edinger's soap bubbles will detect easily. Considerations of this kind have been an important factor in the design of the following experiment.

3. A SIMULTANEOUS EULERIAN-TIME, EULERIAN-SPACE, AND LAGRANGIAN TURBULENCE EXPERIMENT

The Brookhaven National Laboratory, on eastern Long Island, maintains a permanent micrometeorological installation including a completely instrumented, 410-ft. high meteorological tower. A general description of this site has been given by Smith and Singer [25]. For wind fluctuation measurements, there are available at Brookhaven permanently mounted bidirectional wind vanes, or *bivanes*, at three levels on the tower, and Bendix-Friez *Aerovane* anemometers at these three as well as several other levels. The bivanes give the inclination of the wind, and the Aerovanes provide the wind speed and horizontal direction. Instrumental characteristics of the Brookhaven bivanes were described in a study by Mazzarella [13]; and the manner of their use in wind fluctuation measurement, in connection with the Aerovanes, is covered in several papers of Panofsky [17], [18], Panofsky and McCormick [19], and McCormick [11]. The technique may now be considered to be a tested, reliable one for obtaining low-level wind fluctuation data, possessing well understood properties and limitations.

It was accordingly proposed to make simultaneous observations of turbulent fluctuations in the lower atmosphere, with both the fixed bivane-Aerovane apparatus on the Brookhaven tower and visual double theodolite observations of neutral balloons, released from the tower. The cooperation of Mr. Andrew Bunker, Woods Hole Oceanographic Institution, was also obtained; his group agreed to attempt airplane measurements of vertical accelerations at the tower level during the same period. In this way, all three of the space, time, and Lagrangian fluctuations could be estimated at the same time.

Standard practice with the bivane data, because of the instrument's resonant period, is to work with 5- or 10-second averages of the velocity fluctuations, depending on the prevailing wind speed. Resulting spectra often show two peaks, or at least two more or less distinct high energy regions, one at a frequency of about 200 c. p. h., attributed to mechanical turbulence, and another at about 40 c. p. h., due presumably to convection. Since visual triangulation on small balloons any more often than every 10 seconds seemed an impossible prospect, it was thought best to choose a period for the experiment when the convective turbulence would be reasonably well developed. Based on the climatological expectancy of clear days, the period June 14 to 18, 1954, was chosen. This choice was also influenced by the requirement that all three of the different systems for measuring turbulence should provide comparable information. All three systems have a limited ability to respond to high-frequency fluctuations, and so it was thought best to concentrate on the lower, convective frequencies.

4. OBSERVATIONAL TECHNIQUE OF NEUTRAL BALLOONS

Bivane-Aerovane turbulence observations are a standard technique at Brookhaven and have been described in several papers referred to previously. The airplane gust measuring system used by Bunker is described in [2]. The technique of making neutral balloon runs, although familiar, was modified sufficiently in the present experiment to deserve some additional comment.

The basic double theodolite procedure is described in a Weather Bureau Circular [28], and by Lange [10] or Mildner, Hänsch, and Griessbach [14], for example. A number of standard precautions concerning orientation of the baseline, collimation of the theodolites, and so on, are customarily advocated; all these were taken. The instruments used were standard, Gurley-type theodolites, on which the fractional portion of the angular readings appears on the knobs of the tangent screws. The baseline was 1250 ft. long, one station being on top of the Brookhaven meteorological building and the other in an adjacent open field.

Since angular readings each 10 seconds were desired, in order to obtain as detailed a record of the turbulent fluctuations as possible, each theodolite was manned by three people, an observer and two reader-recorders. Timing was provided from a central point, by radio and telephone, to the two stations. The exact manner of making the angular readings is believed to have been quite important in the success that was obtained. The theodolite observer's duty was to keep the instrument's cross hairs centered on the balloon at all times, with no pauses during readings. The two readers, one for azimuths and one for elevation angles, made their readings in the following order: tenths, then hundredths (estimated), and finally whole degrees. This procedure permitted readings to be synchronized closely with the time signals and is strongly recommended to future users of the technique.

Balloons were released from the 300-ft. level of the tower. They were inflated to a nearly neutral condition with the help of a wire hoop of predetermined size, and were fastened to the tower in an exposed position for several minutes. This preliminary exposure was done in a position sheltered from the wind, as much as possible, but exposed to the sun's rays, in order to approximate as nearly as could be done the conditions experienced during balloon flight.

Final weighing off was done by a method suggested to the writer by Mr. Paul Humphrey (U. S. Weather Bureau), and one that also is strongly indorsed in the light of our experience. A length of twine tied to the balloon's neck was allowed to rest on a shelf (inside the elevator cage of the tower, so as to be sheltered from the wind). When the balloon came to rest, the twine was cut where it touched the shelf. This process took less than 30 seconds, usually; and it could be repeated if necessary up to within a few seconds of the release of the balloon. Excess twine was merely wrapped around the balloon's neck and tied. Since this weighing off was to within a precision of two or three inches of string, which weighed .05 gm./in., the balloons were estimated to possess at most $\pm .2$ gm. free lift on release.

5. SIMULTANEOUS BIVANE-AEROVANE, NEUTRAL BALLOON, AND AIRPLANE TURBULENCE MEASUREMENTS AT BROOKHAVEN

During the week of operations at Brookhaven, 35 neutral flights were made. Of these, 16 lasted less than 5 minutes and 3 longer than 20 minutes. The skill of the observers naturally increased as more experience was gained, the longer flights being achieved only after several days' practice. For this and various other reasons, mainly having to do with occurrence of optimum meteorological conditions and with the orientation of the wind with respect to the tower, the 6 flights made on the final day, June 18, were selected for study.

Corresponding to each of these 6 runs there is a so-called bivane "speed run", i. e., a bivane-Aerovane observation at accelerated chart speed, providing detailed vertical and horizontal velocity recordings. The wind instruments mounted at the 300-ft. level of the tower, from which level the neutral balloons were released, have been used for comparison with the neutral observations.

The airplane gust measurements, being at the mercy of weather conditions all the way from Woods Hole to Brookhaven, resulted during this period in only one clear-cut run that corresponded to a successful neutral flight, the very last one, number 35.

REDUCTION OF THE NEUTRAL BALLOON OBSERVATIONS

For maximum detail velocities were calculated from the neutral runs on a non-overlapping basis; that is, each 10-second interval was considered separately, the positions of the balloon being interpreted as due to a 10-second averaged velocity fluctuation. Otherwise, the standard method for working up double theodolite runs was used (cf [28]). Some effects of this on the resulting spectra will be considered in the following section.

The inspection of the balloon observations that takes place during the reduction process was used to detect one possible source of observational error in the readings. It is quite easy, when attempting the difficult job of reading theodolite azimuth and elevation angles every 10 seconds, to make an error of 1 in the units figure (a gross error of 10° or 100° is, of course, immediately detected). Such a unit error is, however, fairly easy to detect, inasmuch as all the angular readings should form relatively smooth progressions. It is indicative of the high quality of these observations that only two errors of this kind were found.

COMPUTED VERTICAL VELOCITY ENERGY SPECTRA

Both the bivane-Aerovane and neutral balloon observations indicate all three components of the velocity fluctuations, whereas the airplane gust measurements give the vertical fluctuations only. Also, the mean vertical velocity may for our purpose always be assumed to be zero; the mean horizontal velocity in the atmosphere is, on the other hand, a conception that has been questioned by several workers in the turbulence field. Largely for these reasons, only the spectra of vertical velocity fluctuations have been considered in this study.

In trying to analyze such a complicated and irregular pattern as a turbulent flow, one is quite naturally lead to represent it as the sum of many harmonic components of various frequencies, each possessing some share of the total turbulent kinetic energy. We think of the energy spectrum of turbulence as the distribution of this energy over all the various frequencies. In order to obtain the energy spectrum, a harmonic analysis of the turbulent flow might in fact be attempted; but this approach would in practice usually turn out to be very difficult. We may obtain a fully equivalent result by first forming the autocorrelation function and then performing on it the Fourier analysis. In short, knowledge of the correlation is equivalent to knowledge of the spectrum and vice versa.

The technique of the numerical analysis of the observed vertical velocity fluctuations is that developed by the Atmospheric Turbulence Project at The Pennsylvania State University; it has been outlined briefly in the papers of Panofsky [17], [18] and described in all relevant detail by Van der Hoven [29]. This technique involves, essentially, forming the lagged products of the observed series of velocity components (autocorrelation) and performing a harmonic analysis on these (Fourier transformation). The resulting values, one corresponding to each of the lags, are known as spectral estimates of the energy density at the various frequencies. Following suggestions of Tukey [26], [27] various secondary operations are also performed: the original velocity series is first of all "pre-whitened." Pre-whitening involves first a certain transformation of the original data (see eq. (5), Appendix); and then at an appropriate point the spectrum of the pre-whitened variable is transformed back to the true spectrum (see eq. (13), Appendix). At a strategic point in the numerical process an averaging is done, to compensate for the finite length of the observational record; and finally the effect of the averaged nature of the original velocity values is taken into account.

There are a moderate number of alternatives to the above numerical process for spectral analysis, including the use of electronic, mechanical, or graphical harmonic analyzers. The numerical method has certain advantages, principally that it is always directly reproducible and does not itself introduce any new uncertainty or subjectivity. Furthermore, the sampling theory for spectral estimates obtained in this way has been worked out by Tukey [27], so that it is possible to form some notion of their reliability.

The chief disadvantage of the numerical method of spectral analysis is the fact that for detailed analysis of the lower frequencies many lags are needed, and the numerical work becomes prohibitive for desk calculators. Instead, the 10-second averages may themselves be averaged, into 20- and 40-second averages, and so on; analysis of these averaged series will, for a given number of lags (that is, a given amount of hand calculation) provide spectral estimates of correspondingly lower frequencies. It has been shown (see, for example, Van der Hoven's dissertation [29]) that, by neglecting in each case the 20 to 30 percent of the spectral estimates of highest frequency, a series of spectra corresponding to increasing averaging periods of one set of velocity observations may be combined to form a composite spectrum, giving the required detail in the low frequencies.

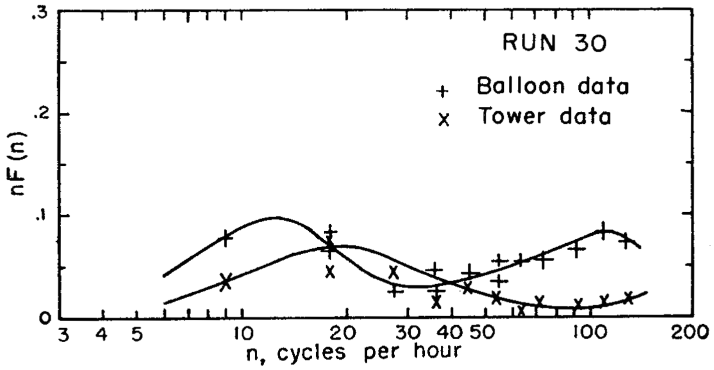

FIGURE 1.—Computed vertical velocity energy spectra for tower and neutral balloon runs number 30.

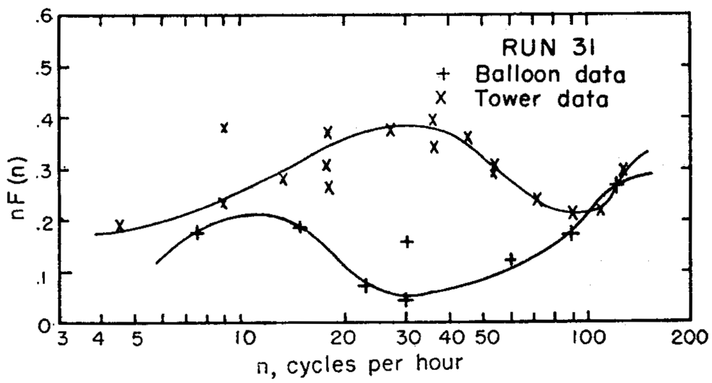

FIGURE 2.—Computed vertical velocity energy spectra for tower and neutral balloon runs number 31.

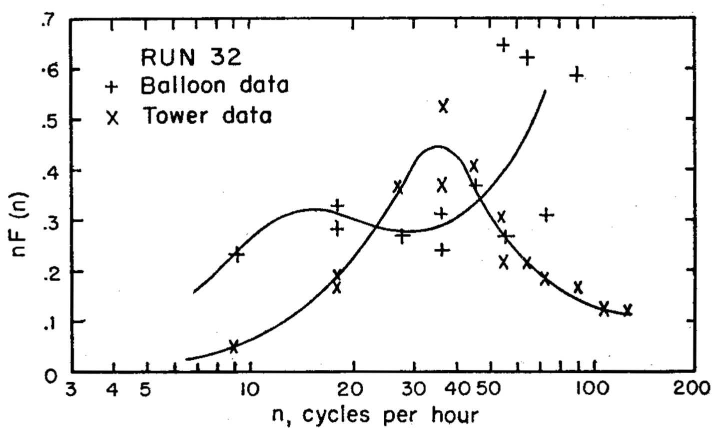

FIGURE 3.—Computed vertical velocity energy spectra for tower and neutral balloon runs number 32.

6. RESULTS

Figures 1 through 6 display the calculated vertical velocity spectra for both bivane speed runs and neutral balloon flights numbers 30 through 35, calculated by the numerical technique. Individual spectral estimates are indicated and suggested continuous spectral curves drawn. Figure 7 is the spectral analysis of Bunker's airplane run number 772, made during the progress of our run number 35. In order to portray the lower frequencies adequately, the logarithm of frequency is used as abscissa

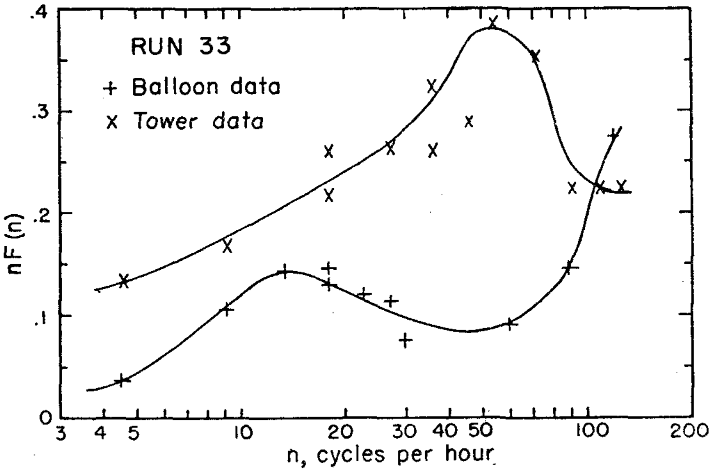

FIGURE 4.—Computed vertical velocity energy spectra for tower and neutral balloon runs number 33.

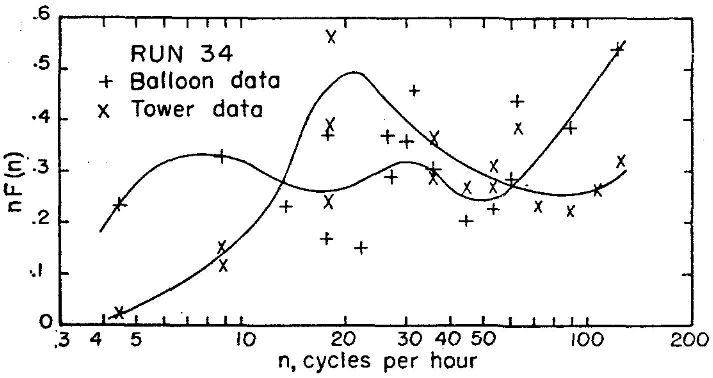

FIGURE 5.—Computed vertical velocity energy spectra for tower and neutral balloon runs number 34.

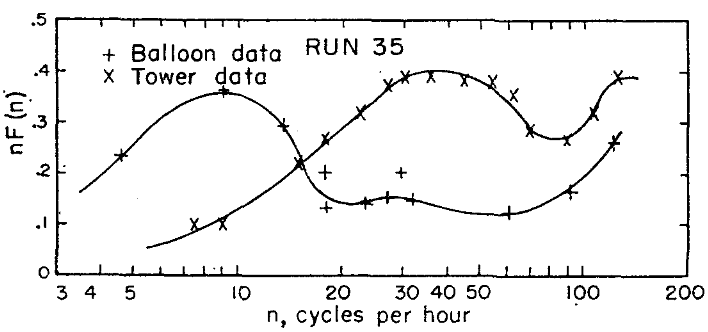

FIGURE 6.—Computed vertical velocity energy spectra for tower and neutral balloon runs number 35.

in these plots. The spectral energy estimate at each frequency is accordingly multiplied by frequency ([29] may be consulted for a discussion on this point).

These results are the first direct, simultaneous measures of Eulerian and Lagrangian turbulent fluctuations in the atmosphere to be reported. Although limited as to frequency range and imperfect in many respects, they are certain to be of considerable interest to students of the turbulence problem. The results are quite satisfying in a qualitative sense. The general picture that is conveyed

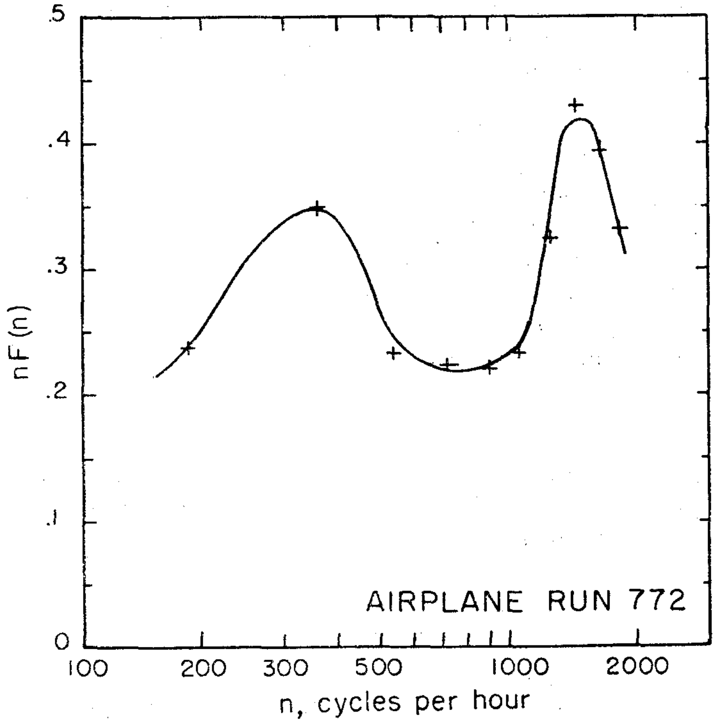

FIGURE 7.—Computed vertical velocity energy spectrum for PBY airplane run number 772.

by a comparison of the six pairs of tower (Eulerian) and neutral balloon (Lagrangian) spectra, figures 1 through 6, is that the Lagrangian spectra are generally similar in shape to the Eulerian spectra but displaced from them toward lower frequencies. If one forms only a very crude mental picture of the turbulence, say that it consists of a series of alternately positive and negative fluctuations, it might be conceived to be acting at a point that is moving along with the mean flow in this way (using an obvious symbolism):

$$+++---+++---+++---+++---.$$

If such a pattern is translated past a point with some speed, an observer at this point might perceive something like this:

$$+-+-+-+-+-+-+-+-+-+-.$$

In other words, the effect of translation past a point by the mean wind of some turbulent pattern is, qualitatively, to shift the significant fluctuations toward higher frequencies, with respect to observations made at a point. This is just what is observed, according to figures 1 through 6.

A quantitative comparison of the Lagrangian and Eulerian time spectra is afforded by Ogura's model of isotropic turbulence [16]. Ogura supposes that "the time variation of the wind velocities at a fixed point is caused both by the passage of the turbulent element and by the decay and rebuilding process of that element without any translation due to mean flow". Inoue [8] clearly has the same idea in mind when he states that ". . . the time correlation for the wind fluctuations observed with an anemometer [moving with the mean wind] is not the Eulerian but the Lagrangian coefficient . . .". On the other hand, Frenkiel [5] calls this system psuedo-Eulerian. Nevertheless, the identification of turbulence statistics obtained in this way with the true Lagrangian (parcel) statistics seems to the writer to be quite reasonable.

According to Ogura's model, the contribution, n_I, to the frequency fluctuations at a point arising from the translation of eddies by the mean flow is

$$n_I = kU \tag{1}$$

where k is wave number and U is the (constant) mean wind speed. The contribution due to decay and rebuilding, n_{II}, is assumed to be

$$n_{II} = k\left[\overline{w^2}\int_k^\infty F(k)dk\right]^{1/2} \tag{2}$$

where $\overline{w^2}$ is the mean square turbulent velocity, and F is the spectrum, following von Weizsäcker's [30] model of turbulence. The total frequency fluctuation at a point, n, is then

$$n = n_I + n_{II} \tag{3}$$

If the Lagrangian contribution is identified with n_{II}, and if the spectral maxima are chosen for the sake of comparison, then the difference in maximum frequency between the Eulerian-time and Lagrangian spectra is given by $n - n_{II}$. Thus the difference between the tower and balloon spectral maxima, Δn, should be proportional to the mean wind. Figure 8 shows this comparison for runs 30 through 35, and a linear relation is certainly indicated. The factor of proportionality should equal the wave number of the spectral maximum. If one assumes that Bunker's airplane run (fig. 7) is equivalent to the Eulerian-space data, then the observed spectral maximum is at .00175 cycles per meter. This corresponds to an airspeed of 57 meters per second. The wave number computed from

$$n - n_{II} \equiv \Delta n = kU$$

where for run 35 the mean wind is 5.5 meters per second and Δn equals 31 cycles per hour, turns out to be .00156 cycles per meter, in good agreement with the actual value.

If Ogura's suggested interpolation formula for the wave number spectrum,

$$F\left(\frac{k}{k_0}\right) = \frac{2}{3}\frac{k/k_0}{\{1+(k/k_0)^2\}^{4/3}} \tag{4}$$

is introduced into equation (2), it follows after integration that

$$n_{II} = \sqrt{\overline{w^2}}\cdot\frac{k}{\{1+(k/k_0)^2\}^{1/6}} \tag{5}$$

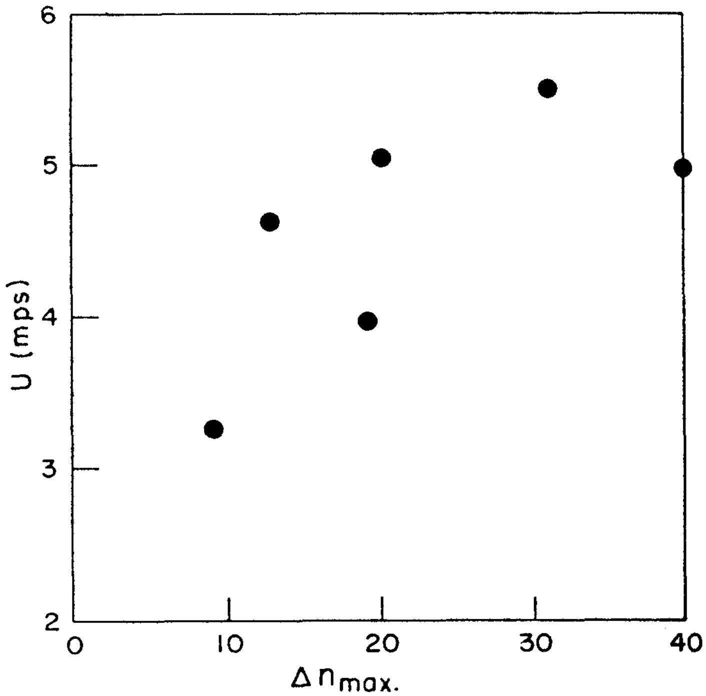

FIGURE 8.—Observed differences between frequencies of the tower and balloon spectral maxima, Δn, as a function of mean wind speed, U.

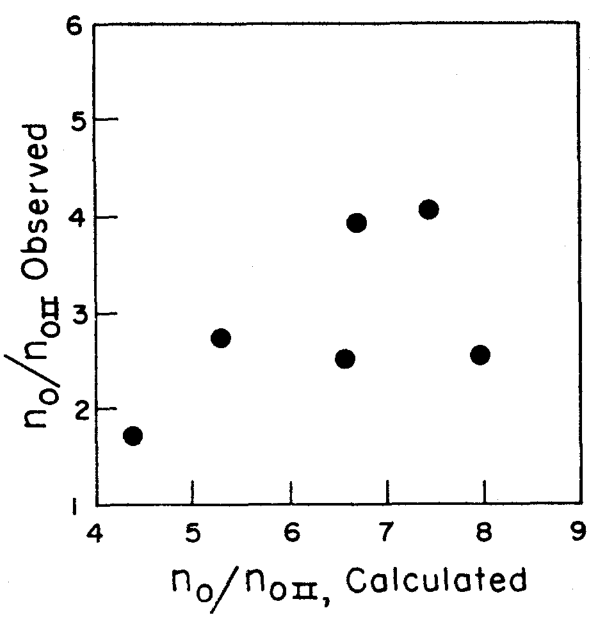

FIGURE 9.—Observed and calculated ratios of the tower and balloon spectral maximum frequencies.

where k_0 is an arbitrary reference wave number, for convenience chosen to be the spectral maximum. Consequently,

$$n_{0II} = \frac{k_0\sqrt{\overline{w^2}}}{(2)^{1/6}}$$

If we again identify this with the Lagrangian contribution, and form the *ratio* of the Eulerian-time spectral frequency maximum, $n_0 = n_{0I} + n_{0II}$, to the Lagrangian, we obtain

$$\frac{n_0}{n_{0II}} = \frac{1.12U}{(\overline{w^2})^{1/2}} + 1 \tag{6}$$

i. e., the ratio of the maxima should depend linearly on $U/(\overline{w^2})^{1/2}$. The spectral maximum ratios calculated by equation (6) are compared, in figure 9, with the observed values from figures 1 through 6. The comparison suggests the possible importance of the turbulence intensity, $(\overline{w^2})^{1/2}/U$, in the Lagrangian, Eulerian-time relationship, and there is some indication of the desired linear relationship.

Although these comparisons leave much to be desired, they may serve as a basis for further study of this difficult problem.

ACKNOWLEDGMENTS

A number of people have given their time and skill freely in order to make this work possible, and the writer is sincerely grateful for their help. The difficult observational program was the joint accomplishment of four separate meteorological groups. Participants from the Brookhaven National Laboratory meteorological staff included M. E. Smith, I. A. Singer, R. M. Brown, F. Bartlett, G. W. Potts, G. S. Raynor, and Miss C. M. Smith. Their ready cooperation, as well as the availability of the excellent micrometeorological facilities at Brookhaven, were both fundamental to the success of the program. The assistance of R. A. McCormick and G. DeMarrais, U. S. Weather Bureau, and D. L. Jones and I. Van der Hoven, Department of Meteorology, The Pennsylvania State University, was likewise invaluable. The airplane gust measurements were obtained by Mr. Andrew Bunker, Woods Hole Oceanographic Institution. The writer feels deeply indebted to each participant, and hopes that all have derived some satisfaction from the successful completion of this unusual observational program. Thanks are also due to Mrs. R. Spaulding and Mrs. V. Mares for performing the spectral computations, and to Prof. H. A. Panofsky for many helpful discussions of the work. Finally, the writer is grateful to Weather Bureau reviewers for a critical review of the manuscript and many helpful suggestions.

APPENDIX

EFFECTS OF AVERAGING AND PRE-WHITENING ON CALCULATED SPECTRA

Any series of turbulent wind velocity fluctuations must necessarily undergo several modifications before appearing as an analyzed spectrum. Observations of velocity fluctuations have usually been subjected to two modifications prior to, or after, Fourier analysis. *Averaging* of the velocity fluctuations in the case of the bivane observations is necessary because of the bivane's resonant period. The neutral balloon observations of wind velocities are also averages, because of the interval between successive readings. The process known, in Tukey's picturesque terminology, as *pre-whitening* is another modification that has been found useful in spectral analysis. Although these processes have been applied in many of the recent studies, it seems appropriate to indicate here how their influence enters the analysis. For pre-whitening, in particular, no discussion seems to be available, at least in meteorological literature.

Averaging.—The effect of averaging a function, prior to harmonic analysis, is a well known aspect of this venerable tool of research; see, for example, Jeffreys and Jeffreys [9], page 450. Ogura has recently discussed the problem in energy spectral terms. Here a somewhat less complicated derivation will be presented. When a Fourier expansion of the velocity fluctuation, $u(t)$, is made,

$$u(t) = \int \phi(n) e^{int} dn \tag{1}$$

where n is frequency.

If, however, the velocity record is an average over an interval of length, say, $2a$, equation (1) becomes

$$\overline{[u(t)]}_{2a} = \int \phi(n) \cdot \frac{1}{2a} \int_{t-a}^{t+a} e^{int} d\tau dn. \tag{2}$$

By a straightforward integration, we obtain

$$\overline{[u(t)]}_{2a} = \int \phi(n) \frac{\sin (na)}{na} e^{int} dn. \tag{3}$$

Here the true amplitudes, $\phi(n)$, are suppressed by a factor $(\sin na)/na$. Squaring the amplitude to obtain the spectrum, it is found that the observed and true spectra are related by

$$F_{true}(n) = \frac{(na)^2}{\sin^2 (na)} F_{avg}(n). \tag{4}$$

Pre-whitening.—In general, the numerical process for the spectral analysis of the velocity fluctuations does not work well where the spectrum varies rapidly as a function of frequency; in fact, spectra computed by Tukey's method sometimes show negative energy at high frequencies. To get around this difficulty, Tukey suggested the following: if u_j represents the original set of averaged velocity values, then define a new set of values such that

$$y_j = u_j - b u_{j-1} \tag{5}$$

where b is some constant, $0 < b < 1$. In practice one would first have removed the mean velocity, i. e.,

$$y_i = (u_j - \overline{u_j}) - b(u_{j-1} - \overline{u_{j-1}})$$

but for simplicity of notation let us consider (5). This has the effect of suppressing long-period fluctuations relative to short-period fluctuations.

Forming the lagged products,

$$y_j y_{j+m} = (u_j - b u_{j-1}) \ (u_{j+m} - b u_{j+m-1}) \tag{6}$$

and expanding and collecting terms, one obtains

(7) $y_jy_{j+m}=u_ju_{j+m}-bu_{j-1}u_{j+m}-bu_ju_{j+m-1}+b^2u_{j-1}u_{j+m-1}$

In functional notation for continuous t, this is equivalent to

(8) $$y(t)y(t+m\tau)=u(t)u(t+m\tau)-bu(t-\tau)u(t+m\tau)-bu(t)u(t+m\tau-\tau)+b^2u(t-\tau)u(t+m\tau-\tau)$$

where τ is the time interval between the original set of values u_j. Putting $h\equiv m\tau$, the ordinary autocorrelation $R(h)$ is formed by summing and averaging terms such as the first one on the right; the terms with coefficients $-b$, however, define slightly different correlations, $R(h+\tau)$ and $R(h-\tau)$. Summing and averaging the terms in (8) therefore gives

(9) $$R_{pre}(h)=(1+b^2)R(h)-bR(h+\tau)-bR(h-\tau)$$

where $R_{pre}(h)$ is the autocorrelation for the pre-whitened variable. The Fourier transform of (9) is obtained by multiplying through by $\frac{1}{2\pi}e^{inh}\,dh$ and integrating. This gives

(10) $$F_{pre}(n)=(1+b^2)F(n)-\frac{b}{2\pi}\int e^{inh}R(h+\tau)dh-\frac{b}{2\pi}\int e^{inh}R(h-\tau)dh.$$

But (10) is equivalent to

(11) $$F_{pre}(n)=(1+b^2)F(n)-\frac{be^{-in\tau}}{2\pi}\int e^{in(h+\tau)}R(h+\tau)d(h+\tau)-\frac{be^{in\tau}}{2\pi}\int e^{in(h-\tau)}R(h-\tau)d(h-\tau)$$

which gives

(12) $$F_{pre}(n)=(1+b^2)F(n)-be^{-in\tau}F(n)-be^{in\tau}F(n).$$

Since $\cos n\tau=\frac{e^{in\tau}+e^{-in\tau}}{2}$, equation (12) reduces to

(13) $$F_{pre}(n)=(1+b^2-2b\cos n\tau)F(n).$$

which shows the relation between the spectrum of the pre-whitened variable, $F_{pre}(n)$, and that of the original, $F(n)$.

This process can clearly be generalized so as to smooth out any unwanted spectral peaks. The true spectrum, $F(n)$, is then obtained, in its original "color", by the use of (13).

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