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COMPUTATION AND USES OF GRADIENT WINDS

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ABSTRACT

The viewpoint is taken that in large portions of the atmosphere, wind speeds (V_{gr}), divergence (D_{gr}), and vorticity (ζ_{gr}) obtained under the gradient wind assumption are considerably more accurate representations of true conditions than those one obtains from the geostrophic assumption. Equations are derived for computing V_{gr}, D_{gr}, and ζ_{gr}. The equation for the vorticity of gradient winds has a considerable resemblance to "balance" equations of Charney [3] and Phillips [17]. Gradient winds and their space derivatives may have advantages over other formulations due to their relative simplicity. The quantities V_{gr}, D_{gr}, and ζ_{gr} may be computed quite easily from geopotential data at grid points by use of high-speed computers. Possible applications of gradient winds in practical and theoretical meteorology are suggested.

1. INTRODUCTION

The geostrophic wind equation, an empirical relation which equates horizontal pressure and Coriolis forces, is a well-known cornerstone of both theoretical and applied meteorology in extratropical latitudes. One might expect that a generalization of the geostrophic equation to include a third term of major importance, namely, the centrifugal force, would lead to more accurate modeling of winds than the simple geostrophic relation provides. However, the reader will immediately realize that the centrifugal force V^2K, where V is wind speed and K is the trajectory curvature, is small in regions of weak winds, and that in regions of straight flow the gradient wind automatically degenerates into the geostrophic. For example, Neiburger et al. [13] compared geostrophic and gradient winds with observed winds on two 700-mb. charts having a mean observed speed of 18 kt. and found that gradient winds were not significantly better than geostrophic. In a study of strong winds (having an average speed of 94 kt.), Endlich and McLean [7] found that in cyclonically-curved jet streams, geostrophic winds overestimated wind speeds measured by aircraft by 28 kt. on the average (18 percent of the geostrophic speeds of the sample) while gradient winds differed only minutely from measured values. In other words, the centrifugal force was 18 percent of the pressure force. The standard deviation of geostrophic departures (after accounting for observational errors) was estimated as 22 kt. while the standard deviation of gradient departures was estimated as 5.5 kt. From these data, one would conclude that in and around jet streams, and probably also in relatively strong currents at other altitudes, consideration of the centrifugal force accounts for the major part of the ageostrophic components. Remaining ageostrophic components are due to cross-contour flow, to vertical motions, and to friction. However, studies of constant-level balloon trajectories (Neiburger and Angell [12], Angell [1]) indicate that cross-contour components are relatively large, being on the average about two-thirds of the magnitude of the along-contour (centrifugal) components. Thus, the question of the relative magnitude of the two ageostrophic components is presently unresolved. Nevertheless, Angell estimates that the percentage error in estimating winds geostrophically exceeds 29 percent half of the time, while the error of the gradient assumption is considerably less, exceeding 11 percent half of the time. In practice, one can include the centrifugal components by use of gradient winds; however, the cross-contour components, though important in initiating disturbances and in subsynoptic phenomena, are very difficult to estimate from conventional data.

The hypothesis is therefore proposed that gradient winds provide an estimate of actual winds which is sufficiently accurate to make them useful in describing the dynamics of synoptic-scale processes. In support of this hypothesis, one may recall that the propagation of cyclone waves was explained (see, for example, the review by Bjerknes [2]) on the basis of divergence of gradient winds in advance of upper-level troughs and convergence behind. Moreover, Phillips [17] has shown that strictly non-divergent winds introduce "meteorological noise" and are not suitable for computing initial data for numerical prediction based on integration of the primitive equations. Gradient winds, which are divergent yet approximately geostrophic, might provide a simple, relatively noiseless balance condition between winds and the geopotential field. It will be shown later that the equation for gradient vorticity (ζ_{gr}) is quite similar to the well-known balance equation (Petterssen [14], Charney [3]) and to a vorticity equation based on a divergent wind (Edelmann [6], Phillips [17]).

One of the main difficulties to be expected in computing winds (and especially divergence fields) from present upper-air data is that random errors caused by observational inaccuracies will be contained in the results. Therefore, smoothing of computed values may be a prerequisite to their use.

In order to determine whether advantages accrue, in practice, from the explicit calculation of gradient winds and their divergence and vorticity fields, in the future it will be necessary to carry out computations for comparison with observed winds and with divergence and vorticity calculated by other techniques. In the present article, a simple method of successive approximations is developed for computing V_{gr}, D_{gr}, and ζ_{gr}. The nature of the computational scheme is such that it can be carried out quite simply and rapidly by use of electronic computers.

2. COMPUTATION OF VELOCITY, DIVERGENCE, AND VORTICITY OF GRADIENT FLOW

The basic assumptions of gradient flow (e.g., Holmboe et al. [11]) are that air parcels maintain constant speed (i.e., do not cross the contours on constant pressure charts toward higher or lower heights) and move horizontally and without friction. The equation which results on introduction of these assumptions into the horizontal equation of motion may be written as follows:

$$V_{gr}^2\,(d\theta/ds)\,\mathbf{n}=-f(V_{gr}-V_g)\,\mathbf{n} \tag{1}$$

where V_{gr} is the gradient speed; θ is the wind direction taken as positive in the counter-clockwise direction; s is distance along the path; $d\theta/ds$ is K the curvature of the path; $\mathbf{n}=-\nabla\phi/|\nabla\phi|$ is a unit vector directed to the left of the path, ∇ operates at constant pressure, and $\phi=gz$; f is the Coriolis parameter; and V_g is the geostrophic wind speed. The term on the left side represents the centrifugal force which is proportional to the difference between V_{gr} and V_g.

If we post-multiply equation (1) vectorially by the vertical unit vector $\mathbf{k}$ we obtain

$$f\mathbf{V}_{gr}=f\mathbf{V}_g-V_{gr}^2\,(d\theta/ds)(\mathbf{n}\times\mathbf{k}). \tag{2}$$

Also
$$f\mathbf{V}_g=\mathbf{k}\times\nabla\phi=|\nabla\phi|(\mathbf{n}\times\mathbf{k}) \tag{3}$$

or
$$\mathbf{n}\times\mathbf{k}=(1/|\nabla\phi|)f\mathbf{V}_g.$$

Therefore
$$\mathbf{V}_{gr}=[1-(V_{gr}^2/|\nabla\phi|)(d\theta/ds)]\mathbf{V}_g. \tag{4}$$

After a further simplification, this form of the gradient wind equation will be used to compute V_{gr}. The simplification involves the use of Blaton's formula

$$\frac{d\theta}{ds}=\frac{dt}{ds}\frac{d\theta}{dt}=\frac{1}{V}\left(\frac{\partial\theta}{\partial t}+V\frac{\partial\theta}{\partial s}+\omega\frac{\partial\theta}{\partial p}\right)\approx\frac{1}{V}\frac{\partial\theta}{\partial t}+\frac{\partial\theta}{\partial s} \tag{5}$$

i.e., we neglect the term containing the vertical advection of θ. If vertical motion is present, actual three-dimensional trajectories tend to be less cyclonic than trajectories estimated on isobaric charts (Danielsen [5]). The difference is greatest in fronts or other phenomena where vertical motions reach maximum values. It is of lesser, and perhaps negligible, importance in larger-scale motions. As there is no simple method for estimating the magnitude of the difference, it will be neglected in the remainder of this paper.

We will now define a parameter $k=(1/|\nabla\phi|)(d\theta/ds)$ or from equation (5)

$$k=(1/|\nabla\phi|)(\partial\theta/\partial s)+(1/|\nabla\phi|)(1/V)(\partial\theta/\partial t) \tag{6}$$

where k represents the trajectory curvature, the first term on the right measures the contour curvature, and the second term measures the local change in wind direction. For cyclonic trajectories $k>0$ and for anticyclonic, $k<0$. Since we have assumed flow parallel to contours, $\partial\theta/\partial t$ may be evaluated as the local change in contour direction. This direction may be measured counterclockwise from east so that

$$\theta=\tan^{-1}(v_g/u_g)=\tan^{-1}(-\phi_x/\phi_y) \tag{7}$$

where the subscripts x and y denote partial derivatives eastward and northward.

To evaluate equation (6) we use $\partial\theta/\partial s=\cos\theta(\partial\theta/\partial x)+\sin\theta(\partial\theta/\partial y)$ where θ is given by equation (7) and, as a matter of convenience, substitute V_g for V. Then

$$k=(\phi_x^2+\phi_y^2)^{-2}[(\phi_y^2\phi_{xx}-2\phi_x\phi_y\phi_{xy}+\phi_x^2\phi_{yy})+f(\phi_x\phi_{yt}-\phi_y\phi_{xt})]. \tag{8}$$

Given the distribution of ϕ at grid points on a constant pressure chart, one can easily compute a finite difference approximation to the first term in the bracket. A shortened version of this term has been used by Phillips [16] in computing contour curvature. The second term in the bracket can be evaluated as the difference between geopotential gradients on charts separated in time, or from the space variations of geopotential tendency.

For progressive, wave-shaped contour patterns, the curvatures of trajectories and contours are of the same sign (unless the waves are traveling faster than the winds); however, the trajectories are less sharply curved than the contours.[1] In other words, the term $(1/V)(\partial\theta/\partial t)$ is generally of opposite sign but smaller magnitude than $\partial\theta/\partial s$. The smoothing effect of a finite-difference evaluation of $\partial\theta/\partial s$ tends to reduce its magnitude toward the magnitude of the trajectory curvature. Therefore, on upper-level charts, especially where V is large, it may prove feasible to neglect the term containing $(1/V)(\partial\theta/\partial t)$.

Having obtained a value of k at each grid point by use of equation (8), we may write equation (4) in vector or scalar form as

$$\mathbf{V}_{gr}=(1-kV_{gr}^2)\mathbf{V}_g \quad \text{or} \quad V_{gr}=(1-kV_{gr}^2)V_g. \tag{9}$$

This quadratic equation may be solved for V_{gr} since k and V_g are known. In the past, the solution has ordinarily been obtained graphically; e.g., Gustafson [9]. If the equation is solved by electronic computer, the square root in the quadratic formula is calculated by use of an initial guess and successive approximations. It appears that a solution to equation (9) may be obtained more directly and simply by the following process of successive approximations. As a first guess, we may substitute V_g^2 for V_{gr}^2 on the right side of (9). Then if we replace the subscript on V_{gr} by the number of the approximation we obtain

$$\begin{aligned} V_1&=(1-kV_g^2)V_g\\ V_2&=(1-kV_1^2)V_g\\ &\cdot \quad \cdot \quad \cdot\\ &\cdot \quad \cdot \quad \cdot\\ &\cdot \quad \cdot \quad \cdot\\ V_n&=(1-kV_{n-1}^2)V_g. \end{aligned} \tag{10}$$

The convergence of V_n to V_{gr} is quite rapid as shown in figure 1. Here we have assumed a typical jet stream condition where $V_g=125$ kt. and $k=2\times10^{-5}$ kt.$^{-2}$ (cyclonic curvature) so that $V_{gr}=100$ kt. The approximations fall alternately on opposite sides of V_{gr} and in this particular case, V_5 approaches V_{gr} within 1 percent. For $V_g=83.3$ and $k=-2\times10^{-5}$ (anticyclonic curvature), the approximations approach V_{gr} asymptotically. In this example, V_3 approached V_{gr} (100 kt.) within 1 percent. It can be shown that for these two examples, convergence will occur if the absolute value of the initial guess is less than 500 kt. In general, the region of convergence is quite broad. The geostrophic speed is a convenient initial guess which is in the neighborhood of the gradient speed and therefore results in an acceptable estimate of the gradient speed after a small number of successive approximations.

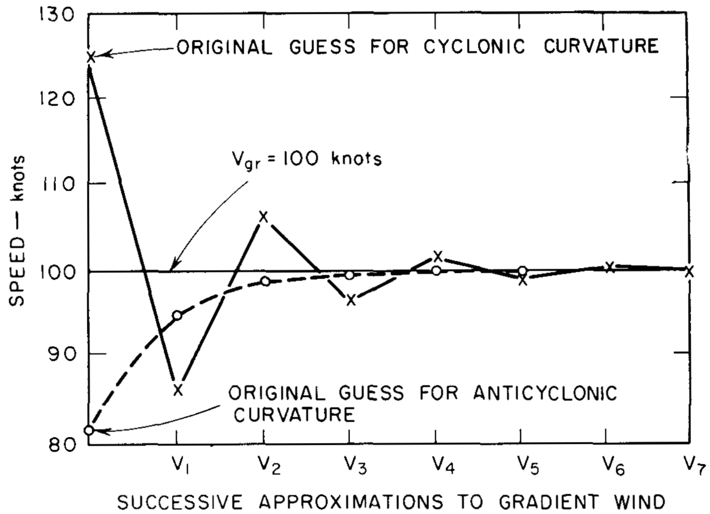

FIGURE 1.—Graph showing the convergence of successive approximations to true gradient wind speed (V_{gr}) using the geostrophic speed (V_g) as the initial guess. Solid lines join approximation for cyclonic curvature; dashed curve joins approximations for anticyclonic curvature. Approximations computed using equation (10) in text.

For anticyclonic flow and for a given radius of curvature, V_{gr} increases with V_g up to a limit of $2V_g$. Beyond this point, the quantity $4kV_g^2<-1$ and equation (9) has complex solutions which have no physical reality. Gradient wind balance does not exist in these "unstable ridges" and the approximations of equation (10) will not converge. Grid points at which this equation diverges may be marked with a special symbol to indicate the absence of gradient flow. Unstable ridges, though rather uncommon, are synoptically important since cut-off Lows sometimes form downstream (Bjerknes [2]).

We will assume that by using equation (10) we have obtained a satisfactory estimate of V_{gr} at each grid point (with the exceptions mentioned above). Now it is convenient to define a quantity

$$B=-kV_{gr}^2$$

so that equation (9) becomes

$$V_{gr}=(1+B)V_g \quad \text{or} \quad B=(V_{gr}-V_g)/V_g. \tag{11}$$

B is a non-dimensional term which expresses the departure

[1] The relationships between streamlines and trajectories are discussed by Holmboe et al. [11] and Haltiner and Martin [10], among others.

of the gradient speed from the geostrophic as a fraction of the geostrophic speed. It may also be thought of as the correction to be made to the geostrophic wind in order to obtain the gradient wind. For convenience, it will be referred to as the "gradient correction." In a roughly sinusoidal pattern of contours in which a jet stream is imbedded, one would expect centers of negative B with central values of about 20 percent where the jet crosses troughs, and similar centers of positive B where the jet crosses ridges.

The "gradient divergence" is defined as

$$D_{gr}=\nabla\cdot\mathbf{V}_{gr}=\nabla\cdot(1+B)\mathbf{V}_g=\mathbf{V}_g\cdot\nabla B+(1+B)\nabla\cdot\mathbf{V}_g. \quad (12)$$

The divergence therefore depends on the geostrophic wind and the field of the gradient correction B. To obtain a quantitative estimate of the first term on the right side of equation (12), we take a typical value of geostrophic wind speed as 30 m. sec.$^{-1}$ To obtain a typical value of ∇B, we assume that B changes from -0.20 in a trough to zero at an inflection point over a distance of 600 km. Then $\mathbf{V}_g\cdot\nabla B$ equals 10^{-5} sec.$^{-1}$ The second term on the right side of (12) is generally an order of magnitude smaller but may be of importance in strong meridional currents in middle or low latitudes. Therefore $D_{gr}\sim 10^{-5}$ sec.$^{-1}$ which is of the order expected for progressive waves.

The relative vorticity of the gradient wind at any point is

$$\zeta_{gr}=\mathbf{k}\cdot\nabla\times\mathbf{V}_{gr}=(1+B)\zeta_g+\mathbf{k}\cdot\nabla B\times\mathbf{V}_g \quad (13)$$

where $\zeta_g=\mathbf{k}\cdot\nabla\times\mathbf{V}_g$ is the geostrophic vorticity. Since B is negative in cyclonic flow and positive in anticyclonic, the term $(1+B)$ on the right side of (13) tends to make ζ_{gr} either less or more than ζ_g. Petterssen [14] has shown that the geostrophic vorticity overestimates the vorticity of cyclones and underestimates it in anticyclones. Therefore, B provides a correction in the proper direction.

Letting $\beta=\partial f/\partial y$ and using equation (3), $\zeta_g=f^{-1}(\nabla^2\phi+\beta u_g)$. On expansion, the last term in (13) is $v_gB_x-u_gB_y$. Using these relations, equation (13) becomes

$$(1+B)\nabla^2\phi-f\zeta_{gr}+(1+B)\beta u_g+fv_gB_x-fu_gB_y=0. \quad (14)$$

In a machine computation of gradient winds and their divergence and vorticity, one might proceed as follows:

1. Tabulate the geopotential at grid points on a constant pressure chart.
2. Using geopotential data at neighboring grid points as finite-difference approximations to partial derivatives, compute u_g, v_g, and V_g (equation (3)), and θ (equation (7)).
3. The curvature parameter k (equation (8)) may be computed in two steps. Compute the first term in the bracket from the grid point data of step 2. The second term in the bracket may be computed as the difference between past and present values of ϕ_x and ϕ_y or as the eastward and northward changes in geopotential tendency.
4. At each grid point, compute V_{gr} from V_g and k (equation (10)).
5. Compute B at each point from V_{gr} and V_g (equation (11)).
6. Approximate ∇B by finite differences and compute D_{gr} at each point (equation (12)). If desired, vertical motions may be obtained from D_{gr} by use of the continuity equation with appropriate boundary conditions.
7. Compute ζ_{gr} from equation (13) or (14).

It should be noted that these calculations do not require solution of first or second order operators over the grid. A rough estimate indicates that all of the above quantities (except vertical motion) can be calculated by a high-speed computer such as the IBM 704, for a 2000-point grid in approximately 30 seconds.

3. COMPARISON OF THE VORTICITY OF GRADIENT WINDS WITH OTHER BALANCE RELATIONS

Equation (14) bears a strong resemblance in its first three terms to corresponding terms in the "balance" equation (Petterssen [14], Charney [3]) written as

$$\nabla^2\phi-f\zeta+\beta u+u_x^2+2v_xu_y+v_y^2=0 \quad (15a)$$

or in terms of a streamfunction as

$$\nabla^2\phi-f\nabla^2\psi-\beta\psi_y-2(\psi_{xx}\psi_{yy}-\psi_{xy}^2)=0. \quad (15b)$$

A slightly different equation has been obtained by Edelmann [6] and Phillips [17]:

$$\nabla^2\phi-f\zeta+\beta u_g-2f^{-2}(\phi_{xx}\phi_{yy}-\phi_{xy}^2)=0. \quad (16)$$

It should be noted that the vorticity specified by this equation generally differs somewhat from that given by equation (15).

The similarities of equations (14), (15), and (16) are evidently due to the fact that they are all obtained from the horizontal equation of motion by approximating the acceleration in one way or another. The gradient wind approximates $d\mathbf{V}/dt$ by the centrifugal force and is equivalent to expressing the geostrophic departure $\mathbf{V}'$ as

$$\mathbf{V}'=\mathbf{V}-\mathbf{V}_g\approx\mathbf{V}_{gr}-\mathbf{V}_g=f^{-1}(\mathbf{k}\times\mathbf{n})V_{gr}^2\,(d\theta/ds). \quad (17)$$

As mentioned earlier, equation (14) is simply the vorticity of the gradient wind. Equation (15) is obtained by taking the divergence of the horizontal equation of motion and then neglecting the terms $d(\nabla\cdot\mathbf{V})/dt$ and $\nabla\omega\cdot(\partial\mathbf{V}/\partial p)$. The wind field specified by this equation is non-geostrophic and non-divergent. Equation (16) gives the vorticity of a divergent, quasi-geostrophic wind which is obtained by expressing the geostrophic departure $\mathbf{V}'$ as $f^{-1}\,\mathbf{k}\times(d\mathbf{V}/dt)$ and by approximating $d\mathbf{V}/dt$ geostrophically (Philipps [15]) so that

$$\mathbf{V}'=f^{-1}\mathbf{k}\times(\partial\mathbf{V}_g/\partial t+\mathbf{V}_g\cdot\nabla\mathbf{V}_g). \quad (18)$$

As a consequence of his demonstration that non-divergent initial wind fields introduce "meteorological noise" into integrations of the primitive equations, Phillips [17] has proposed obtaining initial wind data by solution of three simultaneous equations (not repeated here) which give D, the horizontal divergence. ζ is determined from equation (16), and D and ζ with boundary conditions determine a wind field which is relatively "noiseless."

In the writer's opinion, the relative merits of expressing $\mathbf{V}'$ by equation (17) or by equation (18) are difficult to judge on theoretical grounds. It follows that equations (14) and (16) are equally plausible; however, gradient winds and their divergence and vorticity appear much simpler to compute than the comparable quasi-geostrophic, divergent quantities. It may be logical, therefore, to compute ζ_{gr}, $\nabla^2\psi$, and ζ by using equations (14), (15), and (16), respectively, from actual data for a number of cases. The three vorticity (or corresponding wind) fields might then be compared and tested in actual forecasts.

4. APPLICATIONS OF GRADIENT WINDS

The following applications of gradient winds appear to warrant further investigation:

1. Specifying winds on current or forecast charts. Gradient winds might be used in flight forecasting for aircraft, in determining trajectories, and in locating jet streams (Endlich and McLean [8]) and fronts.

2. Use as a balance condition between winds and the geopotential field; for example, in computing initial vorticity fields for use in forecasting with the vorticity equation or in computing initial wind data for use in integrations of the primitive equations.

3. Determination of a first approximation to horizontal divergence fields and corresponding vertical motions.

4. As a means of introducing wind data into the objective analysis of fields of geopotential or streamfunction in place of the current "geostrophic" use of observed winds (Cressman [4]). A possible method of introducing observed winds according to the gradient assumption would be to compute the gradient correction (B) that corresponds to the geopotential field used as the initial guess. Observed winds divided by $(1+B)$ would give geostrophic winds for use in the objective analysis procedure.

As mentioned in the Introduction, the application of gradient winds to the problems mentioned above is based on the empirically supported hypothesis that they provide a considerably more accurate approximation to true conditions than the more restrictive geostrophic assumption provides; however, this hypothesis must be tested further with observational data before a final judgment of its validity can be made.

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