ANALYSIS OF THE U AND V FIELDS IN THE VICINITY OF THE POLES[1]

ARNOLD APPLETON BARNES, Jr.

Air Force Cambridge Research Laboratories, Bedford, Mass.

[Manuscript received August 7, 1962; revised October 18, 1962]

ABSTRACT

The fields of the eastward (U) and northward (V) components of the wind are investigated mathematically. Maps whose point values are the products of the point values of two maps are also investigated. Rules are given to aid in the analysis of these types of maps.

1. THE PROBLEM

The use of polar coordinate charts means that the maps of U and V, the eastward and northward components of wind, have mathematical singularities at the poles. The existence of these singularities has been either completely ignored or circumvented by some device such as, for example, not analyzing near the poles. A strict mathematical analysis of the U and V fields in the vicinity of one of these poles reveals information which has proved to be of great assistance to the analyst. Application of our rules considerably improved the analysis of a series of Northern Hemisphere maps (Barnes [1]) in the region north of 75° N.

Even though the following mathematical development is around the North Pole, it is applicable to the pole of any polar coordinate system such as, for example, those frequently employed in special studies of hurricanes and extratropical storms where the radial, V, and tangential, U, velocity components are used.

One of the ways of representing the horizontal wind field is to use the south to north component, V, and the west to east component, U. This representation is unique except at the north and south poles. A closer inspection reveals that, under the U, V decomposition, one is mapping a cylinder onto a spherical surface (e.g., 100-mb. surface). This mapping of a cylinder onto a coaxial spherical surface is unique (one-to-one) except at the poles. The top and bottom lines of the cylinder are mapped onto the poles so the mapping is not one-to-one at these points. Thus, special consideration must be given to quantities containing U or V in the neighborhoods of the north pole and the south pole.

2. THE MATHEMATICAL ANALYSIS

Let us consider the wind over a small neighborhood centered at the north pole. If we take this neighborhood small enough, then the quantities such as the horizontal wind velocity, temperature and height of the pressure surface may be considered as constant. This assumption seems to be better justified in the stratosphere than near the surface of the earth where the temperature and horizontal velocity fields may be discontinuous. The fact that one performs continuous, smooth analysis of the maps using discrete data presumes that the gradients are smooth and the point values are finite.

In a sufficiently small neighborhood of the north pole, the horizontal wind velocity can be considered constant and can be defined in the following manner. First, redefine the longitudes so that λ west longitude becomes $(360°-\lambda)$ east longitude. Then the horizontal wind is uniquely defined by the non-negative quantity C, the wind speed, and λ_c, the meridian along which the wind is blowing toward the north pole. If ϵ is the distance from the pole, then ϵ_M is defined as the radius of the largest circle which fits inside the neighborhood of the pole where the horizontal wind velocity can be considered constant. The map of V is now given by $C \cos(\lambda-\lambda_c)$, $0<\epsilon<\epsilon_M$.

If, in this neighborhood, we approach the pole along the meridians $(\lambda_c+90°)$ and $(\lambda_c-90°)$ we find the value of V to be zero, no matter how close we come to the pole along these lines. A map shall be said to be a *pattern 1 map* if it can be expressed by $C \cos(\lambda-\lambda_c)$ for $\epsilon>0$ in the neighborhood of the pole. Thus, figure 1, showing the map of V in the neighborhood of the north pole assuming a wind speed of 10 kt. is a pattern 1 map.

The map of the U component is given by $C \sin(\lambda-\lambda_c) = C \cos[\lambda-(\lambda_c+90°)]$, so the map of U is also a pattern 1 map. Thus, the U map is just the V map rotated 90° to the east about the pole. This means that for the same time period, the zero isopleths on the maps of U and V should be at right angles to each other in the neighborhood of the pole.

As mentioned before, the U and V components of the horizontal wind are not defined at the pole. Hence, the pole has been excluded from figure 1. If the value at the pole were assigned the zero value, then the zero isotach would be continuous and would not have a kink at

[1] The research reported in this document was carried out while the author was at the Massachusetts Institute of Technology, Cambridge, Mass., and was supported by the Geophysics Research Directorate of the Air Force Cambridge Research Laboratories, Office of Aerospace Research, under Contract No. AF19(604)-5223 and by the U.S. Atomic Energy Commission under Contract No. AT(30-1)-2241.

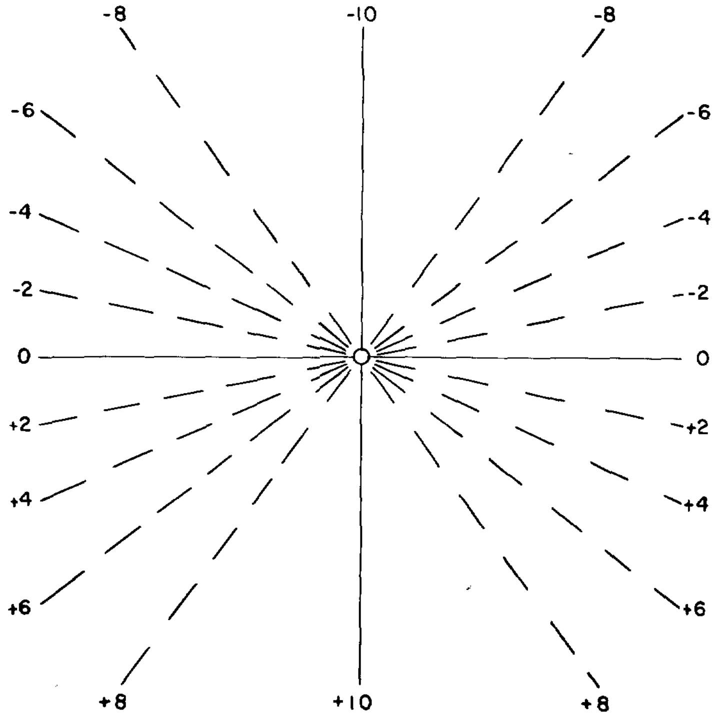

FIGURE 1.—A pattern 1 map, $C \cos(\lambda-\lambda_c)$, for $C=10$ kt.

the pole. If we consider any other contour K $(-C\leq K\leq C)$ and let the pole have the value K, then this isotach would have a kink at the pole. Notice that the equation $U^2+V^2=C^2$ does hold at the poles since U and V are not uniquely defined at the poles.

In the following discussion we shall be concerned with the neighborhood of the pole and not with the value at the pole itself.

As stated before, a pattern 1 map is represented mathematically by $C \cos(\lambda-\lambda_c)$ for $0<\epsilon<\epsilon_M$, where C and λ_c are constants, but is undefined for $\epsilon=0$.

The sum of two maps is defined as the map of the sum of the values at the individual points. Adding two pattern 1 maps, we obtain, for $0<\epsilon<\epsilon_M$,

$$C_1 \cos(\lambda-\lambda_1)+C_2 \cos(\lambda-\lambda_2)=\cos\lambda[C_1\cos\lambda_1+C_2\cos\lambda_2]+\sin\lambda[C_1\sin\lambda_1+C_2\sin\lambda_2].$$

If we now let

$$C_3\cos\lambda_3=C_1\cos\lambda_1+C_2\cos\lambda_2$$

and

$$C_3\sin\lambda_3=C_1\sin\lambda_1+C_2\sin\lambda_2$$

then

$$C_1\cos(\lambda-\lambda_1)+C_2\cos(\lambda-\lambda_2)=C_3\cos\lambda\cos\lambda_3+C_3\sin\lambda\sin\lambda_3=C_3\cos(\lambda-\lambda_3)$$

where

$$C_3=\sqrt{C_1^2+C_2^2+2C_1C_2\cos(\lambda_2-\lambda_1)}$$

Thus, pattern 1 is conserved under addition.

The negation operation is defined by multiplying all point values of the map by minus one. This is equivalent to replacing λ_c by $(180°+\lambda_c)$ giving $C\cos(\lambda-180°-\lambda_c)$ for V, which is only a change of orientation. Thus, pattern 1 is conserved under negation. It should be noted that the orientation is not unique unless C is positive.

Subtraction of map X from map Y is performed by the subtraction of the point values of X from the point values of Y. This is equivalent to the addition of map Y and the negative of map X, so pattern 1 is conserved under the operation of subtraction.

If the value at every point of a pattern 1 map is divided by a finite non-zero number of quantity Z which is a constant over the map, then the resulting map,

$$\frac{C\cos(\lambda-\lambda_c)}{Z}=\left(\frac{C}{Z}\right)\cos(\lambda-\lambda_c)$$

is a pattern 1 map, and pattern 1 is conserved under division by a finite non-zero constant. If Z is negative, there is also a change of orientation by 180°, but the map remains a pattern 1 map.

Since pattern 1 is conserved under addition and division by a finite non-zero constant, the mean map of a finite number of pattern 1 maps is also a pattern 1 map.

Let us consider the field of the product quantity (XV) in the neighborhood of the pole. If X has the same value throughout the neighborhood, $0<\epsilon<\epsilon_M$, then the map of (XV) will be a pattern 1 map in this neighborhood. Most meteorological variables such as temperature, pressure, height of pressure surfaces, ozone concentration, vertical wind speed, and horizontal wind speed are usually considered as continuous quantities in the free atmosphere. The usual analysis of maps of such variables from a finite number of observations presumes that the fields of these variables do not contain discontinuities. Hence, for most purposes maps of (XV) and (XU), $0<\epsilon<\epsilon_M$, will be pattern 1 maps when X is one of the above meteorological variables.

We should look at two of the unusual cases which appear at times. The first is when X is discontinuous in the neighborhood of the pole, and the second is when X has a zero isopleth in the neighborhood of the pole.

Since the fields of U and V may always be considered as continuous except at the poles, the discontinuities in the fields of (XU) and (XV) may occur only at the poles and where X is discontinuous. (At the point where a zero isopleth of a continuous field X crosses a discontinuity in the field of Y, (XY) is continuous.)

Since the product (XY) is zero when and only when either X or Y is zero, the field of (XY) will have zero isopleths where either X or Y have zero isopleths. (This assumes that X and Y take on only finite values.) Thus, if X is discontinuous or has a zero isopleth through the pole, the maps of (XU) and (XV) will not be pattern 1 maps in the neighborhood of the pole.

The covariance of X and Y is:

$$\text{cov}\,(X,Y)\equiv\left(\frac{\Sigma(XY)}{N}-\frac{\Sigma X}{N}\cdot\frac{\Sigma Y}{N}\right)$$

Thus, for a finite N greater than 1, if X and Y are pattern 1 maps, then the map of cov (X, Y) is also a pattern 1 map.

As a corollary, we can prove that there are perpendicular zero isopleths on the maps of (UV) in the neighborhood of the pole for $\epsilon>0$. Since the value at any point of the map (UV) in the neighborhood of the pole is

$$[C\sin(\lambda-\lambda_c)]\cdot[C\cos(\lambda-\lambda_c)]$$

we have

$$C^2\sin(\lambda-\lambda_c)\cos(\lambda-\lambda_c)$$

or

$$\frac{C^2}{2}\sin[2(\lambda-\lambda_c)]\qquad\text{for }\epsilon>0.$$

This pattern, *pattern* 2, is shown in figure 2. In the same manner as above it can be shown that pattern 2 is conservative under addition, subtraction, and the operation of taking the mean.

Maps of the standard deviations of U and of V will have non-negative values everywhere in the neighborhood, $0<\epsilon<\epsilon_M$, but will have indeterminate values at the pole. The standard deviation of V is:

$$S(V)\equiv\sqrt{\frac{1}{N-1}\left\{\Sigma C^2\cos^2(\lambda-\lambda_c)-\frac{1}{N}[\Sigma C\cos(\lambda-\lambda_c)]^2\right\}}.$$

In the neighborhood, not only will $S(U)$ and $S(V)$ be non-negative at every point for $\epsilon>0$, but they will be constant along any meridian since, from above,

$$\frac{\partial}{\partial\phi}[S(V)]=0=\frac{\partial}{\partial\phi}[S(U)]$$

where ϕ=latitude. If the sample is sufficiently large and random, $S(U)$ and $S(V)$ should be almost equal to some positive value throughout the region, $0<\epsilon<\epsilon_M$, and the standard deviations of the quantities X may be taken as positive constants in the neighborhood of the pole.

Maps of the correlation coefficients

$$r(X,Y)\equiv\frac{\text{cov}\,(X,Y)}{S(X)\cdot S(Y)}\quad\text{and}\quad r(U,V)\equiv\frac{\text{cov}\,(U,V)}{S(U)\cdot S(V)}$$

will have their zero isopleths in the same place as the corresponding covariance maps, but in general, will not be pattern 1 or pattern 2 maps. Again we should repeat that the functions of U and V are not defined at the poles.

From both physical and mathematical reasonings it can be shown that this method is not applicable for determining the pattern of the divergence fields, such as $\nabla\cdot Q$, in a small neighborhood of the pole.

From the preceding we have formulated the following rules for aid in analyzing maps.

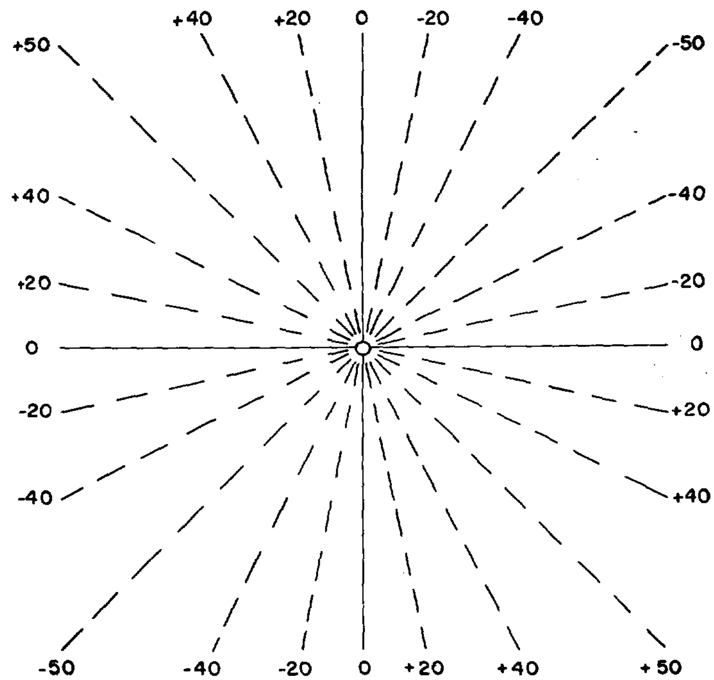

FIGURE 2.—A pattern 2 map, $(C^2/2)\sin[2(\lambda-\lambda_c)]$, for C=10 kt.

3. RULES FOR ANALYSIS OF MAPS OF ALMOST-CONTINUOUS [2] QUANTITIES

1. Zero isopleths of U and V can be drawn as though they pass smoothly through the pole, but the actual values at the pole are not defined.

2. Non-zero isopleths of U and V at the pole are kinked as shown in figure 1.

3. Maps of U and V for the same period have one zero isopleth of U perpendicular to one zero isopleth of V at the pole.

4. The absolute value of any U or V kinked isopleth at the pole can not exceed the value of the wind speed at the pole.

5. At the pole the value of the maximum kinked isopleth is the same for both the U and the V maps.

The following two rules hold anywhere on the maps considered.

6. Any product map (XY) will have zero isopleths where, and only where, the map of X has zero isopleths or the map of Y has zero isopleths, if any exist.

7. A map of the correlation coefficient of X and Y will have its zero isopleths in the same place as the zero isopleths on the map of the covariance of X and Y, if any exist.

[2] Almost-continuous means that there are a countable number of discontinuous points in the field.

The remaining four rules apply only in the neighborhoods of the pole, assuming that $C \neq 0$.

8. In the vicinity of a pole there is one zero isopleth of the map of (XU) which is perpendicular to some one zero isopleth of the map of (XV) at the same pole. If X is non-zero and continuous in the neighborhood of the pole, then these two zero isopleths will be the only zero isopleths, and both the map of (XU) and the map of (XV) will be pattern 1 maps in the neighborhood of the pole.

9. A map of (UV) which includes a pole has perpendicular zero isopleths as shown in figure 2.

10. A map of the covariance of U and V has perpendicular zero isopleths and is a pattern 2 map in the vicinity of the pole.

11. A map of the correlation coefficient of U and V has perpendicular zero isopleths at the pole but is not necessarily a pattern 2 map in the vicinity of the pole.

REFERENCES

1. A. A. Barnes, Jr., *The Energy Balance of the Stratosphere During the IGY*, Ph. D. Thesis, Dept. of Meteorology, Massachusetts Institute of Technology, Cambridge, Mass., 1962.
2. F. B. Hildebrand, *Advanced Calculus for Engineers*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1948.