SOME PRELIMINARY THEORETICAL CONSIDERATIONS OF TROPOSPHERIC WAVE MOTIONS IN EQUATORIAL LATITUDES

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ABSTRACT

Wave solutions to the linearized, quasi-hydrostatic equations for adiabatic, nonviscous flow on an equatorially oriented beta plane are obtained. The basic current is assumed to be zonal and invariant in both space and time. Only solutions for which the meridional wind component is symmetric with respect to the equator are considered. Disturbances with wavelengths on the order of 10^3 km. are found to be very nearly nondivergent. The solutions show the meridional wind component to be very nearly geostrophic even at very low latitudes. The perturbation of the zonal wind, however, is highly ageostrophic at the very low latitudes and significantly ageostrophic even in subtropical latitudes.

1. INTRODUCTION

Charney's [3] scale analysis predicts that nonviscous, adiabatic, synoptic-scale atmospheric motions near the equator should be very nearly nondivergent and, therefore, adequately described by the conservation of absolute vorticity. In the present study, such motions are examined in terms of certain wave solutions obtained from the linearized, quasi-hydrostatic equations for adiabatic, nonviscous flow on a beta plane.

Aside from the assumptions cited above, the basic current is taken to be zonal and invariant in both space and time. The discussion, therefore, pertains only to existing perturbations and provides no information concerning the manner in which they originate. It is not claimed that the theory is applicable to observed equatorial wave motions such as those described by Palmer [6]. It is quite likely that the latter are significantly affected by the release of latent heat and by quasi-barotropic instabilities due to meridional shears of the basic current. However, as will be seen later, there is a fairly reasonable superficial resemblance between the structure of the observed and theoretical waves.

Previous theoretical studies of disturbances in equatorial latitudes have assumed the flow to be nondivergent [10] or have simplified the problem through specification of one of the velocity components [7, 11]. Freeman and Graves [4] make both of these simplifications. Neither is included in the model developed below. On the other hand, Rosenthal [7] and Sherman [11] allowed the basic current to vary with latitude; this is not done here.

2. SOLUTION OF THE PERTURBATION EQUATIONS

The equations which govern nonviscous, adiabatic, quasi-hydrostatic, beta plane flow are,

$$\frac{\partial u^*}{\partial t}+u^*\frac{\partial u^*}{\partial x}+v^*\frac{\partial u^*}{\partial y}+\omega^*\frac{\partial u^*}{\partial p}-\beta y v^*+\frac{\partial \phi^*}{\partial x}=0, \tag{1}$$

$$\frac{\partial v^*}{\partial t}+u^*\frac{\partial v^*}{\partial x}+v^*\frac{\partial v^*}{\partial y}+\omega^*\frac{\partial v^*}{\partial p}+\beta y u^*+\frac{\partial \phi^*}{\partial y}=0, \tag{2}$$

$$\frac{\partial^2 \phi^*}{\partial p\partial t}+u^*\frac{\partial^2 \phi^*}{\partial p\partial x}+v^*\frac{\partial^2 \phi^*}{\partial p\partial y}+\sigma^*\omega^*=0, \tag{3}$$

and

$$\frac{\partial \omega^*}{\partial p}=-\left(\frac{\partial u^*}{\partial x}+\frac{\partial v^*}{\partial y}\right). \tag{4}$$

Here, t is time, x is zonal distance, y is meridional distance measured positive northward from the equator, p is pressure, u^* is the zonal wind component, v^* is the meridional wind component, ω^* is the p-system vertical motion, ϕ^* is the geopotential of isobaric surfaces,

$$\sigma^*=\frac{\partial^2\phi^*}{\partial p^2}+\left(\frac{c_v}{c_p}\right)\left(\frac{1}{p}\right)\frac{\partial \phi^*}{\partial p}$$

is the static stability, $\beta=\frac{\partial f}{\partial y}=$ a constant and f is the Coriolis parameter.

The dependent variables are written,

$$u^*=U+u(x, y, p, t),\ U=\text{a constant} \tag{5}$$

$$v^*=v(x, y, p, t) \tag{6}$$

$$\omega^*=\omega(x, y, p, t) \tag{7}$$

and

$$\phi^*=\bar{\phi}(y, p)+\phi(x, y, p, t) \tag{8}$$

The variables, u, v, ω, ϕ are perturbation quantities. Since the base state must satisfy the governing equations, we find

$$\frac{\partial\bar{\phi}}{\partial y}=-\beta y U \tag{9}$$

and, upon integration,

$$\bar{\phi}(y, p)=\bar{\phi}(0, p)-\frac{\beta y^2 U}{2} \tag{10}$$

Substitution of (5), (6), (7), and (8) into (1), (2), (3), and (4), and utilization of (9) and (10) yields, after linearization by the usual technique,

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}-\beta y v+\frac{\partial \phi}{\partial x}=0 \tag{11}$$

$$\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}+\beta y u+\frac{\partial \phi}{\partial y}=0 \tag{12}$$

$$\frac{\partial^2 \phi}{\partial p \partial t}+U\frac{\partial^2 \phi}{\partial p \partial x}+\bar{\sigma}\omega=0 \tag{13}$$

and

$$\frac{\partial \omega}{\partial p}=-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \tag{14}$$

where

$$\bar{\sigma}=\frac{\partial^2 \bar{\phi}}{\partial p^2}+\left(\frac{c_v}{c_p}\right)\left(\frac{1}{p}\right)\frac{\partial \bar{\phi}}{\partial p} \tag{15}$$

is assumed constant. If, at $t=0$, we require v to attain its maximum value at $x=0$, $p=p_0$, then the system (11), (12), (13), and (14) has solutions of the form

$$u=A(y) \sin k(x-ct) \cos m(p-p_0) \tag{16}$$

$$v=B(y) \cos k(x-ct) \cos m(p-p_0) \tag{17}$$

$$\phi=H(y) \sin k(x-ct) \cos m(p-p_0) \tag{18}$$

$$\omega=W(y) \cos k(x-ct) \sin m(p-p_0) \tag{19}$$

where $p_0=1000$ mb., $k=2\pi/L$, L is the wavelength, c is the wave speed, and

$$m=n\pi/p_0 \tag{20}$$

n is an integer equal to the number of surfaces of nondivergence. From (19) and (20), $\omega=0$ at $p=0$ and $p=p_0$. Substitution of (16), (17), (18), and (19) into (11), (12), (13), and (14) gives

$$k\Delta A-\beta y B+kH=0 \tag{21}$$

$$-k\Delta B+\beta y A+\frac{dH}{dy}=0 \tag{22}$$

$$-mk\Delta H+\bar{\sigma}W=0 \tag{23}$$

and

$$mW=-\left(kA+\frac{dB}{dy}\right) \tag{24}$$

where

$$\Delta=U-c \tag{25}$$

By elimination of variables between (21), (22), (23), and (24), we obtain the following equation for B:

$$\frac{d^2B}{dy^2}-\beta^2\left[\frac{m^2}{\bar{\sigma}}y^2-\frac{m^2\Delta}{\beta\bar{\sigma}}+\frac{(\beta-k^2\Delta)(m^2\Delta^2-\bar{\sigma})}{\bar{\sigma}\beta^2\Delta}\right]B=0 \tag{26}$$

The following new variables are introduced,

$$\lambda=\beta y^2/\gamma \tag{27}$$

$$Q=e^{\lambda/2}B \tag{28}$$

where

$$\gamma=\bar{\sigma}^{1/2}/m \tag{29}$$

From (26), (27), (28), and (29), we obtain

$$\lambda\frac{d^2Q}{d\lambda^2}+\left(\frac{1}{2}-\lambda\right)\frac{dQ}{d\lambda}+\frac{\alpha}{2}Q=0 \tag{30}$$

where

$$\frac{\gamma\left[\beta-k^2\Delta\left(1-\frac{\Delta^2}{\gamma^2}\right)\right]-\Delta\beta}{2\Delta\beta}=\alpha \tag{31}$$

Equation (30) is a special case of the confluent hypergeometric equation ([5] p. 96)

$$s\frac{d^2T}{ds^2}+(b-s)\frac{dT}{ds}-aT=0$$

whose general solution may be written

$$T=K_1M(a, b, s)+K_2s^{1-b}M(a-b+1, 2-b, s)$$

where

$$M(a, b, s)=1+\frac{a}{b}\cdot\frac{s}{1!}+\frac{a(a+1)}{b(b+1)}\cdot\frac{s^2}{2!}+\ldots$$

The general solution to (30) is then

$$Q=K_1M\left(-\frac{\alpha}{2}, \frac{1}{2}, \lambda\right)+K_2\lambda^{1/2}M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, \lambda\right) \tag{32}$$

By use of (27) and (28), equation (32) may be arranged to read

$$B=K_1e^{-\frac{\beta y^2}{2\gamma}}M\left(-\frac{\alpha}{2}, \frac{1}{2}, \frac{\beta y^2}{\gamma}\right)+K_2y\left(\frac{\beta}{\gamma}\right)^{1/2}e^{-\frac{\beta y^2}{2\gamma}}M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, \frac{\beta y^2}{\gamma}\right) \tag{33}$$

The parameter α is an eigenvalue of the problem and must be determined from side conditions imposed upon the motion. If we restrict ourselves to flows in which v is symmetric about the equator (this is the case, for example, with the Palmer waves), then $K_2=0$. If it is further required that v vanish at

$$y=\pm y_w \tag{34}$$

then $\alpha=\alpha_*$ where

$$M\left(-\frac{\alpha_*}{2}, \frac{1}{2}, \frac{\beta y_w^2}{\gamma}\right)=0 \tag{35}$$

In general, numerical methods are required if (35) is to be solved for α_*.

A simpler solution is obtained when the condition $v(\pm y_w)=0$ is replaced by the less stringent restriction that v decay with distance from the equator. This can be achieved by selecting $\alpha=0$ in which case

$$B=K_1e^{-\frac{\beta y^2}{2\gamma}}\equiv v_0e^{-\frac{\beta y^2}{2\gamma}} \tag{36}$$

and, from (31),

$$\left(\Delta^2+\gamma\Delta-\frac{\gamma\beta}{k^2}\right)\left(1-\frac{\Delta}{\gamma}\right)=0 \tag{37}$$

The roots of (37) are[1]

$$\Delta_1=\gamma \tag{38}$$

$$\Delta_2=-\frac{\gamma}{2}\left[1+\left(1+\frac{4\beta}{k^2\gamma}\right)^{1/2}\right] \tag{39}$$

$$\Delta_3=\frac{\gamma}{2}\left[\left(1+\frac{4\beta}{k^2\gamma}\right)^{1/2}-1\right] \tag{40}$$

Table 1 lists some values of Δ_1, Δ_2, and Δ_3 ($\bar{\sigma}=3$ m.t.s. units; a reasonable value for the tropical mid-troposphere). Δ_1 is a pure internal gravity wave which propagates rapidly westward *relative* to the basic current. Δ_2 is an inertia-gravity wave which propagates rapidly eastward relative to the basic current. Δ_3 is also an inertia-gravity wave. However, its propagation relative to the basic current is westward at approximately the Rossby rate of $\Delta_{ND}\equiv\beta/k^2$ (it is shown in the appendix that Δ_3 reduces to Δ_{ND} if the motion studied here is constrained to be nondivergent). According to Palmer's [6] observations, the motion of equatorial waves relative to the basic current is small and, therefore, we assume Δ_3 to be the meteorologically significant root.

Equation (40) may be written,

$$\Delta_3=\frac{\Delta_{ND}}{\frac{1}{2}+\left(\frac{1}{4}+\frac{\Delta_{ND}}{\gamma}\right)^{1/2}} \tag{41}$$

When the basic current is easterly ($E=-U>0$),

$$C_{ND}=-E-\Delta_{ND} \tag{42}$$

and

$$C_3=-E-\frac{\Delta_{ND}}{\frac{1}{2}+\left(\frac{1}{4}+\frac{\Delta_{ND}}{\gamma}\right)^{1/2}} \tag{43}$$

The disturbances, therefore, propagate westward at a speed which exceeds that of the basic current but which is less than would be the case for nondivergent flow. The distribution of convergence and divergence must then be such that westward relative motion is retarded. This implies convergence to the east of zones of cyclonic relative vorticity and divergence to the west of such regions. The reverse is true of zones of anticyclonic relative vorticity and the rule is equally valid in both the Northern and Southern Hemispheres and for easterly and westerly basic currents.

[1] The derivation of equation (30) required division by $k(\bar{\sigma}-m^2\Delta^2)=km^2(\gamma^2-\Delta^2)$. However, when $\Delta=\gamma$, (21), (22), (23), and (24) may be solved to yield $B=v_0\exp(-\beta y^2/2\gamma)$ thus showing that (36) is valid even for this case.

TABLE 1.—*Roots of the frequency equation (37) as given by equations (38), (39), and (40). Values are given in m. sec.* $^{-1}$. *$n=1$ corresponds to one surface of nondivergence; $n=2$ corresponds to two surfaces of nondivergence. The static stability is taken as 3 m.t.s. units.* $\Delta_{ND}=\beta/k^2$ *is the value of* Δ *appropriate to nondivergent motion.*

	$n=1$				$n=2$		
L (km.)	Δ_{ND}	Δ_1	Δ_2	Δ_3	Δ_1	Δ_2	Δ_3
1000	0.58	55.0	−55.6	0.55	27.5	−28.2	0.55
2000	2.3	55.0	−57.2	2.2	27.5	−29.8	2.2
3000	5.2	55.0	−59.7	4.7	27.5	−32.0	4.4
4000	9.3	55.0	−63.0	8.0	27.5	−35.0	7.3
5000	14.4	55.0	−66.7	11.8	27.5	−38.2	10.5

From table 1, it is clear that Δ_3 departs more and more from Δ_{ND} as the wavelength increases. The model convergence and divergence is then of greater significance at the longer wavelengths. This is similar to the so-called "barotropic divergence" found in barotropic, quasi-geostrophic models which allow non-zero vertical integrals of the divergence [1, 2, 8, 9, 12, etc.] and is basically different from the divergence discussed in Charney's [3] scale analysis of low-latitude atmospheric motions. The latter has magnitude

$$|\nabla\cdot\mathbf{V}|\sim\frac{U^3}{gLH^2\frac{\partial\ln\theta}{\partial z}} \tag{44}$$

where U is the characteristic velocity, L is the characteristic scale, g the acceleration of gravity, θ the potential temperature, and H the scale height. This divergence is seen to decrease with increasing wavelength.

By use of (16), (17), (18), (19), (21), (22), (23), (24), and (36), it is a simple matter to complete the solutions for u, v, ϕ and ω. These are

$$u=\frac{\beta y}{k(\Delta_3+\gamma)}v_0e^{-\frac{\beta y^2}{2\gamma}}\sin k(x-ct)\cos m(p-p_0), \tag{45}$$

$$v=v_0e^{-\frac{\beta y^2}{2\gamma}}\cos k(x-ct)\cos m(p-p_0), \tag{46}$$

$$\phi=\frac{\beta\gamma y}{k(\Delta_3+\gamma)}v_0e^{-\frac{\beta y^2}{2\gamma}}\sin k(x-ct)\cos m(p-p_0), \tag{47}$$

and

$$\omega=\frac{\beta\Delta_3 y}{m\gamma(\Delta_3+\gamma)}v_0e^{-\frac{\beta y^2}{2\gamma}}\cos k(x-ct)\sin m(p-p_0). \tag{48}$$

3. DISCUSSION OF THE SOLUTIONS

The divergence may be calculated from (48) as $-\partial\omega/\partial p$. The amplitude of the divergence is then

$$\text{Ⓓ}=-\frac{\beta y\Delta_3}{\gamma(\Delta_3+\gamma)}v_0e^{-\frac{\beta y^2}{2\gamma}} \tag{49}$$

Ⓓ reaches its maximum magnitude at

$$y=\pm\left(\frac{\gamma}{\beta}\right)^{1/2}$$

hence,

$$|\mathfrak{D}|_{\max}=\left(\frac{\beta}{\gamma e}\right)^{1/2}\left(\frac{\Delta_3}{\Delta_3+\gamma}\right)v_0 \tag{50}$$

Table 2 gives $|\mathfrak{D}|_{\max}$ as a function of wavelength for $\bar{\sigma}=3$ m.t.s., $v_0=5$ m. sec.$^{-1}$, $n=1$ and $n=2$. The values are found to be extremely small. Although, as mentioned in the previous section, the divergence found in this model is basically different from that of Charney's [3] discussion, our results do confirm Charney's conclusion that it is necessary to include precipitation and the release of latent heat in order to obtain realistic vertical motion and divergence at equatorial latitudes. Table 2 also shows the divergence increases in magnitude as the wavelength increases. This confirms our conclusions based on a discussion of the frequency equation in the previous section. It also emphasizes the different nature of the divergence of this model and that of Charney's model (also mentioned in section 2).

The relative vorticity, $\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$, may be computed from (45) and (46). This quantity reaches its maximum amplitude

$$|\zeta|_{\max}=\frac{k(\Delta_{ND}+\Delta_3+\gamma)}{\Delta_3+\gamma}v_0 \tag{51}$$

at the equator. Values of $|\zeta|_{\max}$ are given by table 3; the ratio $|\mathfrak{D}|_{\max}/|\zeta|_{\max}$ is shown by table 4. We find the relative vorticity to have an order of magnitude which is meteorologically significant and which is large compared to that of the divergence. Relative to the Palmer waves [6], the model vorticity is of the correct order of magnitude. However, the Palmer waves, which typically have a wavelength of about 2000 km., show divergences on the order of 10^{-6} sec.$^{-1}$. Hence, the model divergence is small compared to that of the Palmer waves. The latter, therefore, should probably be attributed to the release of latent heat in regions of organized convection associated with the waves.

It is of interest to examine the extent to which the perturbation-velocity components are in geostrophic equilibrium. For this purpose, we define the following ratios:

$$R_{0x}=\left|\frac{\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}}{\beta yv}\right|=\frac{\Delta_3}{\Delta_3+\gamma} \tag{52}$$

$$R_{0y}=\left|\frac{\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}}{\beta yu}\right|=\frac{\gamma}{\beta y^2} \tag{53}$$

It is clear that small values of R_{0x} and R_{0y} correspond to near geostrophic flow. Large values indicate highly ageostrophic motion. R_{0x} is independent of latitude while R_{0y} does not vary with wavelength. From table 5, we see that R_{0x} is quite small for the shorter wavelengths. For these wavelengths, therefore, the symmetrical (v) component of the perturbation motion is very nearly in geostrophic equilibrium even at very low latitudes. The behavior of the asymmetrical (u) perturbation component is quite different. R_{0y} tends to infinity as the equator is approached. Table 6 shows that u is markedly ageostrophic even in subtropical latitudes.

The fact that the meridional wind component of our model is very nearly in geostrophic equilibrium helps to explain the different behaviors of the divergence of this model and that of Charney [3]. According to Charney's theory, the horizontal acceleration balances the horizontal pressure-gradient force. The horizontal temperature gradients needed to estimate the vertical motion and divergence are, therefore, computed from vertical shears of the horizontal acceleration. In the model developed here, $\partial\phi/\partial x$ (the only component of $\nabla\phi$ which affects ω in the thermodynamic equation) is given, approximately, by the Coriolis term, βyv, and not by the horizontal acceleration. It is to be noted that part of this discrepency stems directly from the decision to treat only the symmetrical part of (33). It is likely that the asymmetrical solution would give results more like those of Charney.

The role of divergence in the vorticity equation within the framework of our model may be determined by a study of the ratio

$$\left|\frac{f(\nabla\cdot\mathbf{V})}{\beta v}\right|=\frac{\beta y^2\Delta_3}{\gamma(\Delta_3+\gamma)}=\frac{R_{0x}}{R_{0y}} \tag{54}$$

TABLE 2.—*Values of the maximum magnitude of the divergence as computed from equation (50). $\bar{\sigma}=3$ m.t.s. and $v_0=5$ m. sec.$^{-1}$. Values are given in units of sec.$^{-1}$.*

L (km.)	$n=1$	$n=2$
1000	2.0×10^{-8}	5.6×10^{-8}
2000	7.9×10^{-8}	1.9×10^{-7}
3000	1.6×10^{-7}	3.9×10^{-7}
4000	2.5×10^{-7}	5.9×10^{-7}
5000	3.5×10^{-7}	7.9×10^{-7}

TABLE 3.—*Values of the maximum magnitude of the relative vorticity computed from equation (51). $\bar{\sigma}=3$ m.t.s. and $v_0=5$ m. sec.$^{-1}$. Values are given in units of sec.$^{-1}$*

L (km.)	$n=1$	$n=2$
1000	5.0×10^{-5}	5.0×10^{-5}
2000	2.6×10^{-5}	2.7×10^{-5}
3000	1.8×10^{-5}	1.9×10^{-5}
4000	1.4×10^{-5}	1.6×10^{-5}
5000	1.2×10^{-5}	1.4×10^{-5}

TABLE 4.—*Values of the ratio $\frac{|\mathfrak{D}|_{\max}}{|\zeta|_{\max}}$. $\bar{\sigma}=3$ m.t.s.*

L (km.)	$n=1$	$n=2$
1000	4.0×10^{-4}	1.1×10^{-3}
2000	3.0×10^{-3}	7.0×10^{-3}
3000	8.9×10^{-3}	2.0×10^{-2}
4000	1.8×10^{-2}	3.7×10^{-2}
5000	2.9×10^{-2}	5.6×10^{-2}

TABLE 5.—*Values of R_{0x} computed from equation (52). $\bar{\sigma}=3$ m.t.s. units.*

L(km.)	$n=1$	$n=2$
1000	0.01	0.02
2000	0.04	0.07
3000	0.08	0.14
4000	0.13	0.21
5000	0.18	0.28

TABLE 6.—*Values of R_{0y} computed from equation (53). $\bar{\sigma}=3$ m.t.s. units.*

Lat. (deg.)	$n=1$	$n=2$
1	195	97.5
2	49.1	24.5
3	21.8	10.9
4	12.2	6.10
5	7.40	3.70
10	1.95	0.98
15	0.87	0.44
20	0.44	0.22
25	0.31	0.16
30	0.22	0.11

TABLE 7.—*Values of the ratio $\left|\frac{f(\nabla\cdot\mathbf{V})}{\beta v}\right|$ computed from equation (54). $\bar{\sigma}=3$ m.t.s. units*

Lat. (deg.)	$L=2000$ km., $n=1$	$L=5000$ km., $n=2$
0	0	0
5	0.005	0.07
10	0.02	0.22
15	0.05	0.54
20	0.08	1.14
25	0.13	1.79
30	0.19	2.62

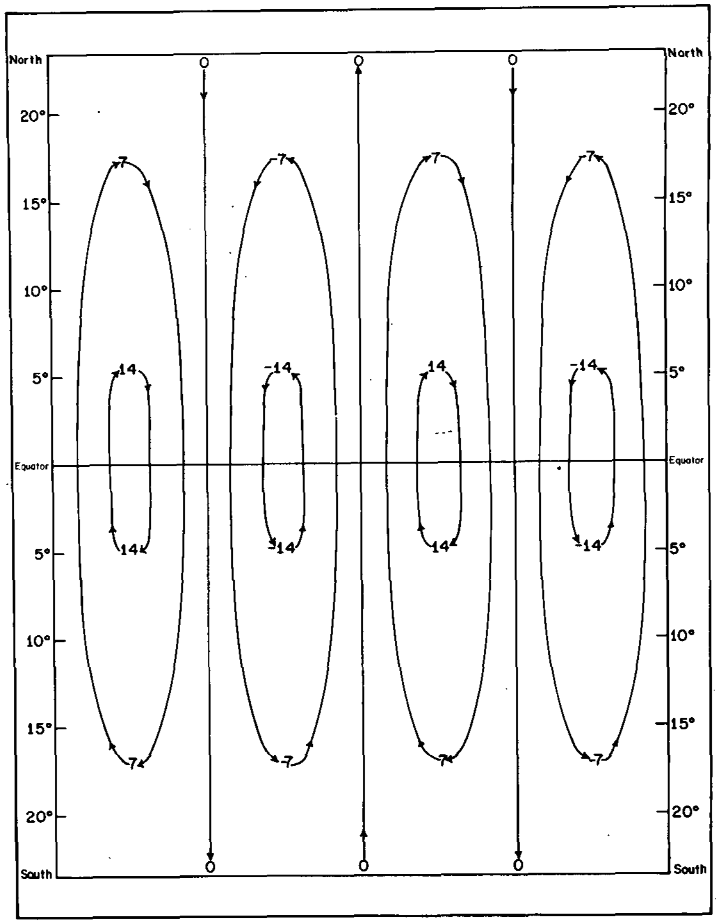

FIGURE 1.—Perturbation-stream function computed from equation (57). Isopleths are labeled in units of 10^5 m.^{2} sec.$^{-1}$, $\bar{\sigma}=3$ m.t.s., $L=2000$ km., $v_0=5$ m. sec.$^{-1}$, $t=0$, $p=p_0$, $n=1$.

where $f(\nabla\cdot\mathbf{V})$ is the amplitude of $f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$ and βv is the amplitude of βV. Values of this ratio for $L=2000$ km., $n=1$ and $L=5000$ km., $n=2$ are listed by table 7. With $L=2000$ km., $n=1$, we find, as predicted by Charney [3], that the divergence term in the vorticity equation is negligible at very low latitudes. On the other hand, with $L=5000$ km., $n=2$, this term becomes significant within 5° to 10° lat. of the equator.

We now turn to a description of some synoptic aspects of the theoretical disturbances. Since the motion is very nearly nondivergent, it is convenient and appropriate to introduce a stream function. From the Helmholtz theorem,

$$u=-\frac{\partial\psi}{\partial y}+\frac{\partial\chi}{\partial x} \tag{55}$$

and

$$v=\frac{\partial\psi}{\partial x}+\frac{\partial\chi}{\partial y} \tag{56}$$

where ψ is the stream function for the perturbation motion and χ is the corresponding velocity potential. Unfortunately, when u and v are given by (45) and (46), ψ and χ cannot be obtained in closed form. Since we require ψ only for the purpose of presenting a pictorial representation of the flow pattern and since the flow is very nearly nondivergent, the following approximate technique will suffice. We neglect the potential flow in (55) and (56), evaluate u and v from (45) and (46), respectively, and integrate the resulting expressions for $\partial\psi/\partial x$ and $\partial\psi/\partial y$. This yields two values for ψ which are only approximately equal due to the neglect of $\partial\chi/\partial x$ and $\partial\chi/\partial y$. A final estimate is obtained by averaging the two approximations. The result is,

$$\psi\approx\frac{1}{2}\left[\frac{\gamma}{\Delta_3+\gamma}+1\right]\frac{v_0}{k}\,e^{-\frac{\beta y^2}{2\gamma}}\sin k(x-ct)\cos m(p-p_0) \tag{57}$$

Figure 1 shows the ψ-pattern for $\bar{\sigma}=3$ m.t.s., $v_0=5$ m. sec.$^{-1}$, $L=2000$ km., $n=1$, $t=0$, and $p=1000$ mb. The motion consists of alternating, north-south elongated cells of clockwise and counterclockwise circulations each of which is centered on the equator. The clockwise cells are cyclonic in the Southern Hemisphere and anticyclonic in the Northern Hemisphere. The reverse is true of the counterclockwise cells. Relative to the field of perturbation-pressure height (computed from (47) and shown by fig. 2), cyclonic perturbation motion is associated with low perturbation pressure-height and the reverse is true of anticyclonic perturbation motion. The field of perturbation pressure-height itself is composed of alternating Highs

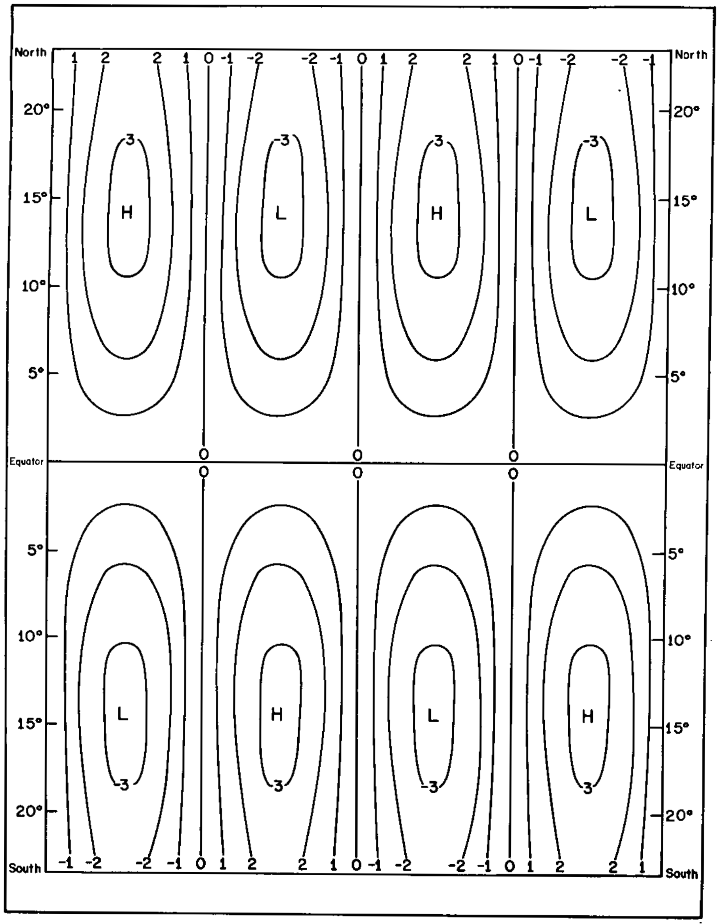

FIGURE 2.—Perturbation pressure-height computed from equation (47). Isopleths are labeled in units of meters. Parameters are the same as for figure 1.

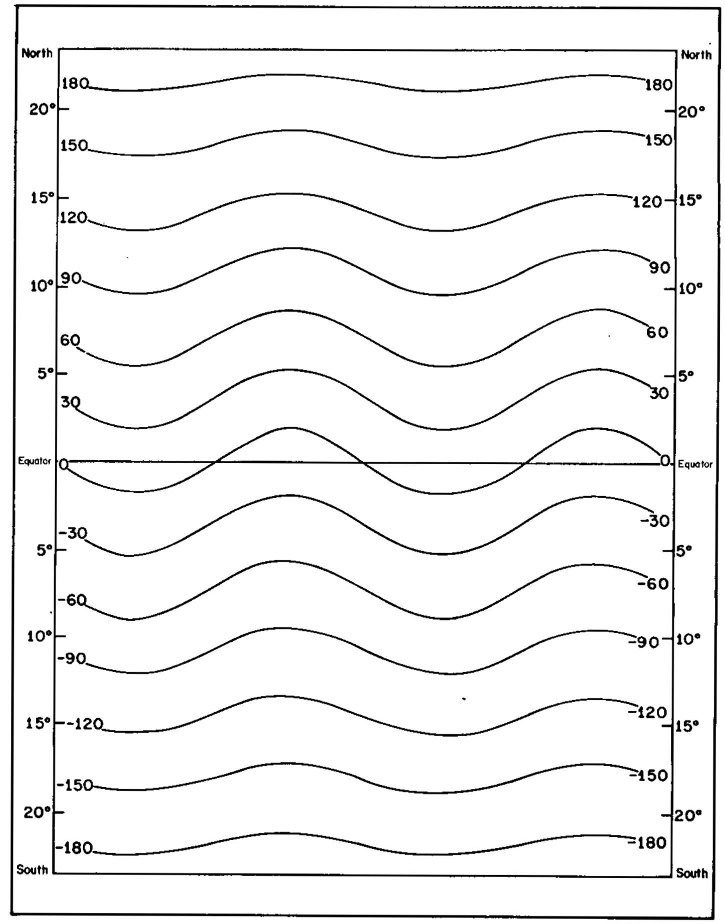

FIGURE 3.—Total-stream function obtained by adding to figure 1 the stream function for a constant easterly current of 7.5 m. sec.$^{-1}$. Isopleths are labeled in units of 10^5 m.^{2} sec.$^{-1}$. Parameters as for figure 1.

and Lows centered asymmetrically at some distance from the equator. The amplitude of the pressure-height perturbation is only a few meters and, therefore, if waves of this type were to exist in the real atmosphere, the variability of pressure-height associated with them would probably go undetected. The total-stream function and pressure-height (figs. 3 and 4, respectively), which consists of the sums of the perturbation and base-state quantities (with $\bar{v}=-7.5$ m. sec.$^{-1}$) shows some resemblance to the streamlines and pressure patterns of Palmer's [6] empirical model (see his figs. 8 and 9).

The 1000-mb. fields of perturbation-relative vorticity and divergence are shown, respectively, by figures 5 and 6. Positive relative vorticity corresponds to counterclockwise turning and, hence, to cyclonic relative vorticity in the Northern Hemisphere and anticyclonic relative vorticity in the Southern Hemisphere. The reverse is true for negative relative vorticity. Thus, as would be expected from our comparison of the perturbation-stream function with the perturbation pressure-height, cyclonic relative vorticity is associated with low perturbation pressure-height and the reverse is true for anticyclonic relative vorticity.

Everywhere, except at the equator itself, convergence occurs to the east of cyclonic relative vorticity and divergence occurs to the east of anticyclonic relative vorticity (fig. 6). This is consistent with the distribution of convergence and divergence previously deduced from the frequency equation. The spatial distribution of convergence and divergence is in good agreement with that of Palmer's empirical model ([6] fig. 10), however the magnitudes given by the theory are far too small.

4. CONCLUSIONS

In equatorial latitudes, nonviscous, adiabatic perturbations superimposed upon an invariant basic current are very nearly nondivergent provided that the wavelength is of the order 10^3 km. and the meridional wind component is symmetric with respect to the equator. Also, the meridional wind component will be very nearly geostrophic even at very low latitudes. In these circumstances, the zonal component of the perturbation wind will be asymmetric with respect to the equator and highly ageostrophic in equatorial latitudes. The order of magnitude of the relative vorticity obtained from the

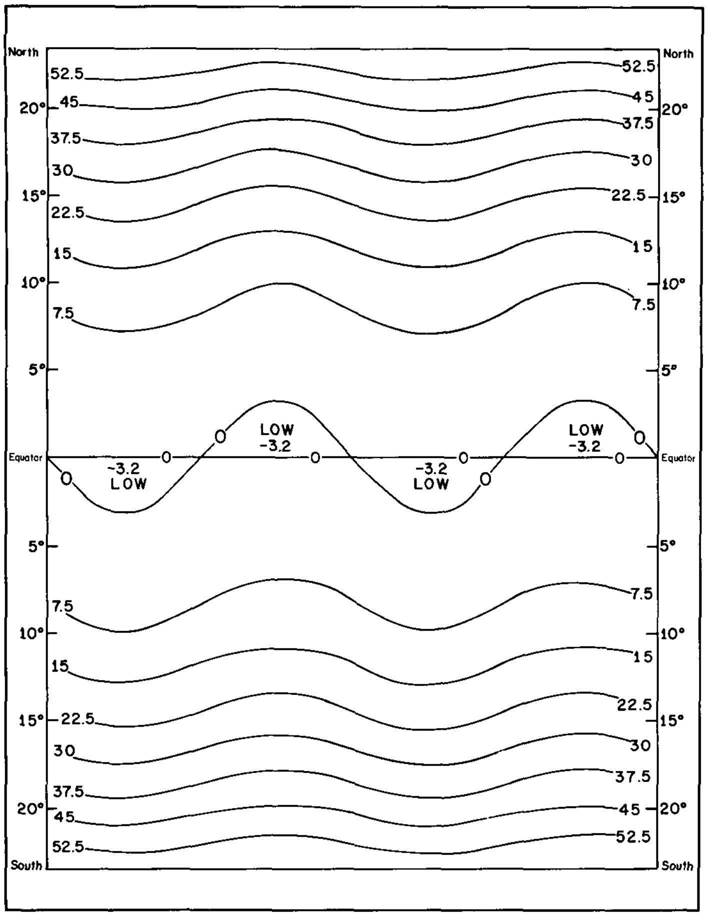

FIGURE 4.—Total pressure-height obtained by adding to figure 2 the pressure-height computed from equation (10) for a basic easterly current of 7.5 m. sec.$^{-1}$. Isopleths are labeled in units of meters. Parameters as for figure 1.

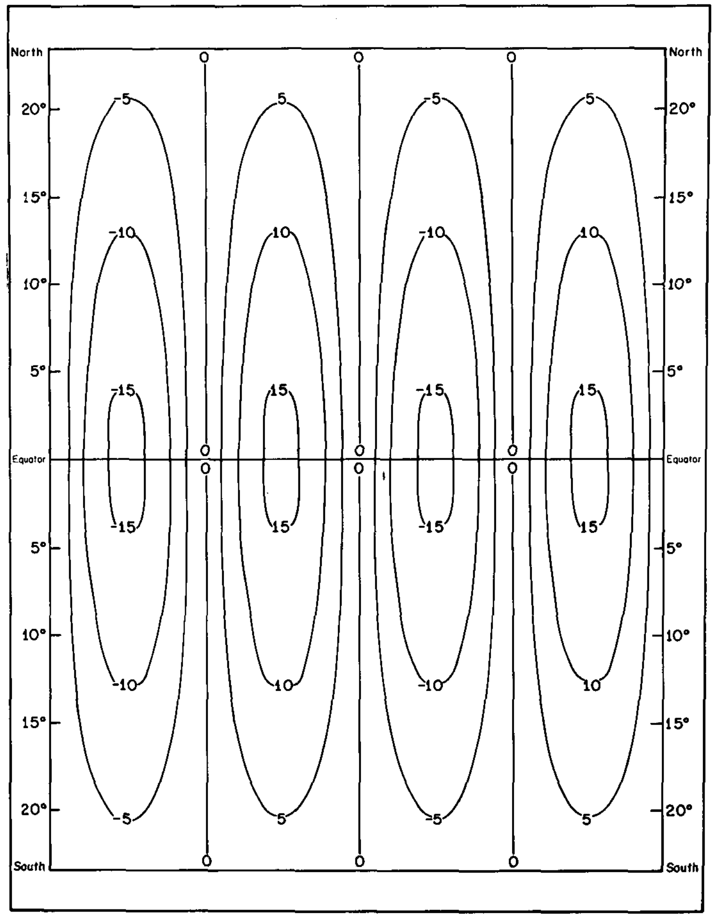

FIGURE 5.—Relative vorticity computed from equations (45) and (46). Positive values indicate counterclockwise rotation. Isopleths are labeled in units of 10^{-6} sec.$^{-1}$. Parameters as for figure 1.

theory is in agreement with that of observed equatorial disturbances as described by Palmer [6]. The order of magnitude of the divergence given by the model is quite small compared to that found empirically by Palmer [6]. This result, taken together with Charney's [3] work, would seem to indicate that models of equatorial flow must take into account the latent heating produced by organized convection in order to provide realistic vertical motions and divergences.

The amplitude of the geopotential perturbation given by the theory is extremely small; with present standards of observation in equatorial latitudes, it would probably be undetectable. As is the case in higher latitudes, the model shows cyclonic flow to be associated with low geopotential and the reverse to be true of anticyclonic flow.

APPENDIX

Inspection of (23) and (24) shows that the solution for nondivergent motion can be obtained by allowing $\bar{\sigma}$ to tend toward infinity while H, Δ, k, and m remain finite. In this case, (40) yields

$$\Delta_3 = \lim_{\gamma\to\infty} \left[\frac{\frac{1}{2}\left[\left(1+\frac{4\beta}{k^2\gamma}\right)^{1/2}-1\right]}{\frac{1}{\gamma}} \right] \tag{A.1}$$

or, if l'Hospital's rule is employed,

$$\Delta_3 = \frac{\beta}{k^2} \equiv \Delta_{ND} \tag{A.2}$$

From (36) the nondivergent solution for B is

$$B = v_0 \tag{A.3}$$

provided that y is not allowed to approach infinity. From (24),

$$A = 0 \tag{A.4}$$

for nondivergent motion. From (21), the nondivergent value of H is

$$H = \frac{\beta y}{k} v_0 \tag{A.5}$$

Equations (A.3) and (A.4) are exactly the assumptions

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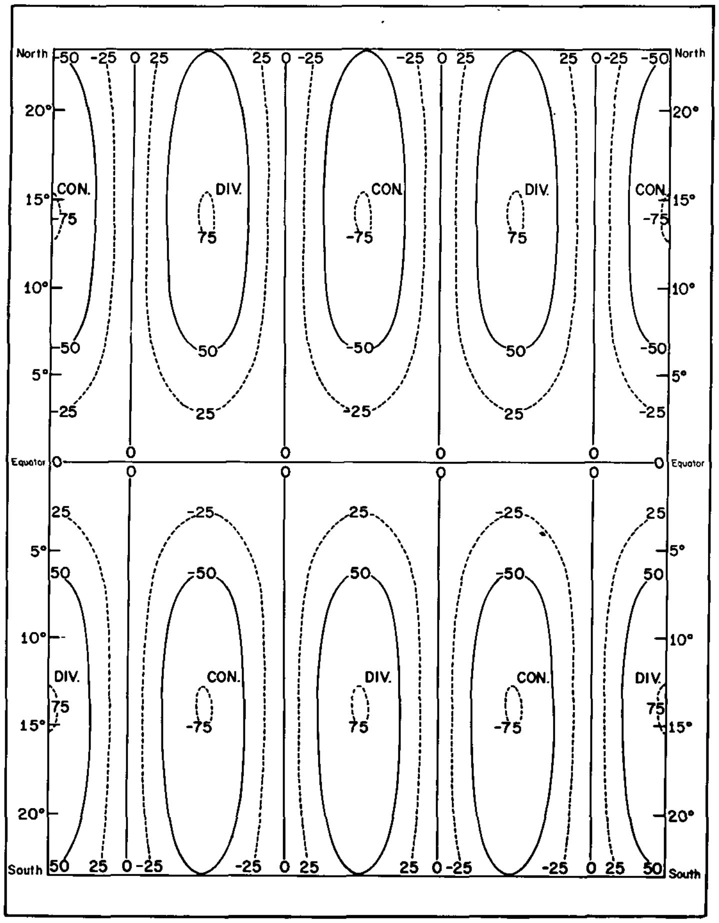

FIGURE 6.—Divergence computed from equations (45) and (46). Isopleths are labeled in units of 10^9 sec.$^{-1}$. Parameters as for figure 1.

employed by Rossby [8]. Furthermore, (A.2), (A.3), (A.4), and (A.5) correspond precisely to Freeman and Graves' [4] recent theory of equatorial Rossby waves.

Alternately, (A.2), (A.3), (A.4), and (A.5) may be obtained by eliminating H between (21) and (22) to form the "vorticity equation" and then imposing the restriction

$$kA+\frac{dB}{dy}=0 \tag{A.6}$$

The latter constrains the motion to be nondivergent. Upon executing this sequence of operations, equation (A.7) is obtained.[2]

$$\frac{d^2B}{dy^2}-\frac{(k^2\Delta-\beta)}{\Delta}B=0 \tag{A.7}$$

The solution for symmetrical B is

$$B=v_0 \cos \frac{(2l+1)\pi}{2y_w} y, \qquad l=\text{an integer} \tag{A.8}$$

provided that

$$\frac{(2l+1)^2\pi^2}{4y_w^2}=\frac{k^2\Delta-\beta}{\Delta} \tag{A.9}$$

If, as was done in the derivations of (36), we require B to have no zeroes over the range of y, then y_w must approach infinity. In this case, (A.8) reduces to (A.3), (A.9) gives (A.2), (A.6) gives (A.4), and (A.5) may be obtained from (21).

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[2] $\Delta=0$ corresponds to a trivial solution in the nondivergent case.