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ON THE THEORY OF THE WINTER-AVERAGE PERTURBATIONS IN THE TROPOSPHERE AND STRATOSPHERE

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ABSTRACT

A general solution is obtained for forced, stationary, quasi-geostrophic perturbations in an atmosphere having the main zonal-average characteristics of the winter troposphere and stratosphere. Special solutions along 45° N. latitude are obtained for idealized representations of forcing due to internal sources and sinks of heat and due to lower boundary airflow over topography. The results show how the solutions depend on the spatial scale of the disturbances. For example, on the long-wave side of a critical vector wave number corresponding to quasi-resonance, the disturbances forced by internal heating tilt eastward with height thereby transporting heat southward, and tend to increase in amplitude above the tropopause into the high stratosphere. The reverse is true for waves smaller than critical. Comparisons with observations suggest that the real atmospheric mean waves are combinations of modes from the two regimes. The energetics of the solutions are discussed.

1. INTRODUCTION

This study is an extension of two previous studies by the writer [15], [18] concerning the linear theory of the time-average perturbations in the westerlies, a subject considered earlier by Charney and Eliassen [5], Smagorinsky [23], and Gilchrist [10], for example, and more recently by the Staff Members of the Academia Sinica [24], Barrett [2], and Döös [8], for example. As in the previous studies, we base the theory on a quasi-geostrophic, β-plane representation of the atmosphere and apply it only to winter conditions along 45° N. latitude. Our aims here are: (1) to improve upon the representation of the basic zonal-average state by adopting a more realistic vertical profile of the basic zonal wind and static stability, including a stratosphere; (2) to extend the calculations to a broader spectrum of harmonic components; and (3) to examine the energetical properties of the solutions. In connection with the first aim, we continue to assume that the basic current is independent of latitude.

The linear equations used here and previously are valid only as a first approximation, and cannot be expected to be capable of accounting for all of the details of the observed mean perturbations, especially in higher latitudes (cf., Saltzman and Rao [20]. Moreover, we make no attempt to treat the effects of lateral deflections of the zonal current around mountain barriers.

Because we are dealing with the stationary component of the motion, we do not expect the long-wave properties treated here to be subject to the full effect of the difficulties described by Burger [4] in connection with the transient behavior of the long waves.

2. THE MATHEMATICAL MODEL

The linearized potential vorticity equation governing the time-average perturbations of the meridional geostrophic wind speed, can be written as follows for a β-plane with no latitudinal variation of the basic state (cf., Saltzman [15]),

$$\frac{\partial^2 v_1}{\partial x^{*2}}+\frac{\partial^2 v_1}{\partial y^{*2}}-\frac{f^2 p}{R\Gamma_0}\frac{\partial^2 v_1}{\partial p^2}-\frac{f^2}{R}\frac{\partial}{\partial p}\left(\frac{p}{\Gamma_0}\right)\frac{\partial v_1}{\partial p}+\left[\frac{\beta}{u_0}+\frac{f^2}{u_0}\frac{\partial}{R\partial p}\left(\frac{p}{\Gamma_0}\frac{\partial u_0}{\partial p}\right)\right]v_1=\frac{F_1}{u_0} \quad (1)$$

where

x^* = distance eastward
y^* = distance northward
p = pressure
t = time
u = eastward wind speed
v = northward wind speed
$\omega = dp/dt$

T = temperature
$\dot{q}$ = rate of heat addition, per unit mass, due to radiation, conduction, water phase changes, and friction
g = acceleration of gravity
R = gas constant for air
c_p = specific heat at constant pressure
f = Coriolis parameter
$\beta = \partial f/\partial y^*$
X = eastward component of viscous force per unit mass
Y = northward component of viscous force per unit mass
$\Gamma = (\partial T/\partial p - RT/c_p p)$ = static stability
$\overline{(\ )} = \Delta t^{-1}\int_0^{\Delta t} (\ )dt$ (time mean over interval or ensemble Δt, such that $\partial\overline{(\ )}/\partial t = 0$)
$(\ )' = (\ ) - \overline{(\ )}$ = transient departure from time mean
$(\ )_0 = K^{*-1}\int_0^{K^*} \overline{(\ )}dx^*$ = zonal average of time mean variable
$(\ )_1 = \overline{(\ )} - (\ )_0$ = departure of time mean variable from zonal average
$F = H + M$
$H = f\partial(Q/\Gamma_0)/\partial p$

$$Q = \left[\frac{\dot{q}}{c_p} - \left(\frac{\partial\overline{u'T'}}{\partial x^*} + \frac{\partial\overline{v'T'}}{\partial y^*} + \frac{\partial\overline{\omega'T'}}{\partial p} + \frac{R}{c_p p}\overline{\omega'T'}\right)\right]$$

$$M = \left\{\frac{\partial}{\partial x^*}\left[Y - \left(\frac{\partial\overline{u'v'}}{\partial x^*} + \frac{\partial\overline{v'^2}}{\partial y^*} + \frac{\partial\overline{v'\omega'}}{\partial p}\right)\right] + \frac{\partial}{\partial y^*}\left[X - \left(\frac{\partial\overline{u'^2}}{\partial x^*} + \frac{\partial\overline{u'v'}}{\partial y^*} + \frac{\partial\overline{u'\omega'}}{\partial p}\right)\right]\right\}$$

At the lower boundary (which we take as the top of a shallow friction layer, designated by the subscript δ) we assume that the mean ω-perturbations can be expressed as the sum of effects due to forced motion over topography and forced vertical motion due to friction (Charney and Eliassen [5]). Thus, defining h as the height of the ground surface above sea level and C as the lower boundary friction coefficient, we have

$$\omega_{1\delta} = -\rho_\delta g\left[\left(\bar{u}\frac{\partial h}{\partial x^*} + \bar{v}\frac{\partial h}{\partial y^*}\right)_1 + C\left(\frac{\partial v_1}{\partial x^*} - \frac{\partial u_1}{\partial y^*}\right)\right]_\delta, \qquad (\rho = \text{density})$$

which leads to the following form of the thermodynamical energy equation,

$$\frac{u_{0\delta}fp}{R}\frac{\partial v_1}{\partial p} - \frac{\partial T_0}{\partial y^*}v_1 + \rho_\delta g\Gamma_0\left[\left(\bar{u}\frac{\partial h}{\partial x^*} + \bar{v}\frac{\partial h}{\partial y^*}\right) + C\left(\frac{\partial v_1}{\partial x^*} - \frac{\partial u_1}{\partial y^*}\right)\right] + Q_1 = 0, \qquad (p = p_\delta) \quad (2)$$

Although C and p_δ are really functions of x^* and y^*, we shall assume the effects of their variability in (2) are small enough that we can adopt some constant average value. Some justification for this is given by Smagorinsky [23].

At the tropopause (which we designate by the subscript R), we assume that v_1 and $\partial T_0/\partial y^*$ are continuous—i.e., $(v_1)_{RI} = (v_1)_{RII}$ and $(\partial T_0/\partial y^*)_{RI} = (\partial T_0/\partial y^*)_{RII}$, in which I denotes the value on the troposphere side and II the value on the stratosphere side. We can thus write the energy equation governing v_{1R} in the form,

$$\left(\frac{\partial v_1}{\partial p}\right)_I - \left[\frac{1}{\alpha}\left(\frac{1}{v_1}\frac{\partial v_1}{\partial p}\right)_{II} + \frac{R(\alpha-1)}{u_0 fp\alpha}\frac{\partial T_0}{\partial y^*}\right]v_1 + \frac{R(\alpha-1)}{u_0 fp\alpha}Q_1 = 0, \qquad (p = p_R) \quad (3)$$

where $\alpha = \Gamma_{0II}/\Gamma_{0I}$.

At the "top" of the atmosphere (which we designate by the subscript T), taken to correspond to some arbitrarily small pressure, we require (cf., Smagorinsky [23]) that

$$v_1 = 0, \qquad (p = p_T). \quad (4)$$

This upper boundary condition is consistent with the requirement that $\omega_1 \rightarrow 0$ as $p_T \rightarrow 0$ providing $\partial v_1/\partial p$ remains finite and $Q_{1T} \rightarrow 0$ (cf., Saltzman [15], eq. (26)), and is also consistent with the requirement that the system be energetically "closed" at the upper boundary (see section 5).

We now specify the basic zonal average state to consist of (1) a troposphere ($p_\delta > p > p_R$) in which the vertical profile of the geostrophic zonal wind is given by

$$u_0(p) = aJ_0(cp^{1/2}) + bN_0(cp^{1/2}) - \frac{\Gamma_0 R\beta}{f^2 c}, \qquad (c^2 > 0) \quad (5)$$

where J_0 and N_0 are the zero order Bessel and Neumann functions, a, b and c are constants, and the static stability is given by a constant, $\Gamma_0 = -A_I$, and (2) a stratosphere ($p_R > p > p_T$) in which $u_0(p)$ is a constant (i.e., $\partial T_0/\partial y^*_{II} = 0$), and the static stability is given by another constant, $\Gamma_{0II} = \alpha\Gamma_{0I} = -A_{II} < -A_I$. With suitable choices of a, b, and c, (5) can give a very good fit to the observed winter profile of u_0 (see fig. 1a) and has the important additional property that it makes the coefficient of the zero order term of (1) a constant (cf., Döös [8]). The representation of Γ_0 is most valid in the troposphere, and is least valid in the stratosphere where a representation of the form $\Gamma_{0II} = -A/p$ corresponding to isothermal conditions would be better (cf., e.g., Gates [9]; Saltzman and Rao [20]). However, the main feature of the static stability variation with height (i.e., a very stable stratospheric layer overlying a less stable troposphere) is adequately represented (see fig. 1b).

If, furthermore, we introduce the coordinate transformations,

$$(x, y) = f(RA_I p_s)^{-1/2} \cdot (x^*, y^*)$$
$$\xi = 2(p/p_s)^{1/2}$$

where s denotes the value at sea level, we can write the governing equations for our system in the form,

$$\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial \xi^2}+\frac{1}{\xi}\frac{\partial v_1}{\partial \xi}+K_I^2 v_1=\left(\frac{RA_I p_s}{f^2 u_0}\right)F_1, \qquad (\xi_\delta>\xi>\xi_R) \tag{6}$$

$$\alpha\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}\right)+\frac{\partial^2 v_1}{\partial \xi^2}+\frac{1}{\xi}\frac{\partial v_1}{\partial \xi}+K_{II}^2 v_1 =\left(\frac{RA_{II} p_s}{f^2 u_0}\right)F_1, \qquad (\xi_R>\xi>\xi_T) \tag{7}$$

$$\frac{\partial v_1}{\partial \xi}+\mathcal{K}_\delta v_1-s\left(\frac{\partial v_1}{\partial x}-\frac{\partial v_1}{\partial y}\right)=-\frac{2R}{u_0 f\xi}S_1, \qquad (\xi=\xi_\delta) \tag{8}$$

$$\frac{\partial v_1}{\partial \xi}+rv_1=-\frac{2R(\alpha-1)}{u_{0R} f\xi\alpha}Q_1, \qquad (\xi=\xi_R) \tag{9}$$

$$v_1=0, \qquad (\xi=\xi_T) \tag{10}$$

where

$$K_I^2=p_s c^2>0$$

$$K_{II}^2=\frac{RA_{II}p_s\beta}{f^2 u_{0R}}$$

$$\mathcal{K}_\delta=-\frac{2R(\partial T_0/\partial y^*)_\delta}{u_0 f\xi_\delta}$$

$$s=\frac{2\rho_\delta g(A_I R/p_s)^{1/2}C}{\xi u_{0\delta}}$$

$$r=-\frac{1}{\alpha}\left(\frac{1}{v_1}\frac{\partial v_1}{\partial \xi}\right)_{RII}$$

$$S_1=S_1^{(h)}+Q_{1\delta}$$

$$S_1^{(h)}=-\rho_\delta g f(A_I/Rp_s)^{1/2}\left(\bar{u}\frac{\partial h}{\partial x}+\bar{v}\frac{\partial h}{\partial y}\right) \approx -\rho_\delta g f(A_I/Rp_s)^{1/2}u_{0\delta}\frac{\partial h}{\partial x}.$$

Notice that the effect of forced airflow over mountains represented by $S_1^{(h)}$ as it appears in the lower boundary condition used here and also previously [18] is that of an equivalent surface heating. This is therefore the converse of the approach used by Stern and Malkus [25] in their treatment of the problem of airflow over a heated surface.

3. THE SOLUTION

We assume that the variations of the forcing functions F_1, Q_1, and S_1 near a given latitude circle can be represented by a double Fourier series over a region formed by the length of the latitude circle and an arbitrary meridional width (let us say, 40° of latitude) centered on the latitude circle. That is,

$$\begin{Bmatrix} F_1 \\ Q_1 \\ S_1 \end{Bmatrix}=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\begin{Bmatrix} \mathcal{F}_{m,n} \\ \mathcal{Q}_{m,n} \\ \mathcal{S}_{m,n} \end{Bmatrix}\exp\left[i(kmx+lny)\right] \tag{11}$$

where

$i=\sqrt{-1}$
$k=2\pi(RA_I p_s)^{1/2}/fK^*$
$l=2\pi(RA_I p_s)^{1/2}/fL^*$
K^*=length of the fundamental region corresponding roughly to the distance around a latitude circle
L^*=zonal width of the fundamental region
m=wave number in the x-direction
n=wave number in the y-direction.

Then we can write the solution of the system (6)–(10) in the form,

$$v_1(x,y,\xi)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}V_{m,n}(\xi)\exp\left[i(kmx+lny)\right] \tag{12}$$

where, for the *troposphere:*

$$V_{m,n}(\xi)=\int_{\xi_\delta}^{\xi_R}G_{m,n}(\epsilon;\xi)\mathcal{F}_{m,n}(\epsilon)d\epsilon +I_{m,n}(\xi)\mathcal{S}_{m,n}(\xi_\delta)+J_{m,n}(\xi)\mathcal{Q}_{m,n}(\xi_R), \tag{13}$$

in which G, I, and J are the complex influence functions for the effects of heat and momentum sources in the interior of the troposphere, at the lower boundary, and at the tropopause, respectively. These are given by

$$G_{m,n}(\epsilon;\xi)=-\frac{\pi RA_I p_s\Phi\epsilon}{2f^2u_0(\epsilon)}\times\begin{cases} g(\xi;\epsilon) & \epsilon<\xi \\ g(\epsilon;\xi) & \epsilon>\xi \end{cases} =G_{m,n}^{(r)}+iG_{m,n}^{(i)} \tag{14}$$

$$I_{m,n}(\xi)=-\frac{2R\Phi}{u_{0\delta}f\xi_\delta}\left[c_3\psi_2(\lambda\xi)-c_4\psi_1(\lambda\xi)\right] =I_{m,n}^{(r)}+iI_{m,n}^{(i)} \tag{15}$$

$$J_{m,n}(\xi)=\frac{2R\Phi}{u_{0R}f\xi_R}\left[c_2\psi_1(\lambda\xi)-c_1\psi_2(\lambda\xi)\right] =J_{m,n}^{(r)}+iJ_{m,n}^{(i)} \tag{16}$$

where

$$g(a;b)=[c_3\psi_2(b)-c_4\psi_1(b)]\times[c_2\psi_1(a)-c_1\psi_2(a)]$$

$$c_1=[\psi_1'(\lambda\xi_\delta)+(\mathcal{K}_\delta-i\gamma)\psi_1(\lambda\xi_\delta)]$$

$$c_2=[\psi_2'(\lambda\xi_\delta)+(\mathcal{K}_\delta-i\gamma)\psi_2(\lambda\xi_\delta)]$$

$$c_3=[\psi_1'(\lambda\xi_R)+r\psi_1(\lambda\xi_R)]$$

$$c_4=[\psi_2'(\lambda\xi_R)+r\psi_2(\lambda\xi_R)]$$

$$\psi'=d\psi/d\xi$$

$$\lambda^2=\left(K^2-\frac{\Gamma_0}{\Gamma_{0I}}\mu^2\right), \; [\lambda_I^2=(K_I^2-\mu^2),\ \lambda_{II}^2=(K_{II}^2-\alpha\mu^2)]$$

$$\mu^2=(k^2m^2+l^2n^2)$$

$$\Phi=(c_2c_3-c_1c_4)^{-1}$$

$$\gamma=\mu^2 s/km$$

and the functions $\psi_1(\lambda\xi)$ and $\psi_2(\lambda\xi)$ are given by,

$$\psi_1=J_0(\lambda\xi),\ \lambda^2>0$$
$$\psi_2=N_0(\lambda\xi),\ \lambda^2>0$$
$$\psi_1=J_0(i\lambda'\xi),\ \lambda^2=(i\lambda')^2<0$$
$$\psi_2=iH_0^{(1)}(i\lambda'\xi),\ \lambda^2=(i\lambda')^2<0$$
$$\lambda_1=1,\ \lambda^2=0$$
$$\lambda_2=\log\xi,\ \lambda^2=0$$

where J_0, N_0, and H_0 are the zero order Bessel, Neumann, and Hankel functions, respectively. Note that for the previous model studied by the writer [18], as well as for the models studied earlier by Smagorinsky [23] and the Staff Members [24], the homogeneous solutions, in terms of which the complete solutions were expressed, were the confluent hypergeometric functions rather than Bessel functions.

Assuming all the forcing is confined below the tropopause (i.e., $F_1=0$ for $\xi<\xi_R$), we have for the *stratosphere:*

$$V_{m,n}(\xi)=\chi_{m,n}(\xi)\cdot V_{m,n}(\xi_R),\qquad(\xi_R>\xi>\xi_T)\qquad(17)$$

where

$$\chi_{m,n}(\xi)=\left[\frac{a\psi_1(\lambda_{II}\xi)+\psi_2(\lambda_{II}\xi)}{a\psi_1(\lambda_{II}\xi_R)+\psi_2(\lambda_{II}\xi_R)}\right],\qquad a=-\frac{\psi_2(\lambda_{II}\xi_T)}{\psi_1(\lambda_{II}\xi_T)}.$$

In the sense that both damping and amplification of the mean tropopause perturbations into the stratosphere are permitted, this solution is more general than the stratosphere solution of Smagorinsky [23] which requires exponential damping in all cases.

Some of the general properties of the solution regarding the positions and slopes of troughs and ridges of the pressure field are readily deducible from the differential equations:·

(*Case I*) *Forcing due to mountains or equivalent boundary heating* ($F_1=Q_1=0$, $S_1^h\neq 0$).—Assuming a harmonic form for v_1 we have applying (6) at the trough or ridge ($v_1=0$),

$$\frac{d}{d\xi}\left(\frac{dv_1}{d\xi}\right)+\frac{1}{\xi}\left(\frac{dv_1}{d\xi}\right)=0$$

which has the general solution

$$\left(\frac{dv_1}{d\xi}\right)=\frac{A}{\xi},\qquad(A\text{ is a constant}).$$

In order to satisfy the tropopause condition (9), for $Q_{IR}=0$, we must set $A=0$ which means that $dv_1/d\xi=0$ for all ξ along a trough or ridge line. Thus we conclude that in this model *trough and ridge lines associated with a boundary-induced perturbation of a given wave number have no slope with height.*

Furthermore, at the intersection of a trough or ridge with the lower boundary, we have from (8)

$$s\left(\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}\right)=\frac{2RS_1}{u_{0\delta}f\xi_\delta}.$$

Thus, in the case of a pure mountain effect, for example, we have the approximate relation

$$C\left(\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}\right)\approx -u_{0\delta}\frac{\partial h}{\partial x}$$

which implies that *the center of cyclonic vorticity must be on the downslope (heated) side of the mountain and the center of anticyclonic vorticity on the upslope (cooled) side.* In the absence of friction (i.e., $C=0$) we would have $\partial h/\partial x=0$ at $v_1=0$, which would mean that troughs and ridges coincide with the valleys and peaks of the topographic harmonics (cf., Barrett [2]).

(*Case II*) *Forcing due to internal sources of heat and momentum* ($F_1\neq 0$, $S_1=0$).—For the intersection of a trough or ridge line with the lower boundary in this case, we can write (8) in the form

$$\frac{\partial v_1}{\partial\xi}=s\left(\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}\right),$$

which means

$$\frac{\partial v_1}{\partial\xi}\begin{cases}>0\ \textit{for a trough}\\<0\ \textit{for a ridge}\end{cases}$$

since $s>0$. Thus, *at the lower boundary all troughs and ridges induced by internal sources of heat and momentum must tilt EASTWARD with height.* In the absence of the surface friction ($C=0$), we have $s=0$ and hence $\partial v_1/\partial\xi=0$ which would imply no tilt of the troughs and ridges with height at the lower boundary. From (13) and (14) we can show that in the absence of friction we have $G^{(i)}=0$ and, hence, the troughs and ridges would continue to show no tilt with height throughout the entire atmosphere if the vertical model surfaces of the forcing function have no tilt with height (cf., Smagorinsky [23]).

As can be seen in the following section (and also in the previous calculations by the writer [18]), when friction is included, the slopes of the troughs and ridges some distance above the lower boundary display systematic variations depending on the wave number of the perturbation. These appear to be in accord with the results of Gilchrist [10].

From (3) we can also deduce some of the properties which the solution for our model must display at the *tropopause.* If, for example, $Q_{1R}=0$, we must have at all points along the tropopause,

$$\left(\frac{\partial v_1}{\partial p}\right)_{RII}=\alpha\left(\frac{\partial v_1}{\partial p}\right)_{RI}$$

or

$$\left(\frac{\partial T_1}{\partial x}\right)_{RII}=\alpha\left(\frac{\partial T_1}{\partial x}\right)_{RI}$$

which implies an increase of the amplitude of the mean temperature waves as we pass from the troposphere side to the stratosphere side.

4. SPECIAL SOLUTIONS CORRESPONDING TO WINTER NORMAL CONDITIONS AT 45° N.

A. PARAMETERS

The following constants defining the physical system are chosen,

$$
\begin{aligned}
K^* &= 3\times10^9 \text{ cm.} \\
L^* &= 4.5\times10^8 \text{ cm.} \\
f &= 10^{-4} \text{ sec.}^{-1} \\
\beta &= 1.7\times10^{-13} \text{ cm.}^{-1} \text{ sec.}^{-1} \\
p_s &= 1000 \text{ mb.} \\
\xi_\delta &= 1.8974 \text{ (900 mb.)} \\
\xi_R &= 1.0000 \text{ (250 mb.)} \\
\rho_\delta &= 1.2\times10^{-3} \text{ gm. cm.}^{-3} \\
C &= 1.6\times10^4 \text{ cm.}
\end{aligned}
$$

and the following constants are taken to describe the mean zonal state,

$$
\begin{aligned}
a &= 0 \\
b &= 33 \text{ m. sec.}^{-1} \\
c &= 0.067 \text{ mb.}^{-1/2} \\
A_I &= 5\times10^{-2} \text{ deg. mb.}^{-1} \\
A_{II} &= 3\times10^{-1} \text{ deg. mb.}^{-1} \qquad (\alpha=6) \\
u_{0\delta}(L/2) &= 4.68 \text{ m. sec.}^{-1}
\end{aligned}
$$

The profiles of $u_0(p)$ and $\Gamma_0(p)$ implied by these constants are shown in figure 1a and 1b along with the profiles used in the previous study (Saltzman [18]) and the *observed* January normal values at 45° N.

Forcing due to two physical effects will be considered: (1) lower boundary heating and cooling due to adiabatic airflow over topography and (2) internal heating and cooling with an assumed maximum near 700 mb. due to a combination of diabatic effects and transient eddy heat fluxes. From previous studies by the writer [17], [18] it appears that forcing due to internal sources of momentum, measured by M_1, is smaller than the above effects, but perhaps is not unimportant if one wishes to account for a small residual percentage of the total variance of the mean map. (Better determinations of M_1 should be possible now with the use of data presented by Crutcher [7].)

B. STRATOSPHERIC AMPLIFICATION, INFLUENCE FUNCTIONS, AND RESONANCE

Before discussing these forcing functions, let us first consider the influence functions $I_{m,n}$, $J_{m,n}$, and $G_{m,n}$, and, also, the quantity $\chi_{m,n}$ which represents the magnification factor for the amplitude of the perturbations above the tropopause. These quantities depend only on the wave number and the physical constants given above.

In table 1 we present some of the derived parameters which depend on wave numbers, for $m=1$ through 6 and $n=0$ and 1 (i.e., for 12 vector wave numbers, measured by μ^2), and in figure 2 we show the stratosphere amplitude function $\chi_{m,n}$ for these same wave numbers. We can see from this figure that only the perturbations having the lowest vector wave numbers can amplify above the tropopause. This characteristic seems to be in good agreement with observations (cf., Teweles [26]; Arctic Meteorology Research Group [1]) even to the extent of implying the observed meridional elongation of the waves at 50 mb. compared to 500 mb., particularly for $m=3$ and 4 (cf., figs. 7 and 8 of Teweles [26]). This result can be understood in terms of the fundamental equation (1) applied to our stratospheric case of $u_0>0$ and $F_1=\partial u_0/\partial p=\partial\Gamma_0/\partial p=0$. It can be seen that for a given double Fourier component in the x^*-y^* plane, the term $\partial(p\partial v_1/\partial p)/\partial p$ (which is proportional to the advection of the thermal part of the potential vorticity) must be opposite in sign to v_1 when the vector wave number is low enough that the β-term dominates over the first two terms of (1) representing the advection of relative vorticity. This requirement can easily be fulfilled by the existence of a maximum or minimum of v_1 within the stratosphere, as is the case in the solution.

The influence functions, $I(=I^{(r)}+iI^{(i)})$, $J(=J^{(r)}+iJ^{(i)})$, and $G(=G^{(r)}+iG^{(i)})$, from which one can compute the troposphere solution, given the forcing functions, are tabulated in tables 2 and 3 at every 100 mb. although actual calculations were made at every 50 mb. In the case of G, tabulations for only four reference levels (900, 700, 500, and 300 mb.) are given although actual calculations were made for every 50 mb. between 900 and 250 mb.

An obvious feature of these tabulated influence functions is the marked increase in magnitude, and the reversal of signs, in the vicinity of wave number (5,0), which imply high amplitude responses and changes of phase of the perturbations near this wave number. This is a reflection of the fact that the vector wave number $\mu^2_{5,0}$ is very close to the "quasi-resonant" vector wave number for the model treated (cf., Gilchrist [10], Smagorinsky [23]). The exact quasi-resonant wave number, which we call $\mu^2_{m^*,n^*}$ $(=km^{*2}+ln^{*2})$ satisfies the condition

$$
c_2{}^{(r)}c_3-{}_1c^{(r)}c_4=0 \tag{18}
$$

which in the absence of surface friction ($C=0$) would imply that $\Phi^{(r)}_{m^*,n^*}$ (and also $I^{(r)}_{m^*,n^*}$, $J^{(r)}_{m^*,n^*}$, and $G^{(r)}_{m^*,n^*}$) approaches infinity and $\Phi^{(i)}_{m^*,n^*}$ (and also, $I^{(i)}_{m^*,n^*}$, $J^{(i)}_{m^*,n^*}$, and $G^{(i)}_{m^*,n^*}$) equals zero. Hence, the responses would be of infinite ampli-

TABLE 1.—*Parameters dependent on wave number.*

m	n	μ^2	λ^2_I	λ^2_{II}	γ	r	$\Phi^{(r)}$	$\Phi^{(i)}$
1	0	0.0630	4.4261	6.1663	0.1273	0.5011	0.5463	0.0061
2	0	.2518	4.2372	5.0332	.2547	.3775	.6272	.0117
3	0	.5666	3.9225	3.1447	.3820	.1542	.8811	.0102
4	0	1.0072	3.4818	.5008	.5093	−.0394	1.8466	−.1117
5	0	1.5738	2.9153	−2.8985	.6368	−.0478	−.6451	−10.9261
6	0	2.6620	2.2228	−7.0532	.8975	−.3600	−.7300	−.0700
1	1	2.8606	1.6285	−10.6193	5.7863	−.4533	−.1785	−.2153
2	1	3.0494	1.4396	−11.7524	3.0842	−.4812	−.2773	−.1702
3	1	3.3642	1.1249	−13.6409	2.2683	−.5252	−.2643	−.1122
4	1	3.8048	.6842	−16.2848	1.9241	−.5828	−.2199	−.0741
5	1	4.3714	.1177	−19.6841	1.7685	−.6502	−.1735	−.0505
6	1	5.0638	−.5748	−23.8388	1.7072	−.7252	----------	----------

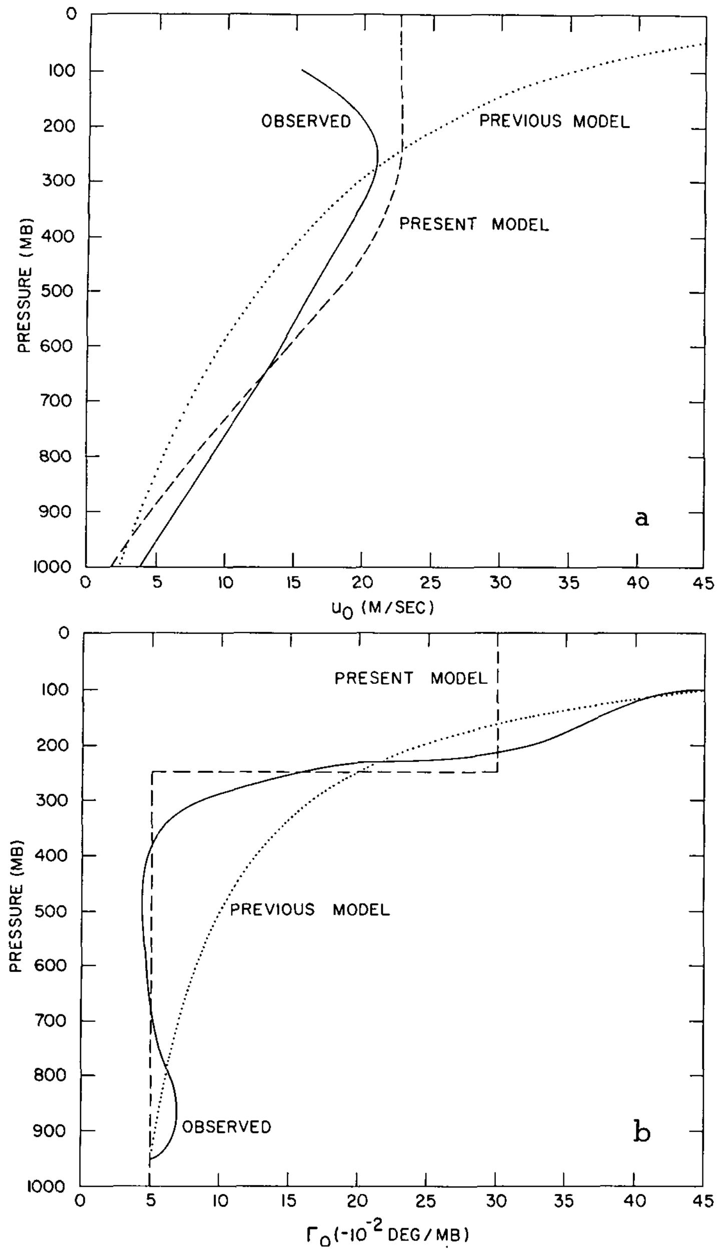

FIGURE 1.—(a) Winter profile of u_0 at 45° N. used in the present study (dashed line), in the previous study (Saltzman [18]) (dotted line), and actually observed according to Döös [8] and Crutcher [7], (solid line). (b) Winter profile of Γ_0 at 45° N. used in the present study (dashed line), in the previous [18] study (dotted line), and actually observed according to Gates [9]), (solid line).

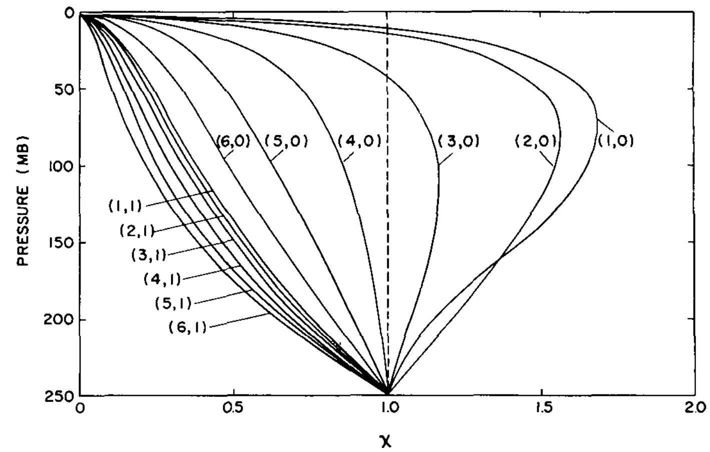

FIGURE 2.—Stratospheric amplification factor $\chi_{m,n}$

tude. In the presence of surface friction, however, the condition (18) implies that $\Phi^{(r)}_{m^*,n^*}$ (and also $I^{(r)}_{m^*,n^*}$, $J^{(r)}_{m^*,n^*}$, and $G^{(r)}_{m^*,n^*}$) equals zero and $\Phi^{(i)}_{m^*,n^*}$ (and also $I^{(i)}_{m^*,n^*}$, $J^{(i)}_{m^*,n^*}$, and $G^{(i)}_{m^*,n^*}$) is a large but finite number. Thus, in our frictional case, internal sources and sinks of heat and momentum of scale (m^*,n^*) having no tilt with height will induce large-amplitude perturbations also having no tilt with height.

The existence of quasi-resonant modes must have an important bearing on the form assumed by the mean waves in the atmosphere. In fact, we actually do observe mean waves on the scale $m=5$ which are perhaps greater than we might have expected (cf., Saltzman and Peixoto [19]). For two main reasons, however, we must be cautious about discussing solutions near the quasi-resonant point even though these may be most significant physically. The first is that a very high-amplitude response would be inconsistent with the hypothesis of small perturbations—i.e., the appearance of strong non-linear effects may invalidate the theory near (m^*, n^*). The second is that even if the solutions near (m^*, n^*) are roughly correct, they are so sensitive to the choice of parameters that we cannot be sure of their correspondence to real atmosphere conditions. Needless to say, these questions concerning resonance can have important implications for the subject of global modification of climate.

C. RESPONSE TO TOPOGRAPHY

Let us represent the topographic surface in the neighborhood of 45° N. by a double Fourier expansion of the form

$$\begin{aligned} h(x,y) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} E_{m,n} \exp\,[i(mkx+nly)] \\ &= \mathcal{E}^{(r)}_{0,0} + \sum_{n=1}^{\infty} [2\mathcal{E}^{(r)}_{0,n} \cos nly + 2\mathcal{E}^{(i)}_{0,n} \sin nly] \\ &\quad + \sum_{m=1}^{\infty} \frac{C_{m,0}}{2} \cos\ m(kx-\epsilon_{m,0}) \\ &\quad + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [C_{m,n} \cos m(kx-\epsilon_{m,n}) \cos nly \\ &\qquad + D_{m,n} \cos m(kx-\theta_{m,n}) \sin nly], \end{aligned} \tag{19}$$

where

$$E_{m,n} = \frac{1}{KL} \int_0^K \int_0^L h(x,y) \exp\,[-i(kmx+lny)]dxdy$$
$$= \mathcal{E}^{(r)}_{m,n} - i\mathcal{E}^{(i)}_{m,n}$$

TABLE 2.—*Influence functions, $I_{m,n}$ and $J_{m,n}$ in units of 10^4 cm. deg.$^{-1}$*

p(mb.)	(1,0)	(2,0)	(3,0)	(4,0)	(5,0)	(6,0)	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
					$I^{(r)}(p)$						
300	−2,103	−2,445	−3,513	−7,511	2,633	3,074	760	1,184	1,134	950	757
400	−1,682	−2,017	−3,049	−6,826	2,437	3,038	773	1,215	1,181	1,008	821
500	−1,188	−1,492	−2,421	−5,761	2,126	2,860	755	1,200	1,187	1,037	869
600	−688	−948	−1,733	−4,524	1,758	2,597	718	1,156	1,168	1,048	906
700	−217	−425	−1,047	−3,235	1,368	2,287	669	1,094	1,131	1,046	936
800	204	53	−396	−1,966	976	1,950	611	1,018	1,083	1,035	960
900	565	472	1,967	−765	597	1,601	549	935	1,025	1,017	980
					$I^{(i)}(p)$						
300	−23	−45	−41	454	44,587	721	916	727	481	320	220
400	−19	−38	−35	413	41,269	712	932	746	501	340	239
500	−13	−28	−28	349	36,000	670	911	737	504	350	253
600	−8	−18	−20	274	29,776	609	866	710	496	353	263
700	−2	−8	−12	196	23,168	536	806	671	480	353	272
800	2	1	−5	119	16,537	457	737	625	459	349	279
900	6	9	2	46	10,118	375	662	574	435	343	285
					$J^{(r)}(p)$						
300	−1,269	−1,519	−2,279	−5,186	5,702	2,687	1,471	1,456	1,295	1,104	930
400	−1,334	−1,567	−2,286	−5,015	4,975	2,363	1,205	1,204	1,058	883	723
500	−1,252	−1,457	−2,091	−4,492	4,086	1,992	954	969	848	697	560
600	−1,085	−1,254	−1,783	−3,779	3,141	1,609	720	751	658	537	425
700	−869	−1,001	−1,414	−2,973	2,200	1,229	505	551	486	395	311
800	−631	−725	−1,020	−2,136	1,298	863	307	366	330	269	211
900	−387	−445	−625	−1,310	457	518	127	197	187	156	123
					$J^{(i)}(p)$						
300	16	43	130	778	34,147	233	211	153	88	49	28
400	13	35	113	707	31,605	230	215	157	92	52	30
500	9	26	89	596	27,570	217	210	155	92	54	32
600	5	17	64	468	22,804	197	200	149	91	54	33
700	2	7	39	335	17,743	173	186	141	88	54	34
800	−2	−1	15	204	12,665	148	170	132	84	53	35
900	−4	−8	−73	79	7,749	121	153	121	80	53	36

$$C_{m,n}=2[(\mathcal{E}_{m,n}^{(r)}+\mathcal{E}_{m,-n}^{(r)})^2+(\mathcal{E}_{m,n}^{(i)}+\mathcal{E}_{m,-n}^{(i)})^2]^{1/}$$

$$D_{m,n}=2[(\mathcal{E}_{m,-n}^{(r)}-\mathcal{E}_{m,n}^{(r)})^2+(\mathcal{E}_{m,n}^{(i)}-\mathcal{E}_{m,-n}^{(i)})^2]^{1/2}$$

$$\epsilon_{m,n}=\frac{1}{m}\arctan\left[\frac{\mathcal{E}_{m,n}{}^{(i)}+\mathcal{E}_{m,-n}{}^{(i)}}{\mathcal{E}_{m,n}{}^{(r)}+\mathcal{E}_{m,-n}{}^{(r)}}\right]$$

$$\theta_{m,n}=\frac{1}{m}\arctan\left[\frac{\mathcal{E}_{m,-n}{}^{(r)}-\mathcal{E}_{m,n}{}^{(r)}}{\mathcal{E}_{m,n}{}^{(i)}-\mathcal{E}_{m,-n}{}^{(i)}}\right]$$

$E_{m,n}$ is the complex Fourier coefficient, and ϵ and θ are the phase angles of the harmonics along a latitude circle indicating the first longitude at which a maximum of h occurs. From (19) and the definition of $S_1{}^{(h)}$ we can now write the following expressions for the real and imaginary parts of the complex Fourier coefficients of the boundary forcing function, providing we take $u_{0\delta}$ as a constant,

$$\mathcal{S}_{m,n}=\mathcal{S}_{m,n}{}^{(r)}-i\mathcal{S}_{m,n}{}^{(i)}$$

$$\mathcal{S}_{m,n}{}^{(r)}=-\Xi\mathcal{E}_{m,n}{}^{(i)}$$

$$\mathcal{S}_{m,n}{}^{(i)}=\Xi\mathcal{E}_{m,n}{}^{(r)}$$

where

$$\Xi=\rho_\delta gfu_{0\delta}(A_I/Rp_s)^{1/2}mk.$$

It is instructive first to show the general effects of the *scale* of the topography only by computing the response v_1 on the assumption that all of the topographic harmonics have equal amplitude and phase along a given latitude (45° N., in our case). We also assume that $u_{0\delta}$ is uniform with y. The calculations were made for all the vector wave numbers included by $m=1$ through 6 and $n=0$ and 1, but because of the similarities of the results for $n=1$, we shall delete $(m,n)=(4,1)$, (5,1), and (6,1) from the presentation.

In particular, we have set $C_{m,0}/2=-C_{m,1}=200$ m., $D_{m,n}=0$, and $\epsilon_{m,0}=\epsilon_{m,1}=0$, which from (19) is equivalent to setting $\mathcal{E}_{m,0}{}^{(i)}=\mathcal{E}_{m,1}{}^{(i)}=\mathcal{E}_{m,-1}{}^{(i)}=0$, $C_{m,0}/2=2\mathcal{E}_{m,0}{}^{(r)}$, and $C_{m,1}=4\mathcal{E}_{m,1}{}^{(r)}=4\mathcal{E}_{m,-1}{}^{(r)}$. Thus, we shall present results for all components except (4,1), (5,1) and (6,1) included by the expansion,

$$h(x,y)=\sum_{m=1}^{6}[h^{(m,0)}+h^{(m,1)}]$$

where

$$h^{(m,0)}=\frac{C_{m,0}}{2}\cos mkx$$

$$h^{(m,1)}=-C_{m,1}\cos mkx\cos ly$$

Theva lues of $I_{m,n}$ from which the responses are easily obtained using (13) are given in table 2. The solutions

TABLE 3.—*Influence functions, $G_{m,n}$, in units of 10^{10} cm. sec.*

	(m, n)										
p(mb)	(1, 0)	(2, 0)	(3, 0)	(4, 0)	(5, 0)	(6, 0)	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)
					$G^{(r)}$ (p; 900 mb.)						
300	−119	−139	−200	−427	150	175	43	67	64	54	43
400	−119	−143	−216	−484	173	215	54	86	84	71	58
500	−110	−138	−223	−532	196	264	70	111	110	96	80
600	−85	−117	−215	−561	218	322	89	143	145	130	112
700	−38	−75	−185	−572	242	405	118	193	200	185	166
800	55	14	−107	−530	263	526	165	275	292	279	259
900	268	224	93	−363	283	759	260	443	486	482	465
					$G^{(i)}$ (p; 900 mb.)						
300	−1	−3	−2	25	2,532	40	52	41	27	18	12
400	−1	−3	−2	29	2,927	50	66	52	35	24	16
500	−1	−3	−3	32	3,321	61	84	68	46	32	23
600	−1	−2	−2	33	3,689	75	107	88	61	43	32
700	0	−1	−2	34	4,098	94	142	118	84	62	48
800	1	0	−1	32	4,462	123	198	168	124	94	75
900	3	4	1	22	4,798	178	314	272	206	162	135
					$G^{(r)}$ (p; 700 mb.)						
300	−268	−313	−451	−969	719	414	172	188	167	137	109
400	−268	−322	−489	−1,099	831	511	219	241	217	181	147
500	−246	−310	−505	−1,207	943	626	278	310	284	243	203
600	−191	−264	−486	−1,273	1,047	764	355	401	375	329	284
700	−86	−169	−419	−1,299	1,163	960	472	542	519	469	418
800	−96	−187	−461	−1,424	1,047	1,029	437	549	537	487	434
900	−103	−202	−496	−1,534	649	1,085	317	519	537	496	444
					$G^{(i)}$ (p; 700 mb.)						
300	1	2	12	109	5,799	58	63	48	30	18	11
400	1	2	13	123	6,702	72	80	61	39	24	16
500	0	2	13	136	7,606	88	10	79	51	33	22
600	0	2	13	143	8,448	107	13	102	67	45	31
700	0	1	11	146	9,383	135	17	139	93	64	45
800	0	0	6	135	10,217	176	24	197	136	96	71
900	−1	−4	−5	92	10,988	254	38	318	227	167	12
					$G^{(r)}$ (p; 500 mb.)						
300	−386	−455	−668	−1,464	1,335	672	325	331	291	241	196
400	−386	−468	−724	−1,661	1.543	829	413	425	379	320	265
500	−354	−451	−748	−1,824	1,742	1,015	525	546	495	428	365
600	−412	−522	−856	−2,060	1,808	1,101	532	568	516	442	372
700	−471	−594	−969	−2,314	1,808	1,201	533	594	544	465	388
800	−522	−656	−1,066	−2,536	1,627	1,286	494	602	563	483	403
900	−564	−708	−1,148	−2,733	1,008	1,356	358	569	563	492	412
					$G^{(i)}$ (p; 500 mb.)						
300	3	8	28	194	9,011	73	71	53	31	18	11
400	3	8	30	220	10,414	90	91	67	41	24	15
500	3	8	32	242	11,819	110	115	87	53	32	20
600	2	7	30	255	13,127	134	145	112	71	44	28
700	0	4	26	260	14,581	169	196	152	98	63	42
800	−1	−1	15	241	15,876	220	273	216	143	96	66
900	−6	−1	−13	165	17,075	318	431	349	239	165	119
					$G^{(r)}$ (p; 300 mb.)						
300	−391	−474	−728	−1,689	1,864	906	501	498	445	382	325
400	−513	−611	−911	−2,040	2,030	995	513	514	454	382	315
500	−627	−739	−1,085	−2,378	2,170	1,091	528	538	473	392	318
600	−730	−854	−1,242	−2,686	2,239	1,183	536	561	493	405	324
700	−834	−974	−1,406	−3,017	2,239	1,291	536	586	520	426	338
800	−924	−1,075	−1,547	−3,306	2,015	1,383	497	594	538	443	351
900	−997	−1,160	−1,666	−3,562	1,249	1,458	360	561	538	451	359
					$G^{(i)}$ (p; 300 mb.)						
300	5	13	41	253	11,161	78	72	52	30	17	9
400	5	13	44	287	12,898	97	91	67	39	22	13
500	4	13	46	315	14,639	118	116	86	51	30	18
600	3	11	44	332	16,258	144	148	111	67	40	25
700	1	7	38	339	18,059	182	197	150	94	58	37
800	−2	−1	22	315	19,663	236	275	213	137	88	58
900	−11	−21	−19	215	21,148	341	434	344	228	151	104

are shown in the form of cross-sections along 45° N. in figures 3 through 11 with the mountain profiles shown at the bottom. It can be seen that the trough and ridge properties are in accord with the discussion of section 3 (case 1), and that the vertical variations of the amplitudes into the stratosphere are in accord with the discussion of section 4B. It can also be seen in figures 3 through 6 that the horizontal node decreases in height with increasing wave number, and disappears altogether between $(m,n)=(3,0)$ and $(4,0)$ coincident with a phase shift of the perturbations relative to the topographic profile. All of these properties are consistent with the fact that for decreasing vector wave number (μ^2) and basic zonal wind speed (u_0) the effect of advection of relative vorticity becomes reduced compared with the β-effect. The consequence of the quasi-resonant condition near (5,0) is clearly in evidence.

In figures 12 and 13, the fields of ω_1 and T_1 associated with the solutions for $(m,n)=(2,0)$ and $(2,1)$ respectively are shown, as these are representative of the two regimes separated by the quasi-resonant wave number. It can be seen that in all cases $|\omega_1|_{\max}$ coincides with $|v_1|_{\max}$, and $|T_1|_{\max}$ coincides with $v_1=0$ (in agreement with the fact that the troughs and ridges have no slope with height—cf. section 3).

D. RESPONSE TO SOURCES AND SINKS OF HEAT

The distribution of $Q_1(x, y, p)$ is very poorly known. For want of a better estimate, we shall adopt a distribution similar to that assumed previously by Smagorinsky [23] and the writer [18]—i.e.,

$$Q_1(x, y, p)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathcal{Q}_{m,n}(p)\exp[i(mkx+nly)] \tag{20}$$

where

$$\mathcal{Q}_{m,n}(p)=A(p)N_{m,n},$$

the vertical distribution $A(p)$ having the form shown in figure 14. If we wish, we can expand (20) further in the manner of (19).

As in the mountain case, we shall examine the effect of the scale only by computing the response v_1 assuming the amplitudes and phases are the same for all forcing harmonics. The following uniform magnitudes of $N_{m,n}$ $(=N_{m,n}^{(r)}-i\,N_{m,n}^{(i)})$ were adopted, all referred to a phase origin at $kx=\epsilon$,

$$N_{m,0}^{(r)}=-2N_{m,1}^{(r)}=-2N_{m,-1}^{(r)}=10^{-5}\text{ deg. sec.}^{-1}$$

$$N_{m,0}^{(i)}=N_{m,1}^{(i)}=N_{m,-1}^{(i)}=0$$

which implies that

$$Q_1(x, y, p)=\sum_{m=1}^{\infty}[Q_1^{(m,0)}+Q_1^{(m,1)}]$$

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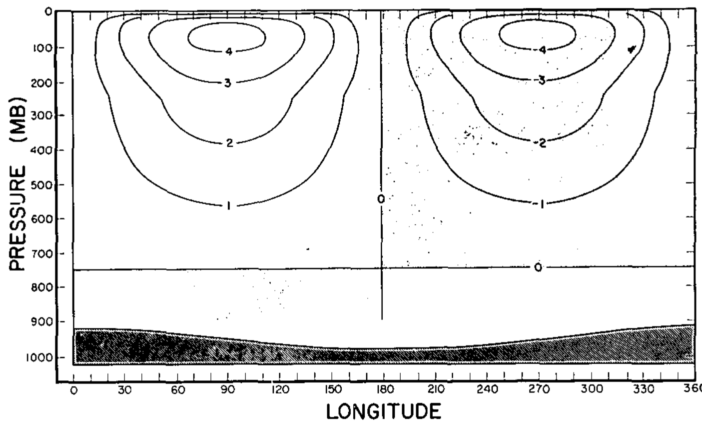

FIGURE 3.—Mean meridional wind response, v_1, to uniform airflow over topographic harmonic, $h^{(1,0)}$, along $y=L/2$ in units of 10^{-1} m. sec.$^{-1}$

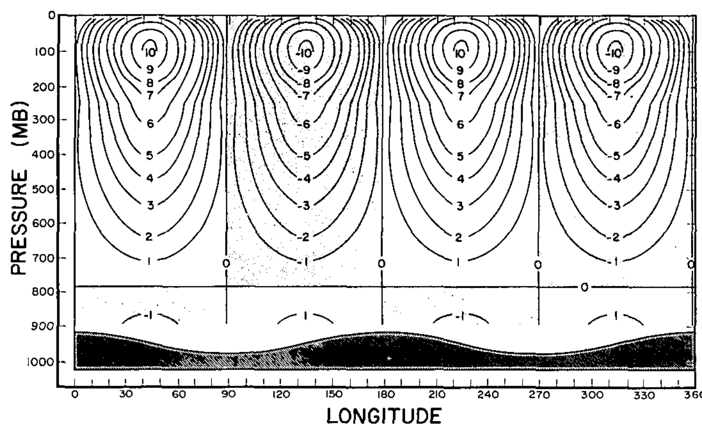

FIGURE 4.—Same as figure 3 for $h^{(2,0)}$

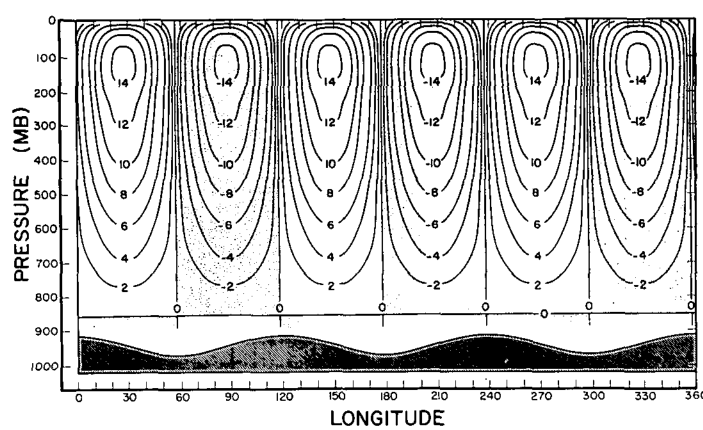

FIGURE 5.—Same as figure 3 for $h^{(3,0)}$

where

$$Q_1^{(m,0)}=2N_{m,0}^{(r)}\,A(p)\cos m(kx-\epsilon_{m,0})$$

$$Q_1^{(m,1)}=-4N_{m,1}A(p)\cos m(kx-\epsilon_{m,1})\cos ly. \tag{21}$$

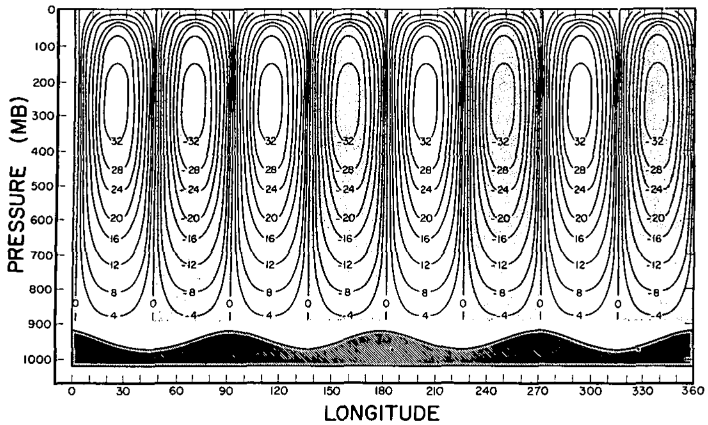

FIGURE 6.—Same as figure 3 for $h^{(4,0)}$

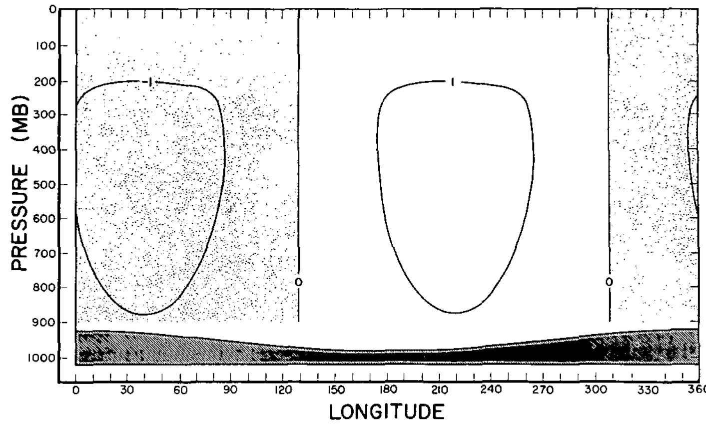

FIGURE 9.—Same as figure 3 for $h^{(1,1)}$, except in units of 10^{-2} m. sec.$^{-1}$

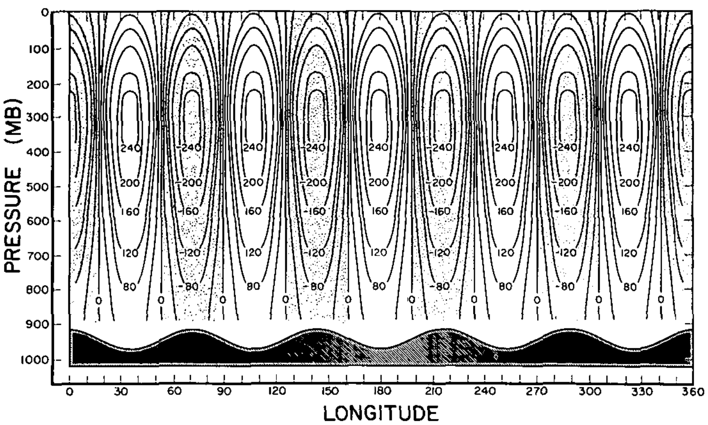

FIGURE 7.—Same as figure 3 for $h^{(5,0)}$

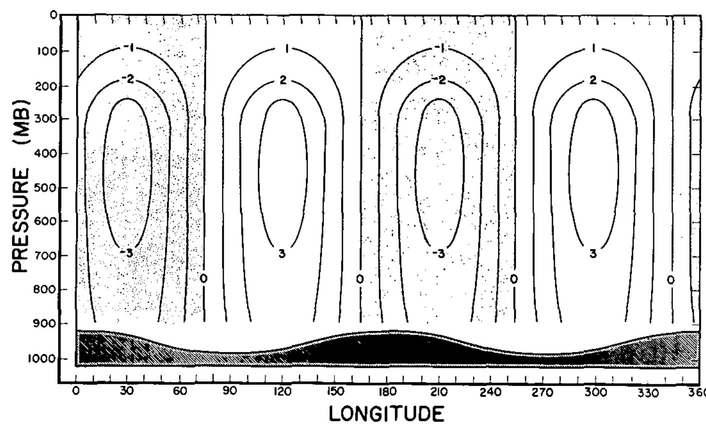

FIGURE 10.—Same as figure 3 for $h^{(2,1)}$

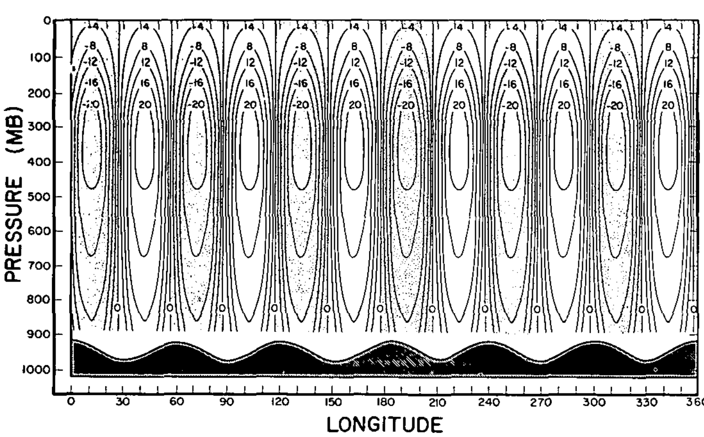

FIGURE 8.—Same as figure 3 for $h^{(6,0)}$

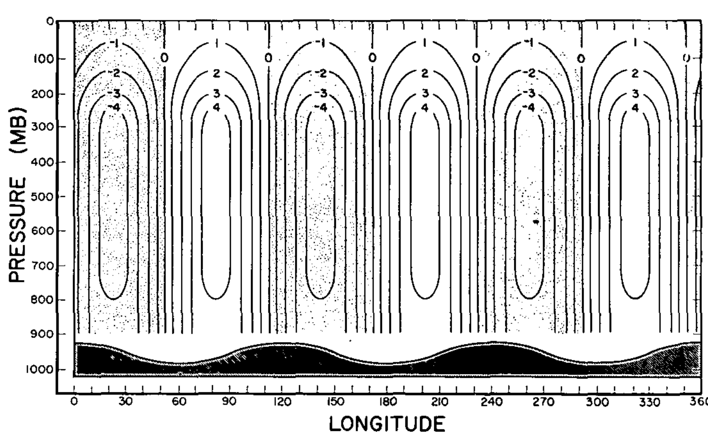

FIGURE 11.—Same as figure 3 for $h^{(3,1)}$

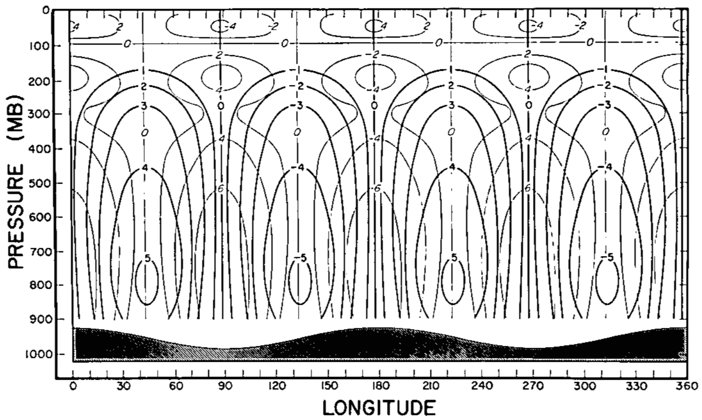

FIGURE 12.—Cross-section along $y=L/2$ of ω_1 (heavy lines) in units of 10^{-2} gm. cm.$^{-1}$ sec.$^{-3}$, and T_1 (thin lines) in units of 10^{-1} deg., corresponding to the solution for $h^{(2,0)}$ shown in figure 4.

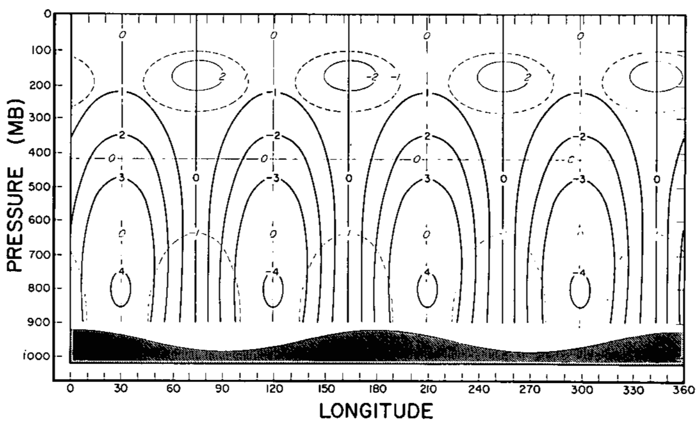

FIGURE 13.—Same as figure 12 for $h^{(2,1)}$, corresponding to the solution shown in figure 10.

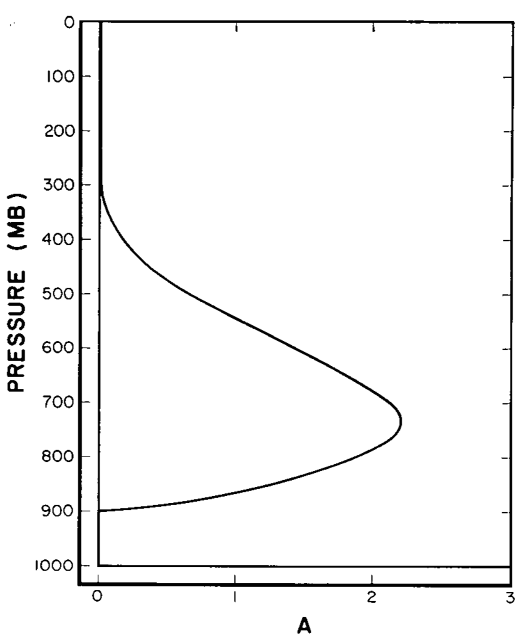

FIGURE 14.—Vertical amplitude function for internal heating, $A(p)$.

For case $\epsilon_{m,0}=\epsilon_{m,1}$ we have

$$Q_1^{(m,1)}=Q_1^{(m,0)} \text{ at } y=\frac{L}{2}\,(45°\text{N.})$$

From these expressions for $\mathcal{Q}_{m,n}$ it is easy to calculate the forcing functions $\mathcal{F}_{m,n}$ and hence to solve for v_1 from (13) given the value of $G_{m,n}$ shown in table 3. The results, again for $m=1$ through 6, and $n=0$ and 1, excluding (4,1), (5,1), and (6,1), are shown in figures 15 through 23 for $\epsilon=0$ (i.e., maximum heating at zero longitude). It can be seen that the distributions are consistent with the discussions of sections 3 and 4. Figures 24 and 25 show the values of ω_1 and T_1 corresponding to the same two harmonics, $(m,n)=(2, 0)$ and $(2, 1)$, used to illustrate the mountain response.

For value of $N^{(r)}_{m,0}=10^{-5}$ deg. sec.$^{-1}$ used here, the maximum vertical-mean heating, given by

$$\{Q_1\}\equiv\frac{1}{p_\delta-p_T}\int_{p_T}^{p_\delta} Q_1 dp$$

is 1.438×10^{-5}deg./sec., which is of the order of observational estimates (e.g., Staff Members, Academia Sinica [24]).

5. ENERGETICS

From the basic perturbation equations and boundary conditions for our study, we can derive the following pair of equations for the energy balance of the mean perturbations:

$$\frac{d}{dt}\{A_1\}=\left\{\frac{R}{p\Gamma_0}\frac{\partial T_0}{\partial y}(v_1T_1)_0\right\}+\left\{\frac{R}{p}(\omega_1T_1)_0\right\}-\left\{\frac{R}{p\Gamma_0}(Q_1T_1)_0\right\}=0 \quad (22)$$

$$\frac{d}{dt}\{K_1\}=\left\{-(u_1\omega_1)\frac{\partial u_0}{\partial p}\right\}+\left\{-\frac{R}{p}(\omega_1T_1)_0\right\}+\{u_1\hat{X}_1+v_1\hat{Y}_1\}-\frac{(\omega_1\phi_1)_{0\delta}}{p_\delta-p_T}+\frac{(\omega_1\phi_1)_{0T}}{p_\delta-p_T}=0 \quad (23)$$

where

$$\{(\quad)\}=\frac{1}{p_\delta-p_T}\int_{p_T}^{p_\delta}(\quad)dp$$

$A_1=\left(-\frac{R}{p\Gamma_0}\frac{T_1^2}{2}\right)$=Available potential energy of the mean perturbations, per unit mass

$K_1=\left(\frac{u_1^2+v_1^2}{2}\right)$=Kinetic energy of the mean perturbations per unit mass

$\phi=gz$

$$\hat{X}_1=X-\left(\frac{\partial\overline{u'^2}}{\partial x^*}+\frac{\partial\overline{u'v'}}{\partial y^*}+\frac{\partial\overline{u'\omega'}}{\partial p}\right)$$

$$\hat{Y}_1=Y=\left(\frac{\partial\overline{u'v'}}{\partial x^*}+\frac{\partial\overline{v'^2}}{\partial y^*}+\frac{\partial\overline{v'\omega'}}{\partial p}\right).$$

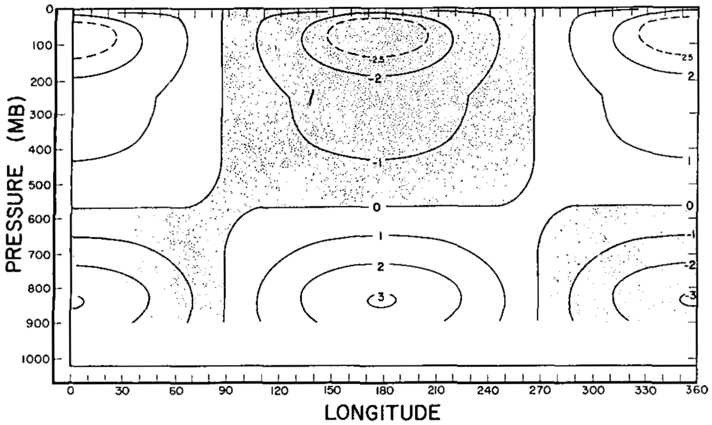

FIGURE 15.—Mean meridional wind response, v_1, to the internal heating function $Q_1^{(1,0)}$, along $y=L/2$, in units of m. sec.$^{-1}$

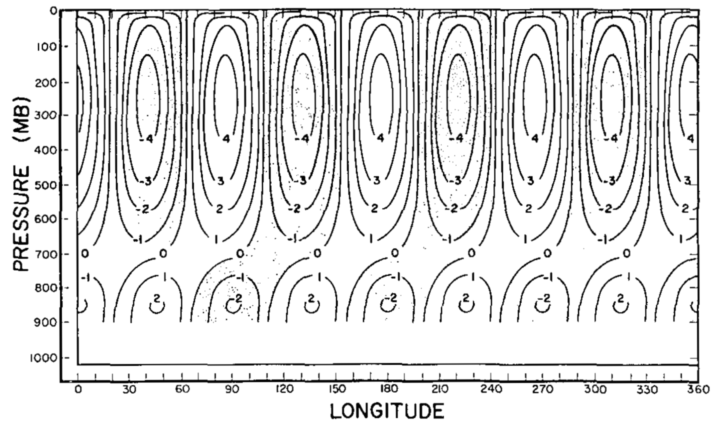

FIGURE 18.—Same as figure 15 for $Q_1^{(4,0)}$

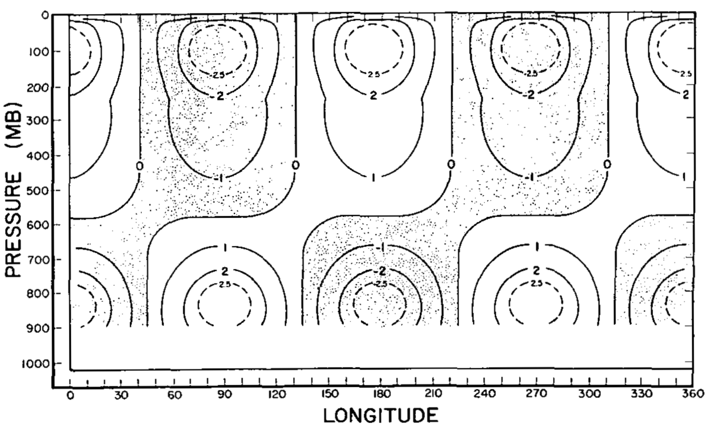

FIGURE 16.—Same as figure 15 for $Q_1^{(2,0)}$

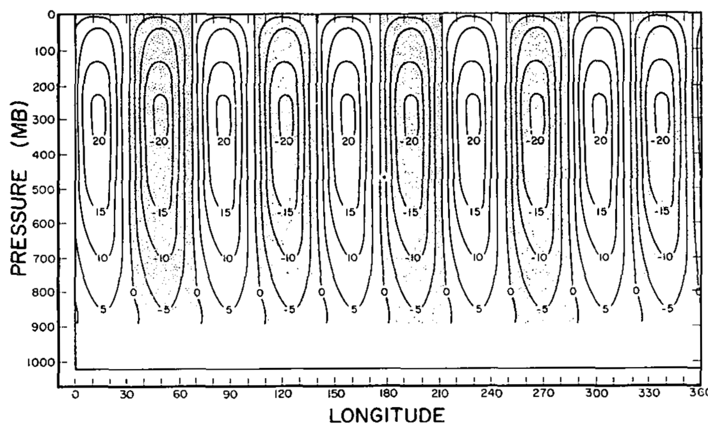

FIGURE 19.—Same as figure 15 for $Q_1^{(5,0)}$

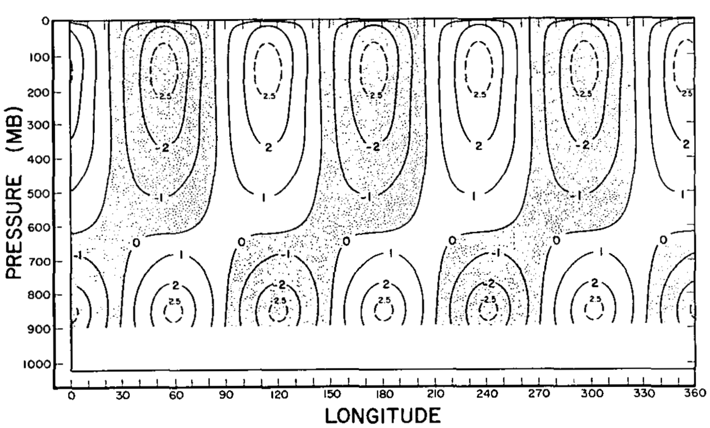

FIGURE 17.—Same as figure 15 for $Q_1^{(3,0)}$

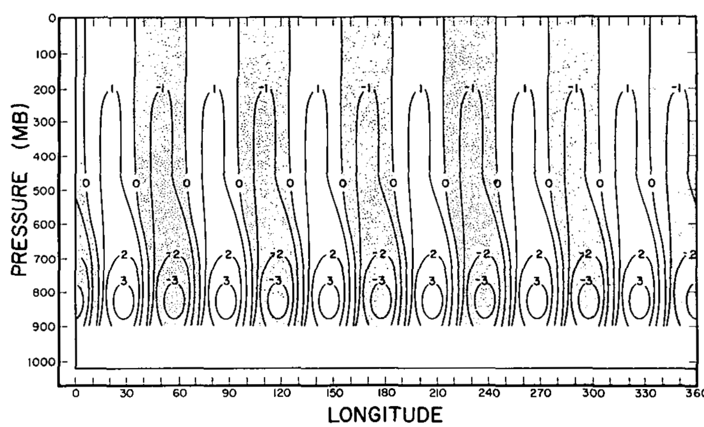

FIGURE 20.—Same as figure 15 for $Q_1^{(6,0)}$

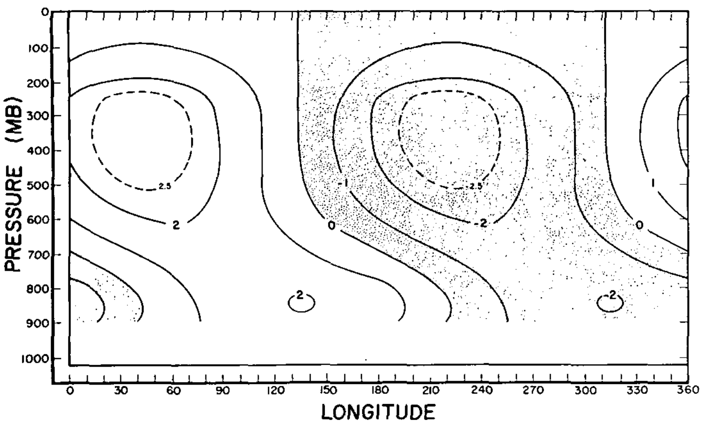

FIGURE 21.—Same as figure 15 for $Q_1^{(1,1)}$

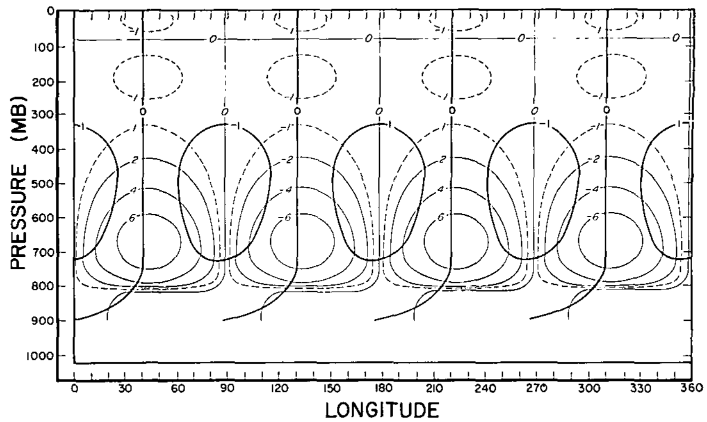

FIGURE 24.—Cross section along $y=L/2$ of ω_1 (heavy lines) in units of 10^{-1} gm. cm.$^{-1}$ sec. $^{-3}$, and T_1 (thin lines) in units of deg., corresponding to the solution for $Q_1^{(2,0)}$ shown in figure 16.

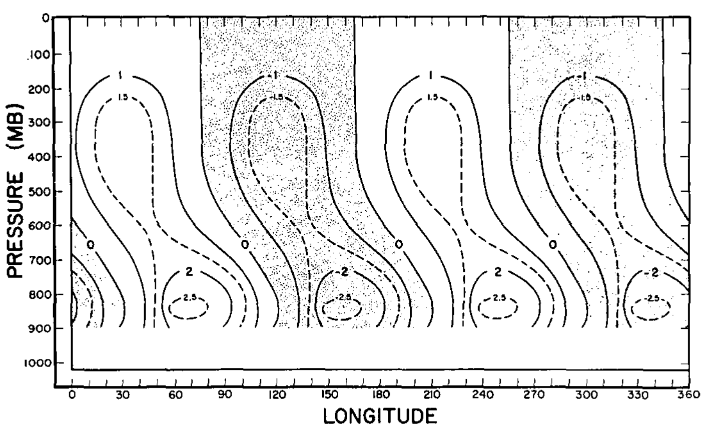

FIGURE 22.—Same as figure 15 for $Q_1^{(2,1)}$

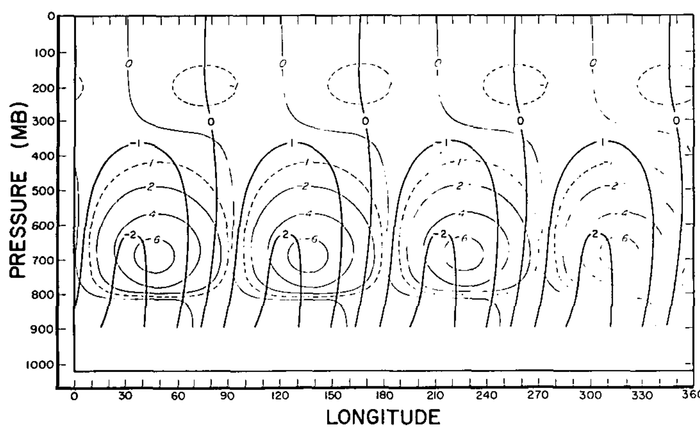

FIGURE 25.—Same as figure 24 for $Q_1^{(2,1)}$

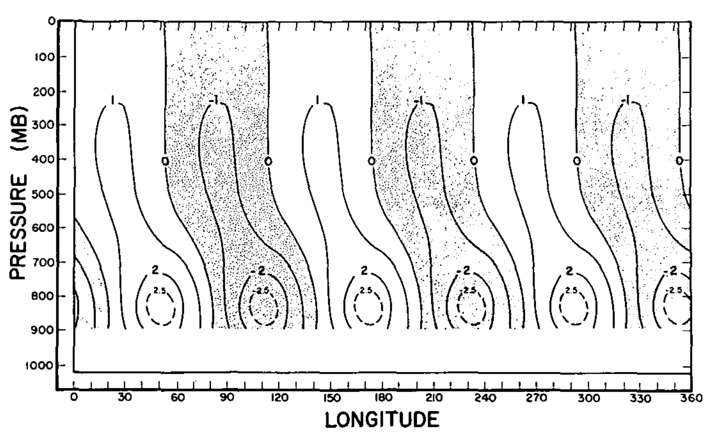

FIGURE 23.—Same as figure 15 for $Q_1^{(3,1)}$

These equations are a special case of the more general system of energy equations presented by Murakami [12].

In our study, we have been concerned with modelling the mean state along a single parallel of latitude (45° N.), and not with the properties of the solution far away from the given latitude. For this reason, we have not been concerned with assuring that the latitude band over which we made our Fourier expansion is energetically "closed." We are concerned, however, with the "contribution" which events near the given latitude circle can make to a closed system, on the assumption that conditions at this latitude circle are somewhat representative of average conditions over the whole atmosphere.

For the Fourier expansions used in the previous section, we have

$$u_1=0 \text{ at } y=\frac{L}{2} \qquad (45° \text{ N.}),$$

and we have taken

$$\hat{X}_1=\hat{Y}_1=0.$$

From the lower boundary condition, we have

$$\omega_{1\delta}=\omega_{1f}+\omega_{1h}$$

where

$$\omega_{1f}=-\rho_\delta g C\left(\frac{\partial v_1}{\partial x^*}-\frac{\partial u_1}{\partial y^*}\right)$$

$$\omega_{1h}\approx -\rho_\delta g u_{0\delta}\frac{\partial h}{\partial x^*}.$$

From the upper boundary condition, we have

$$\phi_{1T}=0.$$

With these relations, the kinetic energy equation (23) becomes

$$\frac{d}{dt}\{K_1\}=\left\{-\frac{R}{p}(\omega_1 T_1)_0\right\}+\frac{(\omega_{1f}\phi_{1\delta})_0}{p_T-p_\delta}+\frac{(\omega_{1h}\phi_{1\delta})_0}{p_T-p_\delta}=0. \qquad (23')$$

The first term on the right of (23′), which has a counterpart with opposite sign in (22), represents the rate of conversion of available potential energy into kinetic energy of the mean perturbations. The second term represents the rate of frictional dissipation of kinetic energy, and the third represents the rate of generation of perturbation kinetic energy due to the presence of mountains at the lower boundary. In (22), the first term represents the rate of gain of mean perturbation potential energy from the mean zonal potential energy, and the third term represents the rate of generation of mean perturbation potential energy due to diabatic processes and heat transfers by transient eddies of all frequencies.

If we make a double Fourier expansion of ϕ_1, T_1, and ω_1 in the manner of (11), denoting the corresponding complex Fourier coefficients by

$$\Phi_{m,n}=\Phi^{(r)}_{m,n}-i\Phi^{(i)}_{m,n}$$

$$B_{m,n}=B^{(r)}_{m,n}-iB^{(i)}_{m,n}$$

$$\Omega_{m,n}=\Omega^{(r)}_{m,n}-i\Omega^{(i)}_{m,n},$$

respectively, we can write the following transforms of the energy equation for the individual harmonics (m, n),

$$\frac{d}{dt}\{A_1\}_{m,n}=\{A_0\cdot A_1\}_{m,n}-\{A_1\cdot K_1\}_{m,n}+\{Q_1\cdot A_1\}_{m,n}=0 \qquad (24)$$

where

$$\{A_0\cdot A_1\}_{m,n}=\left\{\frac{2R}{p\Gamma_0}\frac{\partial T_0}{\partial y^*}[V^{(r)}B^{(r)}+V^{(i)}B^{(i)}]\right\}$$

$$\{A_1\cdot K_1\}_{m,n}=\left\{\frac{2R}{p}[\Omega^{(r)}B^{(r)}+\Omega^{(i)}B^{(i)}]\right\}$$

$$\{Q_1\cdot A_1\}_{m,n}=\left\{-\frac{2R}{p\Gamma_0}[\mathcal{Q}^{(r)}B^{(r)}+\mathcal{Q}^{(i)}B^{(i)}]\right\}$$

and

$$\frac{d}{dt}\{K_1\}_{m,n}=\{A_1\cdot K_1\}_{m,n}+\{D\}_{m,n}+\{h\cdot K_1\}_{m,n}=0 \qquad (25)$$

where

$$\{D\}_{m,n}=\frac{2}{p_T-p_\delta}[\Omega_f^{(r)}\Phi^{(r)}+\Omega_f^{(i)}\Phi^{(i)}]$$

$$\{h\cdot K_1\}_{m,n}=\frac{2}{p_T-p_\delta}[\Omega_h^{(r)}\Phi^{(r)}+\Omega_h^{(i)}\Phi^{(i)}].$$

From the geostrophic and hydrostatic relations, we have

$$V_{m,n}=\frac{2\pi m}{fK^*}i\Phi_{m,n} \qquad (26)$$

$$U_{m,n}=-\frac{2\pi n}{fL^*}i\Phi_{m,n} \qquad (27)$$

$$B_{m,n}=-\frac{p}{R}\frac{\partial\Phi_{m,n}}{\partial p} \qquad (28)$$

and from the energy equation, we have

$$\Omega_{m,n}=\frac{fu_0 p}{\Gamma_0 R}\frac{\partial V_{m,n}}{\partial p}-\frac{1}{\Gamma_0}\frac{\partial T_0}{\partial y^*}V_{m,n}+\frac{\mathcal{Q}_{m,n}}{\Gamma_0}. \qquad (29)$$

With the use of (26) and (27), we can write the lower boundary work terms representing the effects of friction and mountains as follows

$$\{D\}=\frac{2fc\rho g}{p_T-p_\delta}\left(1+\frac{n^2K^{*2}}{m^2L^{*2}}\right)[V^{(r)2}+V^{(i)2}]_\delta \qquad (30)$$

$$\{h\cdot K_1\}=\frac{2f\rho g u_{0\delta}}{p_T-p_\delta}[E^{(r)}V^{(r)}+E^{(i)}V^{(i)}]_\delta. \qquad (31)$$

It follows from (13), (15), (28), and (29) that for a single harmonic response, only to *airflow over mountains*, there can be no internal conversion of energy; i.e., $\{A_1\cdot K_1\}$ must vanish. Hence, there must be an exact balance between $\{D\}$ and $\{h\cdot K_1\}$. On the other hand, for a harmonic response only to *internal heating* we must have an exact balance between frictional dissipation $\{D\}$ and the conversion of available potential energy into kinetic energy $\{A_1\cdot K_1\}$. (By setting $\hat{X}_1=\hat{Y}_1=0$, we have excluded the transfer of kinetic energy between the transient perturbations and mean perturbations, a subject discussed theoretically by the writer [14].)

As an example, in table 4 we present the values of the energy conversion integrals at $y=L/2$ for $(m, n)=(2, 0)$ and (2, 1) for the internal heating solutions only. The possible importance of conversions on these scales has

TABLE 4.—*Energy integrals, evaluated along $y=L/2$, corresponding to the solutions for $Q_1^{(2,0)}$ and $Q_1^{(2,1)}$, in units of 10^{-3} ergs gm.$^{-1}$ sec.$^{-1}$*

Forcing Function	$\{A_0 \cdot A_1\}$	$\{A_1 \cdot K_1\}$	$\{Q_1 \cdot A_1\}$	$\{D\}$
$Q_1^{(2,0)}$	−111	75	186	−75
$Q_1^{(2,1)}$	1025	719	−306	−719

been discussed recently by Saltzman and Teweles [21]. It can be seen that the rates of dissipation and transformation of energy in the solution for (2, 1) are about an order of magnitude larger than in the (2, 0) solution, both being within the range of the observed rates of energy transformation in the wave number domain (e.g., Saltzman [16]). We also see that for the "ultra long wave" (2, 0), the temperature perturbation leads the pressure perturbation with a resulting negative value of $\{A_0 \cdot A_1\}$. This is in agreement with the results of Wiin-Nielsen [27].

If we combine the solutions for heating and mountains for a given harmonic, or, for example, if we combine two harmonics having the same value of m but different values of n (such as is implicit in some observational studies, e.g., Saltzman and Teweles [21]), we introduce the possibility for a form of "resonance" between the individual solutions which can lead to values of the quadratic energy integrals along $y=L/2$ which are much different from the simple sum of the integrals for the individual harmonic solutions. Thus it is possible that on one scale, the combination of fields of ω_1 and T_1 due to mountains and heating are such as to reinforce and give a large conversion of energy, and on another scale the effect may be the opposite.

6. DISCUSSION AND CONCLUSIONS

In trying to compare these theoretical results with real mean conditions in the atmosphere, we are confronted with serious difficulties due to factors falling into two main categories:

A. LACK OF KNOWLEDGE OF THE TRUE FORCING DISTRIBUTION

Despite recent observational studies (e.g., Clapp [6]), we must admit that the vertical structure and amplitudes and phases of the internal forcing functions are very poorly known. In addition, the mean surface heating due to airflow over the topographic features has not yet been determined accurately, and the assumption used here of a uniform surface zonal wind may be significantly in error. It would be worthwhile to make a serious observational study of $S_1^{(n)}$ using anemometer level wind data for the global station network, as well as to make refinements of such estimates of $Q_{1\delta}$ as have already been given (e.g., Budyko [3]).

B. SENSITIVITY OF THE SOLUTIONS TO THE GEOMETRICAL AND BASIC-STATE PARAMETERS

The artificiality of the β-plane geometry, and the choice of the dimension L^* in particular, makes it difficult to associate the harmonic solutions with the harmonics along parallels of latitude of the real atmospheric variables. In the extreme case, it is possible to introduce quasi-resonant conditions at zonal wave numbers lower than $m=5$ simply by increasing the dimension L^*. For this reason alone, we should be cautious about making comparisons with the real atmosphere.

In spite of these reservations, we should expect that the solutions obtained here do demonstrate at least the gross forms of the possible responses to forcing on different scales, the combination of which in the proper spherical geometry can account for the observations.

As an illustration of the possibilities, let us attempt to assign amplitudes and phases to the forcing functions studied here for $m=2$ (the scale of the major continent and ocean system), based on empirical data given by Peixoto et al. [13] and the Staff Members, Academia Sinica [24]. The forms taken are,

$$h_1(x, y)=\Lambda_1 \cos 2(kx-\epsilon) \cdot (1-\cos ly) \tag{32}$$

where

$$\Lambda_1=\frac{C_{2,0}}{2}=C_{2,1}=100 \text{ m.}$$

$$\epsilon=85° \text{ E. longitude}$$

and

$$Q_1(x, y, p)=\Lambda_2 A(p) \cos 2(kx-\theta) \cdot (1-\cos ly) \tag{33}$$

where

$$\Lambda_2=2|N_{2,0}|=4|N_{2,1}|=1.5\times10^{-5} \text{ deg. sec.}^{-1}$$

$$\theta=155° \text{ E. longitude}$$

and $A(p)$ is given in figure 14.

The solutions are given in figures 26 and 27, respectively, in the form of cross sections of v_1 along $y=L/2$ (45° N.). Comparison with the results of Teweles [26] and Julian [11] indicates that the heating solution in particular shows a good deal of agreement with the observations.

The possibility of achieving plausible solutions for the troposphere was demonstrated in the previous study by the writer [18], as well as by the studies of Smagorinsky [23] and Döös [8], for example. We view the results here as demonstrating the possibility for accounting also for the *stratospheric* mean state as a response to forcing processes within the troposphere.

In conclusion, we refer the reader to the list of general suggestions for the improvement and extension of the problem of the mean waves which were given in the last section of the previous study [18]. Our main concern here has been with item 4 of this list which called for improvement of the representation of the basic zonal state. Much work remains to be done on this as well as the other items. The recent work of Sankar-Rao [22], which is based on numerical rather than analytical methods, seems especially promising.

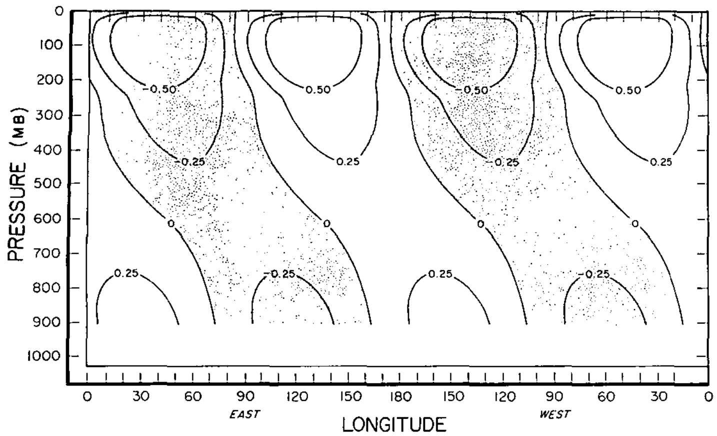

FIGURE 26.—Mean meridional wind response, v_1, along 45° N., to uniform airflow over the topographic representation, equation (32), in units of m. sec.$^{-1}$

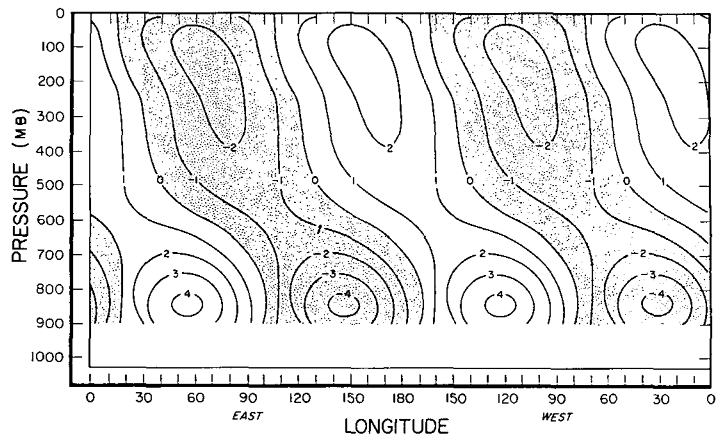

FIGURE 27.—Mean meridional wind response, v_1, along 45° N., to the internal heating function, equation (33), in units of m. sec.$^{-1}$

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