UDC 551.515.11

On the Theory of Atmospheric Development

T. J. SIMONS—*Colorado State University,*[1] *Fort Collins, Colo.*

ABSTRACT—The baroclinic instability problem is formulated as an initial value problem to evaluate the effects of the initial configuration of the wave perturbation. The vertical shape of the initial perturbation is found to be as important as its wavelength in determining the energy conversions during the early stages of its development. The general character of the solution of the initial value problem is compared with normal mode studies of baroclinic instability. It is concluded that the initial value formulation bridges the gap between hydrodynamic stability theory and synoptic studies of cyclone development in the atmosphere.

1. INTRODUCTION

Theoretical studies of cyclone development in the atmosphere generally approach the problem from the viewpoint of the instability of wavelike perturbations imbedded in a zonal current. The mathematical procedure follows the classical normal mode approach to the stability of hydrodynamic flow that dates back to investigations by Thomson, Rayleigh, and Helmholtz, at the end of the last century. The primary goal of such studies is to establish a criterion for instability by finding the conditions under which small periodic disturbances in the fluid display an exponential increase of amplitude with time. Although atmospheric stability studies posed in this fashion undoubtedly provide valuable information concerning the basic mechanisms of cyclone development in the atmosphere, it would seem that this approach tends to highlight one aspect of the problem at the expense of others. It is the purpose of this note to illustrate that, from a meteorological viewpoint, significant information may be gained by posing the atmospheric instability study as an initial value problem.

The theory of atmospheric development aims at increasing our understanding of the formation and evolution of cyclone perturbations in the atmosphere. Instability studies point at the general shearing properties of the baroclinic westerlies as the most likely cause of the growth of small disturbances into mature storms. Actual computations indicate that any zonal current derived from atmospheric data is unstable for a range of cyclone-scale waves, even if this zonal current is computed from climatological mean maps (see, e.g., Simons 1970). Since the major instabilities are found to be of the baroclinic type, we have here an explanation for the role that cyclones are observed to play in the cycle of energy conversion in the atmosphere, at least from a statistical viewpoint. On the other hand, if we look at cyclone development from a more synoptic viewpoint, the question arises as to what causes a particular atmospheric disturbance to develop into a mature cyclone. The answer that the zonal flow is unstable with respect to perturbations of this scale may not satisfy us since the atmosphere at any given time displays an abundance of large-scale eddies, many of which do not develop in spite of satisfying the same criterion. Although we cannot overlook such factors as thermal processes, effects of the earth's topography, and nonlinear interactions, it is shown here that the initial structure or configuration of an observable atmospheric perturbation is a primary agent in determining the short-term development of this disturbance. In other words, the initial value aspect of the atmospheric instability problem may be as important as the normal mode aspect if we want to bridge the gap between stability theory and the theory of cyclone development in the atmosphere.

In general, stability questions can be investigated without formulating the initial value problem, but it is often useful, if not mandatory, to have the latter well in mind. Case (1962) points out that the initial value approach can clarify certain ambiguities that may occur in normal mode solutions of classical hydrodynamic stability problems. With regard to the atmospheric instability question, we may refer to the recent interest in nonlinear studies on this subject. The initial value problem is an essential ingredient of such nonlinear studies, and it is by no means a straightforward matter in view of the multiple unstable modes and the time scales involved. Obviously, proper treatment of the nonlinear problem and interpretation of the results require a knowledge of the linear stability properties of the atmosphere on the same time scales. Thus, if the study is concerned with the instantaneous increase of perturbation kinetic energy as derived, for instance, from conversion of potential to kinetic energy, it should be realized that a distinction between unstable and neutral perturbations may be irrelevant.

2. ATMOSPHERIC MODEL

Most studies concerned with the dynamic instability of the atmosphere have followed the guidelines set by Charney (1947) and Eady (1949) who presented the first mathematical treatment of cyclone waves in terms of a continuous baroclinic atmosphere. For comparing

[1] Present affiliation: Canada Centre for Inland Waters, Burlington, Ontario.

the initial value approach with the normal mode approach to atmospheric instability, it is advisable to keep the model relatively simple. The present study is concerned with the baroclinic model of the atmosphere as formulated essentially by Charney (1947). Thus, the results obtained here are either explicitly or implicitly present in previous papers on atmospheric instability, but the interpretation is directed toward a better understanding of the relationship between the mathematical aspect of the stability problem and the physical problem at hand.

The time-scale of the problem justifies the use of an adiabatic, frictionless model. Considering a quasi-static, quasi-geostrophic atmosphere and using the beta-plane approximation, we obtain the following system of equations:

$$\left(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla\right)\nabla^2\psi+\beta_0\frac{\partial\psi}{\partial x}-f_0\frac{\partial\omega}{\partial p}=0 \tag{1}$$

and

$$\left(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla\right)\frac{\partial\psi}{\partial p}+\frac{\sigma}{f_0}\,\omega=0 \tag{2}$$

where ψ is the stream function, $\omega\equiv dp/dt$ is a measure of the vertical velocity, p is pressure, t is time, and $\mathbf{V}=\mathbf{k}\times\nabla\psi$ where $\mathbf{k}$ is the vertical unit vector. The Coriolis parameter, f_0, and its derivative, $\beta_0\equiv df/dy$, are treated as constants. The static stability, $\sigma=-(\alpha/\theta)\,\partial\theta/\partial p$ where α is specific volume and θ is potential temperature, is considered a function of pressure only to preserve the energy consistency of the system. The boundary conditions for this system of equations are

$$\omega=0 \text{ for } p=0 \text{ and } p=p_0=1000 \text{ mb}. \tag{3}$$

Since the flow field must be periodic in the longitudinal direction, we may write

$$\psi(x,y,p,t)=\bar{\psi}(y,p,t)+\psi'(x,y,p,t)$$

and

$$\omega(x,y,p,t)=\bar{\omega}(y,p,t)+\omega'(x,y,p,t) \tag{4}$$

where the bar denotes a zonal average and the prime denotes the deviation from this average, which is not necessarily small. Further, the deviation may be represented by a trigonometric series, the coefficients of which are functions of y, p, and t. Assuming that initially the perturbation on the zonal flow consists of one single wave of given zonal wavelength, we may write

$$\psi'(x,y,p,t)=\Psi(y,p,t)e^{ikx}+\Psi^*(y,p,t)e^{-ikx}$$

and

$$\omega'(x,y,p,t)=\Omega(y,p,t)e^{ikx}+\Omega^*(y,p,t)e^{-ikx} \tag{5}$$

where k is the wave number and the asterisk denotes the complex conjugate, such that ψ' and ω' are real functions.

After substitution of eq (4) into eq (1) and (2), the equations will involve products of the perturbations that cannot, however, contribute to the time rate of change of the wave of wave number k as long as eq (5) is satisfied. Thus, independently of the magnitude of the perturbation, the initial development of the wave of wave number k can be computed from the following equations:

$$\left(\frac{\partial}{\partial t}+ik\bar{u}\right)\left(k^2\Psi-\frac{\partial^2\Psi}{\partial y^2}\right)+ik\left(\beta_0-\frac{\partial^2\bar{u}}{\partial y^2}\right)\Psi+f_0\frac{\partial\Omega}{\partial p}=0 \tag{6}$$

and

$$\left(\frac{\partial}{\partial t}+ik\bar{u}\right)\frac{\partial\Psi}{\partial p}-ik\frac{\partial\bar{u}}{\partial p}\Psi+\frac{\sigma}{f_0}\,\Omega=0 \tag{7}$$

where

$$\bar{u}=-\frac{\partial\bar{\psi}}{\partial y}.$$

In general, these equations do not completely describe the system since the square of the initial perturbation will contribute toward a change of the zonal flow and will also generate a wave of twice the original wave number. Thereafter, the number of possible interactions increases continuously and a complete spectrum of waves is generated that implies that eq (6) and (7) will also involve products of perturbations. The most familiar effect of the nonlinear processes is a damping of exponentially growing disturbances as compared to results of linear theory. Thus, if the perturbation grows, the shear of the zonal wind decreases, which in turn reduces the growth of the perturbation. A rather complete treatment of the nonlinear system is presented elsewhere (Simons 1970). The present study, however, is confined to the purely baroclinic system arrived at by ignoring the y-dependence of the zonal wind and the initial perturbation. In that case, the nonlinear contributions from the initial perturbation disappear and eq (6) and (7) completely describe the system. The same still holds if the initial perturbation is periodic in y.

The system of eq (6) and (7) may be written in terms of the stream function by eliminating the vertical velocity. Specified for the baroclinic problem, the equation becomes

$$\left[k^2-f_0^2\frac{\partial}{\partial p}\left(\frac{1}{\sigma}\frac{\partial}{\partial p}\right)\right]\frac{\partial\Psi}{\partial t}$$
$$=ik\left[\beta_0-k^2\bar{u}-f_0^2\frac{d}{dp}\left(\frac{1}{\sigma}\frac{d\bar{u}}{dp}\right)\right]\Psi+ik\bar{u}f_0^2\frac{\partial}{\partial p}\left(\frac{1}{\sigma}\frac{\partial\Psi}{\partial p}\right) \tag{8}$$

where Ψ is now a function of t and p, and σ and $\bar{u}$ are functions of p only. The boundary conditions follow from eq (3) and (7); that is,

$$\frac{1}{\sigma}\frac{\partial}{\partial p}\left(\frac{\partial\Psi}{\partial t}\right)+\frac{ik}{\sigma}\left[\bar{u}\frac{\partial\Psi}{\partial p}-\frac{d\bar{u}}{dp}\Psi\right]=0 \text{ for } p=0,p_0. \tag{9}$$

Equations (8) and (9) specify the initial value problem to be discussed in the present paper.

3. METHOD OF SOLUTION

Analytical solutions of somewhat simpler but very similar initial value problems have been presented by Case (1960) and Pedlosky (1964). In general, it will be necessary to employ numerical techniques in the space domain and, in the case of nonlinear equations, also in the time domain. With regard to the former, it is most common to replace the space-derivatives by finite differences. Alternatively, one may apply the spectral technique whereby the dependent variables are represented by series of orthogonal functions with time-dependent coefficients. In either case, the governing equations will be reduced to a large set of ordinary differential equations in the time domain. A comparison of these techniques in the present case does not suggest that the spectral solutions are superior and, therefore, we restrict this discussion to the rather more convenient finite-difference methods.

To apply this technique to eq (8), we divide the interval $0 \geq p \geq p_0$ into N layers of depth $\Delta p = p_0/N$ each. Equation (8) is applied at the centers of these layers, the vertical derivatives are replaced by the usual centered differences, and the boundary conditions given in eq (9) are incorporated into the equations for the lower and upper layer. The result is a system of N equations for the variables $\Psi_n(t) = \Psi(t, p_n)$ where $p_n = p_0 - (n - \frac{1}{2})\,\Delta p$ and n ranges from 1 to N. The new variables are only time dependent, and the system can be written in matrix notation as follows:

$$\mathbf{B}\frac{d\mathbf{P}}{dt} = -ik\mathbf{A}\mathbf{P} \tag{10}$$

where $\mathbf{P}$ is the vector consisting of elements Ψ_n, and $\mathbf{A}$ and $\mathbf{B}$ are square matrices of order N. The matrix elements are real and they are, in the present system, determined completely by the basic state parameters and the wave number of the disturbance. Inverting the matrix $\mathbf{B}$, we obtain

$$\frac{d\mathbf{P}}{dt} = -ik\mathbf{B}^{-1}\mathbf{A}\mathbf{P}. \tag{11}$$

In the case of the general nonlinear problem, it would be necessary at this point to introduce a finite-difference approximation in the time domain since the matrix $\mathbf{A}$ would be time-dependent. The truncation error associated with such time extrapolations can be avoided here since the present linear system allows for an exact solution. The solutions to eq (11) are of the form $\exp(-ikct)$, and we see from eq (5) that c represents the phase speed of the perturbation; and if $c = c + ic_i$, then kc_i is its growth rate. By substituting the exponential solutions into eq (11), we obtain the eigenvalue problem

$$(\mathbf{B}^{-1}\mathbf{A} - c\mathbf{I})\mathbf{P} = 0 \tag{12}$$

where c is the eigenvalue and $\mathbf{I}$ is the unit matrix. Thus, in the first instance, the baroclinic initial value problem becomes an eigenvalue problem. The latter problem, however, represents only the first step of the initial value solution. Once the eigenvalues have been obtained, the eigenvectors have to be determined. The complete solution is

$$\mathbf{P}(t) = \mathbf{D}\mathbf{E}(t) \tag{13}$$

where $\mathbf{D}$ is the square matrix made up of the eigenvectors and $\mathbf{E}$ (t) is the vector consisting of the elements $e_n(t) = d_n \exp(-ikc_nt)$, $n = 1, 2, \ldots, N$. The N constants, d_n, finally are to be determined from the initial values of the perturbation in the N gridpoints. These constants follow immediately from eq (13) after inversion of the matrix $\mathbf{D}$ and evaluation of $\mathbf{P}$ and $\mathbf{E}$ at the initial time.

The outline of the solution of the baroclinic initial value problem has been presented here in some detail to contrast it with the conventional normal mode solution of the baroclinic stability problem. Thus, the usual method of determining the stability of perturbations as described by eq (8) and (9) is the following. Since the coefficients of eq (8) are independent of time, a solution, $\Psi \sim \exp(-ikct)$ can be assumed, and unstable solutions are obtained for complex values of the wave speed c. Actually, in that case, a growing and a decaying mode will exist simultaneously since complex phase speeds can be found only as complex conjugate pairs. Introducing the exponential solution into eq (8) and (9), we obtain

$$(c - \overline{u})\left[k^2 - f_0^2\,\frac{\partial}{\partial p}\left(\frac{1}{\sigma}\frac{\partial}{\partial p}\right)\right]\Psi + \left[\beta_0 - f_0^2\,\frac{d}{dp}\left(\frac{1}{\sigma}\frac{d\overline{u}}{dp}\right)\right]\Psi = 0 \tag{14}$$

with boundary conditions

$$\frac{1}{\sigma}\left[(c - \overline{u})\,\frac{\partial\Psi}{\partial p} + \frac{d\overline{u}}{dp}\,\Psi\right] = 0 \text{ for } p = 0, p_0. \tag{15}$$

This is the baroclinic eigenvalue problem.

Consider now the eigenvalue problem [eq (12)] corresponding to the baroclinic initial value problem as approximated by a multilayered model. The eigenvalues are determined by the condition that the solution be nontrivial; that is, the determinant $|\,\mathbf{B}^{-1}\mathbf{A} - c\,\mathbf{I}\,| = 0$. The eigenvalues are, therefore, simply the roots of the matrix $\mathbf{B}^{-1}\mathbf{A}$. Now, since the matrices are of order N, there will be N eigenvalues. Most of these eigenvalues are real and do not correspond to the normal mode solutions of the continuous eq (8). These real roots can be explained by noting that eq (12) would also be obtained if the space-derivatives in eq (14) were approximated by finite differences. The latter differential equation is singular, however, and a continuous spectrum of real eigenvalues is possible such that $c = \overline{u}$ at any point in the vertical. The spectrum of real eigenvalues of eq (12) apparently corresponds to the continuous spectrum of eq (14). While the continuous spectrum may be just a nuisance in the normal mode problem, it is seen from eq (13) to be an essential aspect of the initial value problem where it becomes necessary to determine the coefficients d_n from the initial values. In other words, the continuous spectrum is required to represent an arbitrary initial perturbation. A discussion of the general character of the eigenvectors corresponding to the eigenvalue problem [eq (12)] may be found in the next section.

Before proceeding to the solutions of the baroclinic initial value problem, we should investigate the truncation error in the space domain resulting from the replacement of eq (8) by (11). For that purpose, eq (8) and (9) may be solved exactly for the time-derivative of the perturbation in terms of the perturbation at a given time, and the solution may be compared with the values obtained from eq (11). For a given profile of the zonal wind, the stability, and the perturbation, eq (8) becomes a nonhomogeneous ordinary differential equation with pressure as the independent variable and the perturbation tendency as the dependent variable. The equation can be easily solved if the inverse static stability is represented by a linear function of pressure, which is a reasonable assumption as we will see in the next section. Now let the perturbation at a given time, together with the zonal wind, be represented by a power series in pressure. It follows then from eq (8) that the perturbation tendency will be made up of another power series plus the solution of the homogeneous equation. The homogeneous equation may be reduced to Bessel's equation, and the appropriate solution is Bessel's function of the first kind of order zero with the argument proportional to the square root of pressure. Solutions have been obtained for a synoptic scale wave and reasonable profiles of zonal wind and perturbations. Comparison of these solutions with solutions computed from eq (11) indicate that the error is of the order of one percent for a 20-layer model. This error must be expected to increase if the perturbation varies irregularly with height such as is observed for short and long waves. Similar calculations have been made by Wiin-Nielsen (1962) to estimate the truncation error of prediction models with low vertical resolution. Recently, this method of solution was used by Sanders (1971) to determine the initial displacement and intensification of an idealized cyclone disturbance.

Another approach to estimating the truncation error of finite-difference solutions consists of an evaluation of the convergence of the solutions as a function of the resolution. This procedure is usually followed in numerical instability studies to determine the accuracy of the eigenvalues corresponding to the unstable modes. We will return to this in the subsequent presentation of the results of the computations. It is, however, useful to mention at this point that a convergence of the normal mode solutions does not necessarily imply a convergence of the numerical solution of the initial value problem. Actual computations show that the contributions from the quasi-continuous spectrum in the layered models do not converge at the same rate as the unstable modes (Simons 1969). However, this is of interest only for long-term integrations in the absence of unstable modes. The present investigation is concerned with short-term integrations for which the convergence of the numerical solution is found to be satisfactory.

4. RESULTS OF NUMERICAL INTEGRATIONS

To put the initial value computations in perspective, we consider first some prominent results of barcolinic instability studies based on eq (14) and (15). This system

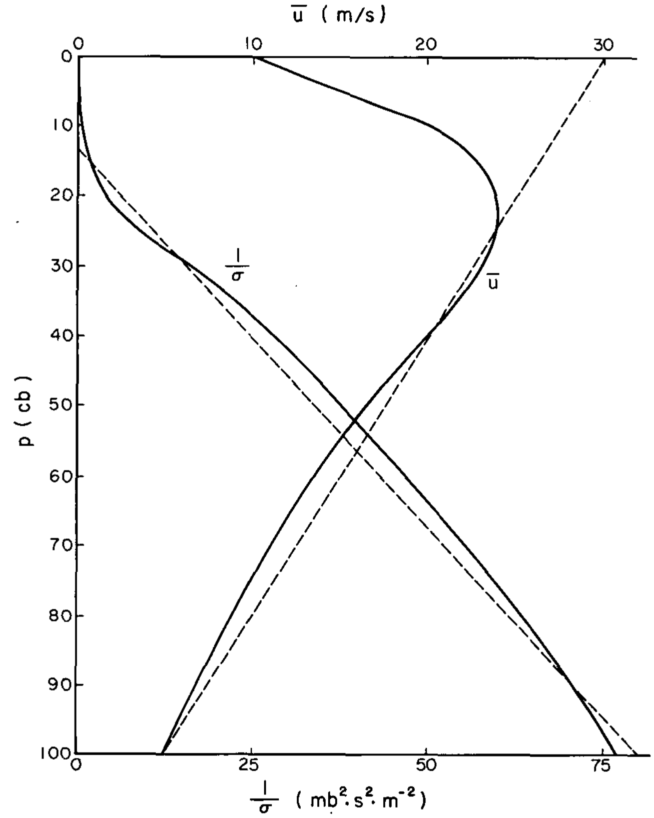

FIGURE 1.—Basic state zonal flow and inverse static stability. Profiles shown by solid lines were used in all calculations unless stated otherwise.

of equations has been solved numerically for the profiles of zonal wind and static stability shown in figure 1. The inverse static stability shown is obtained by averaging the values for summer and winter presented by Gates (1961). The zonal wind is a wind profile for middle latitudes obtained from climatological-mean January data. The dashed lines are linear approximations that have been used in certain calculations carried out for comparison, but the profiles shown by the solid lines have been used in all subsequent computations unless stated otherwise. The real and imaginary parts of the wave speeds of the unstable synoptic scale waves for this basic state are presented in figure 2. The imaginary parts have been multiplied by the wave number to obtain the growth rate, kc_i. Results are shown for numerical models with both low and high vertical resolution. The two-layer model shows the familiar shortwave cutoff to instability, which is markedly reduced in the three-layer model and tends to disappear for high vertical resolution. Weak instabilities are also found for long waves in models of sufficiently high resolution.

The vertical structure of the unstable modes is shown in figure 3A for wavelengths of 2000, 4000, and 6000 km. The relative amplitudes are, of course, arbitrary. The shorter disturbances are shallow waves without variations of phase with height. The longer waves develop in the upper atmosphere and tilt westward. The weakly unstable long waves consist of a lower level wave and an upper

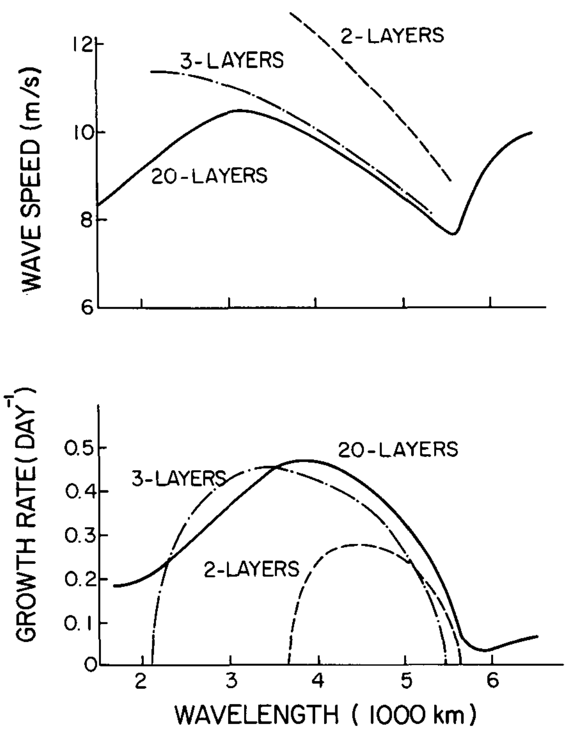

FIGURE 2.—Wave speeds and growth rates of unstable waves as a function of wavelength and vertical resolution of the numerical model.

level wave that are effectively uncoupled and are 180° out of phase. Figure 3A includes also the structure of the 4000-km wave for the linear zonal wind represented by the dashed line in figure 1. Whereas the zonal wind now increases with height in the stratosphere, the perturbation still decreases. This may be attributed to the correct representation of the static stability. The variation of inverse static stability shown in figure 1 together with the governing eq (8) indicate that the stratospheric motion must be nearly horizontal and nondivergent. This would suggest that the upper boundary condition may be applied at a lower level. The curve denoted by 4b in figure 3A represents the 4000-km disturbance if the static stability parameter assumes an infinitely large value above the 125-mb level as shown by the dashed stability curve of figure 1.

The general character of the continuous spectrum eigenvectors corresponding to the eigenvalue problem [eq (12)] is illustrated in figure 3B. Here we have chosen the model with linear profiles of zonal wind and inverse static stability where the singularity $c=\bar{u}$ occurs at only one level for a given root and the roots are equally spaced. The arrows in the figure indicate the location of the singularity where the eigenvector shows a discontinuous first derivative as expected from eq (14). The numerical model consists of 20 layers in the vertical, but the upper three layers are uncoupled ($1/\sigma=0$) which results in a total of 17 eigenvectors instead of 20. Fifteen of these eigenvectors correspond to real eigenvalues; the other two vectors are associated with the pair of complex conjugate roots. The amplitude profile of the latter has been included in figure 3B (see also figure 3A).

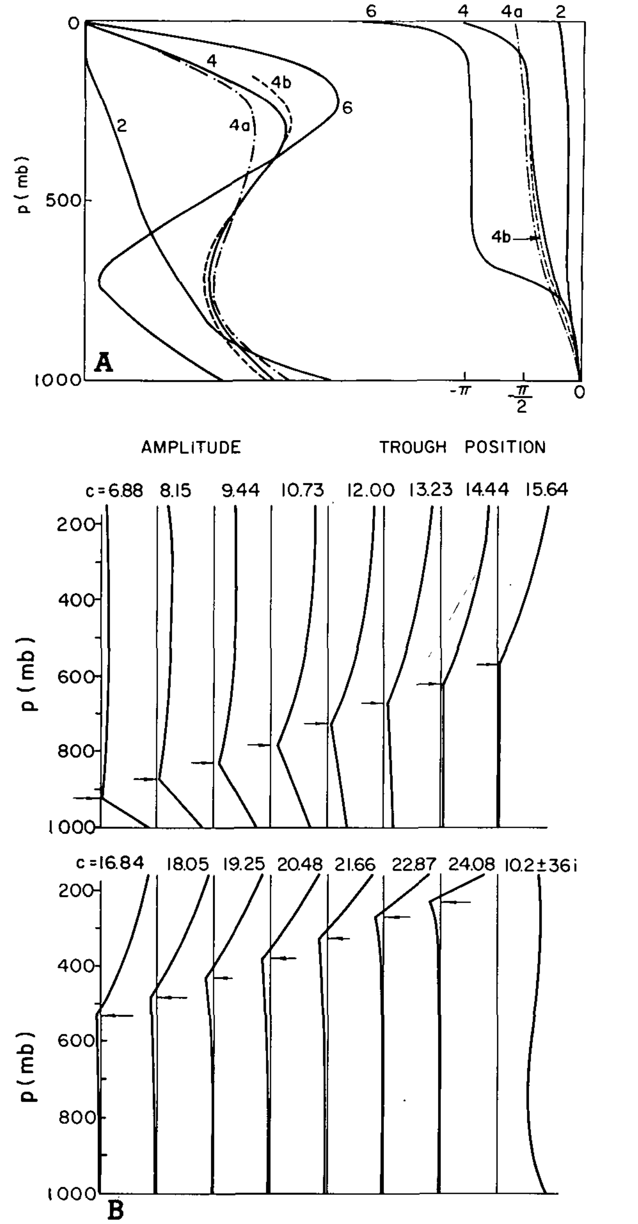

FIGURE 3.—(A) structure of unstable mode for wavelengths of 2000, 4000, and 6000 km (denoted by 2, 4, and 6, respectively). Curve 4a is for the dashed wind profile and 4b for the dashed static stability profile shown in figure 1. (B) eigenvectors for quasi-continuous spectrum and amplitude of normal mode eigenvector. Basic state parameters given by dashed lines of figure 1, wavelength L is 4000 km. Truncation of numerical model is at $N=20$, but upper three layers are uncoupled ($1/\sigma=0$), resulting in 17 eigenvectors.

The preceding review of the normal mode instabilities is restricted to those results which may serve as a background for the subsequent discussion. Far more elaborate computations have been presented in the meteorological literature. We will not attempt to include a complete listing of pertinent references here, but we should mention the recent papers by Hirota (1968) and Brown (1969) on numerical studies of the atmospheric stability problem.

Returning now to the initial value problem, we must interpret the results above with reference to the time scales of atmospheric cyclones. Clearly, then, the emphasis is on

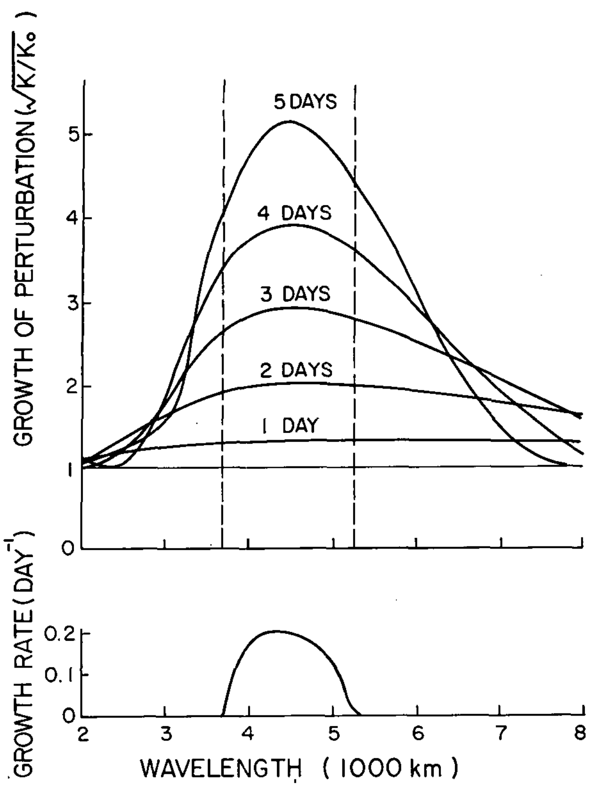

FIGURE 4.—Growth rate of unstable mode and actual growth of lower layer perturbation in a two-layer model. Values of zonal wind were taken from dashed curve of figure 1. Growth rate is measured by kc_i and actual growth by $\sqrt{K/K_0}$ where K is the perturbation kinetic energy.

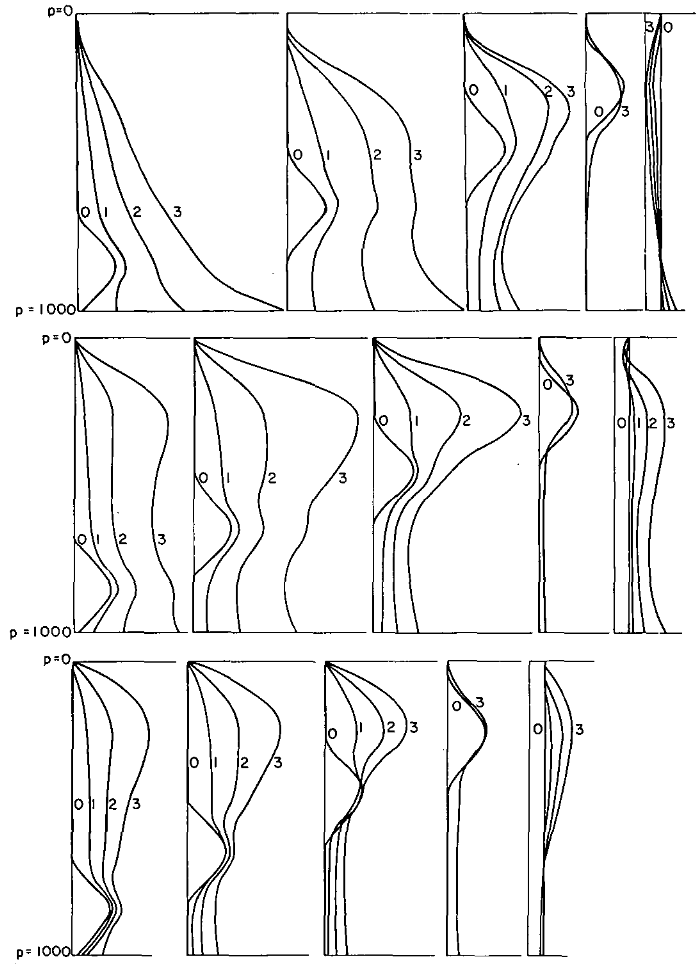

FIGURE 5.—Amplitudes of perturbations after 1, 2, and 3 days as a function of the initial vertical configuration (denoted by 0). From top to bottom the wavelengths are $L=2000$ km, $L=4000$ km, and $L=6000$ km.

the development of a perturbation during a relatively short time period, rather than the asymptotically different behavior of neutral and unstable waves. Figure 4 illustrates how the time scale enters into the instability problem. The lower part of figure 4 shows the normal mode instability in a two-layer model for the zonal wind shear corresponding to the dashed line of figure 1. The actual growth of any perturbation introduced in the lower layer can be read from the upper part of figure 4. Here the perturbation is measured by the square root of the vertically averaged kinetic energy of the wave. The pronounced distinction between neutral and unstable waves is not visible in the upper figure, quite similar to results obtained from nonlinear computations. The growth of the waves for real values of the wave speeds occurs because any perturbation is made up of more than one normal mode. The growth rates of the individual modes are equal to zero, but the different phase speeds cause the amplitude variations of the sum of the modes.

It may be of interest to recall here that unstable waves with large growth rates may actually lose kinetic energy if their initial vertical configuration is not favorable for development. A typical example is the case of the short waves in a model of high vertical resolution. The growth rates of these waves are sufficiently large as shown by figure 2. Nevertheless, the waves will not develop for a period of many days if the initial perturbations extend through the whole depth of the atmosphere. This was also indicated by Wiin-Nielsen's (1962) calculations of initial energy tendencies for a vertically continuous model.

It should be stressed that the preceding does not contradict the general information obtained from the normal mode studies. Thus, it is seen from figure 4 that these studies give an excellent indication of the most unstable waves, even for the time periods considered here. However, since any nonlinear study is essentially an initial value problem with finite initial perturbations, a meaningful evaluation of nonlinear effects can be made only if the inherent properties of the linear initial value problem are borne in mind.

The vertical shape of the initial perturbation was briefly mentioned. This is the most interesting aspect of the initial value formulation of the stability problem. The normal mode instabilities are determined completely by the basic state parameters together with the horizontal scale of the perturbation. The following discussion shows that the initial configuration of the perturbation is just as important, if not more important, in a complete study of cyclone development.

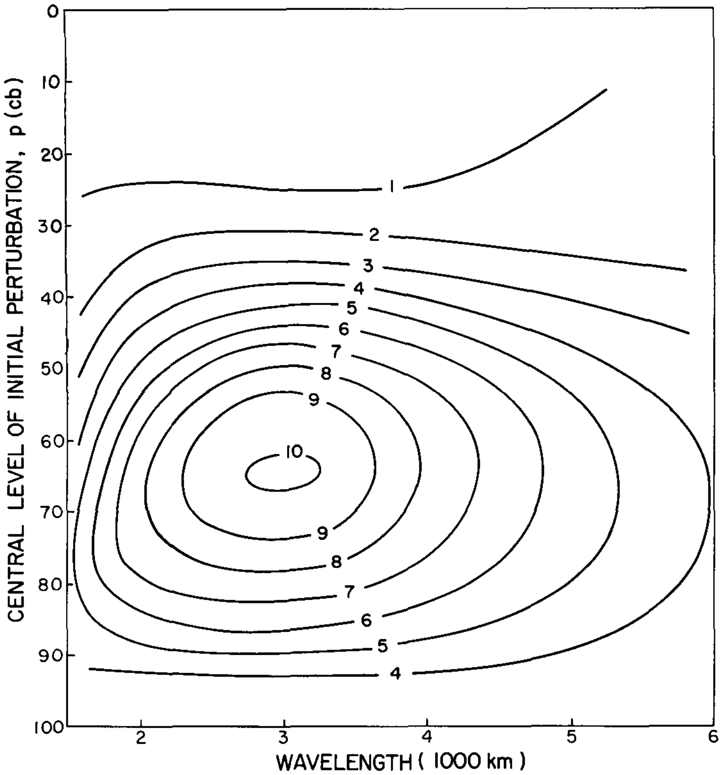

FIGURE 6.—Growth of wave perturbations after 3 days as a function of wavelength and initial perturbation level. Depths of initial perturbations are 300 mb as in figure 5. Growth is measured by $\sqrt{K_3/K_0}$ where K_n is the perturbation kinetic energy after n days.

Figure 5 shows the growth of a wave disturbance in the atmosphere as a function of the level where the perturbation is introduced. The initial perturbations shown here are rather shallow, but similar patterns of development were obtained for deeper perturbations. Also included is a perturbation extending initially through the whole depth of the atmosphere. All computations employed the basic state parameters shown by the solid lines in figure 1. Comparing figure 5 with the structure of the unstable mode shown in figure 3, we see that the quasi-continuous spectrum of neutral modes determines the shape of the perturbations for our time scales to a large extent.

The results of similar calculations for various wavelengths have been combined into figure 6 where the growth of a perturbation after 3 days is shown as a function of the wavelength and the level at which the perturbation is introduced. The initial shape of the perturbations is as shown in figure 5, and the growth is measured again by the square root of the kinetic energy of the perturbation. It is noteworthy that a wave perturbation at the jet stream level does not amplify, but a disturbance introduced at midlevels feeds immediately into the jet stream as seen from figure 5. It may also be noted that the short waves and the low-level disturbances develop strongly at the surface and therefore will be more sensitive to frictional effects. For that reason, one must expect the maximum in figure 6 to move upward and to the right for the actual atmosphere.

The vertical amplitude variation is only one aspect of the initial configuration of the disturbance, and one would expect its vertical tilt to be just as important. No attempt has been made to solve this complex problem except for the simple case of a two-parameter or two-layer model. For such models, exact solutions can be obtained for the amplitude ratio and the phase difference between the initial upper and lower waves that are most favorable for subsequent generation of perturbation kinetic energy. It is found (Simons 1970) that the energy conversion reaches a maximum if the amplitude of the initial lower level wave exceeds that of the upper level wave and if the upper wave is initially behind by a phase lag greater than $\pi/2$. This may be contrasted with the well-known solution for the structure of the unstable normal mode for such models where the upper wave perturbation is larger than the lower one and stays behind by less than $\pi/2$.

In the present context, we can also consider the recent work by Sanders (1971), who computed the instantaneous three-dimensional field of geopotential tendency for a model cyclone perturbation. In the notation of the present paper, Sanders' initial wave disturbance is of the form

$$\psi'(x,y,p)=\psi_s \cos k(x+l)+\psi_T\, Z(p) \cos kx$$

where ψ_s is the amplitude of the surface stream function perturbation, ψ_T is the wave amplitude of the "thermal wind" stream function, and l is the phase lag of the surface disturbance relative to the temperature wave. The pressure function $Z(p)$ describes the vertical structure of the geopotential field corresponding to a logarithmic vertical variation of the temperature field. The same function specifies the vertical profile of the zonal wind similar to that of the concepts of the two-parameter models. For this model, the maximum deepening of the storm center is found to occur if l equals one-quarter of a wavelength.

5. CONCLUSIONS

It is concluded that the atmospheric instability problem formulated as an initial value problem has many interesting aspects that have been only partly explored in the past. Since the nonlinear stability study is an initial value problem, the linear stability problem must be approached in the same manner to allow an evaluation of the nonlinear effects. The most interesting aspect of the baroclinic initial value problem is the initial configuration of the perturbation. The vertical shape of the initial perturbation has been found to be as important as its wavelength in determining its growth over a time period that is consistent with the time scale of meteorological development.

ACKNOWLEDGMENTS

The author thanks Ferdinand Baer and Desiraju B. Rao for their careful criticism of this paper.

This research was supported by the National Science Foundation, Grant No. GA-11637.

REFERENCES

Brown, John A., Jr., "A Numerical Investigation of Hydrodynamic Instability and Energy Conversion in a Quasi-Geostrophic Atmosphere: Part I," *Journal of the Atmospheric Sciences*, Vol. 26, No. 3, May **1969**, pp. 352–365.

Case, K. M., "Stability of Inviscid Plane Couette Flow," *The Physics of Fluids*, Vol. 3, No. 2, Mar.–Apr. **1960**, pp. 143–148.

Case, K. M., "Hydrodynamic Stability and the Initial Value Problem," in: *Hydrodynamic Instability, Proceedings of the Symposium in Applied Mathematics*, Vol. 13, American Mathematical Society, Providence, Rhode Island, **1962**, pp. 25–33.

Charney, Jule G., "The Dynamics of Long Waves in a Baroclinic Westerly Current," *Journal of Meteorology*, Vol. 4, No. 5, Oct. **1947**, pp. 135–162.

Eady, E. T., "Long Waves and Cyclone Waves," *Tellus*, Vol. 1, No. 3, Stockholm, Sweden, Aug. **1949**, pp. 33–52.

Gates, W. Lawrence, "Static Stability Measures in the Atmosphere," *Journal of Meteorology*, Vol. 18, No. 4, Aug. **1961**, pp. 526–533.

Hirota, Isamu, "On the Dynamics of Long and Ultra-Long Waves in a Baroclinic Zonal Current," *Journal of the Meteorological Society of Japan*, Vol. 46, No. 3, Tokyo, June **1968**, pp. 234–249.

Pedlosky, Joseph, "An Initial Value Problem in the Theory of Baroclinic Instability," *Tellus*, Vol. 16, No. 1, Stockholm, Sweden, Feb. **1964**, pp. 12–17.

Sanders, Frederick, "Analytic Solutions of the Nonlinear Omega and Vorticity Equations for a Structurally Simple Model of Disturbances in the Baroclinic Westerlies," *Monthly Weather Review*, Vol. 99, No. 5, May **1971**, pp. 393–407.

Simons, T. J., "Baroclinic Instability and Atmospheric Development," *Atmospheric Science Paper* No. 150, Colorado State University, Fort Collins, Oct. **1969**, 36 pp.

Simons, T. J., "The Nonlinear Dynamics of Cyclone Waves," *Atmospheric Science Paper* No. 162, Colorado State University, Fort Collins, June **1970**, 138 pp.

Wiin-Nielsen, Aksel, "On Truncation Errors Due to Vertical Differences in Various Numerical Prediction Models," *Tellus*, Vol. 14, No. 3, Stockholm, Sweden, Aug. **1962**, pp. 261–280.

[*Received May 7, 1971; revised August 6, 1971*]