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ELEMENTS

OF

GEOMETRY;

81

WITH THEIR

Application to the Menſuration of SUPERFICIES and SOLIDS,

TO THE

Determination of the MAXIMA and MINIMA of Geometrical Quantities,

117

RARE BOOK
QA
453
S6
1760

AND TO THE

Conſtruction of a great Variety of GEOMETRICAL PROBLEMS.

By *THOMAS SIMPSON*, F. R. S.
And Member of the Royal Academy of Sciences at STOCKHOLM.

The SECOND EDITION,
With large ALTERATIONS and ADDITIONS.

LONDON,
Printed for J. NOURSE at the *Lamb* oppoſite *Katherine-Street* in the *Strand*. M.DCC.LX.

1760

National Oceanic and Atmospheric Administration

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TO THE HONOURABLE

Charles Frederick, Esq;

Surveyor-General of his MAJESTY'S ORDNANCE, &c. &c. &c.

HONOURABLE SIR,

THE subject of the sheets which I here beg leave to lay before You, is of so much consequence to mankind, as justly to claim the regard and sanction of the Great. Geometry is, not only a most accurate, but a very extensive science, whose application and great utility, as well in the arts of peace as of war, are well known to You.

But though this work, if the manner in which it is executed be correspondent to the importance of the subject, may not want sufficient merit to render it worthy of the approbation of a Gentleman, who, amidst a multiplicity of publick employments, preserves an undiminish'd ardor for the sciences

and a knowledge of the works of art and nature; yet I have, Sir, ſtill farther motives for this addreſs: Your great influence and zeal to promote the good of an inſtitution under which I am placed; and the favours that I have received at your hands, make me earneſt to embrace this opportunity of teſtifying publickly, that I am,

HONOURABLE SIR,

With great reſpect,

Your much obliged,

and moſt obedient

humble ſervant,

Royal Academy,
March 3, 1760.

Thomas Simpſon.

PREFACE.

MY design in writing upon the subject of Geometry, was to open an easy way for young beginners to arrive at a proficiency in that useful science; without either being obliged to go thro' a number of unnecessary propositions, or having recourse to the ungeometrical *methods of demonstration, that abound in most modern compositions of this nature.*

The difficulty of the undertaking, I was not unapprised of; and objections occurred that were not easy to be removed: Nevertheless, I have grounds to hope, from the reception my first attempt has met with, that my endeavours have not been entirely unsuccessful. No pains have, indeed, been spared to render the work useful: And I flatter myself, that the spirit and rigour of demonstration, so essential to the subject, are also tolerably well preserved; though I have not been so intent to guard against the attack of Criticks, *as to lose sight of my main design of furnishing a plain, easy institution for learners: Yet I have strong hopes, that there will not be found in these sheets, any inaccuracies, or oversights, that are absolutely unpardonable. To expect a faultless piece is impossible: And I well know, that the most elaborate and best-approved systems of Geometry extant, are not without many imperfections. But, were the smallest imperfection to be a real fault, my ambition would rather be, to shew some degree of judgment, by avoiding a multitude of such faults, than by exposing and magnifying the flaws of other writers. It is more easy to see a fault, than to avoid one: And those men who are the most sanguine to distinguish themselves at the expence of others, are*

generally observed to stand in need of greater indulgencies, than even the persons whom they unmercifully attack. But I shall put an end to this digression by pointing out one objection, that may be brought against this work; which is, that in demonstrations admitting of several cases, the most easy ones are sometimes omitted; and that the converse of some propositions is not at all demonstrated. But this, I conceive, will be found a real advantage to the learner; without which, it would have been impossible to have comprised the Elements in the compass they now take up. Besides, the greatest part of the demonstrations omitted being such as may be inferred from those given, by means of Axioms only; they may, therefore, be easily supplied by any reader, should they happen to become necessary, which I have scarce ever found to be the case. But, even allowing this to be a defect, it is abundantly compensated by the extensive application given in the three last sections; which is infinitely more useful, in itself, and more necessary to the forming an able Geometrician, *than any thing of the kind we have been speaking of.*

In this, second, edition (which is, in a manner, a new work) many considerable alterations and additions have been made. The order of some of the first propositions is changed: And some difficult propositions in the second book are rendered more plain. In the fourth book several new Theorems on proportions are added. The solid Geometry is now connected with the plane, *and is demonstrated with the same accuracy. The mensuration of Superficies and Solids is also more explicitly handled; and the demonstration of the several rules is here established on a better foundation, than even in authors who have wrote professedly on the subject. The* Maxima *and* Minima, *and the construction of Geometrical Problems, are likewise considerably extended and improved. And, at the end,*

Notes

Notes geometrical and critical, very useful to improve the judgment of young students, are now added.

But, whilst I am talking of improvements and matters of criticism, I am called upon to answer to a charge, which, should it appear to deserve credit, would indeed leave me but little room to pass myself upon the world for a judge in these matters. As the gentleman by whom I stand accused, is known to the world by his holding one of the most considerable mathematical posts in the kingdom; I shall, in order to do all due honour to the manner and importance of his writing, give you his own words.

" *There has lately been published a book under the* " *title of Elements of Plane Geometry, designed for* " *the use of schools, which is an incorrect copy of the* " *first eight sections of this work, lent the pretended* " *author on a particular occasion, and printed in a* " *spurious manner, without my knowledge or consent;* " *an action too scandalous for any man of honour to* " *be guilty of. The Editor imagined, I suppose, that* " *the changing some propositions, and mangling the* " *demonstrations of others, was a sufficient disguise* " *to make it pass for his own performance; but how* " *far this will justify such a piece of pyracy, must be* " *left to the judgment of the publick.*"

Were I to attempt to describe the ideas excited in my mind by the singular modesty *of this important and solemn appeal to the publick, I should be at a loss for fit words to express them, without transgressing the bounds of decency. But I hope that I have not deserved so ill of the publick, to be thought capable of acting so very humble a part, as that of copying from* this *author, and of mangling* his *demonstrations, in order to make them pass for* my own.—*That a manuscript of his (containing between* 20 *and* 30 *of the principal*

principal Theorems in Geometry, extremely ill digested) came into my hands, is indeed true; but it was not lent me, *but forced upon me, by himself (the very first night after my removal to* Woolwich) *in virtue of an* article *in the original rules and instructions for the Academy; whereby it is ordered, that the second master shall teach Geometry under the direction of the first master. But this well-intended* article, *which has been made subservient to the purposes of ignorant tyranny, and daring calumny, has since, in consequence of a publick examination, been annulled by an express order of the Master-General of the Ordnance.—I could mention some particulars, supported by good authority, that occurred in the course of that examination, which would but ill agree with the importance he assumes in his confident accusation; but I do not think it worth while: This Gentleman has, himself, by his different publications, so well convinced the world of his abilities, as to render any farther comment on that head intirely unnecessary and ineffectual.*

ERRATA.

~~Page 19. l. 4. for *n*, read *m*; p. 47. l. 23. for CAE, *r.* DAE; p. 48. l. 31. *r.* semi diameters; p. 58. for Corollary 2. *r.* Corollary~~; p. 59. l. 12. *dele* the comma; p. 70. l. 15. for DE, *r.* DF; ~~l. 31. for greater, *r.* greater than; p. 71. l. last, for PT, *r.* QT; p. 101. l. 6. for 3, *r.* 4; p. 110. in the quotation l, for 19, *r.* 18~~; p. 124. l. 8. *dele* KF; p. 127. *l.* 4. *dele* the whole parenthesis; and in the quotation [e], *r.* 27. 4. p. 135. l. 25. after BD, *r.* as well as AD; p. 140. l. 14. for GEA, *r.* GEB; p. 154. beg. of l. 16. for CE, *r.* CD; p. 174. l. 25. for OHI, *r.* OHK; l. 34. *r.* annulus; p. 187. l. 21. *r.* frustum's height; p. 221. l. 20. for AG, *r.* AF; p. 226. l. prec. last, for exceeds, *r.* is less than; p. 248. l. 12. for less, *r.* greater; p. 267. l. 27. dele *not*; p. 271. l. 32. for which &c. *r.* when taken less.

ADVER-

ADVERTISEMENT.

AS in every work of this nature, designed to contain whatever may be most requisite to the forming of a regular and complete system of Geometry, a number of propositions must necessarily have a place, whose chief use and application lie in the higher branches of the Mathematicks; and there being many persons, particularly young gentlemen in publick schools, who want to learn so much Geometry only, as is necessary to give them a proper introduction into the practical and most common applications thereof; such as Mensuration, Trigonometry, Navigation, Fortification, Perspective, *&c.* For these reasons, I thought that it might be of service, to point out to such Readers, what propositions in these Elements may be omitted, as least useful to them; without either hurting the connection, or taking away from the evidence of the other demonstrations. The numbers of these propositions, in the several books, are as follow.

In Book I. the 6, 17, 19, 21, 22, 23, and 29th.

In Book II. the 4, 5, 10, 11, 12, 13th, and the 2d Corol. to the 9th.

In Book III. the 4, 5, 6, 7, 8, 9, 15, 18, 19, 20, 25, 26, 27, and 28th.

In Book IV. the 4, 5, 6, 9, 11, 13, 16, 17, 20, 21, 22, 23, 25, 26, 27, 28, and 29th.

In

In Book V. the 1, 2, 16, 17, 18, 19, 20, 25, 26, 28, and 31ſt.

In Book VI. the two or three firſt propoſitions only, need be read; except by thoſe who are concerned in ſurveying and dividing of lands; to whom the whole book will be highly uſeful.

Alſo, with regard to the ſeventh book, if *Perſpective* be the only application in view (which I have known frequently to be the caſe) the 1ſt, 2d, 4th, and 12th propoſitions may ſuffice. But if a more general idea of the properties of interſecting planes ſhould be required, ſuch as is neceſſary in the doctrine of ſolids and ſpheric geometry; then all the propoſitions, to the 12th, ought to be taken.

The 17th, 19th, 20th, 21ſt, 22d, and 23d propoſitions of this ſeventh book ſhould alſo be read by thoſe who would be able to find the content and proportion of ſolid bodies; as ſhould, likewiſe, the whole eighth book; except, perhaps, the firſt and ninth propoſitions, together with the three firſt lemmas; which may be thought too plain, by thoſe who are not very ſolicitous about geometrical rigour, to need a demonſtration.

An

An INDEX *or* TABLE *referring to the places in these Elements, where all the most material propositions in the first six, and in the eleventh and twelfth books of* Euclid, *are demonstrated.*

Euclid.	These El.	*Euclid.*	These El.	*Euclid.*	These El.	*Euclid.*	These El.
Book I.	P. B.	B. II.	P. B.	B. V.	P. B.	B. XI.	P. B.
Prop. 5	12. 1	P. 11	19. 5	P. 16	2. 4	Pr. 5	Cor. to 2. 7
6	18. 1	12	10. 2	17	3. 4	6	4. 7
8	14. 1	13		18		8	5. 7
9	5. 5	14	9. 6	19		9	8. 7
10	6. 5	B. III.	P. B.	22	5. 4	10	9. 7
11	3. 5	Pr. 3	2. 3	24	6. 4	12	3. 7
12	4. 5	7	5. 3	25	3. 4	14	7. 7
13	1. 1	8		B. VI.	P. B.	15	10. 7
15	3. 1	11	8. 3	Pr. 1	7. 4	16	11. 7
18	13. 1	12	7. 3	2	12. 4	18	6. 7
19		14	3. 3	3	18. 4	19	13. 7
21	23. 1	15	4. 3	4	14. 4	24	16. 7
22	8. 5	17	21. 5	5	17. 4	25	21. 7
23	7. 5	20	10. 3	6	15. 4	28	17, 19 and 20 of the 7th, and Cor. 3. to 8. 8.
24	21. 1	21	11. 3	7	16. 4	29	
26	15. 1	22	17. 3	8	19. 4	30	
27	8. 1	25	18. 3	9	11. 5	31	
28		31	16. 3	10	15. 5	32	21. 7
29	7. 1	32	14. 3	11	12. 5	33	9. 8
31	9. 5	33	22. 5	12	13. 5	34	23. 7
32	9. 1	34	25. 5	13	14. 5	35	14. 7
	10. 1	35	21. 3	14	Cor. to 25. 4	37	14. 7
33	26. 1	36	22. 3	15			25. 7
34	24. 1	37		16	10. 4	B. XII.	P. B.
35	2. 2	B. IV.	P. B.	17		Pr. 2	3. 8
36	Cor. to 2. 2	Pr. 2	25. 3	18	11. 6	5	Cor. 4 to 8. 8
37		3	26. 5	19	24. 4	6	
38		4	24. 5	20	26. 4	7	7. 8
41		5	23. 5	22	28. 4	8	9. 8
42	6. 6	10	28. 5	23	25. 4	9	Cor. 3, 4 and 6, to 8. 8
43	3. 2	11		25	13. 6	10	
44	6. 6	12	30. 5	27	17. 5	11	
45	7. 6	15	29. 5	28		12	
46	10. 5	B. V.	P. B.	29	18. 5	13	
47	8. 2	4	4. 4	30	19. 5	14	
B. II.	P. B.	12	6. 4	31	29. 4	15	
Pr. 1	5. 2	15	1. 4	B. XI.	P. B.	18	Cor. to 11. 8
4	6. 2			Pr. 3	1. 7		
5	7. 2			4	2. 7		

The following BOOKS are all written by Mr. THOMAS SIMPSON, F. R. S. and printed for J. NOURSE.

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ELEMENTS

OF

GEOMETRY.

BOOK I.

DEFINITIONS.

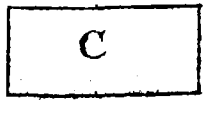
1. GEOMETRY is that ſcience, by which we compare ſuch quantities together as have extenſion.

Extenſion is diſtinguiſhed into length, breadth, and thickneſs.

2. A Line is that, which has length without breadth.

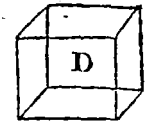
The terms, bounds, or extremes of a Line, are points.

3. A Surface is that, which has length and breadth, only, as C.



The bounds of a Surface are lines.

4. A Solid is that, which has length, breadth, and thickneſs, as D.

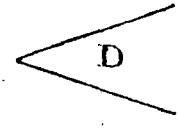


The bounds of a Solid are ſurfaces.

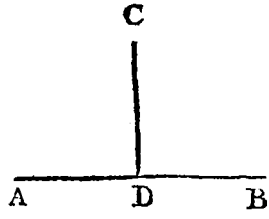
5. A Right (or ſtrait) line is that, which lies evenly between its extremes, or which every-where tends the ſame way, as AB. A——————B

6. A Plane-ſurface is that, which is every-where perfectly flat and even, or which touches, in every part, any right-line extended between points anywhere taken in that ſurface.

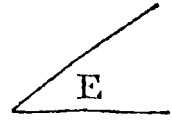
7. An Angle is the inclination, or opening of two right-lines meeting in a point, as D.



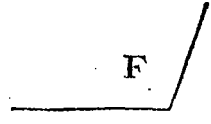
8. When one right-line DC, ſtanding upon another AB, makes the angles on both ſides equal, thoſe angles are called right-angles; and that line CD is ſaid to be perpendicular to the other AB on which it inſiſts.



9. An Acute-angle is that, which is leſs than a right-angle, as E.



10. An Obtuſe-angle is that, which is greater than a right-angle, as F.

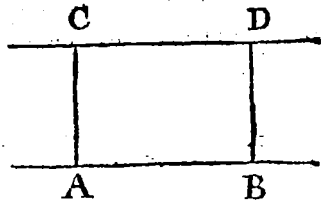


11. The

11. The diſtance of two points, is the Right-line reaching from the one to the other.

12. The diſtance of a point from a line, is a Right-line drawn from that point, perpendicular to, and terminating in, the line given.

13. Parallel (or equidiſtant) right-lines AB, CD are ſuch, which being in the ſame plane-ſurface, if infinitely produced, would never meet.

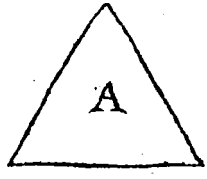


14. A Figure is a bounded ſpace, and is either a ſurface, or a ſolid.

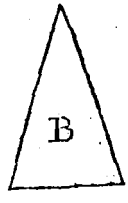
15. A right-lined plane Figure is that, formed in a plane-ſurface, whoſe terms, or bounds, are right-lines.

16. All plane Figures bounded by three right-lines, are called Triangles.

17. An equilateral Triangle is that whoſe bounds or ſides are all equal; as A.

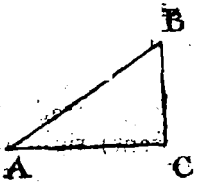


18. An iſoſceles Triangle is, when two ſides are equal, as B.



19. A ſcalene Triangle is, when all the three ſides are unequal, as C.

20. A right-angled Triangle is that, which has one right-angle, as ACB; whereof the ſide AB oppoſite to the right-angle, is called the Hypothenuſe.



21. An obtuſe-angled Triangle is that, which has one obtuſe angle.

22. An acute-angled Triangle is that, which has all its angles acute.

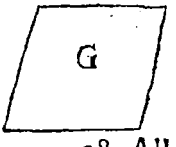
23. Every plane Figure bounded by four right-lines, is called a Quadrangle, or Quadrilateral.

24. Any Quadrangle, whoſe oppoſite ſides are parallel, is called a Parallelogram, as D.

25. A Parallelogram, whoſe angles are all right-ones, is called a Rectangle, as E.

26. A Square is a parallelogram whoſe ſides are all equal, and its angles all right-ones, as F.

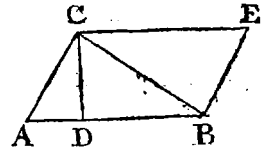
27. A Rhombus is a parallelogram whoſe ſides are all equal, but its angles not right, as G.



28. All

28. All other four-ſided figures, beſides theſe, are called Trapeziums.

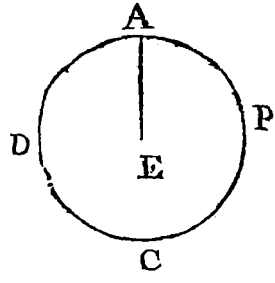
29. A right-line joining any two oppoſite angles of a four-ſided figure, is called a Diagonal.



30. That ſide AB upon which any parallelogram ACEB, or triangle ACB is ſuppoſed to ſtand, is called the baſe; and the perpendicular CD falling thereon from the oppoſite angle C, is called the altitude of the parallelogram, or triangle.

31. All plane figures contained under more than four ſides, are called Polygons; whereof thoſe having five ſides, are called Pentagons; thoſe having ſix ſides, Hexagons; and ſo on.

32. A Regular Polygon is one whoſe angles, as well as ſides, are all equal.



33. A Circle is a plane figure, bounded by one curve-line APCD, called its circumference, everywhere equally diſtant from a point E within the circle, called the center thereof.

34. The Radius of a circle, is the diſtance of the center from the circumference, or a right-line EA drawn from the center, to the circumference.

AXIOMS, *or* Self-evident Truths.

1. Things, equal to one and the ſame thing, are alſo equal to each other.

2. Every whole is greater than its part.

3. Every whole is equal to all its parts taken together.

4. If to equal things, equal things be added, the wholes will be equal.

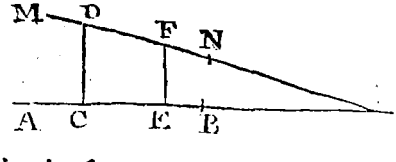
5. If from equal things, equal things be taken away, the remainders will be equal.

6. If to, or from unequal things, equal things be added, or taken away, the ſums, or remainders, will have the ſame difference, as the unequal things firſt propoſed.

7. All right-angles are equal to one another.

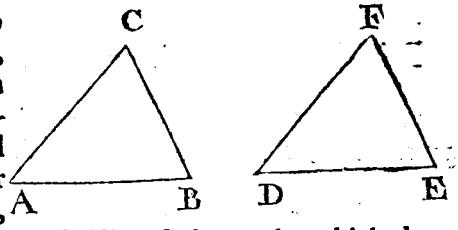
8. More than one right-line cannot be drawn from one given point A to another given point B.

9. If two points D, F, in a right-line MN, are poſited at unequal diſtances DC, FE, from another right-line AB in the ſame plane-ſurface; thoſe two lines, being infinitely produced, on the ſide of the leaſt diſtance EF, will meet each other.



10. If

10. If two right-lines CA, CB, making an angle C, be respectively equal to two other right-lines FD, FE, making an angle F, and the angles which they make C, and F be likewise equal; the right-lines AB, DE joining their extremes will be equal, and the two triangles ACB, DFE equal in all respects.



If this should not appear sufficiently evident for an axiom; conceive the triangle DFE to be removed, and so applied to the triangle ABC, that the point F may coincide with C, and the side FD fall upon the side CA; then, because FD is supposed equal to CA, the point D will also fall upon A. And, the angle F being equal to the angle C, the side FE will fall upon CB; and consequently the point E upon the point B, because FE is supposed equal to CB. Therefore, seeing all the bounds of the two triangles coincide, it is manifest, that not only the bases AB, DE, but the angles opposite to the equal sides, are also equal.

When all the four lines CA, CB, FD, FE are equal; the triangle DFE, being *contrariwise* applied to ACD so that FE may coincide with CA, will, *also*, agree with the triangle ACB (as is manifest from the reasoning above); and so, the angle E (as D did before) now coinciding with the angle A, the two angles E and D must necessarily be equal to each other, in this case, where the triangle DFE is an isosceles one.

POSTULATES, *or* PETITIONS.

1. That, from any given point, to any other given point, a right-line may be drawn.

2. That, a right-line may be produced, or continued out, at pleasure.

3. That, from any point as a center, with a radius equal to any right-line assigned, a circle may be described.

4. That, a right-line may be drawn perpendicular to another, at any point assigned; and that it is also possible for to make a right-line, or a right-lined angle, equal to any right-line, or right-lined angle assigned, or to the half thereof.

This fourth Postulate *is added, more for the sake of making the proper references, than through absolute necessity: since, what is here barely assumed as possible, is effected, and actually demonstrated, in the beginning of the Fifth Book, intirely independent of every thing but* Axioms *and the other Postulates, above laid down. It may also be proper to note here, that, though these* Postulates *are not always quoted, it will be easy to perceive where, and in what sense, they are to be understood.*

NOTES *and* OBSERVATIONS, *with the significations of Signs used in this Tract.*

A PROPOSITION is, when something is, either, proposed to be done, or to be demonstrated, and is either a Problem, or a Theorem.

A PROBLEM is, when something is proposed to be done.

A THEO-

A THEOREM is, when something is proposed to be demonstrated.

A LEMMA is, when some premise is demonstrated, in order to render the thing in hand the more easy.

A COROLLARY is, a consequent truth, gained from some preceding truth, or demonstration.

A SCHOLIUM is, when remarks and observations are made upon something going before.

The signification of SIGNS.

The sign =, denotes that the quantities betwixt which it stands, are equal.

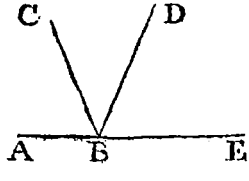
The sign >, denotes that the quantity preceding it, is greater than that which comes after it.

The sign <, denotes that the quantity preceding it, is less than that which comes after it.

The sign +, denotes that the quantity which it precedes, is to be added.

The sign —, denotes that the quantity which it precedes, is to be taken away or subtracted.

A figure, or number, prefixed to any quantity, shews how often that quantity is to be taken, or repeated; as 5A shews, that the quantity represented by A, is to be taken 5 times.



When several angles are formed about the same point (as at B), each particular angle is described by three letters, whereof the middle one shews the angular point, and the other two, the lines that form the angle: thus CBD or DBC signifies the angle formed by the lines CB and DB.

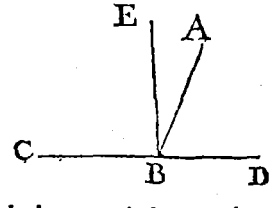
When,

When, in any demonstration, you meet with several quantities joined the one to the other continually by the mark of equality (=), the conclusion drawn from thence, is always gathered from the first and last of them; which are equal to each other, by virtue of the first axiom. Thus if A = B = C = D, then will the first (A) and the last (D) be equal to each other.

Also, when in the quotations you meet with two numbers, the first shews the proposition, and the second the book. Moreover, Ax. denotes axiom; Post. postulatum; Def. definition; Hyp. hypothesis. Note also, that, when-ever the word *Line* occurs, without the addition of either *right*, or *curved*, a right-line is always understood: and that, when a line is said to be drawn to, or from an angle, the angular point is meant.

THEOREM I.

A line (AB) *standing upon another line* (CD) *makes with it two angles* (ABC, ABD) *which, taken together, are equal to two right-angles.*



If the angles ABC, ABD are equal, it is plain they make two right-angles [a]; if unequal, let BE be perpendicular to CD [b], dividing the greater of them (ABC) into the parts EBC, EBA; then the former part EBC being a right-angle [a], and the remaining part EBA together with the whole lesser angle ABD, equal to another right-angle EBD [c]; the whole, of both the proposed angles, taken together, must necessarily be equal to two right-angles [d].

[a] Def. 8.
[b] Post. 4.
[c] Ax. 3.
[d] Ax. 4.

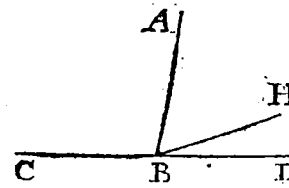
COROL-

COROLLARY.

Hence all the angles at the same point (B) on the same side of a right-line (CD) are equal to two right-angles [c]. [c] Ax. 3.

THEOREM II.

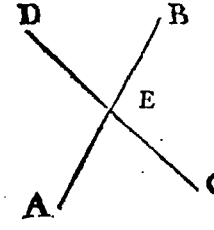
If one line (AB) *meeting two others* (BC, BD) *in the same point* (B), *makes two angles with them* (ABC, ABD) *which together are equal to two right-angles; these lines* (BC, BD) *will form one continued right-line.*



For, *if possible*, let BH, and not BD, be the continuation of the right-line CB: then the angles ABC and ABH being = two right-angles [e] = ABC and ABD [f]; if from these equal quantities, ABC, common to both, be taken away, there will remain ABH = ABD [g]; *which is impossible* [h]. [e] 1. 1. [f] Hyp. [g] Ax. 5. [h] Ax. 2.

THEOREM III.

The opposite angles (DEB, AEC), *made by two lines* (DC, BA) *intersecting each other, are equal.*



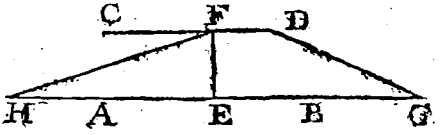
For DEB + DEA = two right-angles [i] = AEC + DEA; whence, by taking away DEA, common, there remains DEB = AEC [k]. [i] 1. 1. [k] Ax. 5.

THEO-

THEOREM IV.

Two right-lines (AB, CD) *perpendicular to one and the same right-line* (EF), *are parallel to each other.*

If you say, they are not parallel; then let them, when produced out, meet in some point, as G.



In EA, produced (if necessary) let there be taken EH = EG [l], and let the right-line FH be drawn [m]. The triangles EHF and EGF, having EH = EG, the angle HEF = GEF [n], and EF common, are therefore equal in all respects [o]: and so, the angle EFH being = EFG (EFD) = a right-angle [p], HFDG (as well as HEG) must be one continued right-line [q]: *which is impossible* [r]. Therefore AB and CD are parallels.

[l] Post. 4. [m] Post. 1. [n] Def. 8. [o] Ax. 10. [p] Hyp. [q] 2. 1. [r] Ax. 8.

SCHOLIUM.

In this theorem, the *possibility* of parallel lines (or such, which being infinitely produced, in the same plane, can never meet) is demonstrated: for EF may be drawn perpendicular to AB [l]; and CFD, again, perpendicular to EF [l]; which last, it is demonstrated, will be parallel to AB.

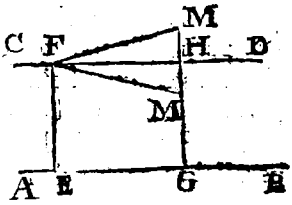
THEOREM V.

Perpendiculars (EF, GH) *to one* (AB) *of two parallel lines* (AB, CD) *terminated by those lines, are equal to each other; and also perpendicular to the other of the two parallels* (CD).

For, AB and CD being parallel to each other, GH can neither be greater, nor less than EF [s]; and therefore must be equal to EF. If you say, that EF is not perpendicular to CD; then let FM be perpendicular to EF [t], meeting GH produced (if be-

[s] Ax. 9. and Def. 13. [t] Post. 4.

necessary) in M: so shall FM be parallel to AB [u]; and consequently GM = EF [w] = GH; *which is impossible* [x]. Therefore EF is perpendicular to CD. And, by the same argument, GH is perpendicular to CD.



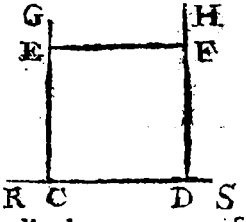
[u] 4. 1.
[w] 5. 1.
[x] Ax. 2.

COROLLARY.

Hence, through the same point F, more than one parallel cannot be drawn to the same line given AB.

SCHOLIUM.

From the preceding proposition, the consistence of the twenty-fifth definition, or the possibility, that all the properties ascribed to a rectangle, can subsist together in the same figure, will appear, together with the method of construction. For, at any two points C, D in a right-line RS, two perpendiculars CG, DH may be erected [y]; and a perpendicular to one of these, at any point E, meeting the other in F, may be drawn. The figure CEFD thus constructed will be a rectangle: for CE and DF are parallel [z]; as are also CD and EF [z]: therefore the angle F (as well as C, D, and E) is a right-angle [a]. If CE be made = CD, then will the rectangle CEDF have all its sides equal [b]. Which answers to the definition of a square.



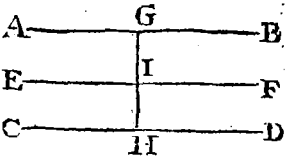
[y] Post. 4.
[z] 4. 1.
[a] 5. 1.
[b] 5. 1. and Ax. 1.

THEOREM VI.

Right-lines (AB, EF) *parallel to the same right-line* (CD) *are parallel to each other.*

For

For let the line HIG be perpendicular to CD: then, that line being alſo perpendicular to both AB and EF [c], theſe laſt are parallel to each other [d].

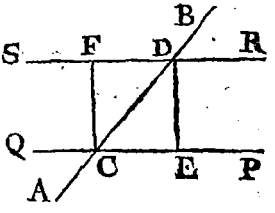


[c] 5. 1.
[d] 4. 1.

THEOREM VII.

A line (AB) *interſecting two parallel lines* (SR, QP) *makes the alternate angles* (SDC, PCD) *equal to each other.*

Let CF and DE be perpendicular to QP, and SR [e]; then theſe lines FC and DE are likewiſe parallels [f]; and ſo the triangles CFD and CDE, having the ſide CF = DE [c], FD = CE [c], and the angle F = E [g], they will alſo have the angle FDC = ECD [h].



[e] 5. 1.
[f] 4. 1.
[g] Ax. 7.
[h] Ax. 10.

COROLLARY I.

Hence, a line interſecting two parallel lines, makes the angles (BDR, BCP) on the ſame ſide, equal to each other: for BDR (= CDS [i]) = BCP [k].

[i] 3. 1.
[k] 7. 1.

COROLLARY II.

Hence, alſo, a line falling upon two parallel lines, makes the ſum of the two internal angles (SDC + QCD) on the ſame ſide of it, equal to two right-angles: for the angle SDC being = PCD, and PCD + QCD = two right-angles [l]; thence is SDC + QCD = alſo to two right-angles [m].

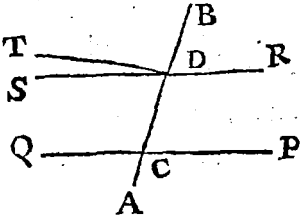
[l] 1. 1.
[m] Ax. 4.

THEOREM VIII.

If a line (AB) *interſecting two other lines* (PQ, RS), *makes the alternate angles* (DCP, CDS) *equal to each other; then are thoſe two lines parallel.*

For,

For, if possible, let some other line DT, and not DS, be parallel to PQ [n]; then must CDT = DCP [o] = CDS [p]: *which is impossible* [q].



[n] Sch. to 4. 1.
[o] 7. 1.
[p] Hyp.
[q] Ax. 2.

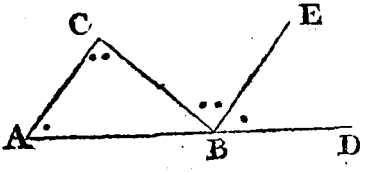
COROLLARY.

Hence, if a line falling on two others, makes the angles (BDR, BCP) above them, on the same side, equal to each other; then those two lines are parallels: because SDC = BDR [r]. [r] 3. 1.

THEOREM IX.

If one side (AB) *of a triangle* (ABC) *be produced, the external angle* (CBD) *will be equal to both the internal opposite angles* (A, C) *taken together.*

For, let BE be parallel to AC [s]; then will the angle C = CBE [t], and the angle A = DBE [u]; therefore C + A = CBE + DBE [x] = CBD [y].



[s] Sch. to 4. 1.
[t] 7. 1.
[u] Cor. to 7. 1.
[x] Ax. 4.
[y] Ax. 3.

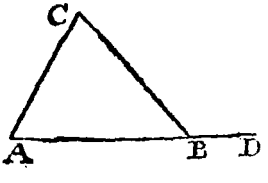
COROLLARY.

Hence the external angle of a triangle is greater than either of the internal, opposite angles.

THEOREM X.

The three angles of any plane triangle (ABC) *taken together, are equal to two right-angles.*

For, if AB be produced to D, then C + A = CBD [z], to which equal quantities let the angle CBA be added, then will C + A + CBA = CBD + CBA [a] = two right-angles [b].



[z] 9. 1.
[a] Ax. 4.
[b] 1. 1.

COROL-

COROLLARIES.

1. If two angles in one triangle, be equal to two angles in another triangle, the remaining angles will also be equal [c].

[c] Ax. 5.

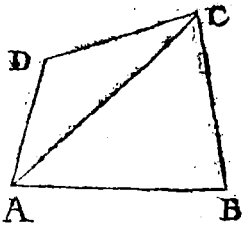
2. If one angle in one triangle, be equal one angle in another, the sums of the remaining angles will be equal [c].

3. If one angle of a triangle be right, the other two taken together, will be equal to a right-angle.

4. The two least angles, of every triangle, are acute.

THEOREM XI.

The four inward angles of a quadrangle (ABCD) taken together, are equal to four right-angles.



Let the diagonal AC be drawn; then the three angles of the triangle ABC being = two right-angles [d], and those of the triangle ACD equal also to two right-angles [d]; it follows that the sum of all the angles of both triangles, which make the four angles of the quadrangle, must be equal to four right-angles [e].

[d] 10. 1.

[e] Ax. 4.

COROLLARY I.

Hence, if three of the angles be right-ones, the fourth will also be a right-angle.

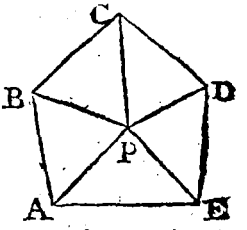
COROLLARY II.

Moreover, if two of the four angles, be equal to two right-angles, the remaining two together will likewise be equal to two right-angles.

SCHO-

SCHOLIUM.

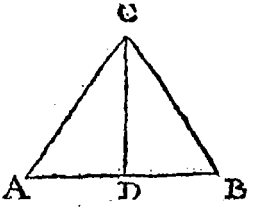
If from any point P, within a polygon ABCDE, lines be drawn to all the angles, so as to divide the whole into as many triangles, APB, BPC, CPD, DPE, EPA, as the polygon has sides; the sum of all the angles of these triangles (which together make up, or compose the angles of the polygon, over and above those about the point P) will be equal to twice as many right-angles as the polygon has sides (by 10. 1.) Therefore, seeing all the angles about the point P, whereby the angles of all the triangles exceed those of the polygon, are equal to four right-angles, it is manifest, that all the angles of the polygon, taken together, will be equal to twice as many right-angles, wanting four, as the polygon has sides.



THEOREM XII.

The angles (A, B) *at the base of an isosceles triangle* (ABC) *are equal to each other.*

For, let the line CD bisect, or divide the angle ACB into two equal parts ACD, BCD, and meet AB in D: then the triangles ACD, BCD, having AC = BC[f], CD common, and the angle ACD = BCD[g], will also have the angle A = B[h].



[f] Def. 18.
[g] Hyp.
[h] Ax. 10.

COROLLARY I.

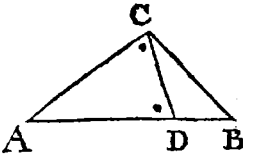
Hence, the line which bisects the vertical angle of an isosceles triangle, bisects the base, and is also perpendicular to it[h].

COROLLARY II.

Hence it appears alſo, that every equilateral triangle is likewiſe equiangular.

THEOREM XIII.

In any triangle (ABC) *the greateſt ſide ſubtends the greateſt angle.*



Let AB be greater than AC; in which let there be taken AD = AC; drawing CD. The triangle ADC being iſoſceles, the angles ACD and ADC are therefore equal [i]; whence ACB, which exceeds the former of them, muſt alſo exceed the latter ADC [k], and conſequently, much more exceed B, which is leſs than ADC [l].

[i] 12. 1.
[k] Ax. 2.
[l] Cor. to 9. 1.

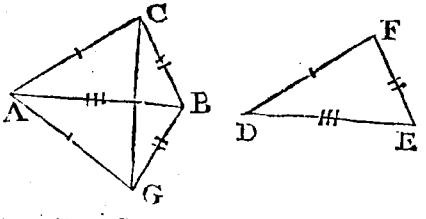
COROLLARY.

Hence, in any triangle, the ſide that ſubtends the greateſt angle, is the greateſt; becauſe ACB cannot be greater than B, unleſs AB is greater than AC [m].

[m] 13. 1.

THEOREM XIV.

If the three ſides (AB, AC, CB) *of one triangle, be equal to the three ſides* (DE, DF, FE) *of another triangle, each to each reſpectively; then the angles oppoſed to the equal ſides will alſo be equal.*



Let the angle BAG = D, AG = DF, and let GB and GC be drawn; ſo ſhall the triangles ABG and DEF be equal in all reſpects [m]: therefore, AG being = DF = AC [n], and BG = EF = BC [n], the

[m] Ax. 10.
[n] Hyp.

the angle ACG is also = AGC [o], and BCG = BGC [o]; and consequently ACB = AGB [p] = DFE: therefore the triangles ABC, DEF are equal in all respects [n].

[o] 12. 1.
[p] Ax. 4 or 5.

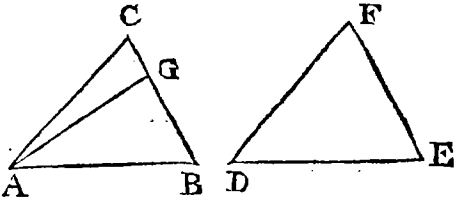
SCHOLIUM.

The demonstration of the last theorem, in obtuse-angled triangles, may admit of another case; which, however, is not necessary: because, if the triangle AGB (equal to DEF) be conceived to be formed on the longest side of ABC; then, all the angles CAB, CBA, GAB, GBA being acute [q], the line CG will, always, fall within the figure ACBG [r], as in the present case.

[q] Cor. 4. to 10. 1.
[r] Ax. 2.

THEOREM XV.

If two triangles (ABC, DEF) *mutually equiangular, have two corresponding sides* (AB, DE) *equal to each other, the other corresponding sides will also be equal.*



If you say BC is greater than EF; from BC let a part BG be taken = EF [s], and let AG be drawn. The triangles ABG, DEF having AB = DE, BG = EF, and B = E *(by hypothesis)*, will also have BAG = D [t]; but D = BAC [u]; therefore BAG = BAC [w]; *which is impossible.*

[s] Post. 4.
[t] Ax. 10.
[u] Hyp.
[w] Ax. 1. and 2.

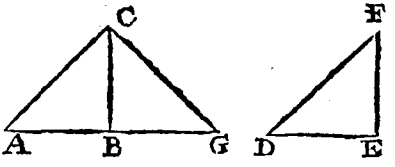
COROLLARY.

Hence, equiangular triangles, having any two corresponding sides equal, are equal to each other [x].

[x] Ax. 10. 1.

THEOREM XVI.

If two right-angled triangles (ABC, DEF) *having equal hypothenuses* (AC, DF), *have two other sides* (BC, EF) *likewise equal; the remaining sides* (AB, DE) *will be equal, and the two triangles equal in all respects.*



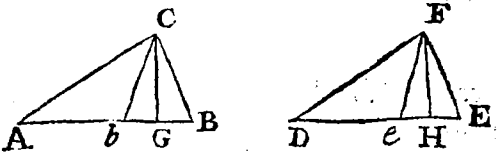
In AB produced, take BG = ED, and let GC be drawn: then, the triangles BCG and DEF, having BG = ED, BC = EF [y], and the angle CBG = E [z], will also have the angle G = D, and CG = DF [a] = AC [y]: whence, the triangle ACG being isosceles, the angle G, or D, will be = A [b]; and consequently F also = ACB [c]: therefore the triangles ABC and DEF, being mutually equiangular, and having AC = DF, they are equal in all respects [d].

[y] Hyp.
[z] Ax. 7.
[a] Ax. 10.
[b] 12. 1.
[c] Cor. 1. to 10. 1.
[d] 15. 1.

THEOREM XVII.

If two triangles (ABC, DEF) *having two sides* (AC, BC) *of the one equal to two sides* (DF, EF) *of the other, respectively, have also the angles* (A, D) *subtended by two of the equal sides* (BC, EF) *equal to each other; and if the angles* (B, E) *subtended by the other equal sides, be either, both acute or both obtuse; then will the two triangles be equal in all respects.*

Let CG and FH be perpendicular to AB and



DE: then, the angle AGC being = DHF [e],

[e] Ax. 7.

A = D, and the side AC = DF [f], CG will also be = FH [g]; whence, CB being = FE [f], the angles GBC and HEF are likewise equal [h]: and so, the triangles ABC and DEF, being mutually equiangular [i], and having the sides AC and DF equal, are equal in all respects [g].

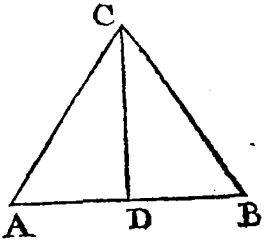
[f] Hyp.
[g] 15. 1.
[h] 16. 1.
[i] Cor. 1. to 10.

The demonstration is the same, when both the angles are obtuse, as in the triangles A*b*C, D*e*F: for, if C*b* (= CB = FE) = F*e*, the angles G*b*C and H*e*F being equal *(as before)*, the angles A*b*C and D*e*F will likewise be equal [k].

[k] 1. 1. and Ax. 5. 1.

THEOREM XVIII.

If two angles (A, B) *of a triangle* (ABC) *be equal, the sides* (BC, AC) *subtending them will likewise be equal.*

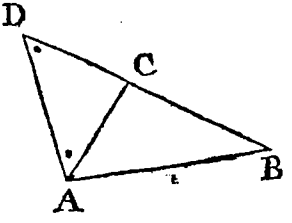


Let CD bisect the angle ACB, and meet AB in D: then the triangles ACD, BCD being equiangular [k], and having CD common to both, they will also have AC = BC [l].

[k] Cor. 1. to 10.
[l] 15. 1.

THEOREM XIX.

Any two sides (AC, BC) *of a triangle* (ABC) *taken together, are greater than the third side* (AB).

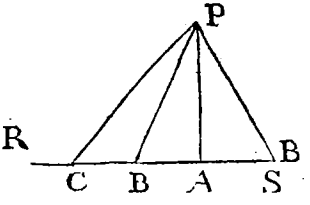


In BC produced, let there be taken CD = CA, and let AD be drawn. The angles D and DAC are equal [m]; therefore BAD, which exceeds the latter [n], must also exceed the former D; and consequently BD (or BC + AC) must exceed AB [o].

[m] 12. 1.
[n] Ax. 2.
[o] Cor. to 13. 1.

THEOREM XX.

Of all the right-lines (PA, PB, PC) *falling from a given point* (P) *upon an infinite right-line* (RS), *that* (PA) *is the least which is perpendicular to it; and, of the rest, that* (PB) *which is the nearest the perpendicular is less than any other* (PC) *at a greater distance.*



For BAP being a right-angle [p], ABP will be acute [q], and therefore AP ¬ BP [r].

[p] Hyp.
[q] Cor. 4. to 10. 1.
[r] Cor. to 13. 1.

Again, when PB and PC are both on the same side of the perpendicular PA; then is CBP ⌐ right-angle [s] ⌐ BCP [q], and consequently PC ⌐ PB.

[s] Cor. to 9. 1.

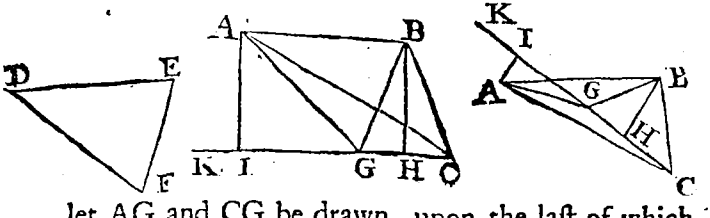
If PB be on the contrary side of the perpendicular to PC; from AC, let AB be taken = AB; then the two lines PB, PB will be also equal [t]; and therefore PC, which exceeds the one (by the preceding case) will also exceed the other.

[t] Ax. 10.

THEOREM XXI.

Of two triangles (ABC, DEF) *having two sides* (AB, BC) *of the one, equal to two sides* (DE, EF) *of the other, each to each respectively, the base of that* (ABC) *will be the greatest, which is subtended under the greatest angle.*

Let the angle ABG = E, BG = EF (= BC) also



let AG and CG be drawn, upon the last of which, produced,

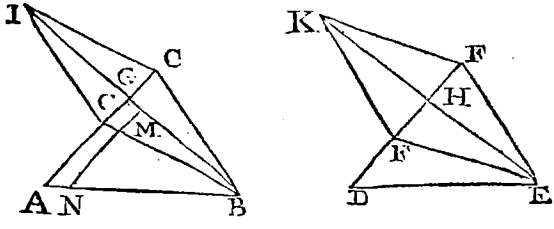
produced, let fall the perpendiculars BH and AI [u]. Since BG = BC [w], and, consequently, GH = HC [x], it is evident, that GI (whether the point I be considered as falling between G and K, or between G and H) will be less than CI [z]; and therefore AG, or its equal DF [a], also less than AC [b].

[u] 4. 1.
[w] Hyp.
[x] 16. 1.
[z] Ax. 2.
[a] Ax. 10.
[b] 20. 1.

THEOREM XXII.

Of two triangles (ABC, DEF), *having one angle* (BAC) *in the one equal to one angle* (EDF) *in the other, and the sides* (BC, EF) *opposed to them also equal, that* (ABC) *will have the greatest base, whereof the opposite angle* (ACB) *differs the least from a right-angle.*

Let BG and EH be perpendicular to AC and DF, in which produced, take HK = HE, GI = GB, and BM = EH; also let MN be parallel to GA, meeting AB, produced if necessary, in N; and let CI and KF be drawn.



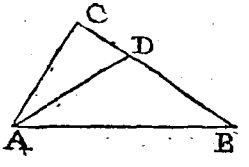
The angle ICG being = BCG [d], and the latter of these greater than EFH [e] (or KFH [d]), thence is ICB ⊏ KFE; and consequently BI ⊏ EK [e]; whence also BG ($\frac{1}{2}$BI) ⊏ EH ($\frac{1}{2}$EK) or its equal BM [d]; and therefore BA ⊏ BN, because AG and MN being parallels, both the points M and N will fall on the same side of AG. But BN (as the triangles NBM, DEH are equiangular, and have BM = EH [f]) is = DE [g]: therefore BA is also greater than DE.

[c] Hyp.
[d] Ax. 10.
[e] 21. 1.
[f] Hyp. and 7. 1.
[g] 15. 1.

THEOREM XXIII.

If, of two triangles (ABC, ABD) *ſtanding upon the ſame baſe* (AB), *the one be wholly included within the other, the two ſides* (AD, BD) *of the included one, taken together, will be leſs, and the angle* (D) *contained by them greater, reſpectively, than the two ſides* (AC, BC), *and the contained angle* (C) *of the other.*

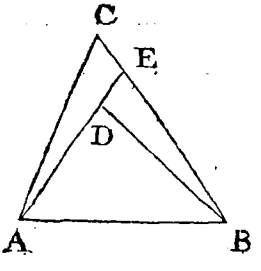
CASE I. *If the vertex of the contained triangle be in one ſide of the other:*



Then, AD is leſs than AC + CD [h]; whence, by adding BD common, AD + BD will alſo be leſs than AC + CD + BD [i], or than its equal AC + BC [k]. But the angle ADB is ⊏ ACB [l].

[h] 19. 1.
[i] Ax. 6.
[k] Ax. 3.
[l] Cor. 9. 1.

CASE II. *If the vertex be within the other triangle:*



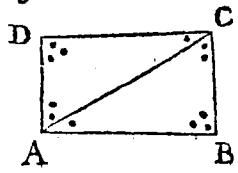
Let AD be produced to meet BC in E: then *(by the preceding caſe)* the ſum of AD and BD is leſs than the ſum of AE and BE; which laſt ſum, and conſequently the former, is, again, leſs than the ſum of AC and BC. Moreover, the angle ADB ⊏ BED ⊏ C.

THEOREM XXIV.

The oppoſite ſides (AB, DC) *of any parallelogram* (ABCD) *are equal, as are alſo the oppoſite angles* (B, D); *and the diagonal* (AC) *divides the parallelogram into two equal parts.*

For,

For, AB, DC, and AD, BC being parallels [n], the angle BAC is = DCA [o], and BCA = DAC [o]; therefore the equiangular triangles ABC, ADC [p] having AC common, are equal in all respects [q].



[n] Def. 24.
[o] 7. 1.
[p] Cor. 1. to 19. 1.
[q] 15. 1.

COROLLARY.

Hence, if one angle (B) of a parallelogram be a right-angle, all the other three will be right ones: for D, being = B, is a right-angle; and BCD is = B, and DAB = D, *by Theor. V.*

THEOREM XXV.

Every quadrilateral (ABCD) *whose opposite sides are equal, is a parallelogram.* (See the preceding scheme.)

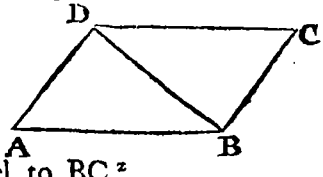
Let the diagonal AC be drawn; then the triangles ABC, ADC being mutually equilateral [r], they will also be mutually equiangular [s]; consequently AB will be parallel to DC, and AD to BC [t].

[r] Hyp.
[s] 14. 1.
[t] 8. 1.

THEOREM XXVI.

The lines (AD, BC) *joining the corresponding extremes of two equal, and parallel lines* (AB, DC) *are themselves equal and parallel.*

Let the diagonal BD be drawn. Because AB and DC are parallel [u], the angle ABD is = CDB [w]; therefore, BA being = DC [x] and BD common, the remaining sides and angles will likewise be respectively equal [y]; and consequently AD parallel to BC [z].



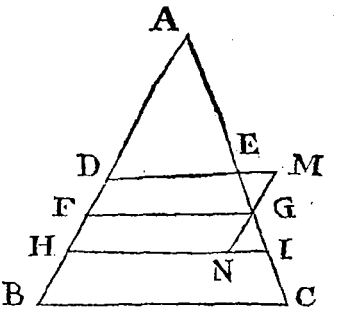
[u] Hyp.
[w] 7. 1.
[x] Hyp.
[y] Ax. 10.
[z] 8. 1.

THEO-

THEOREM XXVII.

If, in one side (AB) *of a triangle* (ABC), *from three points* (D, F, H) *at equal distances* (DF, FH), *lines* (DEM, FG, HI) *be drawn parallel to the base, the parts* (EG, GI) *of the other side* (AC) *intercepted by them, will also be equal to each other.*

Let NGM be parallel to AB, intersecting HI and DE in N and M. Then, the triangles IGN, MGE, having the
[a] 3. 1. angle IGN = EGM[a],
[b] 7. 1. ING = M[b], and GN
[c] 24. 1. (= FH[c] = FD[d]) =
[d] Hyp. GM[e], will also have
[e] 15. 1. GI = GE[e].



COROLLARY I.

Hence it appears, that, if one side of a triangle be divided into any number of equal parts, and from the points of division lines be drawn parallel to the base, cutting the other side, they will also divide it into the same number of equal parts.

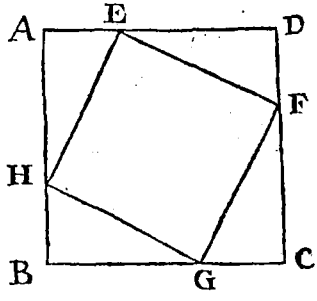
COROLLARY II.

Hence, also, if two lines FG, HI, cutting the sides of a triangle, be parallel to each other, and another line DE be so drawn as to cut off FD = FH and GE = GI, this line DE will be parallel to the two former.

THEO-

THEOREM XXVIII.

If in the sides of a square (ABCD), *equally distant from the four angular points, there be taken four other points* (E, F, G, H) *the figure* (EFGH) *formed by joining those points, shall also be a square.*

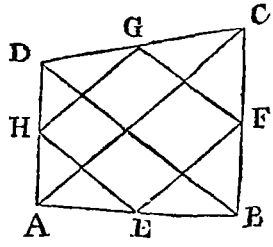


For the wholes AD, DC, CB, BA being equal [f], and also the parts AE, DF, CG, BH [g], the remaining parts ED, FC, GB, HA must consequently be equal [h]; whence, all the angles D, C, B, A being equal [i], the sides EF, FG, GH, HE will be equal likewise [k], and the angle DEF = AHE [k]. Therefore, because DEH is = A + AHE [l], if from these, the equal angles DEF, AHE be taken away, there will remain HEF = A [h] = a right-angle [f]. By the same argument (or by Theor. 25th, and the Corol. to the 24th) the other three angles will be right-angles.

[f] Def. 26.
[g] Hyp.
[h] Ax. 5.
[i] Ax. 7.
[k] Ax. 10.
[l] 9. 1.

THEOREM XXIX.

If all the sides of any quadrilateral (ABCD) *be bisected, the figure* (EFGH) *formed by joining the points of bisection, will be a parallelogram.*



Draw the diagonals AC and BD. Because EF and HG are parallel to AC [n], they are also parallel to each other [o]. After the same manner is FG parallel to EH; therefore EFGH is a parallelogram [p].

[n] Cor. 1. to 27. 1.
[o] 6. 1.
[p] Def. 24.

The End of the FIRST BOOK.

ELEMENTS

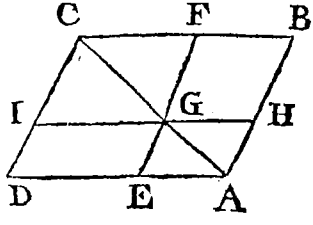
OF

GEOMETRY.

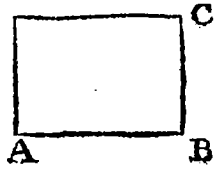
BOOK II.

DEFINITIONS.

1. IN a parallelogram ABCD, if two right-lines EF, HI, parallel to the ſides, interſecting the diagonal in the ſame point G, be drawn, dividing the parallelogram into four other parallelograms; thoſe two GD, GB through which the diagonal does not paſs, are called Complements; and the other two, HE, FI, Parallelograms about the diagonal.



2. Every rectangle is ſaid to be contained under the two right-lines AB, BC that are the baſe and altitude thereof.

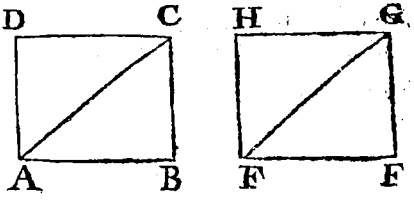


The

The rectangle contained under two right-lines AB *and* BC *is often, for brevity sake, denoted by* AB×BC. *But when the figure is a square, it is usually represented by placing the number* 2 *over the letter, or letters expressing the side thereof: thus* AB^2 *denotes the square made upon the line* AB.

THEOREM I.

The rectangles (BD, FH) *contained under equal lines, are equal.*

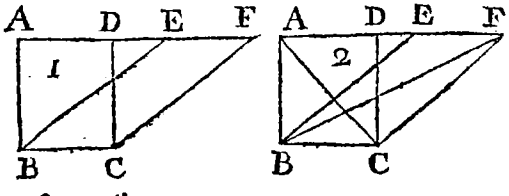


For, let the diagonals AC, EG be drawn: then, because AB = EF, BC = FG, and B = F [a], the triangles ABC, EFG are equal [b]. And, in the very same manner will ADC and EHG appear to be equal. Therefore the whole rectangle ABCD is also equal to the whole rectangle EFGH [c].

[a] Hyp.
[b] Ax. 10.
[c] Ax. 4. 1.

THEOREM II.

Parallelograms (ABCD, BCFE) *standing upon the same base* (BC) *and between the same parallels* (BC, AF) *are equal.*



For, since (in *Fig.* 1.) the angle F = BEA [d], and CDF = A [d], the triangles FDC, EAB are equiangular [e]; they are also equal [f], because CF = BE [g]: therefore, if each be taken from the whole figure ABCF, there will remain ABCD = EBCF [h].

[d] Cor. 1. to 7. 1.
[e] Cor. 1. to 10 1.
[f] 15. 1.
[g] 24. 1.
[h] Ax. 5.

COROL-

COROLLARY I.

Hence, triangles BAC, BFC (*Fig.* 2.) ſtanding upon the ſame baſe, and between the ſame parallels, are alſo equal, being the halves of their reſpective parallelograms [i].

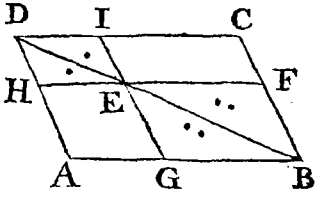
[i] 24. 1.

COROLLARY II.

Hence all parallelograms, or triangles, whatever, whoſe baſes and altitudes are equal, are equal among themſelves; becauſe all ſuch parallelograms are equal to rectangles ſtanding on the ſame baſes, and between the ſame parallels; and theſe laſt are equal, by the preceding propoſition.

THEOREM III.

The complements (EC, EA) *of any parallelogram* (AC) *are equal.*



For, the whole triangle DCB being equal to the whole triangle DAB [k], and the parts DIE, EFB reſpectively equal to the parts DHE, EGB [k], the remaining parts EC, EA muſt likewiſe be equal [l].

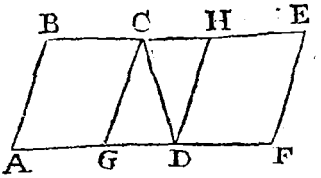
[k] 24. 1. and Def. 1.

[l] Ax. 5. 1.

THEOREM IV.

A trapezium (ABCD) *whereof two ſides* (AD, BC) *are parallel, is equal to half a parallelogram, whoſe baſe is the ſum of thoſe two ſides, and its altitude, the perpendicular diſtance between them.*

For,

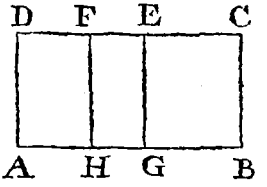


For, in AD produced, take DF = BC; and let CG, DH and FE be all parallel to AB, meeting AF and BC produced, in G, H and E. Then AE is a parallelogram of the same altitude with ABCD, having its base AF equal to the sum of AD and BC [n]: but this parallelogram, because BG = HF [o], and CGD = CHD [p], is equally divided by the line CD [q]; and so ABCD is the half thereof.

[n] Constr.
[o] Cor. 2. to 2. 2.
[p] 24. 1.
[q] Ax. 4.

THEOREM V.

The sum of all the rectangles contained under a given line (AD), *and all the parts* (AH, HG, GB) *of another* (AB), *any how divided, is equal to the rectangle contained under the two whole lines.*



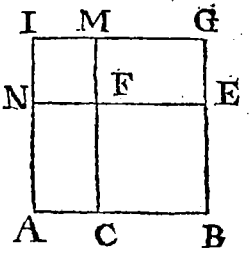
Let ABCD be the rectangle contained under the two whole lines, and let HF, GE be parallel to AD, meeting DC in F and E. Then will AF, HE, GC be rectangles [r] of the same altitude with AC [s]; therefore AF = AD × AH, HE = AD × HG, and GC = AD × BG [t]; and consequently AD × AB (AC = AF + HE + GC) = AD × AH + AD × HG + AD × BG [u].

[r] Cor. to 24. 1.
[s] 24. 1. & Ax. 1.
[t] 1. 2.
[u] Ax. 3. & 4. 1.

THEOREM VI.

If a right-line (AB) *be, any-wise, divided into two parts* (AC, BC), *the square of the whole line will be equal to the squares of both the parts, together with two rectangles under the same parts.*

Let



Let ABGI be the ſquare of AB, and CBEF that of BC, and let EF and CF be produced to meet the ſides of the ſquare ABGI in M and N.

From the equal quantities CM, EN [w] take the equal quantities CF and EF, and there remains FM = FN [x]; therefore, all the angles of the figure being right ones [y], NM is a ſquare [z] upon FN (= AC); and AF, FG are equal to two rectangles under BC and AC [a]: but AG = BF + AF + FG, or $AB^2 = BC^2 + AC^2 + 2AC \times BC$ [b].

[w] 24. 1. and Defin. 26.
[x] Ax. 5. 1.
[y] Cor. to 24. 1.
[z] Def. 26.
[a] 1. 2.
[b] Ax. 3. 1.

COROLLARY I.

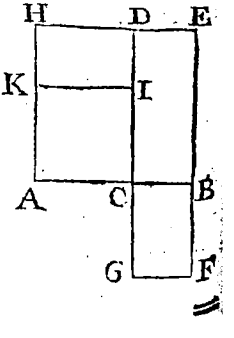
Hence, the ſquare of any line is equal to four times the ſquare of half that line.

COROLLARY II.

Hence, alſo, if two ſquares be equal, their ſides muſt be equal; becauſe unequal lines BA, BC have not equal ſquares.

THEOREM VII.

The difference of the ſquares (ABEH, ACIK) *of any two unequal lines* (AB, AC), *is equal to a rectangle under the ſum and difference of the ſame lines.*



In EB, produced, take BF = AC; let FG be drawn parallel to EH, and let CI be produced both ways, to meet EH and FG in D and G. It is evident that DF is a rectangle [c], whoſe baſe GF (= CB [d]) = the difference of the given lines AB, AC; and whoſe altitude FE (becauſe BE

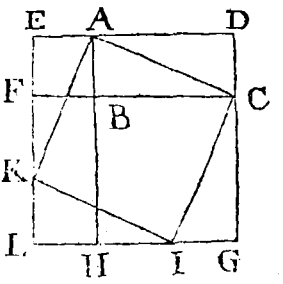
[c] Cor. 24. 1.
[d] 24. 1.

= BA [e], and BF = AC [f]) is = the sum of the same lines: but this rectangle DF is = DB + GB [g] = DB + DK (because DK [h] = GB) = the square AE — the square AI.

[e] Def. 24.
[f] Hyp.
[g] Ax. 3.
[h] 1. 2.

THEOREM VIII.

The square made upon the side (AC) *subtending the right-angle of a plane triangle* (ABC), *is equal to both the squares* (BE, BG) *made upon the sides* (AB, BC) *containing that angle.*



Let the sides of the squares BE, BG be produced to meet each other in L and D; in which take KL and IG each equal to AE (or AB); and let CI, IK, and KA be drawn.

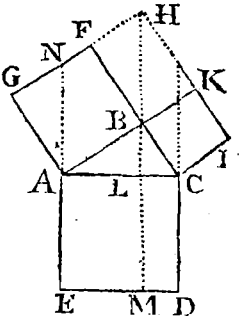
Since ABH and FBC (which are continued right-lines [i]) are equal to each other [k], EL, DG, ED, and LG will be all equal among themselves [l]; and so, the angles E, D, G and L being all right-ones [m], EDGL will be a square, and consequently ACIK a square likewise [n]. Now, if from the square DL, the four equal [o] triangles ADC, CGI, ILK, and KEA be taken away, there will remain the square AI: and, if from the same DL, the two equal [p] parallelograms DB, BL (which are equal to the said four triangles, because DB = two of them [l]) be taken away; then there will remain the two squares BE and BG. Consequently the square AI is = the two squares BE and BG [q].

[i] 2. 1.
[k] Ax. 4.
[l] 24. 1.
[m] Hyp. & 5. 1.
[n] 28. 1.
[o] Ax. 10.
[p] 1. 2.
[q] Ax. 5.

The same demonstrated otherwise.

Let AD be the square on the hypothenuse AC, and BG, BI the two squares on the sides AB and BC:

BC: let MBH be parallel to AE, meeting GF (produced) in H; and let EA be produced to meet GH in N.



[r] Ax. 7. If from the equal [r] angles GAB, CAN, the angle NAB, common to both, be taken away, there will remain NAG [s] Ax. 5. = BAC [s]; whence, as the angle G is also = ABC [t], and the [t] Def. 26. side AG = AB [t], the sides AN and AC (= AE) are likewise [u] 15. 1. equal [u]; and therefore the parallelogram AM = the parallelogram AH [w]; which last, and [w] Cor. to 2. 2. consequently the former, is equal to the square BG [x] [x] 2. 2. standing on the same base AB, and between the same parallels. By the same argument, the parallelogram CM is = the square BI: and, consequently, the square AD (= AM + CM) = both the [y] Ax. 4. squares BG and BI [y].

COROLLARY.

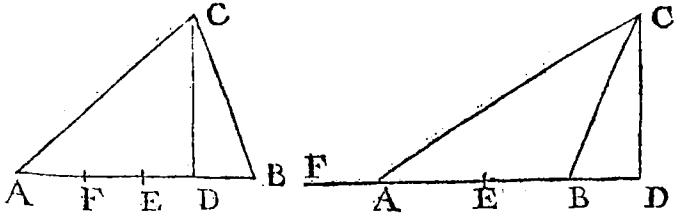
Hence, the square upon either of the sides including the right-angle, is equal to the difference of the squares of the hypothenuse and the other [z] Ax. 5. side [z]; or, equal to a rectangle contained under the sum and difference of the hypothenuse and the other [a] 7. 2. side [a].

THEOREM IX.

The difference of the squares of the two sides (AC, BC) *of any triangle* (ABC) *is equal to the difference of the squares of the two lines, or distances* (AD, BD) *included between the extremes of the base* (AB) *and the perpendicular* (CD) *of the triangle.*

For, since $AC^2 = DC^2 + AD^2$, and $BC^2 = DC^2 + BD^2$ *(by the precedent)*, it is evident that the difference

difference of AC^2 and BC^2 will be equal to the difference between $DC^2 + AD^2$ and [b] $DC^2 + BD^2$, or [b] Ax. 5.



between AD^2 and [c] BD^2, by taking away DC^2, [c] Ax. 6. common, from both.

COROLLARY I.

Since the rectangle under the sum and difference of any two unequal lines, is equal to the difference of their squares [d], it follows, that the difference of [d] 7. 2. the squares (or the rectangle under the sum and difference) of the two sides of any triangle, is equal to the rectangle under the sum and difference of the distances included between the perpendicular and the two extremes of the base.

COROLLARY II.

It follows, moreover, that the difference of the squares (or the rectangle under the sum and difference) of the two sides of a triangle, is equal to twice a rectangle under the whole base, and the distance of the perpendicular from the middle of the base.

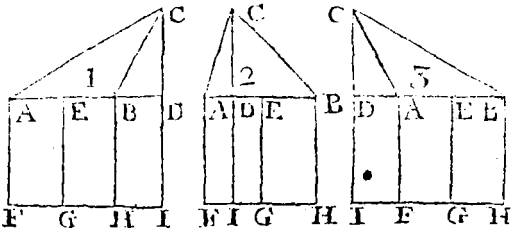
For, let E be the middle of the base, and let EF be made $= ED$; then AF being $= BD$ [e], the excess of AD above BD (or AF) will (*in Fig.* 1.) be [e] Ax. 5. $= DF = 2DE$; therefore the rectangle under the sum and difference of AD and BD ($= {}^{g}AC^2 - {}^{g}$ 9. 2. BC^2) is $= AB \times 2DE$. Again (*in Fig.* 2.) $AD +$ BD being $= AD + AF$ [h] $= FD = 2ED$, and [h] Ax. 4. $AD - BD = AB$, we have, also, in this case [g], $AC^2 - BC^2 = AB \times 2DE$.

THEOREM X.

The ſquare of one ſide (AC) *of a triangle* (ABC) *is greater, or leſs than the ſum of the ſquares of the baſe* (AB) *and of the other ſide* (BC), *by a double rectangle under the whole baſe* (AB) *and the diſtance* (BD) *of the perpendicular from the angle* (B) *oppoſite to the ſide firſt mentioned; that is, greater, when the perpendicular falls beyond the ſaid angle (as in Fig.* 1.); *but leſs, when it falls on the contrary ſide (as in Fig.* 2. *and* 3).

Let the ſquare ABHF, on the baſe AB, be divided into two equal[i] rectangles EF and EH by

[i] 1. 2.



the line EG, biſecting AB in E; and let the perpendicular CD be continued out to meet FH (produced) in I.

In Fig. 1. $AC^2 - BC^2 =$ twice the rectangle EI[l] $= 2EH + 2BI$[m] $= AH\ (AB^2) + 2BI\ (2AB \times BD)$; therefore, if from the firſt and laſt of theſe equal quantities, AB^2 be taken away, then AC^2 *leſs* both BC^2 and $AB^2 = 2AB \times BD$[n].

[l] Cor. 2. to 9. 2.
[m] Ax. 3.
[n] Ax. 5.

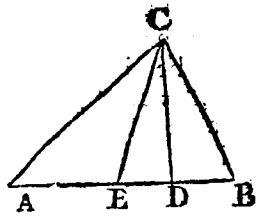
In Fig. 2 *and* 3. $BC^2 - AC^2 = 2EI$[l] $= 2BI - 2BG$[m] $= 2AB \times BD - AB^2$; and ſo, by adding AB^2 to the firſt and laſt of theſe equal quantities, we have here $AB^2 + BC^2 - AC^2 = 2AB \times BD$[o].

[o] Ax. 4.

THEO

THEOREM XI.

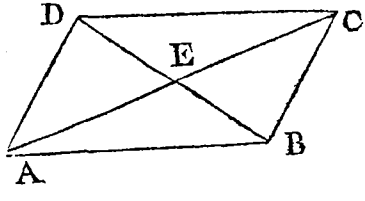
The double of the square of a line (CE) *drawn from the vertex to the middle of the base of any triangle* (ABC), *together with double of the square of the semi-base* (AE), *is equal to the squares of both the sides* (AC, BC) *taken together.*



For, let CD be perpendicular to AB: then, because *(by the precedent)* AC^2 exceeds the sum of the two squares AE^2 and CE^2 (or BE^2 and CE^2) by the double rectangle $2AE \times ED$ (or $2BE \times ED$); and because BC^2 is less than the same sum by the same double rectangle; it is manifest that both AC^2 and BC^2 together, must be equal to that sum twice taken; the excess on the one part making up the defect on the other.

THEOREM XII.

The two diagonals (AEC, BED) *of a parallelogram* (ABCD) *bisect each other; and the sum of their squares is equal to the sum of the squares of all the four sides of the parallelogram.*

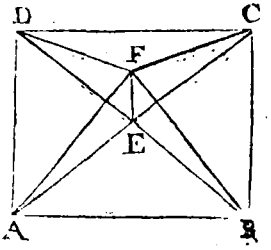


For, the triangles AEB, DEC being equiangular [p], and having AB = DC [q], will also have AE = CE, and BE = DE [r]. Moreover, because $2AE^2 + 2ED^2 =$ [s] $AD^2 + CD^2$, by taking the double of these, we have $4AE^2$ ([t] AC^2) $+ 4ED^2$ (DB^2) = [u] $AD^2 + BC^2 + CD^2 + AB^2$.

[p] 3. and 7. 1.
[q] 24. 1.
[r] 15. 1.
[s] 11. 2.
[t] Cor. 1. to 6. 2.
[u] Ax. 4. and 24. 1.

THEOREM XIII.

If from any point (F), *to the four angles of a rect-angle* (ABCD) *four lines be drawn; the sums of the squares of those drawn to the opposite angles will be equal (I say, that* $FA^2 + FC^2 = FB^2 + FD^2$).



For, let the diagonals AC and BD be drawn, bisecting
x 12. 2. each other in E [x], and let E, F be joined; then the triangles ABC, BAD being
w 24. 1. and Ax. 10. equal in all respects [w], thence will AE ($\frac{1}{2}$AC) = DE ($\frac{1}{2}$DB).
But $FA^2 + FC^2 =$ [y] $2AE^2$
y 11. 2. $(2DE^2) + 2EF^2 =$ [y] $FB^2 + FD^2$.

The End of the SECOND BOOK.

ELEMENTS

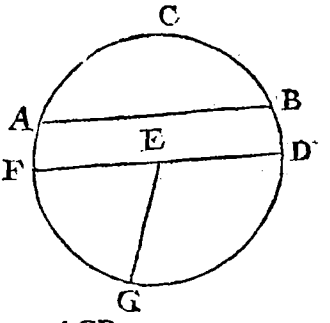
OF

GEOMETRY.

BOOK III.

DEFINITIONS.

1. ANY right-line FD, passing through E the center of a circle, and terminating in the circumference at both ends, is called a Diameter.



2. An arch of a circle, is any portion of the periphery, or circumference, as ACB.

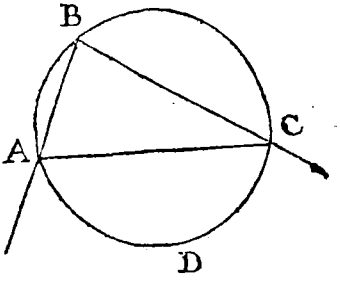
3. The chord, or subtense of an arch ACB, is a right-line AB joining the two extremes of that arch.

4. A semi-circle is a figure contained under any diameter and either part of the circumference cut off by that diameter.

5. A ſegment of a circle is a figure contained under an arch ACB and its chord AB.

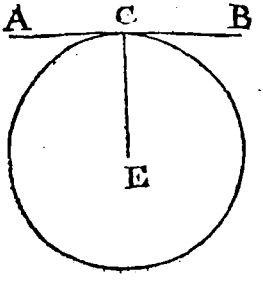
6. A Sector of a circle is a figure contained under two right-lines EF, EG, drawn from the center to the circumference, and the arch FG included betwixt them. When the two lines EF, EG ſtand perpendicular to each other, then the Sector is called a Quadrant.

7. An angle ABC is ſaid to be in a ſegment of a circle ABC, when, being in the periphery thereof, the right-lines BA, BC by which it is formed, paſs through the extremes of the chord AC bounding that ſegment.



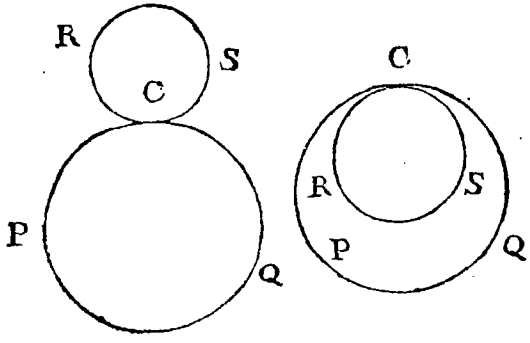
8. An angle ABC in the periphery, comprehended by two right-lines BA, BC, including an arch of the circle, ADC, is ſaid to ſtand upon that arch.

9. A right-line AB is ſaid to touch a circle, when, paſſing through a point (C) in the circumference thereof, it cutteth off no part of the circle.



10. Two

10. Two circles (PCQ, RCS) are ſaid to touch each other, when the circumferences of both paſs



through one point (C) and yet do not cut each other.

11. Two circles, in the ſame plane, are ſaid to cut one another, when they fall partly within, and partly without each other; or, when their circumferences cut each other.

12. A right-line is ſaid to be applied, or inſcribed in a circle, when both its extremes are in the periphery thereof.

13. A right-lined figure is ſaid to be inſcribed in a circle, when all its angles are in the circumference of the circle.

14. A circle is ſaid to be deſcribed about a right-lined figure, when the periphery of the circle paſſes through all the angles of that figure.

15. A right-lined figure is ſaid to be deſcribed about a circle, when all the ſides thereof touch the circle.

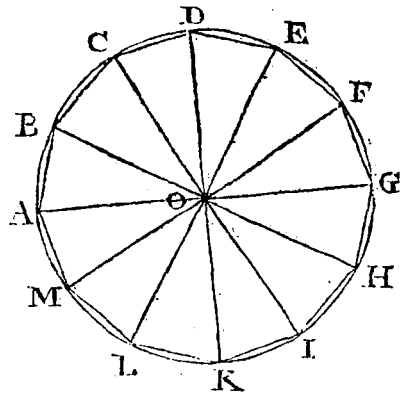
16. A

16. A circle is ſaid to be inſcribed in a right-lined figure, when it is touched by all the ſides of the right-lined figure.

17. A right-lined figure is ſaid to be inſcribed in a right-lined figure, when all the angles of the former are ſituate in the ſides of the latter.

THEOREM I.

If the ſides (AB, BC, CD &c.) *of a polygon inſcribed in a circle, be equal, the angles* (AOB, BOC, COD &c.) *at the center of the circle, ſubtended by them, will likewiſe be equal.*



For AO, BO, CO &c. being equal to each other [a], as well as AB, BC, CD &c. the triangles AOB, BOC, COD, are mutually equilateral; and therefore have all the angles AOB, BOC &c. equal to each other [b].

[a] Def. 33. of 1.

[b] 14. 1.

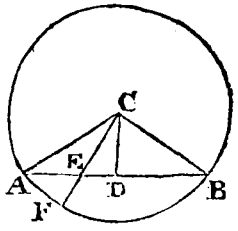
SCHOLIUM.

On this propoſition depends the diviſion of mathematical inſtruments for taking and meaſuring of angles. For, if, by repeated trials, or any other means, the circumference of a circle deſcribed about a center O, be divided into any number of parts AB, BC, CD &c. ſo that the chords be equal; then it is evident, from hence, that all the angles AOB, BOC, COD &c. which make up the four right-angles AOD, DOG, GOK, KOA at the center, will alſo be equal to each other, let the

the radius OA of the *instrument* be what it will.— In the division of the circle for practical uses, the number of parts into which the circumference is thus divided, or the number of equal angles at the center, is 360; which equal angles are called *degrees*; so that a right-angle, consisting of 90 of these equal angles, is said to be an angle of 90 degrees; every angle being denominated, from the degrees, and parts of a degree, contained therein; each degree being conceived to be subdivided into 60 equal parts, called *minutes*; each minute again into 60 equal parts, called *seconds*; and so on to thirds, fourths, fifths, *&c.* at pleasure.

THEOREM II.

Any chord (AB) *of a circle, falls wholly within the same: And a perpendicular* (CD) *let fall thereon, from the center of the circle, will divide it into two equal parts.*



Let C, A, and C, B be joined; and thro' any point E in the chord AB, let the right-line CEF be drawn, meeting the circumference in F.

It is evident, because CA = CB [c], that these equal lines are on different sides of the perpendicular CD [d]: And so, CE being ⊐ CA or CF [d], the point E (take it where you will in the line AB) and consequently the line AB itself, will fall within the circle [e]. Moreover, because the triangles ACD, BCD have CA = CB and CD common, thence will AD be also = BD [f].

[c] Def. 33. 1.
[d] 20. 1.
[e] Ax. 2.
[f] 16. 1.

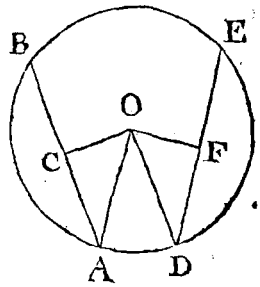
COROLLARY.

Hence a line bisecting any chord at right-angles, passes thro' the center of the circle.

THEO-

THEOREM III.

Any two chords (AB, DE) *equally distant from the center* (O) *of a circle, are equal to each other.*

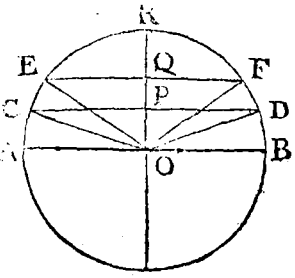


Let the perpendiculars OF, OC be drawn, and let O, D and O, A be joined. Because OF = OC [e], OD = OA [f], and F and C are both right-angles [g], therefore is DF = AC [h], and consequently DE = 2DF [i] = 2AC [k] = AB [l].

[e] Hyp.
[f] Def. 33. of 1.
[g] Constr.
[h] 16. 1.
[i] 2. 3.
[k] Ax. 4. 1.

THEOREM IV.

In a circle (AEFB) *the greatest line* (AB) *is the diameter; and, of all others terminating in the circumference, that* (CD) *which is nearest the center* (O), *is greater than any other* (EF) *further from it.*



1. Draw OC and OD; then it will appear that AB (or OC + OD) ⊏ CD [m].

2. Let OP be the distance of CD from the center, and OQ that of EF, both taken in the same radius OR; draw OE and OF. Because the triangles DOC, OFE, have two sides equal each to each [n], and have the contained angle DOC ⊏ the contained angle FOE [o]; therefore, also, will the base DC be greater than the base FE [p]; and, consequently, greater than any other chord at the same distance with EF [q].

[m] 19. 1.
[n] Def. 33. 1.
[o] Ax. 21.
[p] 21. 1.
[q] 3. 3.

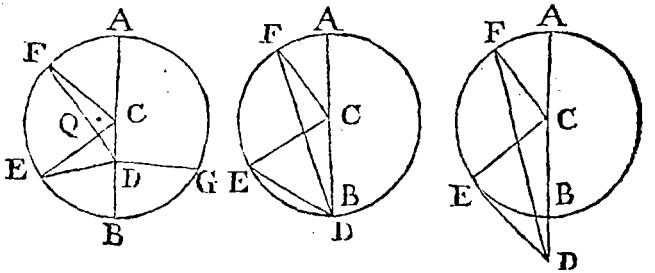
COROLLARY.

Hence, a right-line greater than the diameter, drawn from any point within a circle, will cut the circumference.

THEO-

THEOREM V.

If, to the circumference of a circle (AFEB), *from any point* (D) *which is not the center, right-lines* (DA, DF, DE) *be drawn, the greateſt of all* (DA) *ſhall be that which paſſes through the center* (C); *and, of the reſt, that* (DF) *whoſe other extreme* (F) *is placed neareſt, in the circumference, to the extreme* (A) *of the greateſt, will exceed any other* (DE) *whoſe extreme* (E) *is at a greater diſtance.*



From the center C, let CE and CF be drawn.

1. AD (= DC + CF [r]) ⊏ DF [s]. [r] Ax. 4. [s] 19. 1.

2. Since DC is common, CF = CE, and DCF ⊏ DCE [t], therefore is DF ⊏ DE [u]. [t] Ax. 2. [u] 21. 1.

COROLLARY I.

Becauſe no two lines, DE, DF, drawn from D, on the ſame ſide of the diameter AB, can be equal to each other [w], three equal right-lines cannot poſſibly be drawn from the periphery to any point, beſides the center of the circle: and, therefore, if from a point in any circle, three equal right-lines can be drawn to the periphery, that point is the center of the circle. [w] 5. 3.

COROLLARY II.

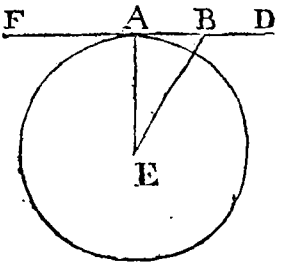
Hence it alſo follows, that no circle can be deſcribed to cut another FBG in more points than two: for, if it were poſſible to cut it in three points G, E, F, then right-lines drawn from the center

Q,

[x] Def. 33. 1. [y] Cor. 1. to 4. 3. Q, to those points, would be all equal [x], which is shewn to be impossible [y], unless when the center Q coincides with C; and then the circles themselves will neither cut, nor touch, but coincide, and become one circle [z].

THEOREM VI.

A right-line (FD) *drawn through any point* (A) *in the circumference of a circle, at right-angles to the radius* (EA) *terminating in that point, will touch the circle.*



From any point in FD, to the center E, let the right-line BE be drawn; which being greater than AE [a], the point B must, necessarily, fall out of the circle [b]: and therefore, as the same argument holds good with regard to every other point in the line FD (except A), it is manifest that this line cuts off no part of the circle, but touches it, in one point only.

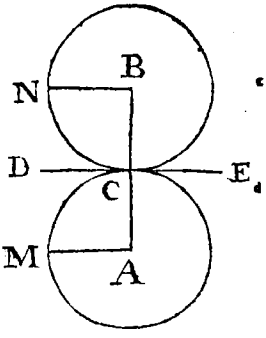
[a] 20. 1. [b] Def. 33. and Ax. 2. of 1.

THEOREM VII.

If the distance (AB) *of the centers of two circles, be equal to the sum of the two semi-diameters* (AM, BN), *the circles will touch each other, outwardly; and the right-line* (AB) *joining their centers, will pass through the point of contact.*

In

In AB, take AC = AM, and let DCE be drawn perpendicular to AB: then, BC being alſo = BN [c], the circumferences of both circles will paſs through the point C [d]: but the right-line DE *(by the precedent)* falls wholly above the one, and wholly below the other; therefore the circles themſelves fall wholly without each other, and touch in one point C only.



[c] Conſtr. and Ax. 5.

[d] Def. 33. 1.

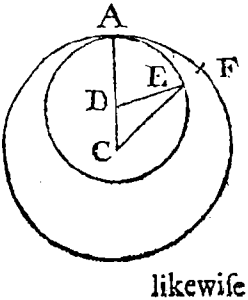
COROLLARY.

Hence, if the centers of two circles be placed at a diſtance, from one another, leſs than the ſum of the two ſemi-diameters, a part, at leaſt, of the one will be contained within the other: but, if the diſtance be greater than that ſum, the two circles will then neither touch, nor cut each other.

THEOREM VIII.

If the diſtance (CD) *of the centers of two circles* (CAF, DAE) *be equal to the difference of the two ſemi-diameters* (CA, DE), *then will thoſe circles touch inwardly; and that radius* (CA) *of the greater, which is drawn through the center* (D) *of the leſſer, will meet the two peripheries in the point of contact.*

From any point E in the circumference of the leſſer, to the two centers, let EC and ED be drawn. Becauſe CA exceeds DE by the line DC [e], or becauſe DE + DC = CA [f] = DA + DC [g], therefore is DA = DE [h]; and ſo the circumference of the circle D



[e] Hyp.

[f] Ax. 4.

[g] Ax. 3.

[h] Ax. 5.

likewiſe

likewise passes through A: but CA is greater than CE[i]: therefore every point in the periphery of the circle D (except A only) falls within the circle C: *which was to be demonstrated.*

[i] 5. 3.

COROLLARY I.

Hence, if the centers of two circles be placed at a distance from each other, greater than the difference of the two semi-diameters, a part, at least, of the one will fall without the other; but, if the distance be less than that difference, the lesser circle will then be contained wholly in the greater, but without touching it.

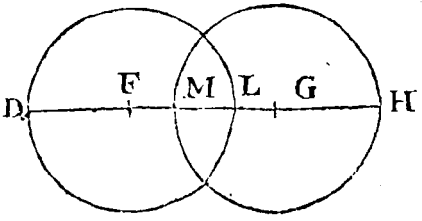
COROLLARY II.

Hence, and from *the precedent*, it likewise appears, that if two circles touch, either inwardly or outwardly, a right-line, drawn through their two centers, will also pass through the point of contact: because they can only touch, when the distance of their centers is equal to the sum, or to the difference of their semi-diameters[k].

[k] Cor. of 7. and Cor. 1. of 8.

THEOREM IX.

If the distance of the centers (F, G) *of two circles* (DL, MH) *be less than the sum, and greater than the difference of the two semi-diameters* (FL, GM), *those circles will cut each other.*



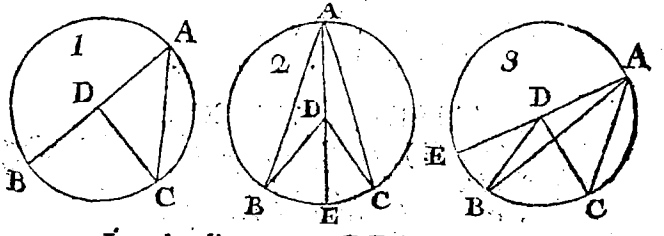
For, since the distance of the two centers is supposed less than the sum of the semi-diameters, a part of the one circle MH, falls within the other DL[l]; and since that distance is greater than the difference of those semi-diameters, a part of the same circle MH also falls without the circle DL[m]: *which was to be proved*[n].

[l] Cor. to 7. 3.
[m] Cor. to 8. 3.
[n] Def. 11. of 3.

THEO-

THEOREM X.

The angle (BDC) *at the center of a circle, is double to the angle* (BAC) *at the circumference, when both angles stand upon the same arch* (BC).



Let the diameter ADE be drawn.

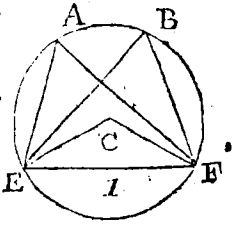
In the first case (where AB passes through the center) BDC = A + C [o] = 2A [p]. [o] 9. 1.

In the second case, BDE = 2BAE (*by case* 1.); [p] 12. 1. to which adding CDE = 2CAE, we have BDC = 2BAC [q]. [q] Ax. 4.

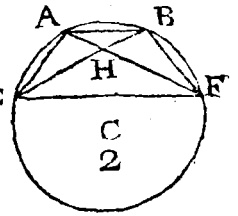
In the third case, CDE = 2CAE (*by case* 1.) from whence subtracting BDE = 2BAE, there remains BDC = 2BAC [r]. [r] Ax. 5.

THEOREM XI.

All angles (EAF, EBF) *in the same segment* (EABF) *of a circle, are equal to each other.*



CASE I. *If the segment be greater than a semi-circle*; from the center C draw CE and CF; then EAF and EBF being each of them = to half ECF [s], they must necessarily be equal to each other. [s] 10. 3.

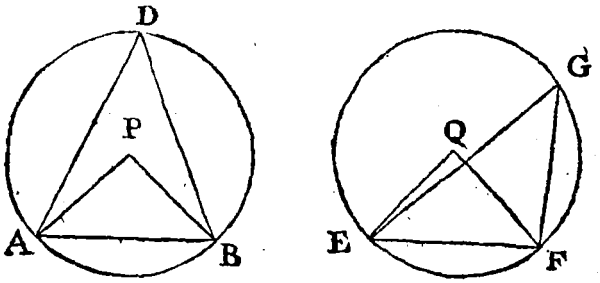


CASE II. *If the segment be less than a semi-circle*; let H be the intersection of EB and AF: then the triangles AEH and BFH, having the angle AHE = BHF [t], and AEH = BFH (*by case* 1.) they will also have EAH = FBH [u].

[t] 3. 1.
[u] Cor. 1. to 10. 1.

THEOREM XII.

Angles (D, G) *in the circumferences, standing upon equal subtenses* (AB, EF) *of circles having equal diameters, are equal to each other. And the subtenses of equal angles, in the circumferences of circles having equal diameters, are also equal.*



From the centers P, and Q, let PA, PB, QE, QF be drawn.

1. Hyp. Since AB = EF [w], and AP = BP [x] = EQ = FQ [w]; therefore is P = Q [y], and consequently D ($= \frac{1}{2}P$ [z] $= \frac{1}{2}Q$) = G.

2. Hyp. Because D = G, therefore P = Q [z]; whence, PA being = QE, and PB = QF [w], AB will also be = EF [a].

[w] Hyp.
[x] Def. 33. of 1.
[y] 14. 1.
[z] 10. 3.
[a] Ax. 10. of 1.

COROLLARY.

Hence angles in the circumference, standing upon equal chords of the same circle, are equal.

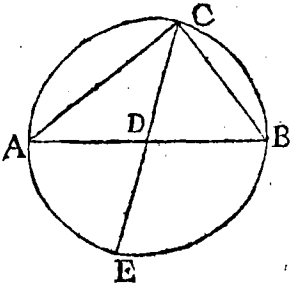
THEOREM

THEOREM XIII.

The angle (ACB) *in a semi-circle, is a right angle.*

Let the diameter CDE be drawn.

Because ACD = half ADE, and BCD = half BDE [b], therefore is ACD + BCD (= ACB) = half of ADE and BDE [c] = half two right angles [d] = one right angle.



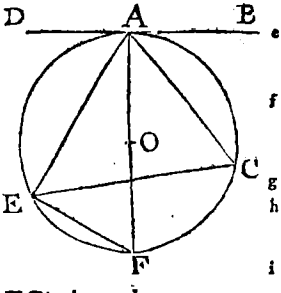
[b] 10. 3.
[c] Ax. 4. 1.
[d] 1. 1.

THEOREM XIV.

The angle (CAB) *included by a tangent to a circle and a chord* (AC) *drawn from the point of contact* (A), *is equal to the angle* (AEC) *in the alternate segment.*

Let the diameter AOF be drawn, and E, F be joined.

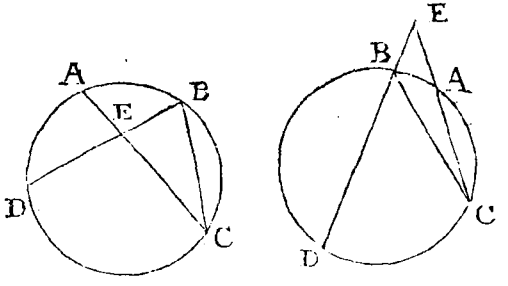
The line DB falling wholly above the circle [e], OA is the least line that can be drawn to it from the center O [f]; and OAB is therefore a right-angle [g]: but FEA is also a right-angle [h]: therefore, if from these equal angles, the equal [i] angles FAC, FEC (standing on the same arch FC) be taken away, there will remain BAC = AEC [k].



[e] Def. 9. 3.
[f] Def. 33. 1. and Ax. 2.
[g] 20. 1.
[h] 13. 3
[i] 11. 3.
[k] Ax. 5.

THEOREM XV.

The angle (DEC) *made by two lines* (DEB, CEA) *intersecting each other within, or without a circle, is, in the former case, equal to the sum, and in the latter, equal to the difference, of two angles in the circumference, standing on the two arcs* (DC, AB) *intercepted by those lines.*



Let the chord CB be drawn.

[g] 9. 1. Then DEC = DBC + ACB[g], *in the first case.*

[h] 9. 1. and Ax. 5. And DEC = DBC — ACB[h], *in the second case.*

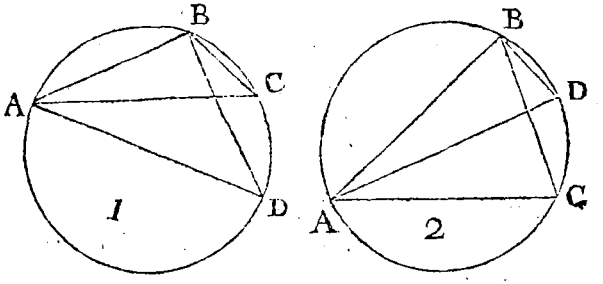
COROLLARY.

Hence an angle (E) formed below, or above the circumference of a circle, is greater, or less than an angle in the circumference, standing on the same arch.

THEOREM XVI.

The vertical angle (ABC) *of any oblique-angled triangle* (ACB) *inscribed in a circle* (ABCD) *is greater, or less than a right-angle, by the angle* (CAD) *comprehended*

prehended under the base (AC) *and the diameter* (AD) *drawn from the extremity of the base.*



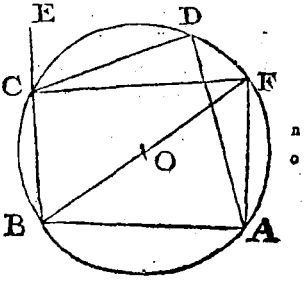
For, BD being drawn, ABD will be a right-angle [i], and CAD = CBD [k]; therefore, in the first case, ABC = right-angle + CAD [l]; and in the second, ABC = right-angle — CAD [m].

[i] 13. 3.
[k] 11. 3.
[l] Ax. 4.
[m] Ax. 5.

THEOREM XVII.

If any side (BC) *of a quadrilateral* (ABCD) *inscribed in a circle, be produced out of the circle, the external angle* (ECD) *will be equal to the opposite, internal angle* (BAD).

Let the diameter BF be drawn, and let AF and CF be joined: then the angle BAF being a right angle (= BCF) = ECF [n], and DAF also = DCF [o], standing both on the same arch DF; thence will the remainders BAD and ECD be also equal [q].



[n] 13. 3.
[o] 11. 3.
[q] Ax. 5.

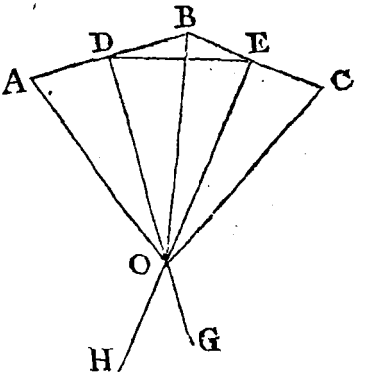
COROLLARY.

Hence the opposite angles BAD, BCD of any quadrilateral inscribed in a circle, are together, equal to two right-angles For, since BAD = ECD, therefore is BAD + BCD = ECD + BCD[r] = two right-angles[s].

[r] Ax. 4.
[s] 1. 1.

THEOREM XVIII.

Through any three points (A, B, C) *not situate in the same right line, the circumference of a circle may be described.*



Draw AB and BC, which let be bisected by the perpendiculars DG and EH, infinitely produced on that side of AB or BC, on which the angle ABC is formed.

These perpendiculars, I say, will intersect each other; and the point of intersection O, will be the center of the circle.

For, if DE be drawn, it is plain, that the angles GDE, HED are less than two right-angles[t]; therefore DG, EH, not being parallels[u], they will meet each other[w]. Hence, if from the point of intersection O, the right-lines OA, OB, OC be drawn, the triangles ADO, BDO, having two sides equal, each to each[x], and the angles ADO, BDO, contained by them, equal[y], will likewise have AO = BO[z]. After the very same manner is CO = BO; therefore AO = BO = CO[a]: whence the circumference of a circle described from the center O, at the

[t] Ax. 2.
[u] Cor. 2. to 7. 1.
[w] Def. 13. of 1.
[x] Constr.
[y] Ax. 7.
[z] Ax. 10.
[a] Ax. 1.

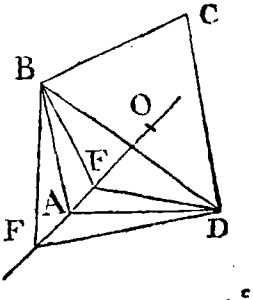
the distance of AO, will also pass through B and C [b]. [b] Def. 33. 1.

SCHOLIUM.

Hence the method of describing the circumference of a circle through three given points, is manifest.

THEOREM XIX.

If the opposite angles (BAD, BCD) *of a quadrilateral* (ABCD) *be equal to two right-angles, a circle may be described about that quadrilateral.*

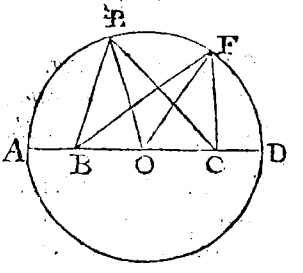


For the circumference of a circle may be described thro' any three points B, C, D, (*by the precedent.*) But, if you deny that it passes thro' A; then, thro' the center O, let OAF be drawn, and let it (if possible) pass through some other point F in the line OAF, (for it must cut this line somewhere [c]); also let BF and DF be drawn. Because BFD + BCD = two right-angles [d] = BAD + BCD [e]; therefore must BFD = BAD [f]: *which is impossible* [g]. Therefore the circumference of the circle described through B, C and D, must also pass through A.

[c] Cor. to 4. 3.
[d] Cor. to 17. 3.
[e] Hyp.
[f] Ax. 5.
[g] 23. 1.

THEOREM XX.

If from two points (B, C) *in the same diameter* (AD), *equally distant from the center* (O) *of a circle, right-lines* (BE, CE; BF, CF) *be drawn to meet, two by two, in the circumference; the sum of the squares of any two corresponding ones, will be equal to the sum of the squares of any other two, meeting in like manner.*

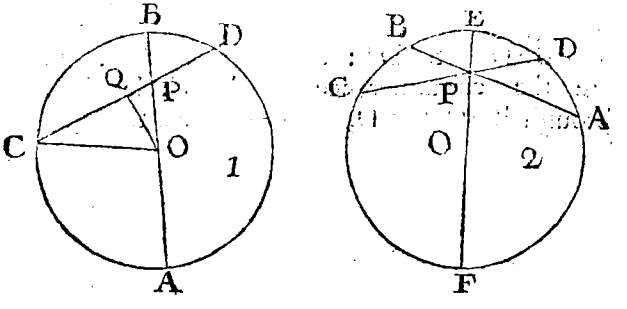


For, if OE and OF be drawn; then will $BE^2 + CE^2 =$ [h] $2BO^2 + 2OE^2$ $(2OF^2) =$ [h] $BF^2 + CF^2$.

[h] 11. 2.

THEOREM XXI.

If two lines (AB, CD), *terminated by the periphery on both sides, cut each other within a circle, the rectangle* (AP × BP) *contained under the parts of the one, will be equal to the rectangle* (CP × DP) *contained under the parts of the other.*



CASE I. If one of the two lines (AB) passes through the center O; then let OQ be drawn perpendicular to the other CD, and let OC be joined.

It

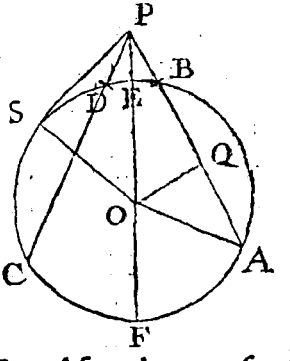
It is plain, because QD = QC [i], that DP is equal to the difference of the segments CQ and PQ [k]: but the rectangle under the sum and difference of the two sides OC, OP, of any triangle COP, is equal to the rectangle under the whole base CP, and the difference of its two segments [l]; therefore, the sum of the two sides OC, OP being (= OA + OP) = AP, and their difference (= OB — OP) = BP, thence is the rectangle contained under AP and BP equal to the rectangle contained under CP and DP.

[i] 2. 3.
[k] Ax. 5.
[l] Cor. 1. to 9. 2.

CASE II. If neither of the two lines pass through the center: let the diameter EPF be drawn; then, *by the preceding case*, AP × BP = FP × EP = CP × DP.

THEOREM XXII.

If from two points (A, C) *in the circumference of a circle, two lines* (AP, CP) *be drawn, to pass through, and meet without the circle; the rectangle* (AP × BP) *contained under the whole and the external part of the one, will be equal to the rectangle* (CP × DP) *contained under the whole and the external part of the other.*



Through the center O, let PF be drawn, meeting the circumference in E and F; let OQ be perpendicular to AP, and let A, O be joined.

Then, *(by Cor. 1. to 9. 2.)* the rectangle contained under PF (= PO + OA) and PE (= PO — OA) is = the rectangle contained under AP and PB. After the very same manner, PF × PE = CP × DP: therefore AP × BP = CP × DP [m].

[m] Ax. 1.

COROL-

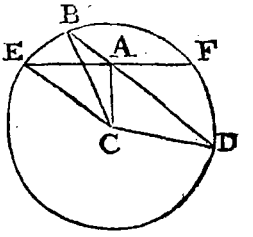
COROLLARY I.

Hence, if PS be a tangent at S, and the radius OS be drawn; then, PF being = the ſum of PO and OS, and PE = their difference; it follows, that $PS^2 = PF \times PE$ [o] $= PC \times PD$.

[o] Cor. to 8. 2.

THEOREM XXIII.

If from the center (C) *of a circle, to a point* (A) *in any chord* (BD), *a line* (CA) *be drawn; the ſquare of that line, together with the rectangle contained under the two parts of the chord, will be equal to a ſquare made upon the radius of the circle.*



Let EAF be another chord, perpendicular to CA, and let C, E be joined.

Since AF = AE [q], thence will $AE^2 = AE \times AF$ [r] $=$ $AB \times AD$ [s]; to which equal quantities adding AC^2, we have [t] $CE^2 = AB \times AD + AC^2$.

[q] 2. 3.
[r] 1. 2.
[s] 21. 3.
[t] 8. 2.

COROLLARY

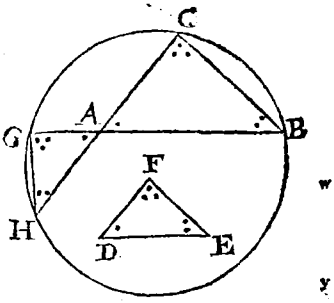
Hence the ſquare of a line (AC) drawn from any point in the baſe of an iſoſceles triangle (BCD) to the oppoſite angle, together with the rectangle of the parts of the baſe, is equal to a ſquare made upon one of the equal ſides of the triangle.

THEOREM XXIV.

The rectangles contained under the correſponding ſides of equiangular triangles (ABC, DEF) *taken alternately, are equal.*

I ſay, if A = D, B = E and C = F, then will $AB \times DF = AC \times DE$.

In



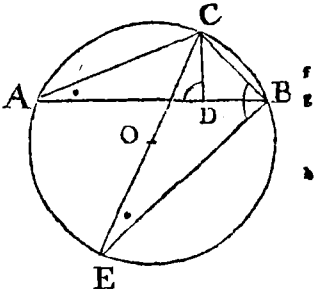
In BA produced, let AG be taken = DF; let GCB be the circumference of a circle passing through the three points B, C, G [w], meeting CA produced in H; and let G, H be joined.

Because the angle H = B [y] = E [z], HAG = [a] BAC = D [z], and AG = DF [z], thence is AH = DE [b]; and therefore AC × DE = AC × AH [c], = AB × AG [d] = AB × DF [e].

[w] 18. 3.
[y] 11. 3.
[z] Hyp.
[a] 3. 1.
[b] 15. 1.
[c] 1. 2.
[d] 21. 3.

THEOREM XXV.

The rectangle under the two sides (AC, BC) *of any triangle* (ABC), *is equal to the rectangle under the perpendicular* (CD) *to the base thereof, and the diameter* (CE) *of the circumscribing circle.*



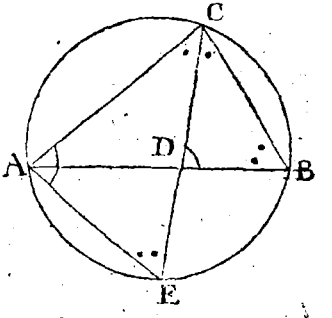
For, B, E being joined, the angles A, E will be equal [f], and ADC, EBC both right-angles [g]; and, consequently, the triangles ACD, ECB equiangular [h]: therefore AC, EC; CD, CB being corresponding sides, opposed to equal angles, the rectangle AC × CB, contained under the first and last of them, will be equal to the rectangle EC × CD contained under the other two [i].

[f] 11. 3.
[g] Hyp. and 13. 3.
[h] Cor. 1. to 10. 1.
[i] 24. 3.

THEO-

THEOREM XXVI.

The square of a line (CD) *bisecting any angle* (C) *of a triangle* (ABC) *and terminating in the opposite side* (AB), *together with the rectangle* (AD × BD) *under the two segments of that side, is equal to the rectangle of the two sides including the proposed angle.*



Let CD be produced to meet the circumference of a circle, described [k] through the points A, C, B, in E; and let AE be drawn.

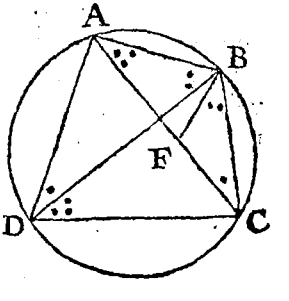
[k] 81. 3.

The angles E and B, standing upon the same segment AC, are equal [l]; and ACE is equal DCB (by hypothesis); therefore the triangles AEC, DCB are equiangular [m]; whereof AC, CD; CE, CB are corresponding sides, opposed to equal angles: therefore AC × CB = CD × CE [n] = CD^2 + CD × DE [o] = CD^2 + AD × DB [p].

[l] 11. 3.
[m] Cor. 1. to 10. 1.
[n] 24. 3.
[o] 5. 2.
[p] 21. 3. and Ax. 4.

THEOREM XXVII.

The rectangle of the two diagonals (AC, BD) *of any quadrilateral* (ABCD) *inscribed in a circle, is equal to the sum of the two rectangles* (AB × DC, AD × BC) *contained under the opposite sides.*



Let BF be drawn, making the angle CBF = ABD, and meeting AC in F.

Because the angle BCF = ADB [q], and CBF = ABD [r], the triangles CBF, DBA are equiangular [s]; and therefore, BC, BD; CF, AD, being corresponding sides, the rectangles BC × AD, and BD ×

[q] 11. 3.
[r] Constr.
[s] Cor. 1. to 10. 1.

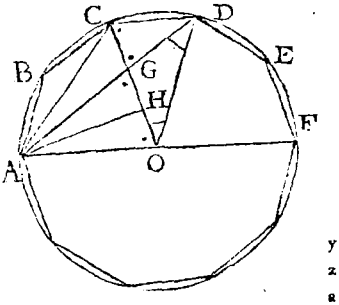
CF

CF will be equal [t]. Again, the angle ABF being = CBD [u], and BAF = BDC [w], the triangles ABF and BDC are, likewise, equiangular; and consequently, AB, BD; AF, DC being corresponding sides, AB × DC = BD × AF [t]; to which adding BC × AD = BD × CF (*so proved above*) we have also AB × DC + BC × AD = BD × AF + BD × CF = BD × AC [x].

[t] 24. 3.
[u] Ax. 4. 1.
[w] 11. 3.
[x] 5. 2.

THEOREM XXVIII.

If the radius of a circle (OADF) *be so divided into two parts, that the rectangle under the whole and the one part shall be equal to the square of the other part; then will this last part be equal to the side* (CD) *of a regular decagon* (ABCDEF, &c.) *inscribed in the circle; and that line whose square is equal to the two squares of the whole, and of the same part, will be equal to the side* (AC) *of a regular pentagon inscribed in the same circle.*



Draw the radii OA, OC, OD, OF; also draw AD, cutting OC in G, and let AH be perpendicular to OG.

The triangle ODG, having the angle COD (= ½DOF [y] = OAD [z]) = ODA [a], is isosceles [b]: moreover the triangle AOG, having AGO (= GDO + DOG [c] = 2DOC [d]) = AOC, is likewise isosceles [d]; as is also the triangle CDG, because, CGD being = AGO [e], and CDG (CDA) = FAD [f], the triangles AOG, CDG are equiangular. Therefore, CD, AO; CG, GO being corresponding sides, we have CG × AO (CG × CO) = CD × GO [g] = GO², because GO = GD

[y] 1. 3.
[z] 10. 3.
[a] 12. 1.
[b] 18. 1.
[c] 9. 1.
[d] 18. 1.
[e] 3. 1.
[f] Cor. to 12. 3.
[g] 24. 3.

[h] 18. 1. $= GD = DC$ [h]: whence the former part of the propoſition is manifeſt.

[i] 16. 1. Again, becauſe $AG = AO$, HG will be $= HO$ [i]; and ſo GC being the difference of the ſegments HO and HC, we have (*by Cor.* 1. *to* 9. 2.) $AC^2 - AO^2 = CO \times CG = OG^2$ (*as above*); and conſequently $AC^2 = AO^2 + OG^2$.

The End of the THIRD BOOK.

ELEMENTS

OF

GEOMETRY.

BOOK IV.

DEFINITIONS.

1. RATIO is the proportion which one magnitude bears to another magnitude of the ſame kind, with reſpect to quantity.

The meaſure, or quantity of a ratio is conceived by conſidering what part, or parts the magnitude referred, called the antecedent, is of the other, to which it is referred, called the conſequent.

2. Three quantities, or magnitudes A, B, C, are ſaid to be proportional, when the ratio of the firſt A to the ſecond B, is the ſame as the ratio of the ſecond B, to the third C. A, B, C. 2. 4. 8.

3. Four quantities A, B, C, D, are ſaid to be proportional, when the ratio of the firſt A to the ſecond B, is the ſame as the ratio of the third C to the fourth D. A, B, C, D. 2. 4. 5. 10.

To

To denote that four quantitites A, B, C, D *are proportional, they are usually wrote thus,* A : B : : C : D; *and read thus, as* A *is to* B, *so is* C *to* D. *But when three quantities* A, B, C *are proportional, the middle one is repeated, and they are wrote thus,* A : B : : B : C.

4. Of three proportional quantities, the middle one is said to be a Mean-proportional between the other two; and the last, a Third-proportional to the first and second.

5. Of four proportional quantities, the last is said to be a Fourth-proportional to the other three, taken in order.

A,B,C,D, E. 1. 2. 4. 8. 16.

6. Quantities are said to be continually proportional (or in continual proportion) when the first is to the second, as the second to the third, as the third to the fourth, as the fourth to the fifth, and so on.

7. In a series, or rank of quantities continually proportional, the ratio of the first and third, is said to be duplicate to that of the first and second; and the ratio of the first and fourth, triplicate to that of the first and second.

8. Any number of quantities, A, B, C, D being given, or propounded, the ratio of the first (A) to the last (D) is said to be compounded of the ratios of the first to the second, of the second to the third, and so on to the last.

9. Ratio of equality, is that which equal quantities bear to each other.

It may be observed here, that ratio of equality, and equality of ratios, are, by no means, synonymous terms: since

ſince two, or more ratios may be equal, though the quantities compared are all unequal. Thus, the ratio of 2 *to* 1, *is equal to the ratio of* 6 *to* 3, (2 *being the double of* 1, *and* 6 *the double of* 3); *yet none of the four numbers are equal.*

10. Inverſe ratio is, when the antecedent is made the conſequent, and the conſequent the antecedent. *Thus, if* 2 : 1 : : 6 : 3; *then, inverſly,* 1 : 2 : : 3 : 6.

11. Alternate proportion is, when antecedent is compared with antecedent, and conſequent with conſequent.

As, if 2 : 1 : : 6 : 3; *then,* by alternation (or permutation) *it will be* 2 : 6 : : 1 : 3.

12. Compounded ratio is, when the antecedent and conſequent, taken as one quantity, are compared, either with the conſequent, or with the antecedent.

Thus, if 2 : 1 : : 6 : 3; *then,* by compoſition, 2 + 1 : 1 : : 6 + 3 : 3, *and* 2 + 1 : 2 : : 6 + 3 : 6.

13. Divided ratio is, when the difference of the antecedent and conſequent is compared, either with the conſequent, or with the antecedent.

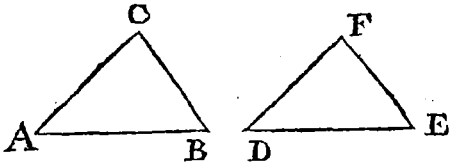
Thus, if 3 : 1 : : 12 : 4; *then,* by diviſion, 3 — 1 : 1 : : 12 — 4 : 4, and 3 — 1 : 3 : : 12 — 4 : 12.

Theſe four laſt definitions, which explain the names given by Geometers *to the different ways of managing and diverſifying of proportions, are put down here for the ſake of order; but are not to be uſed, or referred to, in any ſhape, till thoſe properties and relations are demonſtrated; which is effected in the three firſt Theorems of this book.*

14. Similar (or like) right-lined figures are ſuch, which have all their angles equal, one to another

 reſpec-

respectively, and also the sides about the equal angles proportional.



Thus, if the angle A = D, B = E, C = F; also AC : AB : : DF : DE, BA : BC : : ED : EF, *&c.* then the figures ABC, DEF are said to be *similar*.

AXIOMS.

1. The same quantity being compared with ever so many equal quantities, successively, will have the same ratio to them all.

2. Equal quantities, have to one and the same quantity, the same ratio.

3. Quantities having the same ratio to one and the same quantity, or to equal quantities, are equal among themselves.

4. Quantities, to which one and the same quantity has the same ratio, are equal.

5. If two quantities be referred to a third, that which is the greatest will have the greatest ratio.

6. If two quantities be referred to a third, that is the greatest which has the greatest ratio.

7. Ratios, equal to one and the same ratio, are also equal, one to the other.

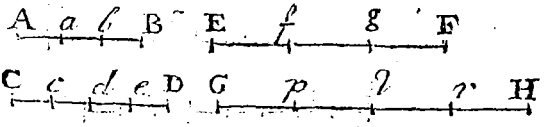
8. If two quantities be divided into, or composed of parts, that are equal among themselves, or all of the same magnitude; then will the whole of the one, have the same ratio to the whole of the other, as the number of the parts in the one, has to the number of equal parts in the other.

9. If the double, treble, or quadruple, *&c.* of every part of any quantity be taken, the aggregate will

will be the double, treble, or quadruple, &c. of the whole quantity propounded.

THEOREM I.

Equimultiples of any two quantities (AB, CD) *are in the same ratio as the quantities themselves.*



Let the ratio of AB to CD be that of any one number M (3) to any other number N (4), or, which is the same, let AB contain M (3) such equal parts (Aa, ab, bB [a]), whereof CD contains the number N (4). Let there be taken Ef, fg, gF any equimultiples of Aa, ab, bB, respectively; and let Gp, pq, qr, rH be the same multiples of Cc, cd, de, eD; so shall the whole EF be the same assigned multiple of the whole AB, and the whole GH of the whole CD, as each part in the one, is of its correspondent in the other [b]. And, since the parts Aa, ab, bB, Cc, cd, &c. are all equal [c], their equimultiples Ef, fg, gF, Gp, pq, &c. will also be equal [d]. Therefore EF is in proportion to GH, as the number of the parts in EF is to the number of equal parts in GH [e], or (which is the same) as the number of parts in AB to the number of parts in CD, that is, as AB is to CD [a]: which was to be demonstrated.

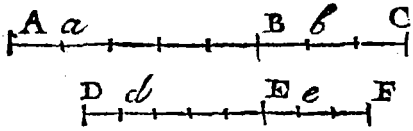
[a] Ax. 8. 4.
[b] Ax. 9. 4.
[c] Hyp.
[d] Ax. 4. 1.

COROLLARY.

Hence, like parts of quantities, have the same ratio as the wholes; because the wholes are equimultiples of the like parts.

THEOREM II.

The two antecedents (AB, DE) *of four proportional quantities, of the same kind* (AB, BC, DE, EF) *are in the same ratio with the two consequents* (BC, EF).



Let the common ratio of AB to BC, and of DE to EF, be that of any one number *M* (5) to any other number *N* (3); then will AB contain *M* (5) such equal parts (A*a*) whereof BC contains *N* (3); and DE will, in like manner, contain *M* (5) such equal parts (D*d*), whereof EF contains *N* (3). And so, AB and DE, as well as BC and EF, being equimultiples of A*a* and D*d*, thence will AB : DE : : A*a* (B*b*) : D*d* (E*e*) : : BC : EF[f]

[e] Ax. 8. of 4.

[f] 1. 4.

COROLLARY.

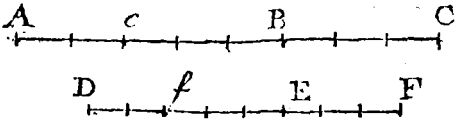
That, the proportionality will subsist, when the consequents are taken as antecedents, and the antecedents as consequents, also appears from hence. For BC : AB : : number of parts in BC (or EF) : number of parts in AB (or DE) : : EF : DE[e].

THEOREM III.

Of four proportional quantities (AB; BC, DE, EF) *the sum, or difference of the first antecedent and consequent* (AB ± BC) *is to the first antecedent, or consequent, as the sum, or difference of the second antecedent and consequent* (DE ± EF) *is to the second antecedent, or consequent.*

Let

Let what was premised in the demonstration of the preceding theorem, be retained here: then will



AC (AB + BC) be in proportion to AB, as the number of parts in AC is to the number of equal parts in AB [g], or as the number of parts in DF [g] (DE + EF) to the number of equal parts in DE, that is, as DF (DE + EF) is to DE [g]. Again, if from AB and DE, be taken away B*c* = BC, and E*f* = EF, then will the difference A*c* be in proportion to AB, as the number of parts in A*c* (or D*f*) is to the number of parts in AB (or DE [g]), that is, as D*f* is to DE. In the same manner it will appear, that AB + BC : BC : : DE + EF : EF; and AB — BC : BC : : DE — EF : EF. [g] Ax. 8. 4.

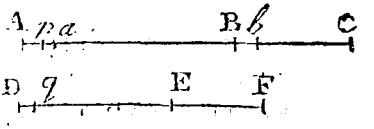
COROLLARY.

It will appear from hence, that the sum of the greatest and least (AB + EF) of four proportional quantities (of the same kind) will exceed the sum (BC + DE) of the two means: because, AB being supposed greater than DE, A*c* will be greater than D*f*, in the same proportion [h]: and, if to these there be added BC + EF (common); then will the sum A*c* + BC + EF (AB + EF) be also greater than the sum D*f* + BC + EF [i] (DE + BC.) [h] 3. 4. [i] Ax. 6. 2.

SCHOLIUM.

In the demonstration of this, and the preceding theorems, the antecedent and consequent are supposed to be divided into parts, all mutually equal among themselves. But it is known to Mathematicians, that there are certain quantities, or magnitudes

nitudes that cannot possibly be divided in that manner, by means of a *common measure.* The Theorems themselves are, nevertheless, true, when applied to these *incommensurables:* since no two quantities, of the same kind, can possibly be assigned, whose ratio cannot be expressed by that of two numbers, so near, that the difference shall be less than the least thing that can be named. But if the matter, viewed in this light, should not appear sufficiently *scientific*, and you will not (in the preceding theorem) allow the ratio of AC to BC, to be *exactly* the same with that of DF to EF, when AB, BC, and DE, EF, are incommensurables; then let it, if possible, be as some quantity *a*C (less than AC) is to BC, so is DF to EF. Let B*b* be a part or measure of BC less than the difference (A*a*) between AB and *a*B; let B*p* be that multiple of B*b*, which least exceeds B*a*, and let *q*E be to EF, as *p*B to BC.



It is evident, that *p*B is less than AB (because *ap* ⊐ B*b* ⊐ A*a*); and that *q*E is also less than DE, because the ratio of *q*E to EF, being equal to that of *p*B to BC [k], it must necessarily be less than that of AB to BC [l], or of DE to EF; and so *q*E less than DE [m].

[k] Hyp.
[l] Ax. 5. 4.
[m] Ax. 6. 4.

Now, if (as is supposed) the ratio of *a*C to BC can be = the ratio of DF to EF, it must, of consequence, be greater the ratio of *q*E to EF [n], or (which is the [o] same) than the ratio of *p*C to BC [l], *which is impossible.* In like manner it will appear, that no quantity, that is greater than AC, can possibly be to BC, as DF is to EF. Therefore AC : BC :: DF : EF. And by the same kind of argumentation (authorised and adopted by *Euclid* himself, in

[n] Ax. 5. and 7. 4.
[o] Proof of 3. 4.

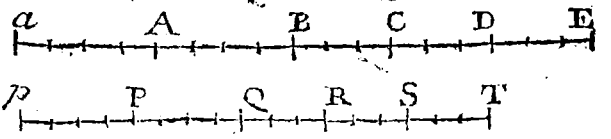
in his twelfth book) any difficulties, or ſcruples, that may be elſewhere brought, from the incommenſurability of quantities, may be obviated and removed.

THEOREM IV.

If, of four proportionals (AB, BC, PQ, QR) *equimultiples of the antecedents* (AB, PQ) *be taken, and compared with any equimultiples of their reſpective conſequents* (BC, QR), *the ratios will be the ſame, and the four quantities proportionals.*

Let the common ratio of AB to BC, and of PQ to QR, be that of any one number *M* to any other number *N*: ſo ſhall AB contain *M* ſuch equal parts whereof BC contains [p] *N*, and PQ, in like manner, *M* ſuch equal parts whereof QR contains *N*. [p] Ax.8. 4.

Let CD, DE be taken each = BC, and RS, ST each = QR, ſo that BE and QT may be equimultiples of BC and QR; and let CD, DE; RS, ST be conceived to be divided, each into the ſame number of parts with BC, or QR. In like man-

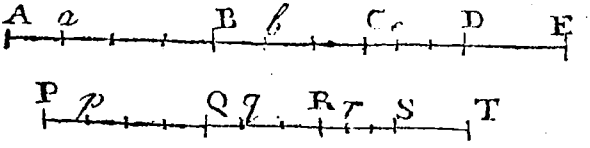


ner, let *a*B and *p*Q be taken as equimultiples of AB and PQ, *&c.* Then will the number of parts in BD = number of parts in QS [p], and the number of parts in BE = number of parts in QT [q]: and ſo likewiſe with reſpect to *a*B and *p*Q. Therefore *a*B is to BE, as the number of parts in *a*B to the number of (equal) parts in BE [p], or, which is the ſame thing, as the number of parts in *p*Q to the number of parts in QT, that is, as *p*Q is to QT, *which was to be demonſtrated.* [q] Ax.4. 1.

THEOREM V.

If, of two ranks of quantities (AB, BC, CD; PQ, QR, RS), *the ratio of the first and second, in the one, be equal to the ratio of the first and second in the other, and the ratio of the second and third, in the one, equal likewise to the ratio of the second and third in the other; then, also, shall the ratio of the first to the third, be the same in the one rank, as in the other.*

Let the common ratio of AB to BC, and of PQ to QR, be still expressed as in the preceding demonstrations. Let, moreover, CD and RS be conceived to be divided, each into the same number of parts with BC and QR.



Because the quantities BC, CD, QR, RS, are proportional [r], their like parts B*b*, C*c*, Q*q*, R*r* (being in the same ratio with the wholes [s]) will also be proportionals; or, because A*a* = B*b* and P*p* = Q*q* [t], it will be [t] A*a* : C*c* : : P*p* : R*r*. But AB and PQ are *equimultiples* of the antecedents A*a* and P*p* [t]; and CD, RS are *equimultiples* of the consequents C*c* and R*r*: therefore [u] AB : CD : : PQ : RS *;

[r] Hyp.
[s] Cor. to 1. 4.
[t] Ax. 2. 4.
[u] 4. 4.

which was to be demonstrated.

---

* *When, in two ranks of quantities, the proportions are* inordinate, *as* AB : BC : : QR : RS, *and* BC : CD : : PQ : QR; *the same thing may be demonstrated; and that in the very same manner, except only, that* QR *must here be divided into the same number of parts with* AB, *and* PQ.

COROL.

COROLLARY I.

If other quantities DE, ST be taken, ſtill proportional to the two next preceding them, ſo that CD : DE : : RS : ST; then, by the ſame argument (regard being had to AB, CD, DE in the one rank, and PQ, RS, ST in the other) it is evident, that AB : DE : : PQ : ST [x]. And thus we may go on, ſtill aſſuming other quantities, as many as we pleaſe; and the ratio of the firſt and laſt, will always be the ſame in one rank, as in the other. Therefore ratios [y] compounded of the ſame number of like, or equal ratios, are equal.

[x] 5. 4.

[y] Def. 8. of 4.

COROLLARY II.

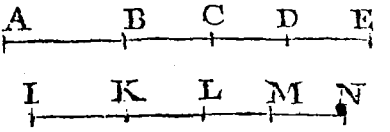
It is alſo evident from hence, that if any two quantities be taken proportional to the two conſequents of an aſſigned proportion, they will alſo be proportionals when compared with antecedents: and *vice verſâ*. For, the two quantities CD and RS, when compared, ſucceſſively, with the conſequents, and antecedents of the given proportion AB : BC : : PQ : QR, appear to be proportional, in the one caſe, as well as in the other [z].

[z] 5. 4.

THEOREM VI.

If, to the two conſequents (BC, KL) *of four proportionals* (AB, BC, IK, KL), *any two quantities* (CD, LM) *that have the ſame ratio to the reſpective antecedents be added; theſe ſums and the antecedents will ſtill be proportionals (I ſay, if* AB : BC : : IK : KL, *and* AB : CD : : IK : LM; *then ſhall* AB : BD : : IK : KM).

For,



For, CD and LM being proportional to the antecedents AB and IK [a], they and the consequents (BC, KL) will also be proportionals *(by Corol. 2. of the precedent)*: whence [b] *(by composition)* BC : BD : : KL : KM. And so again, *(by the same Corol.)* AB : BD : : IK : KM.

[a] Hyp.

[b] 3. 4.

COROLLARY.

From this Theorem it will appear, that, if the ratios of the corresponding quantities of two ranks with respect to the two first, are the same in both ranks (AB : BC : : IK : KL, AB : CD : : IK : LM, *&c.*); then the ratio of all the quantities to the first, will also be the same in the one rank, as in the other. For, by adding DE and MN to the last consequents (BD, KM) there will be had [c] AB : BE : : IK : KN (and so on, to any number of quantities whatever.) Then (by composition) AB : AE : : IK : IN.

[c] 6. 4.

When the quantities, in both ranks, are of the same kind, it will appear (by alternation) that the ratio of the two sums, and *that* of every two corresponding terms, will be the same.

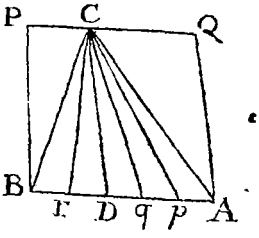
The six Theorems here delivered, on proportions of magnitudes in general, comprehend all that is most useful in that subject.—What relates to the proportions of extended magnitudes, under different limitations, and figures, as far as regards right lines and surfaces, will be the subject of the remaining part of this book.

Note,

Note, when-ever, in any demonstration, you meet with several proportional quantities, connected continually by the sign : : *(like these,* A : B : : C : D : : E : F : : G : H) *the conclusion to be drawn, is always from the first and last of the two equal ratios.*

THEOREM VII.

Triangles (ACD, BCD), *and also parallelograms* (ADCQ, BDCP), *having the same altitude, are to one another in the same ratio as their bases* (AD, BD).



Let the base AD be to the base BD in the ratio of any one number *m* (3) to any other number *n* (2), or, which is the same [c], let AD contain *m* (3) such equal parts whereof BD contains the number *n* (2). Then, the triangles AC*p*, *p*C*q*, BC*r*, *&c.* made by drawing lines from the points of division to the vertex C, being all equal among themselves [d]; the triangle ACD will therefore be in proportion to the triangle BCD, as the number of equal parts in the former to the number of equal parts in the latter, or as the number of parts in AD to the number of parts in BD, that is, as AD to BD [e]. Whence, also, the parallelograms ADCQ, BDCP, being the doubles of their respective triangles [d], are likewise in the same ratio as their bases AD and BD [e].

[c] Ax. 8. 4.

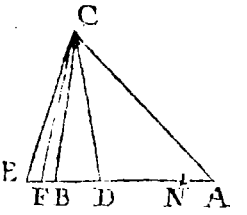
[d] Cor. 2. to 2. 2.

[e] 4. 4.

SCHOLIUM.

If the bases AD *and* BD *are incommensurable to each other, the ratio of the triangles cannot be other than that of their bases.*

For,



For, if possible, let the triangle BCD be to the triangle ACD, not as BD to AD, but as some other line ED, greater than BD, is to AD.

Let AN be a part, or measure of AD, less than BE [e], and let DF be that multiple of AN which least exceeds DB; also let CE and CF be drawn. It is manifest that the point F falls between B and E, because (*by Hyp.*) BF is less than AN, and AN less than BE. Moreover, the ratio of FCD to ACD is the same as that of FD to AD (*by the precedent.*) But the ratio of BCD to ACD (or of ED to AD [f]) is ⊏ the ratio of FD to AD [g], or of FCD to ACD; and consequently BCD ⊏ FCD [h]: *which is impossible* [i]. By the same argument it will appear, that the triangle BCD cannot be to the triangle ACD, as a line, less than BD, is to AD. Therefore BCD : ACD : : BD : AD.

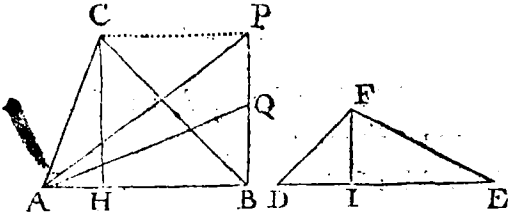
[e] Lem. 1. 8.
[f] Hyp.
[g] Ax. 5. of 4.
[h] Ax. 6. of 4.
[i] Ax. 2. of 1.

If this Scholium should appear difficult to the Learner, it may not be amiss to omit it intirely; since it is only put down for the sake of those who may be scrupulous about the business of incommensurables; to whom it may not be improper to observe, that nothing more is taken for granted herein, than what is effected by means of the first Lemma in the 8th book; which being demonstrated from axioms, and one single theorem in the first book, is referred to here, though not given till hereafter, for reasons already hinted at, in this note.

THEO-

THEOREM VIII.

Triangles (ABC, DEF) *ſtanding upon equal baſes* (AB, DE) *are to one another, in the ſame ratio as their altitudes* (CH, FI.)



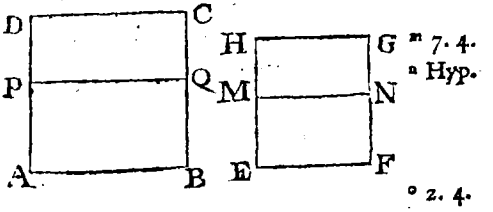
Let BP be perpendicular to AB, and equal to CH; in which let there be taken BQ = FI, and let AP and AQ be drawn.

The triangle ABP is = ABC [k], and ABQ = DEF [k]; but ABP (ABC) : ABQ (DEF) : : [l] BP (HC) : BQ (FI). [k] Cor. 2. to 2. 2. [l] 7. 4.

THEOREM IX.

If, parallel to the baſes of any two parallelograms (AC, EG), *two lines* (PQ, MN) *be drawn, ſo as to cut the ſides proportionally* (AP : AD : : EM : EH), *then will thoſe parallelograms and their correſponding parts* (AQ, EN) *be alſo proportionals.*

For, AQ : AC : : AP : AD [m] : : EM : EH [n] : : EN : EG : [m]; and therefore, *by alternation*, AQ : EN : : AC : EG [o].



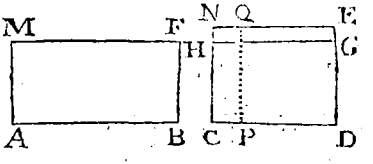
[m] 7. 4.
[n] Hyp.
[o] 2. 4.

THEO-

THEOREM X.

If four lines are proportional (AB : CD : : DE : BF), *the rectangle* (AF) *under the two extremes will be equal to the rectangle* (CE) *under the two means. And, if the rectangles under the extremes and means of four given lines* (AB, CD, DE, BF) *be equal, then are those four lines proportional.*

In DE let DG be taken BF, and let GH, parallel to DC, be drawn.



[p] 7. 4. [q] Hyp. [r] Ax. 1. 4. 1. *Hyp.* AF : CG : : AB : CD[p] : : DE : BF[q] : : DE : DG[r] : : CE : CG[p]; therefore, the consequents of the first and last of these equal ratios being the same quantity CG, [s] Ax. 3. the two antecedents AF and CE must be equal[s].

[t] Ax. 1. 4. 2. *Hyp.* AB : CD : : AF : CG[p] : : CE : CG[t] : : DE : DG[p] (BF.)

SCHOLIUM.

From the same demonstration, and scheme, it will appear, that the two antecedents of four proportional lines (AB, CD, DE, BF) are in the same ratio to each other, as the two consequents: for, if in DC there be taken DP = BF, and PQ be drawn parallel to DE; then AB : DE : : AF : PE[t] : : CE : PE : : CD : PD (BF).

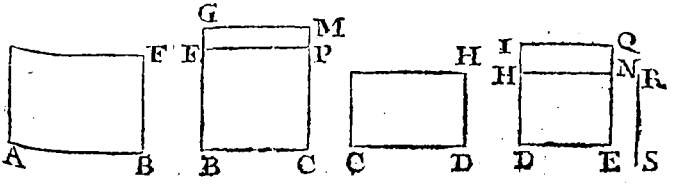
[t] 7. 4.

THEO-

THEOREM XI.

The rectangles under the corresponding lines, of two ranks of proportionals, are themselves proportionals. (*I say, if* AB : BC : : CD : DE, *and* BF : BG : : DH : DI, *then will the rectang.* AF : *rectang.* BM : : CH : *rect.* DQ).

For, in BG and DI (produced if necessary) let there be taken BF = BF, DH = DH, and let



FP be parallel to BC, and HN to DE: then AF : BP : : AB : BC [y] : : CD : DE [z] : : CH : DN [y]; whence *(alternately)* AF : CH : : PB : DN, and so likewise is BM to DQ [a]: whence *(again by alternation)* AF : BM : CH : DQ.

[y] 7. 4.
[z] Hyp.
[a] 9. 4.

COROLLARY I.

Hence, the squares of four proportional lines, are themselves proportional.

COROLLARY II.

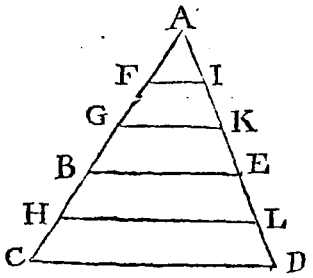
Hence also, the sides of four proportional squares (AB^2, BC^2, CD^2, DE^2) will be proportional. For, let the line RS be taken such, that, AB : BC : : CD : RS; then, since $AB^2 : BC^2 : : CD^2 : RS^2$ (*by Corol.* 1.) and $AB^2 : BC^2 : : CD^2 : DE^2$ (*by supposition*): thence will [b] $RS^2 = DE^2$; therefore [c] RS = DE, and consequently AB : BC : : CD : DE [d].

[b] Ax. 4. 4.
[c] Cor. 2. of 6. 2.
[d] Ax. 1. 4.

THEO-

THEOREM XII.

A line (BE) *drawn parallel to one side* (CD) *of a triangle* (ACD) *divides the other two sides proportionally.* (I say, AB : AC : : AE : AD, AB : BC : : AE : ED, and AC : BC : : AD : ED).



Let AB be to AC, as any one number *m* (3) is to any other number *n* (5), or, which is the same, let AB contain *m* (3) such equal parts whereof AC contains *n* (5). Then, if from the points of division, lines be drawn parallel to the side CD, they will also divide AE and AD into the like numbers of equal parts [e]. Therefore AE is to AD, as the number of equal parts in AE to the number of equal parts in AD, or as the number of equal parts in AB to the number of equal parts of AC, that is, as AB to AC [f]. In the same manner, AE is to ED, as the number of parts in AE to the number of parts in ED, or as the number of parts in AB to the number of parts in BC, that is, as AB to BC. Also, in the same manner, AC : BC : : AD : ED.

[e] Cor. 1. to 27. 1.

[f] Ax. 8.

The same otherwise.

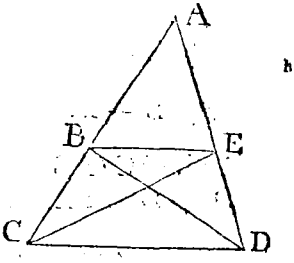
Draw CE and DB. Then will the triangles BEC and EBD be equal to each other [f]; whence, by adding BEA to both, AEC will be also = ABD [g].

[f] Cor. 1. to 2. 2.

[g] Ax. 4. 1.

But

But, AB : AC : : triang. AEB : triangle AEC [h] (ABD) : : AE : AD [h]; and AB : BC : : triang. AEB : triangle CEB [h] (DEB) : : AE : ED.



[h] 7. 4.

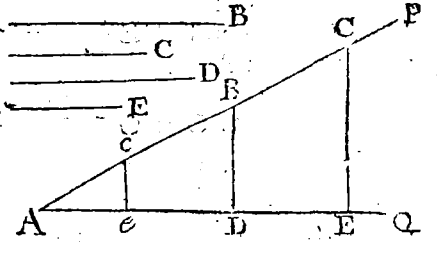
COROLLARY.

Hence a right-line, which divides two sides of a triangle proportionally, is parallel to the remaining side: because AD is divided in the same ratio with AC, when BE is parallel to CD; but not else [i]. [i] Ax. 2. 1, and 5. 4.

SCHOLIUM.

From this last Theorem, whatever relates to the composition and division of ratios, when these respect the comparison of right-lines, will appear exceedingly obvious.

For, let AB, BC, AD, and DE, be proportionals: and from any point A, let two indefinite right-lines AP, AQ be drawn; in which take AB = AB, BC = BC, AD = AD, and DE = DE; also take B*c* = BC, D*e* = DE, and let BD, CE, and *ce* be drawn.



Since AB : BC : : AD : DE [k], thence is EC parallel to DB [l]; and so, B*c* being = BC, and D*e* = DE, *ec* will also be parallel to DB [m]. Therefore,

[k] Hyp.
[l] Cor. of 12. 4.
[m] Cor. 2. to 27. 1.

AC (AB + BC) : AB : : AE (AD + DE) : AD;

AC (AB + BC) : BC : : AE (AD + DE) : DE;

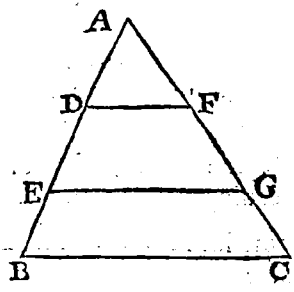
$Ac(AB - BC) : AB :: Ae(AD - DE) : AD$;

$Ac(AB - BC) : BC :: Ae(AD - DE) : DE$;

And, $AC(AB + BC) : Ac(AB - BC) :: AE(AD + DE) : Ae(AD - DE)$.

THEOREM XIII.

The parts (DE, FG) *of the two sides of a triangle, intercepted by right-lines* (DF, EG) *drawn parallel to the base* (BC), *are in the same ratio with the wholes.* (DE : FG :: AB : AC).



For, DF and EG being parallel to each other, thence will DE : AE :: FG : AG [o]; therefore, (by alternation) DE : FG :: AE : AG [p]. In the same manner, AE : AG :: AB : AC [o]. Consequently DE : FG :: AB : AC [q].

[o] 12. 4.
[p] 2. 4.
[q] Ax. 7. 4.

COROLLARY.

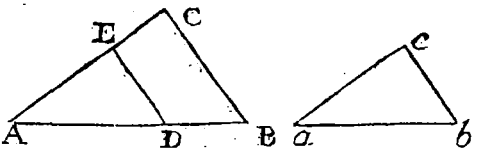
Hence, if ever so many lines be drawn parallel to the base, cutting the sides of a triangle, every two corresponding segments will have the same ratio [q].

THEOREM XIV.

In triangles (ABC, abc) *mutually equiangular, the corresponding sides* (AB, ab, AC, ac) *containing the equal angles* (A, a) *are proportional.*

In

In AB and AC (produced if necessary) take AD = *ab*, and AE = *ac*, and join D, E.



The triangles *abc*, and ADE, having *ab* = AD, *ac* = AE, and the angle *a* = A, have also the angle ADE [r] = *abc* = ABC [s]; whence DE will be parallel to BC [t]; and therefore AB : AD (*ab*) : : AC : AE [u] (*ac*).

[r] Ax. 10. 1.
[s] Hyp.
[t] Cor. to 8. 1.
[u] 12. 4.

The same otherwise.

Because AB × *ac* = *ab* × AC [w], therefore is AB : *ab* : : AC : *ac* [x].

[w] 24. 3.
[x] 10. 4.

COROLLARY.

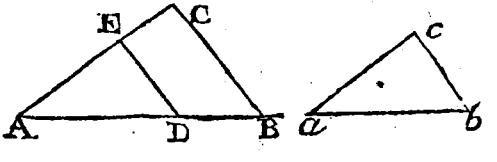
Hence equiangular triangles are similar to each other [y].

[y] Def. 14. of 4.

THEOREM XV.

If two triangles (ABC, abc) *have one angle* (BAC) *in the one, equal to one angle* (bac) *in the other, and the sides* (AB, ab, AC, ac) *about those angles proportional; then are the triangles equiangular.*

In AB and AC, take AD = *ab*, and AE = *ac*, and let DE be drawn.



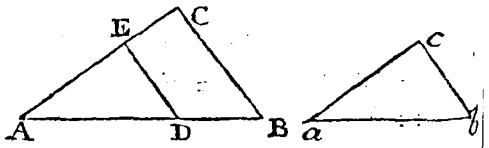
Since AB : *ab* (AD) : : AC : *ac* (AE [b]), therefore is DE parallel to BC [c]; whence the angle B = ADE [d] = *b* [e], and the angle C = AED = *c*.

[b] Hyp.
[c] Cor. to 12. 4.
[d] Cor. 1. to 7. 1.
[e] Ax. 10. 1.

THEOREM XVI.

If two triangles (ABC, abc) *have one angle* (A) *in the one, equal to one angle* (a) *in the other, and the sides* (AB, ab, CB, cb) *about either of the other angles proportional; then will the triangles be equiangular, provided these last angles* (B, b) *be, either, both less, or both greater, than right-angles.*

In AB, let AD be taken = *ab*, and let DE be drawn parallel to BC, meeting AC in E.



Then will the triangles ABC, and ADE, be equiangular[f]; therefore, CB : ED : : AB : AD[g] : : AB : *ab*[h] : : CB : *cb*[i]; and consequently ED = *cb*[k]: whence the triangles *abc* and ADE (having *ab* = AD, *cb* = ED, and *a* = A) will be equal in all respects[l], provided the angles *abc* and ABC (= ADE) are either both less, or both greater than right-angles. Therefore, since the latter of these equal triangles (*abc*, ADE) is equiangular to ABC, the proposition is manifest.

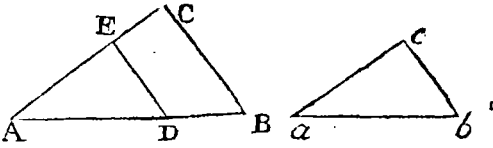
[f] Cor. 1. to 7. 1.
[g] 14. 4.
[h] Ax. 1. 4.
[i] Hyp.
[k] Ax. 4. 4.
[l] 17. 1.

THEOREM XVII.

If two triangles (ABC, abc) *have all their sides respectively proportional* (AC : ac : : AB : ab : : CB : cb) *then are those triangles equiangular.*

In AC and AB, take AE = *ac*, and AD = *ab*, and join E, D.

Since

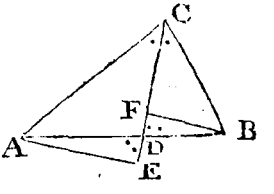


Since AC : AE (*ac*) : : AB : AD (*ab*) [m], the triangles ABC, ADE, are equiangular [n]; hence CB : ED : : AB : AD [o] (*ab*) : : CB : *cb* [m]; and consequently ED = *cb* [p]: therefore the triangles *abc*, ADE, being mutually equilateral, they must also be mutually equiangular [q]; and consequently *abc*, as well as ADE, equiangular to ABC.

[m] Hyp.
[n] 15. 4.
[o] 14. 4.
[p] Ax. 4. 4.
[q] 14. 1.

THEOREM XVIII.

A right-line (CD) *bisecting any angle* (ACB) *of a triangle* (ABC) *divides the opposite side* (AB) *into two segments* (AD, BD) *having the same ratio with the sides* (AC, CB) *containing that angle.*

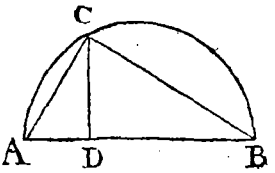


Let AE and BF be perpendicular to CDE. Then the triangles ACE, CBF, and ADE, BDF being, respectively, equiangular [r], it will be AD : BD : : AE : BF [s] : : AC : BC.

[r] Hyp. and 3. 1.
[s] 14. 4. and 2. 4.

THEOREM XIX.

A perpendicular (CD) *let fall from the right-angle* (C) *upon the hypothenuse* (AB) *of a right-angled triangle* (ABC) *will be a mean proportional between the two segments* (AD, BD) *of the hypothenuse: and each of the sides containing the right-angle, will be a mean proportional between its adjacent segment, and the whole hypothenuse.*



For, ſince the angle BDC is = BCA [t], and B common, the triangles BDC, BCA are equiangular [u]: after the ſame manner ADC and ABC appear to be equiangular.

[t] Ax. 7.
[u] Cor. 1. to 10. 1.

Therefore, by Theor. XIV.

BD : CD : : CD : AD
AB : BC : : BC : BD
AB : AC : : AC : AD.

COROLLARY.

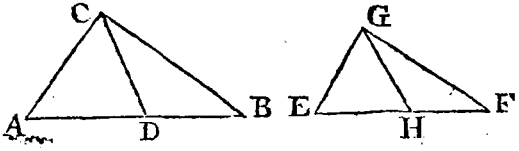
Becauſe the angle in a ſemi-circle is a right-angle [w], it follows, that, if from any point C, in the periphery of a ſemi-circle ACB, a perpendicular CD be let fall upon the diameter AB, and from the ſame point C, to the extremities of that diameter, two chords CA, CB be drawn; the ſquare of that perpendicular will be equal to a rectangle under the two ſegments of the diameter; and the ſquare of each chord, equal to a rectangle under the whole diameter and its adjacent ſegment: for, becauſe of the above proportions, we have $CD^2 = BD \times AD$, $BC^2 = AB \times BD$, and $AC^2 = AB \times AD$. [x]

[w] 13. 3.
[x] 10. 4.

THEOREM XX.

If, in ſimilar triangles (ABC, EFG) *from any two equal angles* (ACB, EGF) *to the oppoſite ſides, two right-lines* (CD, GH) *be drawn, making equal angles with the homologous ſides* (CB, GF); *thoſe right-lines will have the ſame ratio as the ſides* (AB, EF) *on which they fall, and will alſo divide thoſe ſides proportionally.*

For,

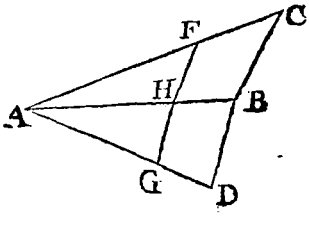


For, the triangles ADC, EHG, and BDC, FHG (as well as the *wholes* ABC, EFG) being equiangular [y], [y] Hyp.
thence is [z] AB : EF (:: AC : EG) :: CD : GH; [z] 14. 4. and Ax. 7.
and AD : EH (:: DC : HG) :: BD : FH.

THEOREM XXI.

If in two triangles (ABC, ABD) *having one side* (AB) *common to both, from any point* H *in that side, two lines* (HF, HG) *respectively parallel to two contiguous sides* (BC, BD) *be drawn, to terminate in the two remaining sides* (AC, AD); *those lines* (HF, HG) *will have the same ratio as the sides* (BC, BD) *to which they are parallel.*

For, AB : AH :: BC : HF [a], and AB : AH :: BD : HG [a]; therefore, by equality, BC : HF :: BD : HG; whence, alternately, BC : BD :: HF, HG [b].



[a] 14. 4.

[b] 2. 4.

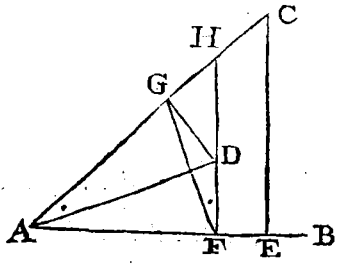
COROLLARY.

Hence, if BC = BD, then also will HF = HG.

THEOREM XXII.

If, at any two points (F, G) *in two lines* (AB, AC) *meeting each other, two perpendiculars* (FD, GD) *be erected, so as to meet each other; the distance* (AD) *of their concourse from that of the proposed lines, will be to the distance* (FG) *of the two points themselves, in the ratio of one of the said lines* (AC) *to a perpendicular* (CE) *falling from the extreme thereof upon the other* (AB).

Let FD be produced to meet AC in H.



Since the angles AFD and AGD are right-ones [c], the circumference of a circle will pass thro' all the four points A, F, D, G [d]; and so the angles GFD, GAD, standing on the same subtense GD, will be equal [e]; and consequently the triangles AHD, FHG equiangular [f]: therefore AD : FG : : AH : HF [g] : : AC : CE.

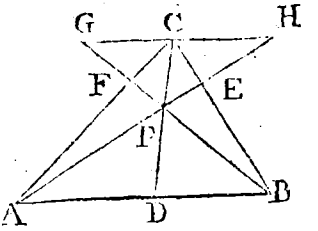
[c] Hyp.
[d] 19. 3.
[e] 11. 3.
[f] Cor. 1. to 10. 1.
[g] 14. 4.

THEOREM XXIII.

If thro' any point (P) *in a triangle* (ABC) *three right-lines* (AE, BF, CD) *be drawn, from the angular points to cut the opposite sides, the segments* (AD, BD) *of any one side* (AB) *will be to each other, as the rectangles* (AF × CE, BE × CF) *under the segments of the other sides taken alternately.*

Let

Let GCH be parallel to AB, and let AE and BF be produced to meet it in H and G.



It is manifest, that the triangles FBA, FCG; EAB, EHC; APB, GPH, are equiangular[h]: [h] 7. and 3. of 1.

Therefore AF : CF : : AB : CG[i], [i] 14. 4.
CE : BE : : CH : AB[i].

Whence AF × CE : CF × BE : : CH × AB : CG × AB[k] : : CH : CG[l]; but CH : CG : : AD : BD[m]; therefore, by equality, AF × CE : CF × BE : : AD : BD. [k] 11. 4. [l] 7. 4. [m] 20. 4.

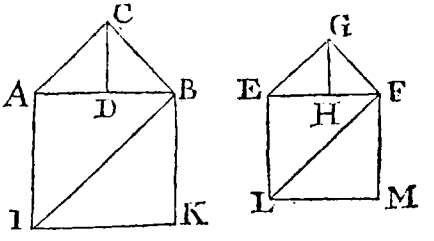
COROLLARY.

Hence, if AD = BD, then also will AF × CE = CF × BE, and therefore AF : CF : : BE : CE[n]. [n] 10. 4.

THEOREM XXIV.

Equiangular triangles (ABC, EFG) *are in proportion to one another, as the squares* (AK, EM) *of their homologous sides.*

Upon AB and EF let fall the perpendiculars CD and GH, and let the diagonals BI, FL be drawn.

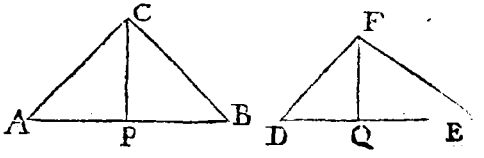


Because ABC : ABI : : CD : AI (AB)[o] : : GH : EF (EL) : : EFG : EFL[o]; therefore, alternately, ABC : EFG : : ABI : EFL[q] : : AK : EM[r]. [o] 8. 4. [p] 20. 4. [q] 2. 4. [r] Cor. 2. to 2. 2.

THEO-

THEOREM XXV.

Triangles (ABC, DEF) *having one angle* (A) *in the one, equal to one angle* (D) *in the other, are in the ratio of the rectangles* (AC × AB, DF × DE) *contained under the sides including the equal angles.*



Upon AB and DE, let fall the perpendiculars CP and FQ. Then AC × AB : CP × AB : : AC : CP [s] : : DF : FQ [t] : : DF × DE : FQ × DE [s]; whence, alternately, AC × AB : DF × DE : : CP × AB : FQ × DE : : triangle ABC : triangle DEF [u].

[s] 7. 4.
[t] 14. 4.
[u] Cor. 1. 4.

COROLLARY.

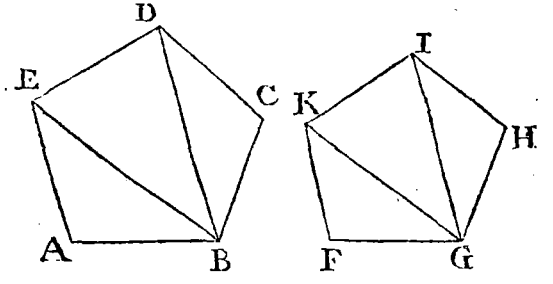
Hence, if the rectangles of the sides containing the equal angles, be equal, or the sides themselves reciprocally proportional [w], the triangles will be equal. The same also holds in parallelograms, being the doubles of such triangles.

[w] 10. 4.

THEO-

THEOREM XXVI.

All similar right-lined figures (ABCDE, FGHIK) *are in proportion to one another as the squares of their homologous sides* (AB, FG).



Draw the right-lines BE, BD, GK, GI.

Because A = F, and AB : AE : : FG : FK [x], the triangles BAE, GFK are equiangular [y]; therefore, if from AED = FKI [x], there be taken AEB = FKG, the remainders BED, GKI will also be equal [z]. Wherefore, since ED : KI (: : EA : KF [x]) : : EB : KG [w], the triangles EBD, KGI are likewise equiangular [y]. In the same manner it will appear, that DBC, IGH are also equiangular.

[x] Def. 14. of 4.
[y] 15. 4.
[z] Ax. 5. of 1.
[w] 14. 4.

Therefore, because ABE : GFK (: : [a] BE^2 : GK^2) : : BED : GKI (: : [a] BD^2 : GI^2) : : [a] BDC : GIH, it is evident, that the sum of all the antecedents (ABCDE) is to the sum of all the consequents (FGHIK) as the first antecedent ABE is to the first consequent GFK [b], or as AB^2 to [a] FG^2; *which was to be demonstrated.*

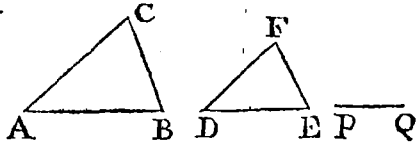
[a] 24. 4.
[b] Cor. to 6. 4.

THEO-

THEOREM XXVII.

If three right-lines (AB, DE, PQ) *are proportional, the right-lined figure* (ABC) *upon the first, will be in proportion to the similar, and similarly described, figure* (DEF) *on the second, as the first line* (AB) *to the third* (PQ).

For, AB : PQ : : AB^2 : AB × PQ [c] ([d] DE^2) : ABC : DEF [e].



[c] 7. 4.
[d] Hyp. and 10. of 4.
[e] 26. 4.

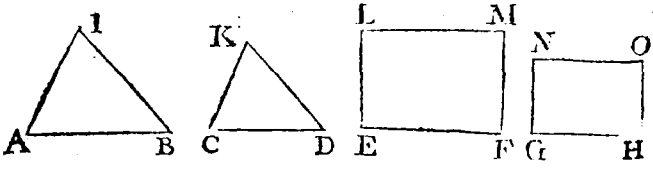
COROLLARY.

Hence, similar right-lined figures, are in the duplicate ratio of their homologous sides [f].

[f] Def. 7. of 4.

THEOREM XXVIII.

If four right-lines (AB, CD, EF, GH) *be proportional, the right-lined figures described upon them, being like, and in like sort situate, shall also be proportional* (ABI : CDK : : EM : GO).



For, ABI : CDK (: : [g] AB^2 CD^2 : : EF^2 : GH^2 [h]) : : EM : GO [g].

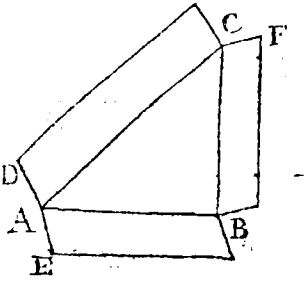
[g] 26. 4.
[h] Cor. 1. to 11. 4.

THEO-

THEOREM XXIX.

If upon the three sides of a right-angled triangle (ABC) *as many right-lined figures* (CD, BE, BF) *like, and alike situate, be described, that* (CD) *upon the hypothenuse* (AC) *will be equal to both the other two* (BE, BF) *taken together.*

For, BE : BF : : AB^2 : [l] BC^2; therefore (by composition) BE + BF : BE : : $AB^2 + BC^2$ (= [m] AC^2) : AB^2 : : CD : BE [l]; and consequently BE + BF = CD [n].



[l] 26. 4.
[m] 8. 2.
[n] Ax. 3. of 4.

The End of the FOURTH BOOK.

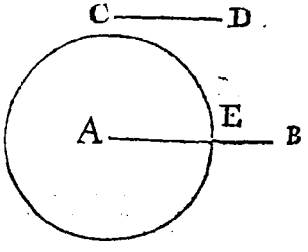
ELEMENTS

OF

GEOMETRY.

BOOK V.

PROBLEM I.



FROM *the greater* (AB) *of two unequal lines* (AB, CD) *to cut off, or take away a part* (AE) *equal to the lesser* (CD).

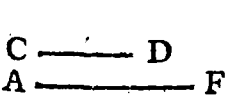
From A as a center, with a radius equal to CD, let the circumference of a circle be described [a], cutting [b] AB in E; *and the thing is done.*

[a] Post. 3.
[b] Ax. 2.

PROBLEM II.

At a given point (A) *to make a line* (AB) *equal to a given line* (CD).

Draw

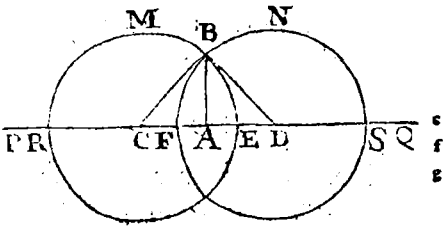


Draw the indefinite line AF[c]; from which take away AB = CD[d]; *and the thing is done.*

[c] Post. 1. and 2.
[d] 1. 5.

PROBLEM III.

At a given point (A) *in an infinite right-line* (PQ) *to erect a perpendicular.*



In the line propounded, take two equal distances AC, and AD[e]; and from the centers C[f] and D[g], with any equal radii greater than AC (or AD), let two circles EMR and FNS be described; which will cut each other: and, if from the point B of their intersection, you draw BA, *the thing is done.*

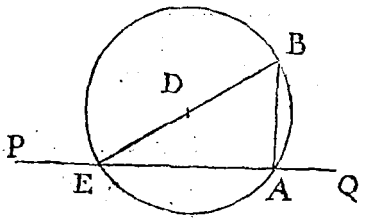
[e] 1. 5.
[f] Post. 3.
[g] Post. 1.

For, let the points R, E, and F, S, be those wherein the infinite line PQ intersects the circumferences of the two circles EMR and FNS[h]: then AF being ⊐ FD[h] (or CR[i]) ⊐ AR[h]; and AS ⊏ DS[h] (or CE[i]) ⊏ AE[h], the point F falls within, and the point S without the circle EMR; and so the two circles cut each other[l]. If therefore, from the point of intersection, BC and BD be drawn; then the triangle CBD being isosceles[m], the angles BCD, BDC at the base thereof, will be equal[n]; and so, CA being = AD[k], and CB = BD[m], the angle CAB is also = DAB[p].

[h] Ax. 2.
[i] Hyp.
[l] Def. 11.
[m] 3. Constr. and Def. 33. of 1.
[n] *Obs. on* Ax. 10.
[k] Constr.
[p] Ax. 10.

Otherwise.

Otherwise.



From any point D above the line PQ, as a center, thro' the given point A, let the circumference of a circle be described[q], intersecting PQ in E[r]; draw the diameter EDB, and also BA[s]; then the angle EAB, being in the semicircle EAB[t], is a right-angle; *which was to be done.*

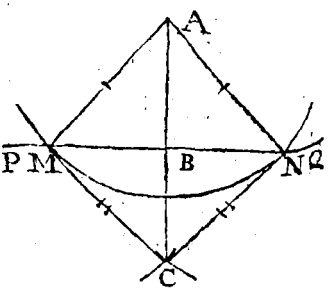
[q] Post. 3.
[r] Cor. to 4. 3.
[s] Post. 1.
[t] 13. 3.

COROLLARY.

From the former of these constructions it appears, that, if from any two points, with two equal radii, greater, each, than half the distance of those points, two circles be described; those circles will cut each other.

PROBLEM IV.

From a given point (A) *upon an infinite right-line* (PQ) *to let fall a perpendicular* (AB).



From the given point A, as a center, let an arch of a circle be described, so as to pass below PQ, and intersect it in M and N; from which points, with any equal radii, greater than half MN, let two other arches be also described, and from the point of

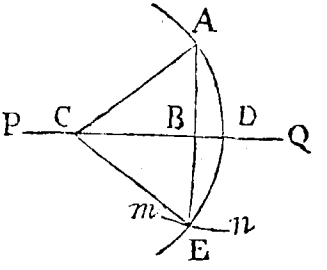
their

their intersection C, let the right-line CBA be drawn; which will be perpendicular to PQ.

For, let AM, AN, CM, and CN be drawn; then AM being = AN [u], and MC = NC [w], the angle AMB is = ANB [x], and CMB = CNB [x]; and consequently AMC = ANC [y]: whence, (as AM = AN, and MC = NC) the triangles AMC, ANC are equal in all respects [z]; and so, the angle MAB being = NAB, the angle MBA is likewise = NBA [z].

[u] Def. 33. of 1.
[w] Hyp. and Def. 33. 1.
[x] *Obs. on* Ax. 10.
[y] Ax. 4.
[z] Ax. 10.

The same otherwise.



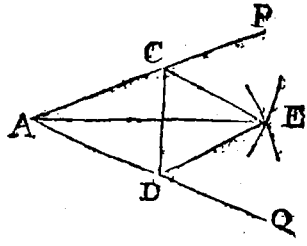
From any point C in the line PQ, as a center, let the circumference of a circle be described thro' the given point A [a], intersecting PQ in D [b]; and from the center D, with a radius equal to the distance of the points A and D, let another circle *m*E*n* be also described, cutting the former ADE in E; then draw ABE for the perpendicular required.

[a] Post. 3.
[b] Constr.

For, conceiving right-lines to be drawn from C and D, to A and E, the triangles ACE, ADE will be both of them isosceles [b]; and so the demonstration is the same with that of the preceding method.

PROBLEM V.

To bisect, or divide into two equal parts, any given right-lined angle (PAQ).



In the lines containing the given angle, take AC = AD [c]; and upon the centers C and D, with any equal radii, let two circles be described [d], so as to intersect each other; and from the point of intersection E draw EA, *and the thing is done.*

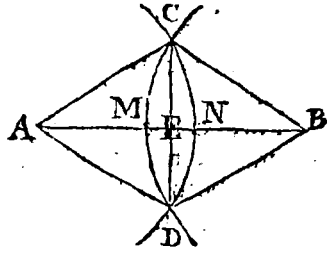
[c] 1. 5.
[d] Post. 3.

For, let CD, CE, and DE be drawn [e]; then, the triangles ACD and ECD being both isosceles [f], the angle ACD will be = ADC, and ECD = EDC [g]; and consequently the whole angle ACE = the whole angle ADE [h]; whence (AC being = AD, and EC = ED), the angle CAE is also = the angle DAE [i].

[e] Post. 1.
[f] Constr. and Def. 33. of 1.
[g] Ax. 10.
[h] Ax. 4.

PROBLEM VI.

To bisect a given right-line (AB).



From the extremes A, B, of the given line, with equal radii, describe two circles, so as to cut each other [l]; and between the two points of intersection draw CD, cutting AB in E; *and the thing is done.*

[l] Post. 3. and Cor. to 3. of 5.

For, if AC, AD, BC and BD be drawn, the triangles ACB, ADB being isosceles [k], thence is the angle CAB = CBA [l], and DAB = DBA [l]; and consequently CAD = CBD [m]: whence the triangles ACD and BCD are equal in all respects [n]; and so the angle ACE being = BCE, AC = BC, and CE common, thence is AE also = BE [n].

[k] Constr. and Def. 33. of 1.
[l] *Obs. on* Ax. 10.
[m] Ax. 4.
[n] Ax. 10.

COROL-

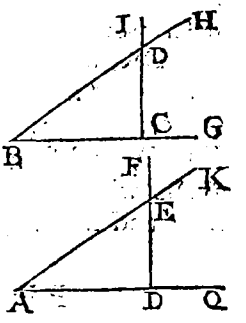
COROLLARY.

Hence, it is manifest, that CD not only bisects AB, but is also perpendicular to it [n]. [n] Ax. 10. and Def. 8. of 1.

PROBLEM VII.

From a given point (A) *in a given right-line* (AQ), *to draw a line* (AK) *which shall make with the former an angle, equal to an angle given* (HBG).

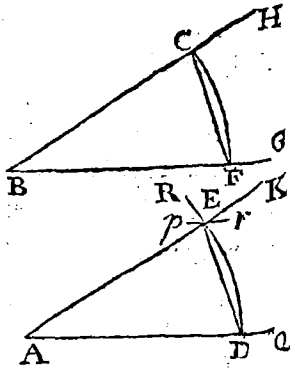
In BG and AQ take two equal distances BC, AD [o]; and at C and D erect the two perpendiculars CI and DF to BG and AQ [p]; and in DF take DE equal to the part CD of the former, intercepted by the lines containing the given angle HBG [q]; then thro' E draw AEK [r], *and the thing is done* [s].



[o] 2. 5.
[p] 3. 5.
[q] 1. 5.
[r] Post. 1.
[s] Ax. 10.

SCHOLIUM.

Having, in the seven preceding problems, effected and demonstrated, *by means of the axioms only*, whatever was assumed in the fourth postulate, *as barely possible*; we are now authorised, by the most rigid laws of geometrical reasoning, to make use of any theorem or conclusion, whatsoever, derived in the preceding books, in virtue of those assumptions, by which the process and result can be rendered the most obvious and eligible.—Accordingly, by having recourse here to Theorem XIV. of the first book, we shall be able to arrive at a construction of the last problem, better adapted to practice than that above laid down.



From the centers A and B, at any equal distances AD, BF, let two arcs of circles, FC, DR, be described [t], intersecting the given lines in D, F, and C; also from D, with a radius equal to the distance of the points F, C, let another circular arch *pEr* be described, cutting the former DR, in E; then draw AEK, *and the thing is done.*

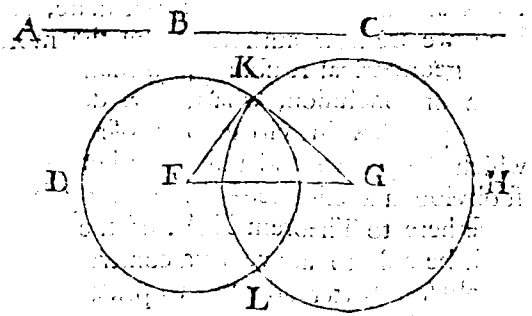
[t] Post. 3.

For, conceiving right-lines to be drawn from F to C, and from D to E, the triangles BFC and ADE will be equilateral to each other, *by construction,* and therefore equiangular also [u].

[u] 14. 1.

PROBLEM VIII.

To describe a triangle, whose three sides shall be equal to three given lines (A, B, C); *provided any two of them, taken together, be greater than the third.*



Make FG = B [w], and from the centers F and G, with the intervals, or distances A and C, let two

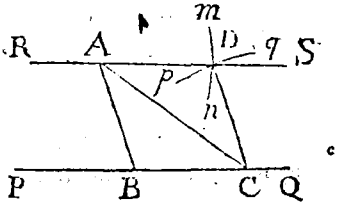
[w] 2. 5.

two circles DKL, HKL be described [x]; which will cut each other; and, if from the point of intersection K, the lines KF and KG be drawn, then will FKG be the triangle required. [x] Post. 3.

For, the distance FG of the two centers, is less than the sum of the radii A and C [y]; and greater than their difference (because B + A being ⊏ C [z], thence is B ⊏ C — A [y]); therefore the two circles cut each other [a]: consequently FK = A, FG = B, and GK = C [b]. [y] Ax. 6. 1. [z] Hyp. [a] 9. 3. [b] Def. 33. of 1.

PROBLEM IX.

Through a given point (A), *to draw a right-line* (RS) *parallel to a given right-line* (PQ).



At any point B in the given line, make BC equal to the distance of the points A and B [c]; and from the centers A and C, with an interval equal to CB, let two circles be described [d], then thro' their intersection D, let the line RS be drawn, *and the thing is done.* [c] 2. 5. [d] Post. 3.

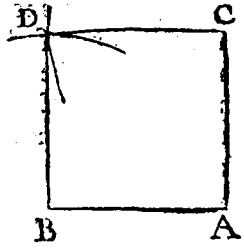
Let AB, and AC be drawn [e]. It is plain that the two circles will cut each other [f], because the sum of their semi-diameters (= AB + BC [g]) is greater than AC [h]: therefore, if ADS and CD be also drawn, then will AB = BC = CD = DA [i], and therefore RS parallel to PQ [k]. [e] Post. 1. [f] 9. 3. [g] Constr. [h] 19. 1. [i] Ax. 1. [k] 25. 1.

The same otherwise.

From A, to any point in PQ, draw AB [l]; make the angle SAB = PBA [m]; and then AS will be parallel to PQ [n]. [l] Post. 1. [m] 7. 5. [n] 8. 1.

PROBLEM X.

Upon a given line (AB) *to describe a square* (ABCD).



Make AC perpendicular, and equal, to AB [o]; and from the centers B and C, let two circles, with the radius AB or AC, be described [p], intersecting each other [q] in D; from which point draw DB and DC, *and the thing is done.*

[o] 3. 5. and 1. 5.
[p] Post. 3.
[q] 9. 3.

For, all the four sides being equal, by construction, the figure is a parallelogram [r]; and therefore, the angle A being a right-angle [s], the other three will be all right-angles [t], and ACDB a square [v].

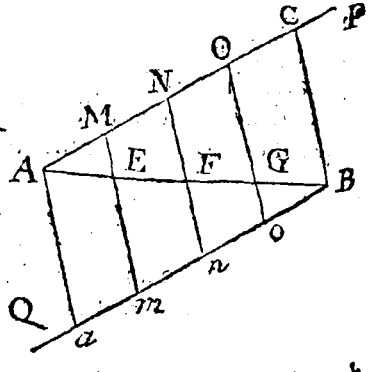
[r] 25. 1.
[s] Constr.
[t] Cor. to 24. 1.
[v] Def. 26.

SCHOLIUM.

By the same method a rectangle may be described, the sides thereof being given.

PROBLEM XI.

To divide a given line (AB) *into any proposed number of equal parts.*



From the extremes of the given line AB, draw two indefinite lines AP, BQ parallel to each other [u]; in each of which lines let there be taken as many equal distances AM, MN, NO, OC; B*o*, *on*, *nm*, *ma*, (of any

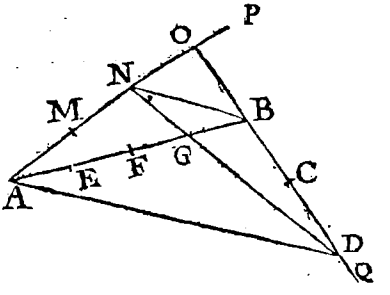
[u] 9. 5.

length

length at pleasure) as you would have AB divided into [w]; then draw M*m*, N*n*, O*o*, intersecting AB in E, F, G, *and the thing is done.* [w] 1. 5.

For, MN and *mn* being equal and parallel [x], FN will be parallel to EM [y]; and in the same manner will GO be parallel to FN: therefore, AM, MN, NO, *&c.* being all equal [z], AE, EF, FG, will likewise be equal [z]. [x] Constr. [y] 26. 1. [z] Cor. 1. to 27. 1.

The same otherwise.

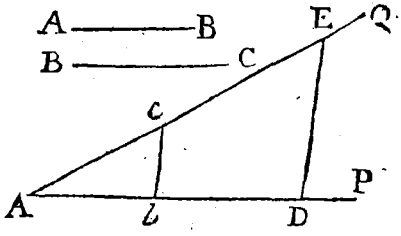


In any right-line AP drawn from A, take as many equal distances (AM, MN, NO) wanting one, as you would have AB divided into; then, having drawn the indefinite line OBQ, in it take an equal number of parts or distances OB, BC, CD, each of the length of OB, and let DN be drawn, cutting AB in G; make GF, FE, each equal to BG, *and the thing is done.*

For, if AD and BN be drawn, they will be parallel [a] (because OA : ON :: OD : OB [b]); and so, the triangles BNG, ADG, being equiangular [c], it will be BG : AG :: BN : AD [d] :: ON : OA [d]. Therefore BG is the same part of AG, as ON is of OA. [a] Cor. to 12. 4. [b] Constr. [c] 3. and 7. 1. [d] 14. 4.

PROBLEM XII.

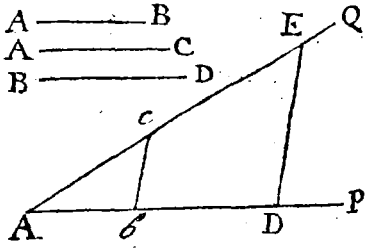
To two given lines (AB, BC) *to find a third proportional.*



From any point A, draw two indefinite lines AP, AQ, in which take A*b* = AB, A*c* = BC, and
[e] 1. 5. *b*D = BC[e]; draw *bc*, and parallel
[f] 9. 5. to *bc*, draw DE[f], cutting AQ in E; then *c*E will be the third-proportional required: for, A*b* (AB) : A*c* (BC)
[h] 12. 4. : : *b*D (BC) : *c*E[h].

PROBLEM XIII.

To three given lines (AB, AC, BD) *to find a fourth-proportional.*



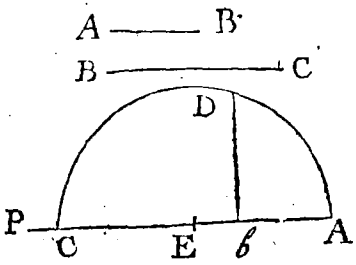
Having drawn AP and AQ, as in the preceding problem, take therein A*b* = AB, A*c* = AC, and
[i] 1. 5. *b*D = BD[i]; draw *bc*, and parallel to
[l] 9. 5. it, draw DE[l], intersecting AQ in E; then is *c*E *the fourth proportional required.*
[m] 12. 4. For, A*b* (AB) : A*c* (AC) : : *b*D (BD) : *c*E[m].

PROBLEM XIV.

Between two given lines (AB, BC) *to find a mean proportional.*

In

In the indefinite line AP, take Ab = AB, and bC = BC [n]; bisect AC in E [o], and from the center E, at the distance of EA, or EC, let a semicircle ADC be described [p]; erect bD perpendicular to AC [q], cutting the circumference in D; then will bD be the mean-proportional required.



[n] 1. 5.
[o] 6. 5.
[p] Post. 3.
[q] 3. 5.

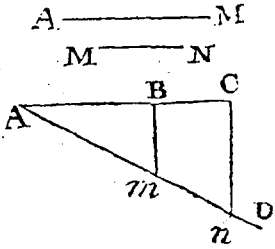
For, Ab (AB) : bD :: bD : bC (BC) [r].

[r] 19. 4. and it's Corol.

PROBLEM XV.

To divide a given line (AC) *into two parts* (AB, BC) *having the same proportion as two given lines* (AM, AN).

From A draw AD, making any angle with AB; in which take Am = AM, and mn = MN [s]; draw nC, and mB parallel thereto [t], meeting AC in B. Then will AB : BC :: Am (AM) : mn [u] (MN). *Which was to be done.*

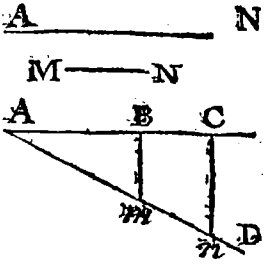


[s] 2. 5.
[t] 9. 5.
[u] 12. 4.

PROBLEM XVI.

To add a line (BC) *to a line given* (AB), *so that the whole compounded line* (AC) *shall be in proportion to the part added, as one given line* (AN) *is to another* (MN).

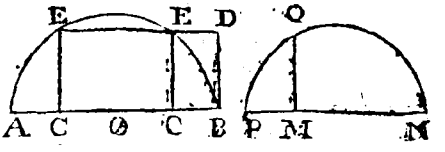
From



From A draw AD, making any angle with BA; in which
[w] 2. 5. take An = AN [w], and nm = NM; draw mB, and nC pa-
[x] 9. 5. rallel thereto [x], meeting AB produced, in C: then will
[y] 12. 4. AC : BC : : An (AN) : mn [y] (MN). *Which was to be done.*

PROBLEM XVII.

To divide a given line (AB) *into two such parts* (AC, BC) *that the rectangle contained under them, shall be equal to the rectangle under two given lines* (PM, MN); *provided that the given rectangle is not greater than the square of half the line* (AB) *to be divided.*



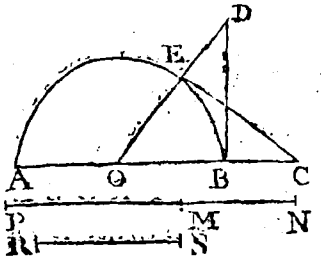
Between PM and MN take a mean-propor-
[z] 14. 5. tional MQ [z]; make BD perpendicular to AB, and equal to MQ; bisect AB
[a] 6. 5. in O [a], from which, as a center, let a semi-circle
[b] 9. 5. be described; and draw DE parallel to BA [b],
[c] Hyp. which (because BD is less than the radius [c]) will meet the circle in some point E; from which, upon AB let fall the perpendicular EC: so shall
[d] Cor. to 19. 4. AC × BC = [d] $EC^2 = DB^2$ ($= QM^2$) = PM × MN [e]. *Which was to be done.*
[e] Hyp. and 10. 4.

PROBLEM XVIII.

To a given line (AB) *to add another line* (BC), *such, that the rectangle under the whole compounded line* (AC) *and the part added, shall be equal to a rectangle under two given lines* (PM, MN).

Between

Between PM and MN take a mean-proportional RS [f]; make BD perpendicular to AB [g], and equal to RS [h]; bisect AB in O [i], draw OD, and take OC = OD [h]: so shall AC × BC = PM × MN, *as was to be done.*



[f] 14. 5.
[g] 3. 5.
[h] 2. 5.
[i] 6. 5.

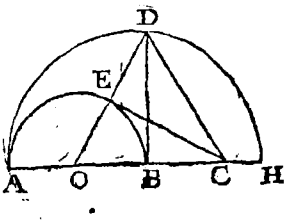
For, if thro' A and B, from the center O, the circumference of a circle be described, cutting DO in E, and E, C be joined; then, the triangles OCE, ODB (having OE = OB, OC = OD [k], and the angle EOB common) will be equal in all respects [l]; and so, EC being a [m] tangent to the circle in E, we have [n] AC × BC = CE^2 = [o] BD^2 = RS^2 = PM × MN [p].

[k] Constr.
[l] Ax. 16.
[m] 6. 3.
[n] Cor. to 22. 3.
[o] 1. 2.
[p] 16. 4.

PROBLEM XIX.

To divide a given line (BH) *into two such parts, that the square of the one* (BC) *shall be equal to the rectangle under the other* (CH) *and a second given line* (AB).

Taking BA in the same strait line with BH, between them let a mean-proportional BD be found [q]; bisect AB in O [r]; draw OD, and make OC = OD; so shall BC^2 = CH × AB, *as was to be done.*



[q] 14. 5.
[r] 6. 5.

For, by the demonstration of the precedent, AC × BC (= CE^2 = BD^2) = AB × BH; from each of which taking away AB × BC, there remains BC^2 = AB × CH [s].

[s] 5. 2. and Ax. 5. 1.

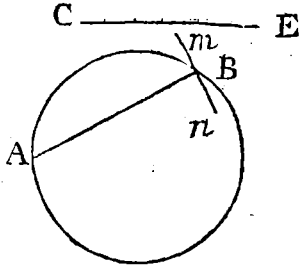
COROL.

COROLLARY.

If AB = BH, then will $BC^2 = BH \times CH$; in which case the line BH is said to be divided according to extreme and mean-proportion.

PROBLEM XX.

In a given circle, to apply, or inscribe a line (AB) *equal to a given line* (CE), *less than the diameter of the circle.*

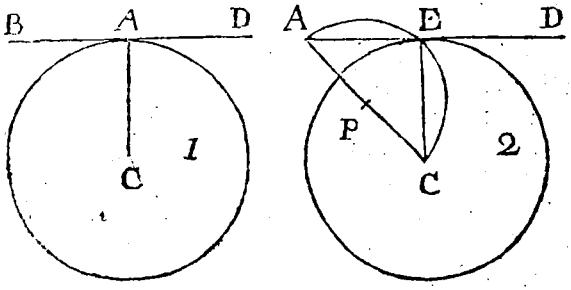


From any point A in the circumference, with the radius CE, let a circular arch *mn* be described [t], cutting the given circle in B [u]; then draw AB, *and the thing is done* [w].

[t] Post. 3.
[u] 9. 3.
[w] Def. 33. of 1.

PROBLEM XXI.

To draw a tangent to a given circle (C) *thro' a given point* (A).



CASE I. If the given point be in the circumference; then, to the center C, draw AC, and perpendicular to AC [y] draw BAD; which will touch the circle at A [z].

[y] 3. 5.
[z] 6. 3.

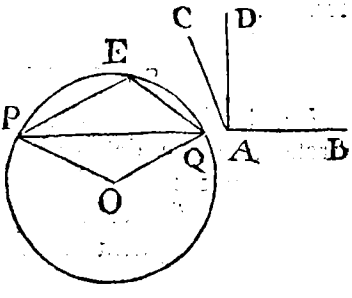
CASE II.

CASE II. If the point A be without the circumference; then draw AC, which bisect in P[b]; and from the center P, at the distance of AP, or CP, let a semi-circle AEC be described[c], cutting the given circle in E[d]; then draw AED, which will be the tangent required[e]; because (CE being drawn) AEC is a right-angle[f].

[b] 6. 5.
[c] Post. 3.
[d] 9. 3.
[e] 6. 3.
[f] 13. 3.

PROBLEM XXII.

Upon a given line (PQ) *to describe a segment of a circle* (PEQ) *to contain an angle* (E) *equal to a given angle* (BAC).



Make AD perpendicular to AB[g]; also make PQO, and QPO, each, equal to DAC[h] (the difference between the given angle and a right one); then upon the point of intersection O, as a center, at the distance of OP (or OQ), let a circle be described; *and the thing is done.*

[g] 3. 5.
[h] 7. 5.

For the angle E = right-angle + QPO[i] = DAB + DAC[k] = BAC.

[i] 16. 3.
[k] Constr.

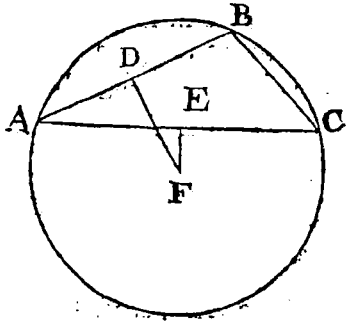
SCHOLIUM.

In the same manner the problem may be constructed, when the given angle is acute; only the lines PO, QO must then be drawn on the other side of (PQ) as is manifest from the 16th theorem of the 3d book.

PRO-

PROBLEM XXIII.

About a given triangle (ABC) *to describe a circle.*



Let any two sides, AB and AC, be bisected by two perpendiculars DF and EF [k]; which will intersect each other in the center (F) of the required circle [l]; from whence the circle may be described.

[k] 6. 5.

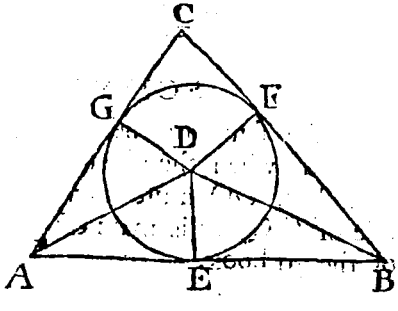
[l] Proof of 18. 3.

SCHOLIUM.

By the same method, the circumference of a circle may be described thro' any three given points, not situate in the same right line: also from hence, the center of a circle may be found, by having a segment of the circle given.

PROBLEM XXIV.

To inscribe a circle in a given triangle (ABC).



Bisect any two of the angles, A and B, by the lines AD and BD [m], meeting each other in D; make DE perpendicular to AB [o]; then, if from the center D, at the distance of DE, a circle be described, it will touch all the sides of the triangle.

[m] 5. 5.

[o] 4. 5.

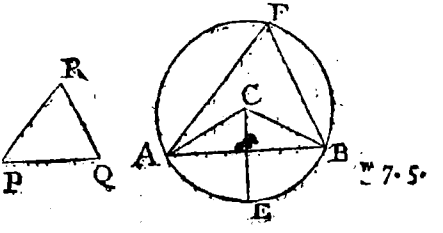
For,

For, let DG and DF be perpendicular to AC and BC [p]; then the triangles ADE, ADG, having two angles equal, each to each (by construction) and AD common, will not only be equiangular [q], but also have DE = DG [r]. By the same argument DE = DF; therefore the circumference of the circle also passes through G and F [s]; but it touches the sides of the triangle in those points [t], because G and F are right-angles [u].

[p] 4. 5.
[q] Cor. 1. to 10. 1.
[r] 15. 1.
[s] Def. 33. of 1.
[t] 6. 3.
[u] Constr.

PROBLEM XXV.

In a given circle (AFB) *to describe a triangle, equiangular to a given triangle* (PQR).



From the center C, draw the radii CA, CE, CB, making the angles ACE and BCE equal, each, to the angle R [w]; join A, B, and make the angle ABF = Q [w], and from the point F, where BF cuts the circle, draw FA; so shall AFB be the triangle required.

[w] 7. 5.

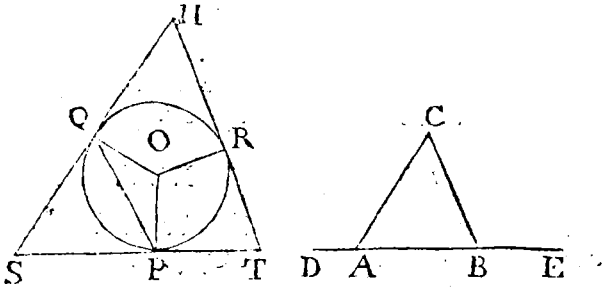
For, ABF = Q [x], F (= ACE [y]) = R [x]; and consequently BAF = P [z].

[x] Constr.
[y] 10. 3. and Constr.
[z] Cor. 1. to 10. 1.

PRO-

PROBLEM XXVI.

About a given circle (O), *to describe a triangle, equiangular to a given triangle* (ABC).



Produce out the side AB both ways; and draw the radii OP, OR, OQ, so as to make the angle
[a] 7. 5. POR = EBC, and POQ = DAC [a]; then draw three right-lines to touch the circle in the points
[b] 21. 5. P, Q and R [b], *and the thing is done.*

For, if PQ be drawn, the angles SQP and SPQ, will be less than the two right-angles SQO and
[c] Ax. 2. 1. and 6. 3. SPO [c]; and so PS and QS, not being parallels [d],
[d] Cor. 2. to 7. 1. and Cor. to 5. 1. they will meet each other [d]; therefore, as the like may be inferred with regard to PT and RT, &c. it is manifest that the three tangents form a triangle STH. Now, POR + T being = two right-
[e] Cor. 2. to 11. 1. angles [e] = ABC + EBC [f], and POR = EBC [g];
[f] 1. 1. thence will T = ABC: and, by the same argu-
[g] Constr. ment, S = BAC; whence also H = C.

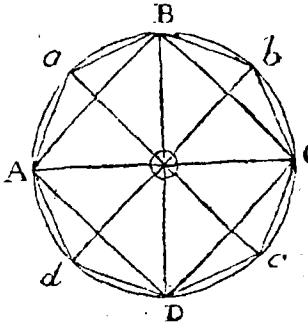
PROBLEM XXVII.

In a circle given (ABCD) *to inscribe a square.*

Draw two diameters AC and BD perpendicular
[h] 3. 5. to each other [h]; then draw AB, BC, CD and DA; so shall ABCD be a square inscribed in the circle.

For,

For, the angles AOB, BOC, DOC and DOA (as well as the sides OA, OB, OC, OD, containing them) being equal[i], the opposite sides AB, BC, CD, DA will likewise be equal[k]: and the angles ABC, BCD, CDA, DAB, are all of them right-angles[l], and therefore are equal.



[i] Ax. 7. 1. and Def. 33. 1.
[k] Ax. 10. of 1.
[l] 13. 3.

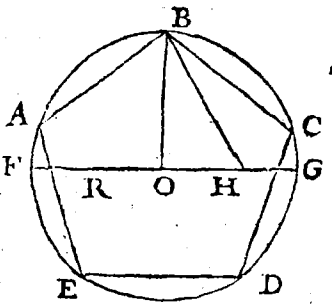
SCHOLIUM.

If two other diameters *ac*, *bd* be drawn (*by prob.* 5.) to bisect the angles AOB, BOC, a regular octagon A*a*B*b*C*c*D*d* may be inscribed in the circle. And if all the angles at the center O, be again bisected, a regular polygon of sixteen sides, may in like manner be determined; and so on, at pleasure.

PROBLEM XXVIII.

In a circle given (ABGE) *to inscribe a regular pentagon.*

At the center O, upon the diameter FG, erect the perpendicular OB[m], meeting the circumference in B; divide OG in H (*by prob.* 19.) so that $OH^2 = GH \times OG$; that is, take $OR = \frac{1}{2}OF$, and RH = dist. RB: Then draw BH; which will be equal to the side of the pentagon[n]; from whence the figure itself may be described.



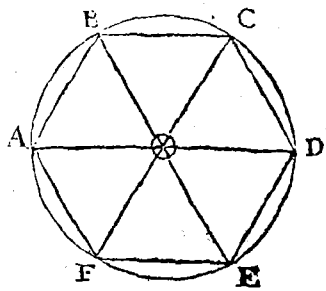
[m] 3. 5.
[n] 28. 3. and 8. 2.

SCHOLIUM.

Hence a regular decagon may be inscribed in
[o] 28. 3. the circle; the side thereof being = OH [o].

PROBLEM XXIX.

In a circle given, to inscribe a regular hexagon (ABCDEF).



From the extremes of any diameter AD, apply AB, AF, DC, and DE equal, each, to
[p] 20. 5. the radius AO [p]; then join B, C, and E, F; *and the thing is done.*

For, if the radii OB, OC, OE, OF be drawn; the triangles AOB and
[q] Constr. DOC, being equilateral [q], will also have the an-
[r] 14. 1. gle OAB = DOC [r]; whence AB is parallel (as
[s] Cor. to 8. 1. well as equal) to OC [s]; and consequently BC and
[t] 26. 1. AO are likewise equal, and parallel [t]: Therefore, seeing the triangles AOB, BOC, COD, *&c.* are equilateral, and alike in all respects; not only the sides, but also the angles ABC, BCD, *&c.* of
[u] Ax. 4 1. the hexagon, will be equal among themselves [u].

COROLLARY.

Hence it appears, that the side of a regular hexagon, inscribed in a circle, is equal to the semi-diameter, or radius.

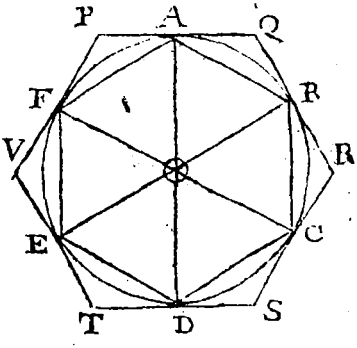
SCHOLIUM.

Besides the figures constructed in the preceding problems, and those arising from thence by continual bisections, or taking the differences, no other regular polygon can be described, from any known method, *purely geometrical*, by means of right-lines and circles only.

PRO-

PROBLEM XXX.

About a given circle to describe a regular polygon, of the same number of sides with a regular polygon (ABCDEF) *inscribed in the circle.*



From the center O, to the angles of the inscribed polygon, draw OA, OB, OC, *&c.* and perpendicular thereto draw PAQ, QBR, RCS [t], *&c.* intersecting [u] in P, Q, R, S, T, V; so shall PQRSTV be the polygon that was to be described.

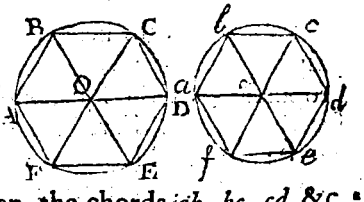
[t] 3. 5.
[u] Proof of 26. 5.

For, by taking away the equal [w] angles OAF, OAB, OBA, OBC, *&c.* from the equal (right) angles OAP, OAQ, OBQ, OBR, *&c.* the remainders FAP, BAQ, ABQ, CBR, *&c.* will also appear to be equal [x]: therefore the triangles FAP, ABQ, BRC, *&c.* (having also FA = [w] AB = BC, *&c.*) are equal in all respects [y]; and so the angles P, Q, R, *&c.* as well as the sides PQ, QR, RS, *&c.* are equal among themselves [z].

[w] Hyp.
[x] Ax. 5. 1.
[y] 15. 1.
[z] Ax. 4. 1.

PROBLEM XXXI.

Any two circles (ACE, ace) *being given, to describe a polygon in, or about the one* (ace) *that shall be similar to any polygon described in, or about the other* (ACE).

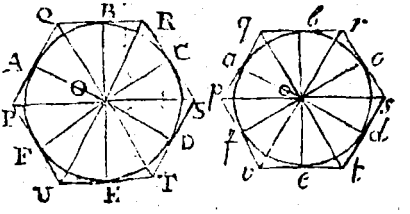


First, having drawn the radii OA, OB, *&c.* to the angles of the given insc. polyg. ABCDEF; make, at the center *o*, the angle *aob* = AOB, *boc* = BOC [a], *&c.* Then, the chords *ab*, *bc*, *cd*, &c. being drawn, I say, the polygon *abcdef* will be similar

[a] 7. 5.

lar to the given one ABCDEF. For, the triangles AOB, *aob*; BOC, *boc*; &c. being equiangular[b], the angles ABC, *abc* must also be equal[c]; and[d] AB : *ab* (: : OB : *ob*) : : BC : *bc*. In the same manner, the other corresponding angles are equal, and the sides containing them proportional: Therefore the two polygons are similar[e].

[b] Cor. 1. to 10. 1. and : 2. 1.
[c] Ax. 4. 1.
[d] 14. 4.
[e] Def. 14. 4.



Again, having drawn the radii OA, OB, OC, &c. to the points of contact of the given circumscribing polygon PQRSTU; draw likewise the radii *oa*, *ob*, *oc*, &c. making the angle *aob* = AOB, *boc* = BOC[f], &c. perpendicular to which[g] draw *pq*, *qr*, *rs*, &c. so shall the polygon *pqrstv* be similar to the given one PQRSTU. For, the angles OAQ, OBQ, *oaq*, *obq*, being all equal[h], and AOB also = *aob*[i]; the remaining angles AQB, *aqb* of the two quadrilaterals AOBQ, *aobq* must be equal[k]; as must likewise their halves OQB, *oqb* (for the right-angled triangles OAQ, OBQ, having OA = OB, and OQ common, have also OQA = OQB[l]). In the same manner is ORB = *orb*, &c. whence, the triangles POQ, *poq*; QOR, *qor*, &c. being equiangular, it follows[m] that PQ : *pq* (: : OQ : *oq*) : : QR : *qr*: And so of the rest. Therefore PQRSTU and *pqrstv* are similar[n].

[f] 7. 5.
[g] 3. 5.
[h] Ax. 7. 1.
[i] Constr.
[k] 11. 1. and Ax. 5.
[l] 16. 1.
[m] 14. 4.
[n] Def. 14. 4.

COROLLARY.

It appears from hence, that the similar inscribed polygons, as well as the circumscribing ones, are in proportion, as the squares of the radii of their respective circles. For, in the former case, $AO^2 : ao^2$: : [o] $AB^2 : ab^2$: : [p] ABCDEF : *abcdef*; and, in the latter, [o] $AO^2 : ao^2$: : $PO^2 : po^2$: : $PQ^2 : pq^2$: : [p] PQRSTU : *pqrstv*.

[o] 14. 4. and Cor. to 11. 4.
[p] 26. 4.

The End of the FIFTH BOOK.

ELEMENTS

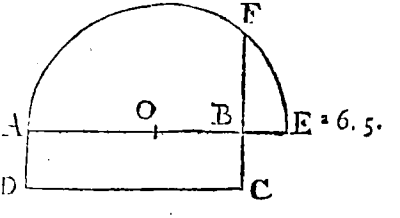
OF

GEOMETRY.

BOOK VI.

PROBLEM I.

To make a ſquare equal to a given rectangle (ABCD).



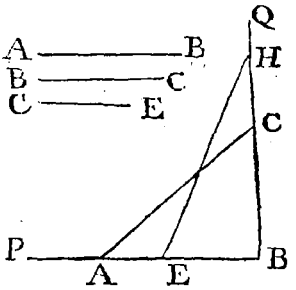
IN one ſide AB of the rectangle, produced, take BE = the other ſide BC; biſect AE in O [a]; and from the center O, at the diſtance of OA, or OE, let a ſemi-circle AFE be deſcribed; and let CB be produced to meet the circumference thereof in F: then a ſquare deſcribed on BF (*by* 10. 5.) will be equal to the given rectangle ABCD [b].

[a] 6. 5.

[b] Cor. to 19. 4.

PROBLEM II.

To make a ſquare equal to the ſum of two given ſquares.



Let AB and BC be the ſides of the two given ſquares.

Draw two indefinite lines BP, BQ, at right-angles to each other [c]; in which take BA = BA, BC = BC, and join A, C; then a ſquare deſcribed on AC *(by* 10. 5.*)* will be equal to the ſum of the two ſquares deſcribed upon AB and BC [d].

[c] 3. 5.

[d] 8. 2.

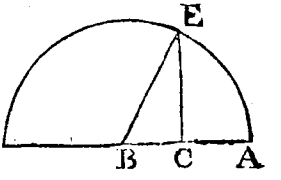
SCHOLIUM.

In the ſame manner a ſquare may be made equal to the ſum of three, or more, given ſquares: for if AB, BC, CE be taken as the ſides of the given ſquares, then, by making BH = AC, BE = CE, and drawing EH, it is evident that a ſquare upon EH will be equal to the ſum of the three ſquares upon AB, BC, and CE; or that, $EH^2 = BH^2 (AC^2) + BE^2 = AB^2 + BC^2 + CE^2$.

PROBLEM III.

To make a ſquare equal to the difference of two given ſquares.

Let



Let AB and BC (taken in the ſame ſtrait line) be equal to the ſides of the two given ſquares.

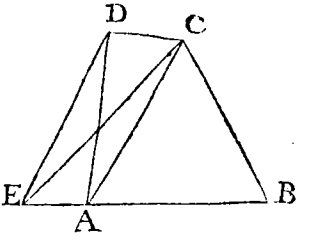
Upon the center B, with the radius BA, let a circle be deſcribed, and make CE perpendicular to BC [e], meeting the circumference thereof in E: ſo ſhall a ſquare deſcribed on CE (*by* 10. 5.) be equal to BE^2 (BA^2) — [f] BC^2.

[e] 3. 5.

[f] Cor. to 8. 2.

PROBLEM IV.

To make a triangle equal to a given quadrilateral (ABCD).



Draw the diagonal AC, alſo draw DE parallel to AC [g], meeting BA produced in E, and join CE; then will the triangle BCE = the given quadrilateral ABCD.

[g] 9. 5.

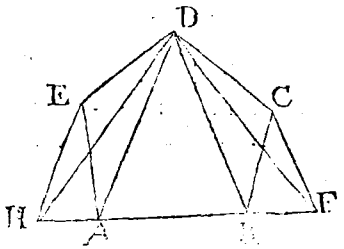
For, the triangles ACE, ACD, being upon the ſame baſe AC, and between the ſame parallels AC and ED, are equal [h]; therefore, if ABC be added to each, then alſo will BCE = ABCD [i].

[h] Co 1 to 2. 2.

[i] Ax. 4. 1.

PROBLEM V.

To make a triangle equal to a given pentagon (ABCDE).

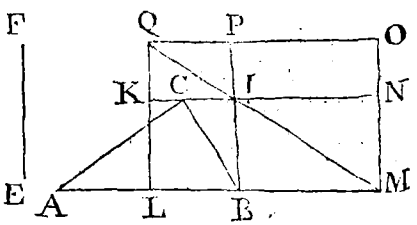


Draw DA and DB, and also EH and CF
[k] 9. 5. parallel to them [k], meeting AB produced in H and F; then draw DH and DF; so shall the triangle DHF = the pentagon ABCDE.

[l] Cor. 1. to 2. 2. For the triangle DHA is = DEA [l], and DFB = DCB [l]; therefore DHF (= DHA + DAB +
[m] Ax. 4. 1. DFB = DEA + DAB + DCB) = ABCDE [m].

PROBLEM VI.

Upon a given line (EF), *to make a rectangle equal to a given triangle* (ABC).



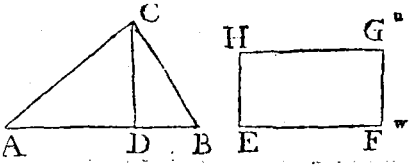
Thro' C, the vertex of the triangle, draw KN parallel to the base
[n] 9. 5. AB [n]; and bisect AB with the per-
[o] 6. 5. pendicular LQ [o], meeting KN in K; also draw BP
[p] 3. 5. perpendicular to AB [p], intersecting KN in I; then in AB, produced, take BM = EF, and draw MIQ cutting LQ in Q; draw QO and MO, parallel to AM and LQ [n], meeting each other in O: then will INOP be the rectangle required.

For,

For, it is evident, that LI, IO and LO are all rectangles [q]: therefore IN = BM [r] = EF [s], and IO = LI [t] = ABC [u].

[q] Cor. to 24. 1.
[r] 24. 1.
[s] Constr.
[t] 3. 2.
[u] Cor. 2. to 2. 2.

The same otherwise.



From the vertex C, upon the base AB, let fall the perpendicular CD [w]; make EH perpendicular to EF [x], and equal to a fourth-proportional to 2EF, AB, and CD [y]: then the rectangle EG contained under EF and EH will be equal to the triangle ABC.

[w] 4. 5.
[x] 3. 5.
[y] 13. 5.

For, since, by construction, 2EF : AB : : CD : EH, therefore is 2EF × EH = AB × CD [z], and consequently EF × EH = ½AB × CD = ABC [a].

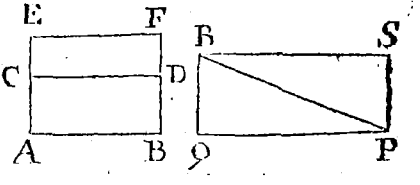
[z] 10. 4.
[a] Cor. 2. to 2. 2.

SCHOLIUM.

By either of the two preceding methods, a parallelogram having a given angle may be described, upon a given line, equal to a given triangle; if, instead of MBP, MLQ, or BDC, FEH being right-angles, you make them all equal to the angle given: the rest of the construction being the same.

PROBLEM VII.

Upon a given line (AB) *to describe a rectangle equal to a given right-lined figure* (PQRS).



Let the given figure be divided into triang. PQR, PRS: and upon the given line AB (by the precedent) let a rectangle ABDC, equal to the triangle PQR, be

be described; also upon CD make the rectangle CDFE equal to the triangle PRS: so shall ABEF, which is a rectangle (because both ACE, and BDE are continued right-lines [b]) be = PQR + PRS [c] = PQRS; *which was to be done.*

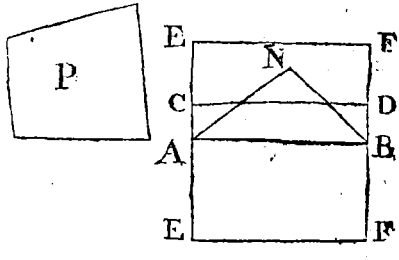
[b] 2. 1.
[c] Ax. 3. and 4. 1.

SCHOLIUM.

When the figure given has not more than five sides, the construction will be more easy, by first finding a triangle equal to it (*by prob.* 4. *or* 5.) and then making a rectangle equal to that triangle. But if the figure be a rectangle, the easiest way of all, will be to take a fourth-proportional BF to the given line AB and the two sides PQ and PS of the given rectangle (*by* 13. 5); which fourth-proportional will be the altitude of the rectangle required. For, since AB : PQ : : PS : BF (*by Constr.*) therefore (*by* 10. 4.) AB × BF = PQ × PS.

PROBLEM VIII.

To describe a rectangle equal to the sum, or difference of two given right-lined figures.



Let the two given figures be ABN and P.

By the precedent, let two rectangles AD and AF, respectively equal to ABN and P, be described, on the same, or different sides of AB, according as the difference, or sum, of the two figures is required: then will the rectangle CF be equal to that sum or difference [c].

[c] Ax. 4. or 5. 1.

COROL-

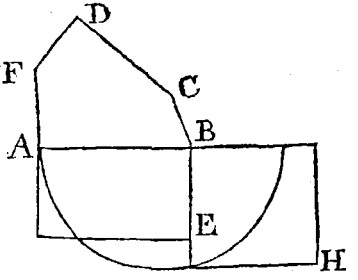
COROLLARY.

Hence two lines having the ſame ratio with two given right-lined figures, are determined: for AC : AE : : AD (ABN) : AF [f] (P). [f] 7. 4.

PROBLEM IX.

To make a ſquare equal to any right-lined figure given (ABCDF).

Upon AB deſcribe a rectangle AE equal to ABCDF [g]; then make a ſquare BH equal to that rectangle [h], *and the thing is done.*



[g] 7. 6.

[h] 1. 6.

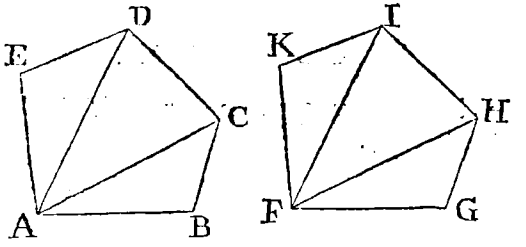
SCHOLIUM.

After the ſame manner (*from prob.* 8.) a ſquare may be deſcribed equal to the ſum, or difference, of any two given right-lined figures.

PRO-

PROBLEM X.

To describe a figure (FGHIK) *equal, and similar to a given right-lined figure* (ABCDE).



[i] 2. 5. Draw AC and AD, and also FG equal to AB [i]
[k] 7. 5. make the angle GFH = BAC [k], HFI = CAD, and IFK = DAE [k]; likewise make FH = AC, FI = AD, and FK = AE [i]; then draw GH, HI, IK and KF, *and the thing is done.*

[l] Ax. 10. 1. For, since the triangle FGH = ABC [l] and FHI = ACD [l], *&c*; therefore is the whole polygon FGHIK, also, equal to the whole polygon
[m] Ax. 4. 1. ABCDE [m].

Moreover, these equal triangles being also equiangular [l], it is manifest, that G = B, GHI = BCD [m], HIK = CDE, and so on; therefore, FG being also = AB, GH = BC, HI = CD [l], *&c.* the two polygons ABCDE, FGHIK are similar to each
[n] Def. 14. 4. other [n].

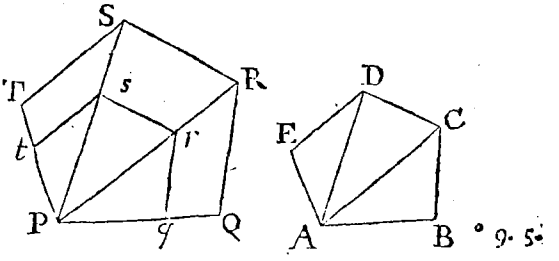
SCHOLIUM.

The figure FGHIK may be otherwise constructed, by making the triangles FGH, FHI, *&c.* respectively equilateral to ABC, ACD, *&c.* as is evident from 14. 1. and Ax. 4.

PRO-

PROBLEM XI.

Upon a given line (AB) *to describe a figure* (ABCDE) *similar to a given right-lined figure* (PQRST).



Draw PR and PS, and in PQ (produced if needful) take P*q* = AB; draw *qr* parallel to QR [o], meeting PR in *r*; also draw *rs* and *st*, parallel to RS and ST [o], intersecting PS and PT in *s* and *t*; then upon AB, *by the precedent*, describe a polygon equal and similar to P*qrst*; *and the thing is done.* [o] 9. 5.

For, since any angle BCD (*qrs*) of the polygon ABCDE, is equal to its correspondent QRS [p]; and also CB (*qr*) : CD (*rs*) : : RQ : RS [q], therefore the two polygons ABCDE, PQRST are like to each other [r]. [p] Cor. 1. to 7. 1. [q] 21. 4. [r] Def. 14. of 4.

SCHOLIUM.

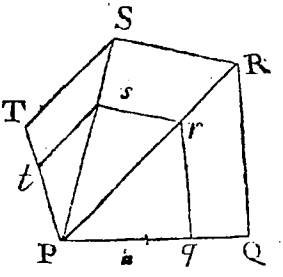
This last problem may be otherwise constructed, by making the triangles ABC, ACD, ADE equiangular to the triangles PQR, PRS, PST, respectively.

For, then the angle BCD being = QRS, CDE = RST [s], *&c*; and also BC : QR (: : AC : PR) : : CD : RS [t], *&c.* the two polygons must therefore be similar to each other [u]. [s] Ax. 4. 1. [t] 14. 4. [u] Def. 14. of 4.

PRO-

PROBLEM XII.

To describe a figure, similar to a right-lined figure given (PQRST), *which shall be to it in a given ratio of one right-line to another.*

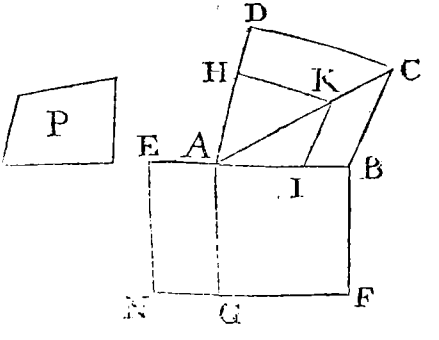


In PQ (produced if necessary) take P*n* to PQ in the given ratio of the figure to be described to the figure
[w] 12. 5. given[w]; and, in the same line PQ, take P*q* equal to a mean-proportional between
[x] 14. 5. P*n* and PQ[x]; upon which *(by the precedent)* let P*qrst*, similar to PQRST, be described, *and the thing is done.*

For, since P*n* : P*q* : : P*q* : PQ *(by Constr.)*;
[y] 27. 4. therefore P*n* : PQ : : P*qrst* : PQRST[y].

PROBLEM XIII.

To describe a figure that shall be equal to one right-lined figure given (P), *and similar to another* (ABCD).



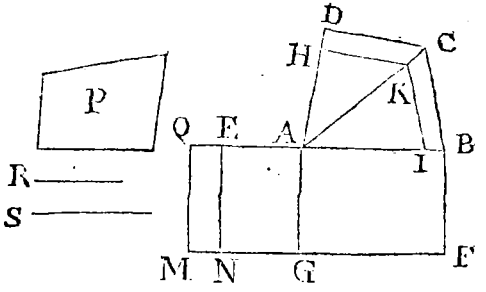
Upon AB make the rectangle ABFG
[z] 7. 6. = ABCD[z], and upon AG make the rectangle AGNE = P[z]; in AB take AI equal to a mean-proportional between
[a] 14. 5. AE[a], let AIKH be described similar,

lar, and alike ſituate, to ABCD [b], *and the thing is done.* [b] 11. 6.

For ABCD (AF) : P (AN) : : AB : AE [c] (: : AB^2 : AB × AE [c] = [d] AI^2) : : ABCD : AIKH [e]; and therefore AIKH = P [f]. [c] 7. 4. [d] Conſtr. and 10. 4. [e] 26. 4. [f] Ax. 4. 4.

PROBLEM XIV.

To deſcribe a figure ſimilar to a given right-lined figure (ABCD), *which ſhall be to another given right-lined figure* (P) *in a given ratio of one right-line* (S) *to another* (R).



Make the rectangle ABFG = ABCD [g], and the rectangle AGNE = P [g]; alſo in AE, produced, take AQ = a fourth-proportional to R, S and AE [h]; then, *by the precedent*, make AIKH ſimilar to ABCD, and equal to the rectangle AG × AQ: then will AIKH (AG × AQ) : P (AG × AE [i]) : : AQ : AE [k] : : S : R [l]; *which was to be done.* [g] 7. 6. [h] 13. 5. [i] Conſtr. [k] 7. 4.

The End of the SIXTH BOOK.

ELEMENTS

OF

GEOMETRY.

BOOK VII.

DEFINITIONS.

1. A Right-line is ſaid to be perpendicular to a plane, when it is perpendicular to all right-lines, that can be drawn in that plane, from the point on which it inſiſts.

2. One plane is ſaid to be perpendicular to another, when all right-lines drawn in the one, perpendicular to the common ſection, are perpendicular to the other.

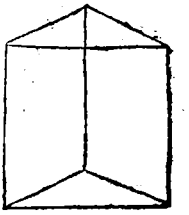
3. Parallel planes are thoſe, which are every where equally diſtant, the one from the other.

4. A Solid is that, which has length, breadth, and thickneſs.

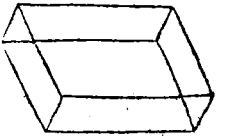
5. Similar ſolids are ſuch, as are bounded by an equal number of ſimilar planes.

6. A Priſm

6. A Prism is a solid, whereof the planes of the sides are parallelograms, and whereof the two ends, or opposite bases, are plane, rectilinear figures, parallel to each other.

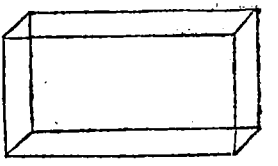


7. A Parallelepipedon is a solid bounded by six parallelograms, whereof the opposite ones are parallel, equal, and like to each other.

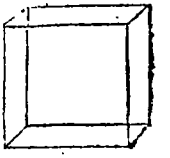


8. An upright prism, or parallelepipedon, is that, whereof the planes of the sides are perpendicular to the plane of the base.

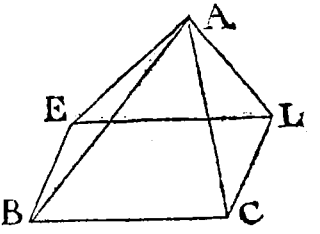
9. A rectangular parallelepipedon is that, whose bounding planes are all rectangles, and which stand at right-angles one to another.



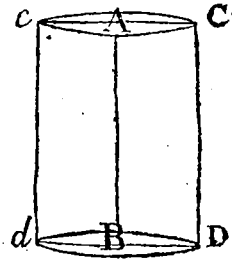
10. When all the bounding planes are squares, the parallelepipedon is called a cube.



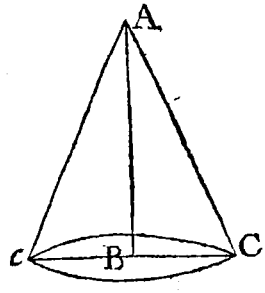
11. A Pyramid is a solid, whose base is any right-lined plane figure, and whose sides are triangles, having all their vertices united in a point, above the base, called the vertex of the pyramid. Thus ABCLE represents a pyramid, whose vertex is A, and base BCLE.



12. A Cylinder (DC*cd*) is a ſolid generated by the rotation of a rectangle ACDB about one of its ſides AB, ſuppoſed at reſt; which quieſcent ſide AB is called the axis of the cylinder.



13. A Cone (AC*c*) is a ſolid generated by the rotation of a right-angled triangle ABC about its perpendicular AB, called the axis of the cone.



14. A Sphere is a ſolid generated by the rotation of a ſemi-circle about its diameter.

15. The Fruſtum of a pyramid, or cone, is that part which remains, when any part next the vertex, cut off by a plane parallel to the baſe, is taken away.

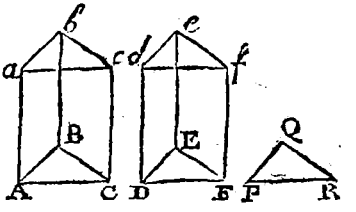
16. The Altitude of a pyramid, or priſm, is the perpendicular diſtance of the vertex, or upper plane thereof, from the plane of the baſe.

17. Every rectangular parallelepipedon is ſaid to be contained under the three right-lines that are the length, breadth, and altitude thereof.

18. A Plane is ſaid to be extended (or to paſs) by a right-line, when every part of the latter is placed in, or touched by, the former.

An AXIOM.

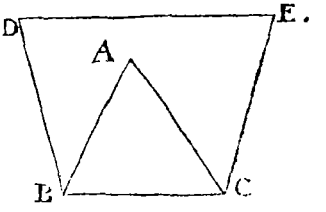
Upright prisms (A*abc*CBA, D*def*FED) of the same altitude, standing upon bases (ABC, DEF) equal and like to each other, are themselves equal.



To see the evidence of this AXIOM *in the strongest light, conceive a right-lined plane figure* PQR *to be formed, equal and like in all respects to the bases* ABC, DEF *of the two prisms; upon which, conceive the prisms to be placed, one after another, so that their bases may coincide therewith: Then, because the planes of the sides stand, in both cases, perpendicular to the plane of the base, upon the same lines* PQ, QR, PR, *and are carried up to the same height, it is manifest, that the bounds of the two solids, when thus placed, have the very same position; and, consequently, that the solids themselves, occupying (successively) the same identical space, are equal the one to the other.*

A POSTULATE.

That by any two right-lines (AB, AC) meeting in a point, a plane may be extended.



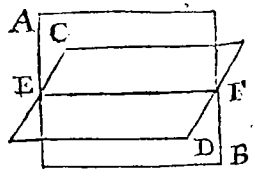
In order the better to comprehend the sense and design of this Postulate, *let a plane* BDEC, *extended by the right-line joining the points* B *and* C, *be conceived to be revolved about upon that line, till it meets with, or takes in, the point* A; *then, the plane including, in that position, all the*

the three points B, C, *and* A, *it also includes, or is extended by, the right-lines* AB, AC, BC, *joining those points; which are in the same plane with their extremes (by Def.* 6. 1).

Hence it appears, that, by any three points, a plane may be extended; and that all the three sides of any right-lined triangle, are in the same plane.

THEOREM I.

The common section of two planes (AB, CD) *is a right-line.*

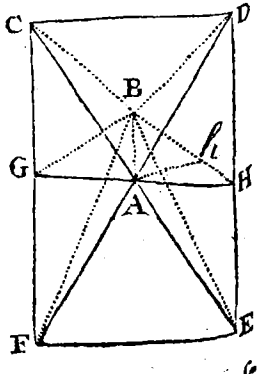


For, between the two extreme points E, F of the common section, let a right-line EF be drawn[a]; then, that line being in the plane AB[b], and also in the plane CD[b], it must, of consequence, be the common section of them both.

[a] Post. 1.
[b] Def. 6. 1. and Ax. 8. 1.

THEOREM II.

If a right-line (AB) *be perpendicular to two other right-lines* (CE, DF) *cutting each other, at the common section* (A), *it will be perpendicular to the plane* (CDEF) *passing by those two lines.*



Take AC, AD, AE, AF all equal to one another; and, having joined CD, DE, EF, CF, let there be drawn thro' A, in the plane CDEF, any right-line GH, meeting CF and DE in G and H; and let BC, BG, BF, BD, BH, and BE be also joined.

Because

Becauſe AC[b] = AE = AD = AF, and CAF = DAE[c], therefore is CF = DE[d], and the angle FCA (or GCA) = DEA (or HEA); and ſo, GAC being likewiſe = HAE[c] and AC = AE[b], thence will AG = AH[e] and GC = HE.

[b] Conſtr.
[c] 3. 1.
[d] Ax. 10. 1.
[e] 15. 1.

Again, ſince the right-angled triangles CAB, DAB, EAB, FAB have their baſes all equal[f], and the perpendicular AB common, their hypothenuſes BC[d], BD, BE, BF will be equal too; and therefore, the triangles CBF, DBE being mutually equilateral, the angle FCB (or GCB) muſt be = DEB (or HEB[g]); whence, GC being alſo = HE, and BC = BE, thence is BG = BH[d]: Therefore, AG being likewiſe (as is proved above) = AH, and AB common, the angles GAB, HAB are equal, and conſequently right-angles[h]. In the ſame manner, AB is perpendicular to every other right-line drawn thro' A in the plane CDEF; *which was to be demonſtrated*[i].

[f] Hyp.
[g] 14. 1.
[h] Def. 8. of 1.
[i] Def. 1. of 7.

COROLLARY.

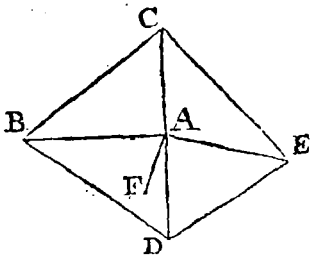
Hence it will appear, that, if one right-line (AB), meeting ſeveral others (AF, AE, *&c.*) in the ſame point (A), is perpendicular to them all; theſe laſt will be all in the ſame plane. Becauſe it is impoſſible for a right-line (A*h*) drawn from A, out of the plane (FEDC) of the two former of theſe, to be perpendicular to AB; ſeeing the angle BA*h* is leſs, or greater, than a right-angle (or BAH[k]), according as A*h* is poſited above, or below the ſaid plane FEDC[l].

[k] 2. 7.
[l] Ax. 2. 1.

THEOREM III.

If thro' any given point (A) *in a plane given* (BCD), *a line* (CD) *be drawn, and perpendicular to that line, at the ſame point* (A), *two other lines* (AB, AE) *be alſo drawn, the one* (AB) *in the plane given* (BCD), *and the other in any other plane* (CDE)

passing by the first line (CD); *then, I say, that a right-line* (AF) *drawn from the given point* (A), *at right-angles to the first perpendicular* (AB) *in the plane* (BAE) *of the two, will be perpendicular to the given plane* (BCD) *at the given point* (A).



For CA being perpendicular both to AB and AE [m], it will likewise be perpendicular to AF [n]; and so FA, being perpendicular to AB [m] (as well as to CA) is also perpendicular to the plane BCD, in which AB and CA are drawn [n].

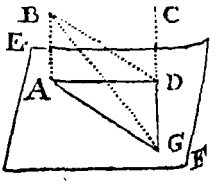
[m] Hyp.
[n] 2. 7.

SCHOLIUM.

In this last Theorem, the manner of erecting a perpendicular to a plane, at a point given, is indicated, and the consistence of the *first definition*, of this book, evinced.

THEOREM IV.

Two right-lines (AB, CD), *perpendicular to the same plane* (EF), *are parallel to each other.*



Draw, in the plane EF, the right-line AD, and also DG perpendicular to AD; make DG = AB, and let AG, BG, and BD be drawn.

The triangles BAD, ADG, having AB = DG [o], AD common, and the angle BAD (= right-angle [p]) = ADG [o], will also have BD = AG [q]: And so, the triangles BDG, BAG being mutually equilateral, the angle [r] BDG must be = BAG = [p] right-angle: But the line CD (as well as BD and AD) being perpendicular

[o] Constr.
[p] Def. 1. 7.
[q] Ax. 10. 1.
[r] 14. 1.

pendicular to DG [p], it is therefore in the same plane (CDAB) with BD and AD [t]; and consequently, as the angles BAD, CDA are both right-ones [p], it must be also parallel to BA [u].

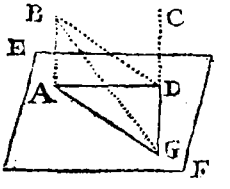
[p] Def. 1. 7.
[t] Cor. to 2. 7.
[u] 4. 1.

COROLLARY.

Hence it follows, that from the same given point, to one and the same plane, more than one perpendicular right-line cannot possibly be drawn: Because all perpendiculars to the same plane, are parallels; but lines drawn from the same point are not parallels.

THEOREM V.

If, of two parallel right-lines (AB, CD) *the one* (AB) *is perpendicular to any plane* (EF), *the other* (CD) *shall also be perpendicular to the same plane* (EF).

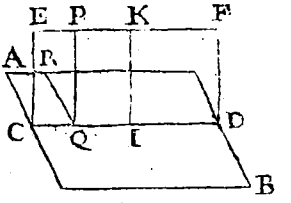


The construction of DG, AG, &c. being supposed the same here as in the preceding Theorem; it appears from thence, that ADG and BDG are both right-angles: And, because BD is in the plane of the proposed parallels BA, CD [w] the angle CDG is also a right-one [x], as is likewise the angle CDA [y]. Therefore CD is perpendicular to the plane ADG [z].

[w] Def. 6. 1.
[x] 2, 7.
[y] 4. 1.
[z] 2. 7. and Def. 1.

THEOREM VI.

If a right-line (PQ) *be perpendicular to a plane* (AB), *any plane* (ED) *passing by that line, will be perpendicular to the same plane* (AB).



In the plane ED, from any point K, draw KI perpendicular to the common section CD: Then, the angle DIK being a right-angle = DQP[y], IK will be parallel to PQ[z], and therefore perpendicular to the plane AB[a]. By the same inference, all other right-lines drawn in the plane ED, perpendicular to the common section CD, are also perpendicular to the plane AB. Therefore the plane ED itself is perpendicular to the plane AB[b].

[y] Def. 1. 7.
[z] 4. 1.
[a] 5. 7.
[b] Def. 2. 7.

COROLLARY I.

Hence it will appear, that the plane AB (according to the sense of the definition) is perpendicular to the plane ED: For a right-line QR drawn in the former, perpendicular to the common section CD, being also[c] perpendicular to PQ, it[d] (and consequently the plane AB in which it is) will be perpendicular to the plane ED[e].

[c] Hyp. and Def. 1. 7.
[d] 2. 7.
[e] 6. 7.

COROLLARY II.

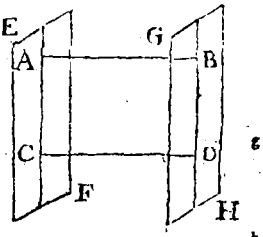
Hence it also appears, that a line standing at right-angles to one of two perpendicular planes at any point (I) in the common section (CD), must be in the other plane: For the line IK, in the plane ED, is perpendicular to the plane AB; besides which line, another perpendicular to AB, from the same point I, cannot be drawn[f].

[f] Cor. to 4. 7.

THEOREM VII.

Planes (EF, GH) *to which one and the same right-line* (AB) *is perpendicular, are parallel to each other.*

From



From any point C, in the plane EF, let CD be drawn parallel to AB; which (as well as AB) will be perpendicular to both the planes [g]; and so the angles A, B, C, D (when AC and BD are joined) being all right-ones [h], the figure ABDC (whereof the sides AC, BD are in the same plane with the parallels AB, CD [i]) will therefore be a rectangular parallelogram [k]; and consequently CD = AB [l]. By the very same argument, all other perpendiculars, terminated by the two planes, are equal among themselves; *which was to be demonstrated* [m].

[g] 5. 7.
[h] Def. 1. 7.
[i] Def. 6. 1.
[k] 4. 1. and Def. 24.
[l] 24. 1.
[m] Def. 3.

COROLLARY.

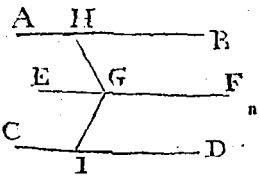
Hence, all right-lines perpendicular to one of two parallel planes, are also perpendicular to the other.

SCHOLIUM.

From the two last Theorems, the sense, and propriety of the two definitions of perpendicular, and parallel planes, appear manifest.

THEOREM VIII.

Right-lines (AB, CD) *parallel to one and the same right-line* (EF), *tho' not in the same plane with it, are also parallel to each other.*



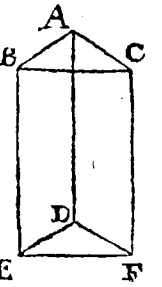
Let GH and GI be drawn perpendicular to EF, in the planes AF and ED of the proposed parallels. Then [n] shall GF be perpendicular to the plane passing by HGI; and HB, ID will also be perpendicular to the same plane [o], and therefore parallel to each other [p].

[n] 2. 7.
[o] 5. 7.
[p] 4. 7.

THEO-

THEOREM IX.

If two right-lines (AB, AC) *meeting each other, be respectively parallel to two other right-lines* (DE, DF) *also meeting each other, and not being in the same plane with them; the angles* (BAC, EDF) *contained by those lines, will be equal.*

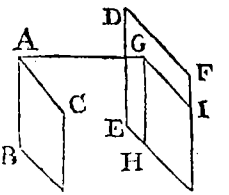


Take AB, AC, DE, DF all equal to each other, and let BE, AD, CF, BC, EF be drawn. Then, AB and ED, as well as AC and DF, being equal, and parallel [q], BE and CF will be both equal, and parallel, to AD [r], and therefore equal, and parallel, to each other [s]; whence BC is also equal to EF [r]; and so, the triangles ABC, DEF being mutually equilateral, the angles BAC, EDF are likewise equal [t].

[q] Hyp. and Constr.
[r] 26. 1.
[s] Ax. 1. and 8. 7.
[t] 14. 1.

THEOREM X.

If two right-lines (AB, AC) *meeting each other, be respectively parallel to two other right-lines* (DE, DF) *also meeting each other, and not being in the same plane with them, the planes* (BAC, EDF) *extended by those lines, will be parallels.*



Let AG be perpendicular to the plane BAC, meeting the plane EDF in G; in which last plane, let GH and GI be drawn parallel to ED and DF; and they will also be parallel to AB and AC [u]; whence, seeing the angles GAB and GAC are both right [w], AGH and AGI must likewise be right-angles [x]; and so AG being

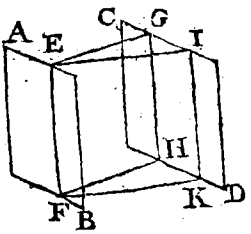
[u] 8. 7.
[w] Constr.
[x] 5. 1.

being perpendicular to the plane EDF [a] (as well as [b] to BAC [b]), the two planes are parallel to each other [c].

[a] 2. 7.
[b] Constr.
[c] 7. 7.

THEOREM XI.

The sections (EF, GH) *made by a plane* (EFHG) *cutting two parallel planes* (AB, CD), *are also parallel, the one to the other.*



Let EG and FH be drawn parallel to each other, in the plane EFHG; also let EI, FK be perpendicular to the plane CD, and let IG, KH be joined: Then, EG being parallel to FH [d], and EI to FK [e], the angle GEI is = HFK [f]; but the angle EIG is also = FKH, being both right-angles [g]; and EI is = FK [h]: Therefore EG will be equal [i] (as well as parallel) to FH; and consequently EF likewise parallel to GH [k].

[d] Constr.
[e] 4. 7.
[f] 9. 7.
[g] Def. 1. 7.
[h] Def. 3. 7.
[i] 15. 1.
[k] 26. 1.

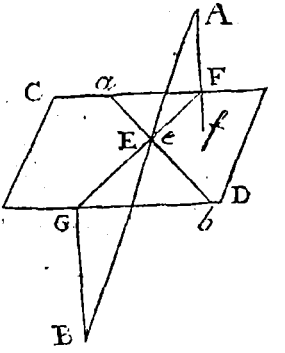
COROLLARY.

It appears from hence, that parallel lines, terminated by the same parallel planes, are equal to each other.

THEOREM XII.

If, from the two extremes of a right-line (AB) *cutting a plane* (CD), *two perpendiculars* (AF, BG) *be drawn to the plane; the right-line* (FG) *joining the points where they meet the plane, will pass thro' the point* (E) *in which the proposed line* (AB) *cuts the plane, so as to be divided by it into two parts* (FE, EG), *having the same ratio to each other as those two perpendiculars* (AF, BG).

For,



For, if AF be produced to *f*, the lines A*f* and BG (which are both perpendicular to the plane CD[l]) will be parallel to each other[m]; therefore AB and FG being both in the same plane with these parallels, in which their extremes are posited[n], they must necessarily (as they are not themselves parallels) intersect each other: And so the alternate angles FAE, GBE being equal[o], as well as the opposite ones FEA, GEA[p], thence will FE : EG :: AF : BG[q]; *which was to be demonstrated.*

[l] Hyp.
[m] 4. 7.
[n] Def. 6. 1.
[o] 7. 1.
[p] 3. 1.
[q] 14. 4.

COROLLARY.

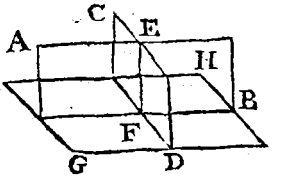
Hence, if in the plane CD, the lines FC, GD be made parallel, the one to the other, and in them be taken F*a* = FA, and G*b* = GB; then will the line (*ab*) joining the points *a* and *b*, cut FG in the very same point in which it is cut by AB. For, if *e* be taken as the intersection of *ab* and FG, the triangles *a*F*e*, G*eb* will be equiangular[r]; whence F*e* : *e*G :: F*a* (FA) : [s]G*b* (GB) :: FE : EG[t]. Therefore, seeing FG is divided in one and the same ratio, both by *e* and E, these points must necessarily coincide[u].

[r] 3. and 7. of 1.
[s] 14. 4.
[t] 12. 7.
[u] Ax. 2. and 5. of 4.

THEOREM XIII.

If two planes (AB, CD) *cutting each other, be both perpendicular to a third plane* (GH), *their common section will also be perpendicular to the same plane* (GH).

For,



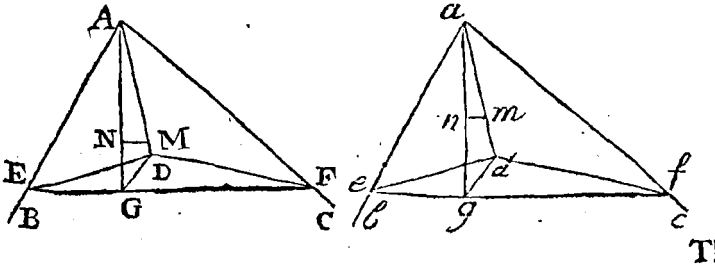
For, from the extreme point F of the common section, let the right-line FE be erected perpendicular to the plane GH: which line being in both the planes AB, CD[l], it must necessarily be their common section. Therefore the common section is perpendicular to the plane GH[m].

[l] Cor. 2. to 6. 7.
[m] Constr.

THEOREM XIV.

If, from the angular points (A, a) *of two equal angles* (BAC, bac), *two right-lines* (AD, ad) *be drawn, or elevated on high, above the planes of the said angles, so as to form equal angles with the lines first given, each to its correspondent* (DAB = dab, DAC = dac), *and if, from any points* (M, m) *in those elevated lines, perpendiculars* (MN, mn) *be let fall upon the planes* (BAC, bac) *of the first-mention'd angles; these perpendiculars will be, in proportion, as the parts* (AM, am) *of the elevated lines included between them and the angular points* (A, a) *first named.*

Make AD and *ad* equal to each other; and in the planes ADB, ADC, *adb*, *adc*, draw DE, DF, *de*, *df* perpendicular to AD and *ad*; and from their intersections with AB, AC, *ab*, *ac*, draw EF and *ef*, meeting AN and *an* (produced) in G and *g*, and let D, G, and *d*, *g* be joined.



The

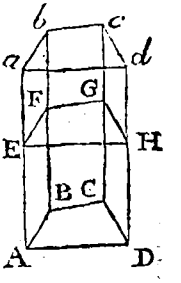
[n] Constr. The angles ADE, ADF being both right-ones [n]; [o] 2. 7. not only the line AD [o], but the plane ADG ex- [p] 6. 7. tended by it, is perpendicular to the plane EDF [p]. But the same plane ADG is also perpendicular to the plane EAF [p]: Therefore the common section EF [q] 13. 7. is likewise perpendicular to the plane ADG [q]; and [r] Def. 1. 7. consequently the angle EGA a right-one [r]. By the very same argument, *ega* is a right-angle. Now the triangles ADE, *ade*; ADF, *adf* being equal in all [s] Hyp. and respects [s], and the angle EAF = *eaf* [t], the triangles 15. 1. AEF, *aef* are also equal and alike [u]; and so, the [t] Hyp. angle AEG being = *aeg*, EGA = *ega*, and AE = [u] Ax. 10. 1. *ae*, thence is AG = *ag* [w], and the angle DAG [w] 15. 1. (MAN) = [x] *dag* (*man*), because ADG, *adg* are both [x] 16. 1. right-angles [o]. Therefore MN : *mn* : : AM : *am* [y]. [y] 14. 4.

COROLLARY.

Hence the two perpendiculars MN, *mn* subtend equal angles at the points (A, *a*) from whence the two elevated (or inclining) lines are drawn.

THEOREM XV.

If any solid (Ac), *having a rectilinear base* (ABCD), *whereof the planes* (Ab, Bc, Cd, Ad) *of the sides are parallelograms, be cut by a plane parallel to the base, the section* (EFGH) *will be equal, and similar to the base.*



For, the plane EFGH being parallel to ABCD [z], EF is therefore parallel to AB [a]; and so, AF being a [z] Hyp. [a] 11. 7. parallelogram [b], EF is equal (as well [b] Def. 24. of 1. as parallel) to AB [c]. In the same [c] 24. 1. manner is FG equal, and parallel to FG, &c. Whence also the angle EFG is = the angle ABC [d]; and so [d] 9. 7. of the rest. Therefore EFGH is both equilateral and equiangular to ABCD.

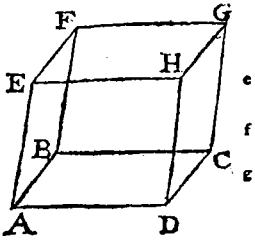
COROL-

COROLLARY.

Hence the oppofite bafes of a prifm are equal and fimilar (as well as parallel) to each other.

THEOREM XVI.

If, from one of the angular points (A) *of any parallelogram* (AC), *a right-line* (AE) *be elevated above the plane of the parallelogram, fo as to make any angles* (EAB, EAD) *with the two contiguous fides thereof, and there be alfo drawn, from the three remaining angular points, three other right-lines* (BF, CG, DH) *parallel, and equal to the former* (AE); *then, the extremes of thofe lines being joined, I fay, the figure* (AG) *thus defcribed, will be a parallelepipedon.*



For AE, BF, CG, DH being all parallel to each other [e], AE and BF are in the fame plane [f], as are alfo AE and DH, &c. Therefore, all thefe lines being [g] equal among themfelves, AF, AH, DG, and BG are parallelograms [h]; and fo, EF being parallel to AB, parallel to DC, parallel to HG [i], EF and HG are in the fame plane [f]; and EG is alfo a parallelogram [k], equilateral to its oppofite AC: But EG is equiangular, and parallel (as well as equilateral) to its oppofite AC; becaufe, EF being parallel to AB, and EH to AD, the angle FEH is therefore = BAD [l], and the plane EG parallel to the plane AC [m]. And in the fame manner, the other oppofite parallelograms appear to be equiangular and parallel (as well as equilateral). Therefore the folid AG, bounded by them, is a parallelepipedon [n].

[e] Hyp. and 8. 7.
[f] Def. 13. 1.
[g] Hyp.
[h] 26. 1.
[i] 8. 7.
[k] 25. 1.
[l] 9. 7.
[m] 10. 7.
[n] Def. 7. of 7.

COROL-

COROLLARY.

If the angle A of the parallelogram AC be a right-one, and AE be erected perpendicular to the plane AC; then will the parallelepipedon be a rectangular one: For, all the three contiguous planes AC, AF, AH being rectangular [o], their opposites will be rectangular likewise [p]: And so, the angles HGF, HGC being right-ones, HG will be perpendicular to the plane GB [q]; and consequently both the planes EG and DG likewise perpendicular to the plane BG [r]. And so of the rest.

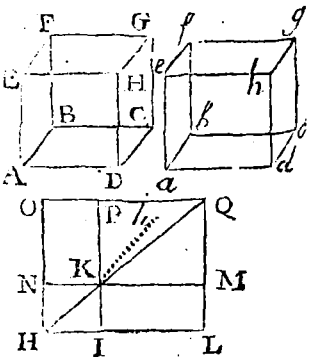
[o] Hyp. and 24. 1.
[p] 16. 7.
[q] 2. 7.
[r] 6. 7.

SCHOLIUM.

In this Theorem, a way to describe a parallelepipedon of any given dimensions, is indicated; and the consistence of the 7th and 9th definitions evinced.

THEOREM XVII.

Rectangular parallelepipedons (AG, ag) *standing upon equal bases* (AC, ac), *and having equal altitudes* (AE, ae), *are equal.*



Let the rectangles OK and KL, equal and like to the bases AC and ac of the two solids, be so formed, that NK may be in the same strait line with KM; then shall PK be also in the same strait line with KI [s]; and the figures NI, PM, OL, formed by producing the sides of the two rectangles, will likewise be rectangles [t].

[s] 2. 1.
[t] Cor. to 24. 1.

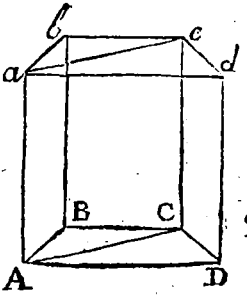
Now,

Now, HK and QK being drawn, the triangle NHK = IHK[u], PKQ = MKQ[u], and the rectangle OK = LK[w]; and consequently (by the addition of equals) OQKH = LQKH. Therefore, HKQ being a diagonal to the rectangle OL[x], dividing it into two equal, and like triangles OQH, LQH[u], if upon these, as bases, two upright prisms be conceived to be erected, of the same common altitude (K*b*) with the proposed solids, these prisms will also be equal[y]. But the former of these is composed[z] of three prisms, on the bases OPKN, NHK, KPQ; and the latter of three others, on KMLI, HIK, KMQ; whereof the second and third, in both ranks, are respectively equal[y]. Therefore the remaining two, on the bases OPKN, KMLI must also be[1] equal. But the former of these is = AG[y], and the latter = *ag*: Therefore, also, is AG = *ag*[a].

[u] 24. 1.
[w] Hyp.
[x] 24. 1. and Ax. 2.
[y] Ax. 1. 7.
[z] Cor. to 18. 7.
[1] Ax. 5.
[a] Ax. 1.

THEOREM XVIII.

If, at the angular points of any given right-lined figure (ABCD), *equal perpendiculars* (Aa, Bb, Cc, Dd) *be erected to the plane thereof, and the extremes of these* (a, b; b, c &c.) *be joined; an upright prism* (AabcdDCBA) *on the given base* (ABCD) *will thereby be formed.*



For, A*a*, B*b*, C*c*, D*d* being all equal[b], and parallel[c], it is evident that A*ab*B, B*bc*C, &c. are parallelograms[d]; and that the planes of these are all perpendicular to that of the base ABCD[e] (since A*a*, B*b*, C*c* are so, by construction). Moreover it will appear, that *abcd* is one plane figure, parallel to

[b] Constr.
[c] 4. 7.
[d] 26. 1.
[e] 6. 7.

ABCD; for, the lines *ac*, AC (when *a*, *c* and A, C
[f] 26. 1. are joined) being parallels [f] (as well as *ab*, AB; *ad*, AD), the plane *abc* is, therefore, parallel to
[g] 10. 7. ABC [g] (or ABCD); and *acd* is likewise parallel to ABCD. But *abc* and *acd* are in one plane; be-
[h] Cor. to 7. 7. cause A*a* being perpendicular to both of them [h],
[i] Def. 1. 7. and [i] consequently to all the lines *ab*, *ac*, *ad*; these
[k] Cor. to 2. 7. must necessarily be all in one [k] plane, parallel to
[l] Def. 6. ABCD; *which was to be demonstrated* [l].

COROLLARY.

It appears from hence, that, if upon all the parts ABC, ACD, into which any rectilineal figure ABCD is divided, upright prisms (A*abc*CB, A*acd*DC) of the same altitude be constituted; these prisms will form one prism, on the (whole) given base ABCD; seeing that *abc* and *acd* form one continued plane
[m] 18. 7. superficies *abcd* [m], parallel to ABCD.

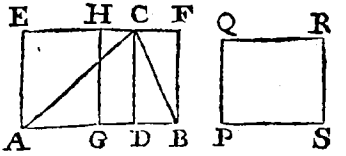
SCHOLIUM.

After the same way, a prism, any how inclining on the given base ABCD, may be constructed; by giving to A*a* the proposed inclination, and then drawing B*b*, C*c*, D*d* parallel, and equal thereto. For A*ab*B, B*bc*C, &c. will *(still)* be parallelograms [f]: And, that *abcd* is one plane, parallel to ABCD, will also appear *(in the same manner)*; if a perpendicular from *a* to the plane ABCD, be conceived to be drawn.

THEOREM XIX.

If, on equal bases (ABC, PQRS), *an upright triangular prism, and a rectangular parallelepipedon be erected, of the same altitude; the two solids, themselves, will be equal.*

Let

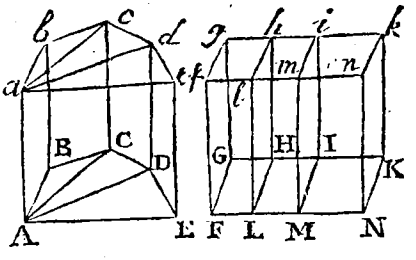


Let CD be perpendicular to AB, and let the rectangles ADCE, BDCF be completed; also let GH be drawn parallel to AE, bisecting AB in G; so shall AGHE = $\frac{1}{2}$ABFE [b] = ACB [c] = PQRS [d]. Now, the two prisms on ADC and BDC, into which, *that* on ABC may be divided [e], will be respectively equal to two others, on the equal and similar [f] bases AEC and BFC [g]: and consequently the prism on ACB = half the prism on AEFB [h] = half the two prisms on AGHE and BGHF [i] = the prism on AGHE = the prism on [k] PQRS *.

[b] 1. 2.
[c] Cor. to 2. 2.
[d] Hyp.
[e] Cor. to 18. 7.
[f] Ax. 24. 1.
[g] Ax. 1. 7.
[h] Ax. 4. 1.
[i] Ax. 3. 1.
[k] 17. 7.

THEOREM XX.

Every upright prism (AaceA) *is equal to a rectangular parallelepipedon* (Fk) *of equal base, and altitude.*



Let AC, *ac*, AD, *ad* be drawn; and in the base FK of the parallelepipedon, let HL, IM, be drawn parallel to FG, in such sort that the rectangles FH, HM, IN may be respectively equal to the triangles ABC, ACD, ADE [k]: Then also shall the prism (A*abc*CBA) [k] 6. 6.

* *In this Theorem, the representations of the prisms are not described; because a great multiplicity of lines, tends to produce confusion in the mind of a learner; especially, where solids are represented. The schemes, however, may be formed, at large, by those who think proper to do it: But every tittle of the demonstration will remain the same.*

on ABC, be equal to the parallelepipedon (F*b*) on
[l] 19. 7. FH[l]; and the prism (A*acd*DCA) on ACD, equal to the parallelepipedon (L*i*) on LI[l]; and consequently the whole prism (A*ace*A) on ABCDE, equal to all the parallelepipedons on FK, which form one parallelepipedon (F*k*); because L*l*, M*m* are
[m] 18. 7. and Cor. 2. to 6. 7. in the plane F*n* [m], and H*b*, I*i* in the plane G*k* [m].

COROLLARY.

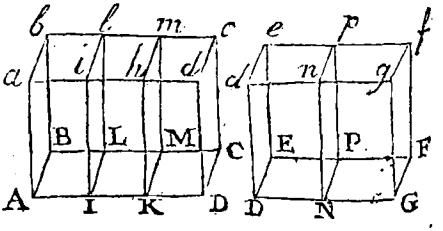
Hence all upright prisms, having equal bases, and altitudes, are equal among themselves.

SCHOLIUM.

In the very same manner, the aggregate of any number of prisms, of one common altitude, will appear to be equal to one single prism, or parallelepipedon, of the same altitude, whose base is equal to the sum of all theirs.

THEOREM XXI.

Rectangular parallelepipedons (Ac, Df) *having equal altitudes* (Aa, Dd) *are in the same proportion as their bases* (AC, DF).



Let the proportion of the base AC to the base DF, be that of any one number *m* (3) to any other number *n* (2).

Let AC be divided into *m* (3) equal parts (or rectangles) AL, IM, KC (by dividing AD into that number of equal parts [m], and drawing IL, KM parallel to AB) [n]: And let DF be divided, in like manner, into *n* (2) equal parts, or rectangles, DP, NF: Which parts, taken singly, will be equal, in magnitude, to those of the former division [o]; and so the

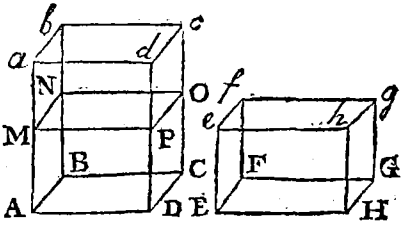
[m] 11. 5.
[n] 9. 5. and 1. 2.
[o] Hyp. and Ax. 8. 4.

the parallelepipedons upon them (A*l*, I*m*, K*c*, D*p*, N*f*) will likewise be all equal[p]: Therefore the solid A*c* is in proportion to the solid D*f*, as the number of parts in A*c* to the number of equal parts in D*f*[r], or as the number of parts in AC to the number of equal parts in DF, that is, as AC to DF[r].—If the bases are supposed to be *incommensurable*, the solids will still be in the same ratio with them; as appears from the reasoning laid down in the *Scholium to Theor.* VII. *Book* IV.; which is equally applicable in this case.

[p] 17. 7.
[r] Ax. 8. 7.

THEOREM XXII.

Rectangular parallelepipedons (Ac, Eg) *standing upon equal bases* (AC, EG) *have the same ratio as their altitudes* (Aa, Ee).



Let AO be a parallelepipedon on the base AC, whereof the altitude AM is equal to that (E*e*) of the parallelepipedon E*g*: So shall the solid AO = the solid E*g*[s]. But (if A*b* and AN be considered as bases) it will be A*c* : AO (or E*g*) : : A*b* : AN[t] : : A*a* : AM[u] (or E*e*): *which was to be proved.*

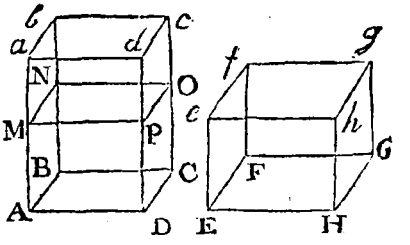
[s] 17. 7.
[t] 21. 7.
[u] 7. 4.

COROLLARY.

Hence, and from the preceding Theorem, it follows, that all upright prisms are, also, as the bases, when the altitudes are equal; and as the altitudes, when the bases are equal; all such solids being (*by Theor.* XX.) equal to rectangular parallelepipedons of equal base and altitude.

THEOREM XXIII.

Upright prisms and parallelepipedons (Ac, Eg) *which have their bases and altitudes reciprocally proportional* (AC : EG : : Ee : Aa), *are equal to each other.*



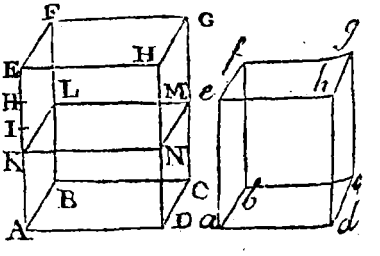
Let AO be a prism on the base AC, whereof the altitude AM is equal to that (E*e*) of the prism E*g*.

Then AO : E*g* :: AC : EG[w] : : E*e* (AM) : [x]A*a* : : AO : [y]A*c* ; and consequently E*g* = A*c*[z].

[w] 21. 7.
[x] Hyp.
[y] 22. 7.
[z] Ax. 4. of 4.

THEOREM XXIV.

Similar upright prisms and parallelepipedons (AG, ag) *are, to one another, in the triplicate ratio of their altitudes* (AE, ae).



Having made AH = *ae*, take AE, AH, AI, AK in continued proportion[a] ; and let AM be a prism on the base AC, whereof the altitude is AK. Then (because of the similar planes) it will be AC : *ac* : : [b]AB² : *ab*² : : [c]AE² : *ae*² (AH²) : : AH² : [c]AI² : : AH (*ae*) : AK[e] ; and so, the bases and altitudes of the solids AM, *ag* being reciprocally proportional, the solids themselves are equal ;

[a] 13. 5.
[b] 26. 4.
[c] Cor. to 11. 4.
[e] 27. 4.

equal [f]; and therefore, AG : *ag* : : AG : AM [g] : : AE : AK [h]: But the ratio of AE to AK is triplicate to that of AE to [i] AH (or *ae*). Therefore, *&c.*

[f] 23. 7.
[g] Ax 1. of 4.
[h] 22. 7.
[i] Def. 7. of 4.

COROLLARY I.

Hence, cubes are in the triplicate ratio of their sides, or altitudes.

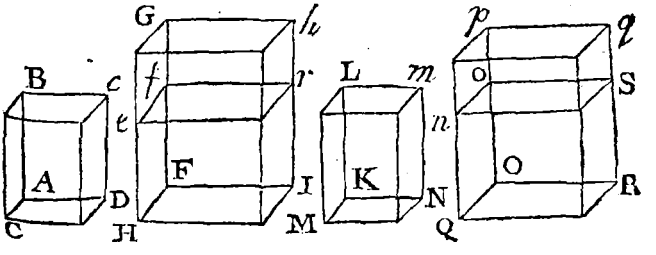
COROLLARY II.

Hence, also, all similar upright-prisms, are to one another, as the cubes of their altitudes; since both prisms and cubes, are in the *same* triplicate ratio of the altitudes.

THEOREM XXV.

Rectangular parallelepipedons, contained under the corresponding lines of three ranks of proportionals, are themselves proportionals.

I say, if { AB : FG : : KL : OP,
AC : HF : : MK : QO,
AD : FI : : KN : OR,



then shall the solid (Cc), *contained under the three first antecedents, be to that* (Hh) *contained under their three consequents; as the solid* (Mm) *contained under the three other antecedents, is to that* (Qq) *contained under the three remaining consequents.*

Let H*r* and Q*s* be parallelepipedons on the bases HI and QR, of the same altitude with C*c* and M*m*, respectively.

21. 7. Then shall C*c* : H*r* : : base CD : base HI [k];

And M*m* : Q*s* : : base MN : base QR [k]. But the four bases, because of the proportionality of

11. 4. their sides, are themselves proportionals [l]: And so, by equality, the ratio of C*c* to H*r*, is the same as the ratio of M*m* to Q*s*. And the ratio of H*r* to H*b* is likewise the same as that of Q*s* to Q*q* (be-

[m] 22. 7. cause H*r* : H*b* : : F*f* (AB) : FG [m] : : KL (O*o*) :

[n] Hyp. O*p* [n] : : Q*s* : Q*q* [m]). Therefore shall the ratio of C*c* to H*b*, be also the same, as the ratio of M*m*

[o] 5. 4. to Q*q* [o].

COROLLARY.

Hence, the cubes of four proportional lines are proportional.

The End of the SEVENTH BOOK.

ELEMENTS

OF

GEOMETRY.

BOOK VIII.

POSTULATES.

1. THAT, of any two unequal magnitudes, of the ſame kind, the leſs may be multiplied ſo often, till it exceed the greater.

2. That, a right-line may be taken ſo ſmall, that the ſquare thereof ſhall be leſs than any ſuperficies aſſigned.

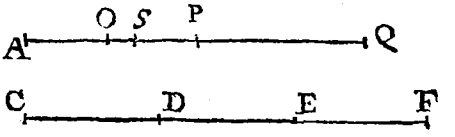
3. That, the circumference of a circle is greater than the perimeter (or the ſum of all the ſides) of any inſcribed polygon; and leſs than the perimeter of any polygon deſcribed about the circle.

What is required to be granted, in the ſecond of theſe three Poſtulates, might be effected and proved, in form, by means of the Firſt; but being itſelf more obvious (if poſſible) than even that, *it ſeemed unneceſſary to make it depend thereon.*

LEM-

LEMMA I.

If from the greater (AQ) *of two unequal magnitudes* (AQ, CD) *there be taken the half* (PQ), *and from the remainder* (AP) *be again taken the half* (PO), *and so on, continually; there shall at length be left a magnitude, less than the least* (CD) *of the two magnitudes first propounded.*



Take DE = CD, EF = CD, and let this be so often done, till the multiple CF exceed AQ [a]. Let the proposed bisections of AQ, AP, *&c.* be continued till the parts PQ, OP, AO be equal in number to the parts EF, DE, CE. Now AP ($\frac{1}{2}$AQ [b]) $\sqsupset$ $\frac{1}{2}$CF [b] $\sqsupset$ CE. And, in the same manner, AO ($\frac{1}{2}$AP) $\sqsupset$ $\frac{1}{2}$CE (CD); *which was to be done.*

[a] Post. 1.

[b] Hyp.

SCHOLIUM.

When the magnitudes given (AQ, CD) are right-lines, a part, or measure (AS) of the one, less than the other, may be found at one operation [c]; by taking AS the same part of AQ, as CD is of CF [c]. For, the whole AQ being less than the whole CF [b], the part AS will also be less than the part CD [d].

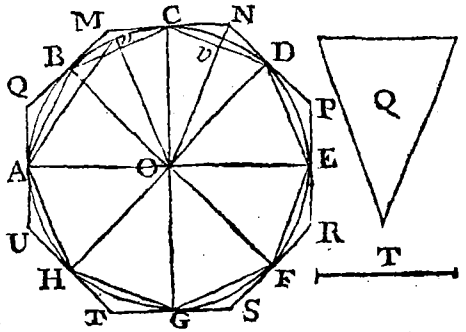
[c] 11. 5.

[d] Cor. 1. 4.

THEOREM I.

Two polygons may be formed, the one in, the other about a given circle, which shall differ less from each other (and consequently from the circle itself) than by any assigned magnitude (Q) *however small.*

Let



Let T be the side of a square equal to, or less than Q [e]; and in the circle apply A*n* = T [f]: and, having drawn the two perpendicular diameters AE, CG, proceed by a continual bisection of the angles at the center, till you arrive at an angle AO*B* less than the angle AO*n* subtended by A*n* [g]: Inscribe the regular [h] polygon ABCDEFGH, by making the angles BOC, COD, *&c.* equal to AOB: and let another regular polygon QMNPRSTU, of the same number of sides, [i] be described about the circle; which will exceed the inscribed one by a magnitude less than Q.

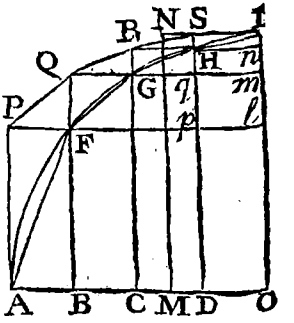
[e] Post. 2.
[f] 20. 5.
[g] 5. 5. and Lem. 1.
[h] Ax. 10. and 4. 1.
[i] 30. 5.

For, if to any angle N of the greater, ON be drawn, it will bisect the same [k], and will cut the side CD of the inscribed polygon at right-angles [l] (in *v*): And so, the triangles OCN, *v*CN being equiangular [m], they (and consequently their doubles OCNDO, CND) will be in proportion to each other, as [n] OC^2 to Cv^2, or as [o] AE^2 to AB^2. And it is manifest, that the whole circumscribing polygon (OCND + ODPE *&c.*) must be to its whole excess (CND + DPE *&c.*) above the inscribed one, in the same [o] proportion of AE^2 to AB^2. But the first *antecedent* is less than the *second*, or than a square described about the circle [p]: Therefore the first *consequent* (CND + DPE *&c.*) is also $\supset$ [q] AB^2 $\supset$ [r] An^2 (T^2) $\supset$ [s] Q.

[k] 16. 1.
[l] Cor. to 12. 1.
[m] 19. 4.
[n] 24. 4.
[o] 1. 4.
[p] Ax. 2.
[q] 2. 4.
[r] 21. 1.
[s] Hyp.

Other -

Otherwise.



Let AIO be one quadrant of the proposed circle; and
[t] 7. 6. on the radius OI, make [t] the rectangle OMNI $= \frac{1}{4}T^2$, (T being as before): Take OD a part of OA, less than
[u] Lem. OM [u]; and, having made AB, BC, *&c.* each $=$ OD, draw AP, BFQ, CGR, DHS
[w] 3. 5. perpendicular [w] to AO, meeting the circumference in A,
[x] 9. 5. F, G, H; through which points, parallel to AO [x], draw PF*l*, QG*m*, RH*n*, meeting AP, BFQ, CGR, in P, Q, R: Join PQ, QR, RS, SI, as also AF, FG, GH, HI. Then will the two polygons OAFGHI and OAPQRSI (whereof one is less, and the other greater than the quadrant) differ less from each other, than by $\frac{1}{4}$ of the proposed quantity Q.

For, that the former OAFGHI is less than the
[y] Ax. 2. quadrant, in which it is inscribed, is manifest [y]: And, that the latter is greater than the quadrant, will also plainly appear; seeing two sides AP, SI,
[z] 6. 3. only, touch the circumference [z]; all the rest PQ, QR, RS, falling wholly above it, as being sides of triangles PFQ, QGR, RHS formed out of the cir-
[a] Constr. cle [a]. Now the excess of the polygon OAPQRSI
and Ax. 2. above OAFGHI, is composed of the triangle PAF ($= \frac{1}{2}$OD*pl*) and of all the parallelograms PFGQ,
[b] Ax. 3. QGHR [b], RHIS (for they are such [c], because PF
[c] 26. 1. (AB), QG (BC) are equal, as well as parallel [d]).
[d] Constr. and 26. 1. Therefore PFGQ being $=$ *plmq* [e], QGHR $=$
[e] Cor. 2. *qmn*H [e], *&c.* the said excess will consequently be
to 2. 2. $= \frac{1}{2}$OD*pl* $+$ *lp*SI [b] $\sqsupset$ ODSI [f] $\sqsupset$ OMNI ($\frac{1}{4}T^2$)
[f] Ax. 2. $\sqsupset \frac{1}{4}Q$ [g]; *which was to be done.*—This last con-
[g] Hyp. struction is equally applicable to other curvilineal figures; the former is peculiar to the circle.

COROL-

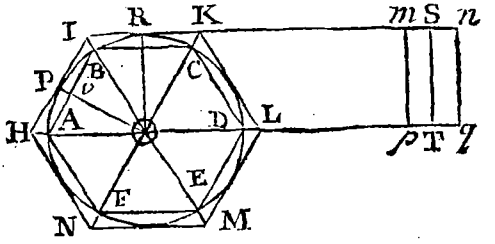
COROLLARY.

It follows from hence, that a magnitude, which is greater than any polygon that can be described in, and less than any polygon that can be formed about a given circle, must be equal to the circle itself: seeing that a polygon may be inscribed, which (as well as that formed about the circle) shall exceed any quantity less than the circle itself, be the difference ever so small; and because a polygon may be formed about the circle, which (as well as that in the circle) shall be less than any quantity that exceeds the circle.

THEOREM II.

Every circle (ACE) *is equal to a rectangle* (ORST) *under the radius thereof* (OR) *and a right-line* (OT) *equal to half the circumference.*

It is evident, in the first place, that the proposed rectangle ORST is greater than any poly-



gon ABCDEF that can be described in the circle: For, drawing OA, OB, *&c.* and also O*v* perpendicular to AB; it is plain, that the triangle AOB ([h] O*v* × ½AB) will be less than [i] OA × ½AB (or OR × ½AB): And, in the same manner, BOC ¬ OR × ½BC, *&c.* Consequently, the whole polygon ABCDEF is less than [k] OR × ½AB + OR × ½BC, *&c.* that is [l], less than a rectangle (O*m*) under OR and O*p* = half the perimeter (AB + BC + CD *&c.*) But this rectangle (O*m*) is, itself, less than

[h] Cor. to 2. 2.
[i] 20. 1. and Ax. 2.
[k] Ax. 2. 1.
[l] 5. 2.

than OS, becauſe O*p* (half the perimeter of the polygon) is leſs than OT [m] (half the circumference of the circle). Conſequently the polygon ABCDEF is leſs than the rectangle OS.

[m] Poſt. 3.

But, ſecondly, it will appear, that the ſame rectangle ORST is leſs than any polygon HIKLMN that can be deſcribed about the circle: For, if OH, OI, *&c.* be joined, and the radius OP be drawn to the point of contact of HI; then will the triangle HOI = [n] OP × ½HI (= OR × ½HI). In the very ſame manner IOK = OR × ½IK, *&c.* and therefore the whole polygon HIKLMN = [o] OR × ½HI + OR × ½IK, *&c.* = [p] a rectangle (O*n*) under OR and O*q* = half the perimeter (HI + IK + KL, *&c.*); which rectangle is, manifeſtly, greater than OS, ſince O*q* (= half the perimeter of the polygon) is greater than OT [q].

[n] Cor. to 2. 2.

[o] Ax. 4. 1.

[p] 5. 2.

[q] Poſt. 3.

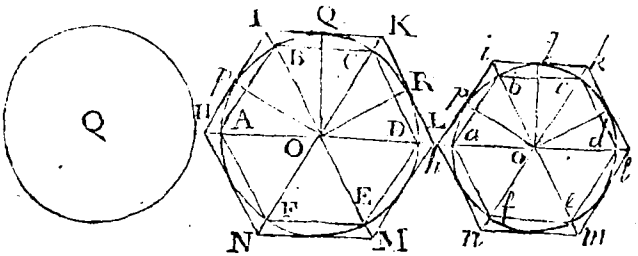
Seeing, therefore, that the rectangle OS is greater than any polygon that can be deſcribed in the circle, and leſs than any polygon that can be deſcribed about the circle; it muſt be equal to the circle [r].

[r] Cor. to 1. 8.

THEOREM III.

All circles (ACE, ace) *are in proportion to one another, as the ſquares of their radii* (AO^2, ao^2).

Let Q : circle *ace* : : AO^2 ao^2; then I ſay, that Q = circle ACE. For, firſt, it is evident that Q is greater than any polygon ABCDEF that can be



deſcribed

described in the circle ACE: Because, if another polygon *abcdef*, similar thereto, be [s] described in the circle *ace*; then will polyg. ABCDEF : polyg. *abcdef* (: : [t] $AO^2 : ao^2$) : : Q : [u] circle *ace*; where the first *consequent* (polyg. *abcdef*) being less than the *second* (or, than the circle in which it is inscribed [w]) it is manifest, that the first *antecedent* ABCDEF must also be less than the second Q [x].

[s] 31. 5.
[t] Cor. to 31. 5.
[u] Hyp.
[w] Ax. 2.
[x] 2. 4.

In the same manner it will appear, that Q is less than any polygon HIKLMN that can possibly be described about the circle ACE: For, if about the other circle *ace*, a similar polygon *hiklmn* be described [y]; then will HIKLMN : *hiklmn* (: : [z] $AO^2 : ao^2$) : : Q : [a] circle *ace*; where the first *consequent* (*hiklmn*) being greater than the second (*ace*) [b], the first *antecedent* HIKLMN must therefore be also greater than the [c] second Q.

[y] 31. 5.
[z] Cor. to 31. 5.
[a] Hyp.
[b] Ax. 2.
[c] 2. 4.

Therefore, seeing that Q is greater than any polygon that can be described in the circle ACE, and less than any polygon that can be described about the circle; it must be equal to the circle [d].

[d] Cor. to 1. 8.

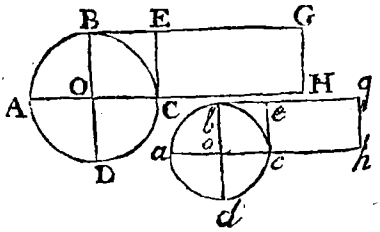
SCHOLIUM.

After the same manner, other similar curvilineal figures are proved to be in proportion, as the squares of their diameters, or other homologous dimensions; by means of the second construction of the first proposition; it being very easy to demonstrate, that the polygons formed from thence, whether both within, or both without two similar figures, will themselves be similar.

THEO-

THEOREM IV.

The circumferences of all circles (ABCD, abcd) *are in the same proportion as their radii* (OB, ob).

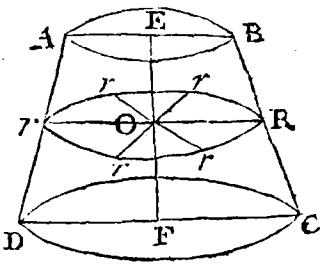


Let OE, *oe* be squares on the radii OB, *ob*; and let OG, *og* be two rectangles contained under the same radii and right-lines OH, *oh*, respectively equal to the semi-circumferences ABC, *abc*. Then, these rectangles being equal to the circles themselves [d], it will therefore be, OE : OG : : *oe* : *og* [e]. And in this same ratio are [f] also the bases OC, OH; *oc*, *oh*: whence *(by equality and alternation)* OC (OB) : *oc* (*ob*) : : OH : *oh* : : 2OH (circumf. ABCD [g]) : 2*oh* (circumf. *abcd*).

[d] 2. 8.
[e] 3. 8.
[f] 7. 4.
[g] Hyp.

LEMMA 2.

If a solid (AC) *generated by the revolution of any plane figure* (EBCF) *about a quiescent axis* (EF), *be cut by a plane perpendicular to the axis; the section will be a circle, having its center in the point* (O) *where it meets the axis.*



For, from O, in the generating plane EBCF, draw OR perpendicular to the axis EF, meeting BC in R.

Then, since this line OR, during the whole revolution, every-where preserves its perpendicu-

larity

larity to the axis EF, it is therefore always in the plane passing through O perpendicular to the said axis [h]: and consequently, as the length thereof also continues the same in every position, the line R*rrrr* described, in that plane, by the extreme point R, by which the section is bounded, must be the circumference of a circle [i], whereof the point O is the center.

[h] Cor. to 2. 7.

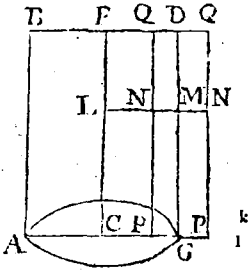
[i] Def. 33. 1.

COROLLARY.

Hence, not only the bases of cylinders and cones, but all sections parallel to them, are circles.

LEMMA 3.

A right-line (PQ) *standing perpendicular to the plane of a cylinder's base (and not exceeding the axis* CF) *falls wholly within, or wholly without the cylinder, according as the point* (P) *on which it insists, is situate within, or without the circumference of the base.*



From the center C, to the given point P, draw CP; take, in CF and PQ, any two equal distances CL, PN, and let LN be drawn, meeting the surface of the cylinder in M.

Because CL and PN are parallel [k], and therefore both in the same plane [l], LN is parallel, and equal to CP [m]. Therefore, when CP is less than the radius CG, LN will be less than CG, or than its equal LM [n]; and so the point N must fall within the cylinder [o]. And the same is equally true with regard to any other point in the line PQ. But, when CP is greater than CG, LN will also be greater than HG (LM); and the point N will then fall out of the cylinder [o].

[k] 4. 7.

[l] Def. 14.

[m] 26. 1.

[n] 24. 1. Def. 12. 7. and

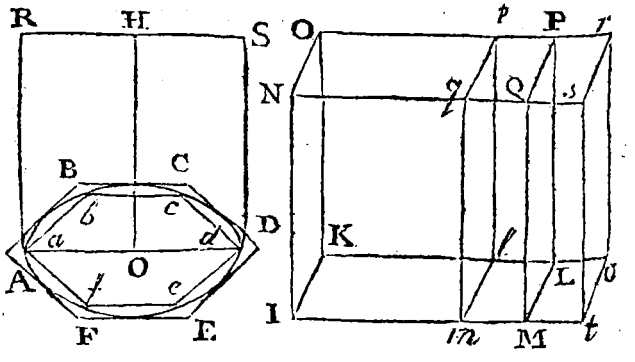
[o] Ax. 2. 1.

THEOREM V.

Every cylinder is equal to a rectangular parallelepipedon of equal base and altitude.

I say, if the base *ace* of the cylinder *a*S be equal to the base IKLM of the rectangular parallelepipedon IP, and the altitude OH of the former be also equal to the altitude KO of the latter; then the two solids will be equal.

For, first, it is evident, that the cylinder exceeds any parallelepipedon (I*p*), of the same given altitude, whose base I*klm* is less than the base (*ace*) of



the cylinder: Because a polygon (*abcdef*) may be
[p] 1. 8. described in the circle *ace*, that shall exceed IK*lm*[p];
upon which, an upright prism (of the given alti-
[q] 18. 7. tude) may be constituted[q]; which will be less than
the cylinder, as being wholly contained therein;
since (*by Lemma* 3.) all right-lines drawn perpen-
[r] Cor. 2. to 6. 7. dicular to the base, in the planes of the sides[r],
from any points in *ab*, *bc* &c. fall wholly within the
cylinder, and consequently the planes themselves,
in which they are. But this contained prism is
[s] 21. 7. greater than the parallelepipedon I*p*[s]: Therefore
the cylinder itself must, necessarily, be greater
[t] Ax. 2. than I*p*[t].

In

In like manner it will appear, that the cylinder is less than any parallelepipedon I*r* (of the same altitude) whose base IK*vt* exceeds that of the cylinder: For a polygon (ABCDEF) may be described about the circle *ace* that shall be less than IK*vt* [u]; upon which a prism may be constituted [w], which, tho' less than I*r* [x], will, nevertheless, exceed the cylinder [y].

[u] 1. 8.
[w] 18. 7.
[x] 21. 7.
[y] Ax. 2.

Therefore, seeing that the cylinder can neither be less, not greater than IP; it must necessarily be equal to it.

COROLLARY.

Hence, whatever is demonstrated in the 21st, 22d, and 23d Theorems of the preceding Book, with respect to the proportions of prisms, holds equally true in cylinders also; being equal to prisms of equal base and altitude [z].

[z] 20. 7. and 5. 8.

SCHOLIUM.

From the same demonstration, it will likewise appear, that every regular solid, whose sections, by planes perpendicular to the base, are all rectangles; is equal to a parallelepipedon of equal base and altitude; and consequently, that all solids of this kind (which may be comprehended under the name of *Cylinderoids)* will be equal among themselves, when their altitudes, as well as bases, are equal.

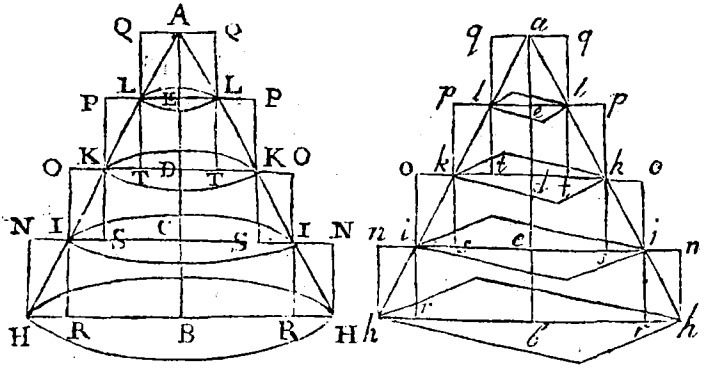
LEMMA 4.

If two solids (HAH, hah) *of the same altitude, have their sections by planes parallel to the bases, at all equal distances therefrom, equal to each other; it is proposed to demonstrate (under certain restrictions specified hereafter) that the solids themselves will be equal.*

Let II, KK &c. *ii*, *kk* &c. be ſections of the two ſolids by planes parallel to the baſes HH, *hh*, dividing the altitudes AB, *ab* into parts BC, CD &c. *bc*, *cd* &c. all mutually equal to each other. Then, every two correſponding ſections being equal [m] (HH = *hh*, II = *ii*, &c.) the upright ſolids HNNH, *hnnh*; IOOI, *iooi* &c. formed thereon, will alſo be, reſpectively, equal one to another [n], whether they be priſms, cylinders, or cylinderoids, that is, whether the ſections themſelves be right-lined figures, circles, or curvilineal figures of any other kind.

[m] Hyp.

[n] 20, 7. and Sch. 5. 8.



Now, if theſe ſections HH, II *&c.* be ſuppoſed to decreaſe, from the baſe upwards, ſo that the ſolids (HNNH, IOOI *&c.*) formed upon them may exceed the correſpondent parts (HIIH, IKKI *&c.*) of the given ſolid HAH; it is manifeſt, that the ſum of all the ſaid ſolids (HNNH + IOOI *&c.*) will likewiſe exceed the whole propoſed ſolid HAH [o]. But, if within HAH, on the ſame ſections (but on contrary ſides thereof) another ſeries of ſuch ſolids IRRI, KSSK *&c.* be formed; the ſum of all theſe will, manifeſtly, be leſs than the propoſed ſolid HAH, in which they are contained [p]: And it is alſo evident, that this laſt ſeries will be leſs than the former (HNNH + IOOI *&c.*) by the greateſt of theſe

[o] Ax. 2.

[p] Ax. 4.

these solids HNNH; because (this one, alone, being excepted) to every other solid of the rank, an equal, in the contained rank, may be assigned, and *vice versâ:* For [q] IRRI = IOOI, KSSK = KPPK, LTTL = LQQL. [q] 20. 7. and Sch. 5. 8.

Now, since the altitude BC, of the solid HNNH whereby the contained, and containing series' differ from each other, may be taken so small a part of BA, that the solid itself shall be less than any assigned magnitude whatever [r]; it is manifest (from the reasoning in Corol. to Theor. I.) that a magnitude, which is greater than any series of solids (of the kind above specified) that can be formed within the proposed solid HAH, and less than any series that can be formed about HAH, must be equal to HAH. But the solid *hah*, being greater than any series of solids (*irri* + *kssk* &c.) contained therein [s], is therefore greater than any series of solids IRRI + KSSK &c.) contained in HAH (*these*, being, respectively, equal to *those*): And the same solid *hah*, being less than any series of solids (*hnnh* + *iooi* &c.) formed about it, is also less than any series of solids (HNNH + IOOI &c.) that can be formed about HAH. Therefore the solid *hah* is equal to HAH. [r] Lem. 1. and 22. 7. [s] Ax. 2.

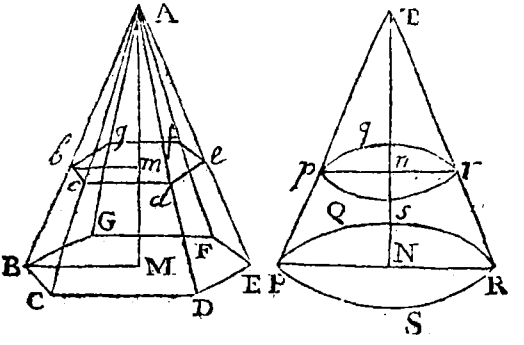
In this demonstration, the sections are supposed to decrease, continually, from the bases upwards; so as to have the sides of the upright solids formed thereon, placed wholly without, or wholly within, the superficies of the given solids HAH, *hah*: Which can only be the case, when all perpendiculars, from any points in the surface of either, to the plane of the base, fall within the limits of the base. If, however, the sections be supposed to decrease to a certain distance, only, and then to increase again; the two solids will, still, appear to be equal: Because the parts of the one, terminated

by ſuch limits of decreaſe, or increaſe, will (by the ſame demonſtration) be reſpectively equal to the correſpondent parts of the other. But, as no ſuch ſolids have a place in the *Elements of Geometry*, to ſay more about them here, would be improper.

LEMMA 5.

If pyramids and cones (ABCDEFG, TPQRS) *having equal altitudes* (AM, TN), *be cut by planes parallel to the baſes; the ſections* (bcdefg, pqrs), *at all equal altitudes* (Mm, Nn), *will be in the ſame proportion as the baſes.*

For, the plane *bcdefg* being parallel to BCDEFG,
[a] 11.7. thence is *bc* parallel to BC [a], *bg* to BG, *&c.* and
[b] 9. 7. conſequently the angle *cbg* = CBG [b], *bcd* = BCD,
[c] 14. 4. *&c.* Alſo *bc* : BC (: : A*b* : AB [c]) : : *bg* : BG. And, in the ſame manner, the ſides about the other equal angles are proportional. Therefore, the two po-



[d] Def. 14. 4.
[e] 26. 4.
[f] Cor. to 11. 4.
[g] Cor. to 7. 7.
[h] 4. 1.

lygons *bcdefg*, BCDEFG being ſimilar [d], they are in proportion [e], as bc^2 to BC^2, or as Ab^2 to [f] AB^2, or, laſtly, as Am^2 to [f] AM^2; becauſe (BM and *bm* being drawn) the angles AMB, A*mb* will be right-ones [g], and *bm*, therefore, parallel to BM [h].

But

But the section *pqrs* is *also* to the base PQRS in the *same* proportion of Am^2 (Tn^2) to AM^2 (TN^2); because *pqrs*, PQRS, being [i] circles, they are as the squares of their radii *pn*, PN [k], and consequently as Tn^2 to [l] TN^2. Therefore, seeing that the two sections have both the same ratio to their respective bases, the proposition is manifest [m].

[i] Lem. 2.
[k] 3. 8.
[l] Cor. to 11 4.
[m] 2. 4

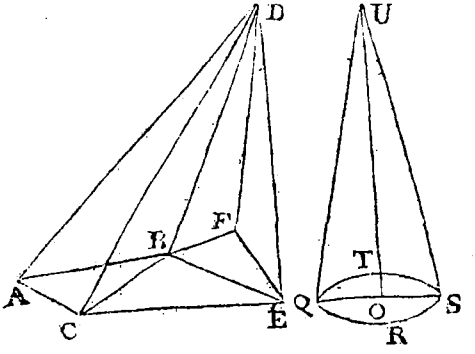
COROLLARY.

It appears from hence, that the section of any pyramid, by a plane parallel to the base, is similar to the base.

THEOREM VI.

All pyramids of the same altitude, standing upon equal triangular bases, are equal among themselves; and every such pyramid (ABCD) *is equal to a cone* (QRSTU) *of equal base and altitude.*

Case I. If the perpendicular, let fall from the vertex D of the pyramid upon the plane of the base ABC, falls not out of the base, or beyond the limits of the triangle: Then it is manifest, *from Lemma* 4, seeing the sections of the solids ABCD,



QRSTU, at all equal distances from the bases, will be equal [t], that the solids themselves will likewise be equal.

[t] Lem. 5.

Case II. If the perpendicular (DE) from the vertex to the plane of the base, falls beyond the limits of the triangle: Then, to the point E where it meets the plane, let BE and CE be drawn; and on BE let a triangle EBF be described equal to ABC (or QRST), and let F, D be joined. So shall the pyramid CBFED, standing on the base CBFE, be equal to the pyramid CABED, standing on the equal base CABE [u]; from each of which, let the common pyramid CBED be taken away; and there will then remain the pyramid BFED = pyramid ABCD [w]: But the former of these is (*by Case* 1.) equal to the cone QRSTU; therefore it is evident, that the latter ABCD will also be equal to the [x] cone QRSTU; and, consequently, that all pyramids of the same altitude, standing on equal triangular bases, will be equal among themselves [x]; seeing every such pyramid is equal to a cone (QRSTU) of equal base and altitude.

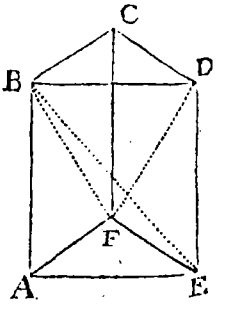
[u] Lem. 4.

[w] Ax. 5.

[x] Ax. 1.

THEOREM VII.

Every prism (ABCDEFA) *having a triangular base* (AFE) *is equal to the triple of a pyramid of the same base and altitude.*



In the planes of the three sides, let the diagonals BE, BF, FD be drawn. Then will the part FBCD of the prism cut off by a plane extended by FB and FD, be a pyramid on the base BCD, having the same altitude with the prism itself [y], both solids being contained between the same [z] parallel planes AFE, BCD. Moreover,

[y] Def. 16. 7.

[z] Def. 6. 7.

over, the remaining part FABDE of the prism, if a plane be extended by FB and FE, will be divided into the two pyramids FBAE, FDBE, which are equal to each other [a], as standing on the equal [b] triangular bases ABE, BDE. But the former of these pyramids FBAE, if B be now considered as the vertex thereof, will appear, also, to be equal to the first-mention'd pyramid FBCD [a], the two bases AFE, BCD (as well as the altitudes) being equal [b]. Therefore, since the three triangular pyramids (FBCD, FABE, FBDE) into which the prism is resolved, are all equal to each other; the proposition is manifest.

[a] 6. 8.
[b] 24. 1.

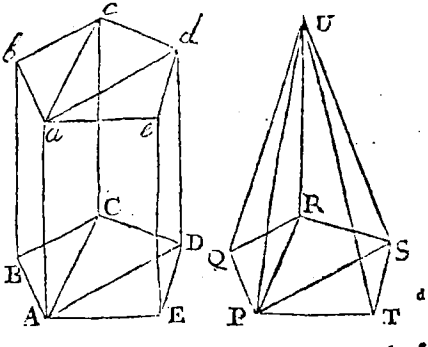
COROLLARY.

Hence, every prism having a triangular base, is equal to the triple of any pyramid of the same altitude, standing upon an equal triangular base [c].

[c] 6. 8. and Ax. 1.

THEOREM VIII.

If a prism (AbcE) *and a pyramid* (PQRSTU) *stand upon equal, and similar bases* (ABCDE, PQRST), *and have both the same altitude; the prism will be equal to the triple of the pyramid.*



If the bases be resolved into triangles, ABC, ACD &c. it is manifest, that B*bac*CA will be a prism, on the base ABC; because C*c* being equal and parallel to [d] A*a*, A*ac*C will be a parallelogram [e] (as well as B*ba*A and B*bc*C [f]). There-

[d] Def. 6. and 8. 7.
[e] 26. 1.
[f] Def. 6. 7.

Therefore B*bac*CA is equal to the triple of the pyramid PQRU [g], ſtanding on an equal [h] baſe (PQR). And, in the ſame manner, the priſm A*acd*DC, on the baſe ACD, is equal to the triple of the pyramid PRSU, on the equal baſe PRS; and ſo on. Therefore, alſo, ſhall the whole priſm A*d*, on the baſe ABCDE, be equal to the triple of the whole pyramid PQRSTU, on the equal baſe PQRST.

[g] Cor. to 7. 8.
[h] Hyp. and Ax. 19.

COROLLARY I.

Hence, all pyramids having the ſame baſe and altitude, are equal; being like parts of one and the ſame priſm.

COROLLARY II.

Hence, alſo, all priſms having the ſame baſe and altitude, are equal; being equimultiples of one and the ſame pyramid.

COROLLARY III.

Therefore it appears, that every priſm inclining on its baſe, as well as every upright one, is equal to a rectangular parallelepipedon of equal baſe and altitude [k]; and, conſequently, that all priſms whatever, having equal baſes, and altitudes, are equal to each other [k]: which muſt be alſo true in pyramids and cones, every ſuch ſolid being ſubtriple to a priſm, or cylinder, of the ſame baſe and altitude [l].

[k] 20. 7. and Ax. 1.
[l] 8. 8. and 6. 8.

COROLLARY IV.

Hence it alſo follows, that whatever is demonſtrated in the 21ſt, 22d, and 23d Theorems of the preceding Book, concerning the proportion of priſms, holds equally in pyramids and cones; *theſe* being like parts of *thoſe* [l].

COROL-

COROLLARY. V.

It follows, moreover, that all correſponding fruſtums of pyramids and cones of the ſame altitude, are *alſo, in proportion, as their baſes.* For, the ſections, at all equal altitudes, being in that proportion [m], the parts cut off (as well the wholes) will be in the ſame proportion [n]; and, conſequently, the remaining parts likewiſe [o].

[m] Lem. 5.
[n] Cor. 4.
[o] 3. 4.

COROLLARY VI.

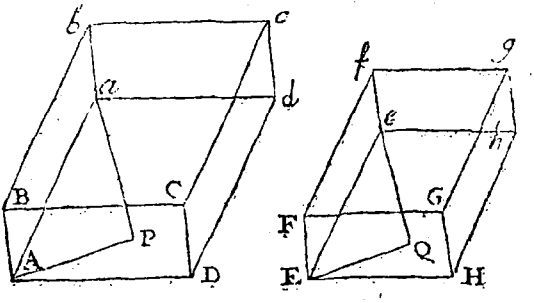
Laſtly, it will appear, that all cones, which have their altitudes and the diameters of their baſes directly proportional, are in the triplicate ratio of their altitudes [p]; being to each other in the ſame proportion with priſms of equal baſe and altitude, whereof they are like parts [q].

[p] 24. 7.
[q] Cor. 3. and Cor. to 1. 4.

THEOREM IX.

All ſimilar priſms, and pyramids, are in the triplicate ratio of their altitudes.

From the extremes of the homologous ſides A*a*, E*e*, upon the baſes ABCD, EFGH of the propoſed ſolids A*c*, E*g*, let fall the perpendiculars



*a*P, *e*Q. The angle BAD being = FEH, BA*a* = FE*e*, and DA*a* = HE*e* [p], thence is *a*P : *e*Q : : *a*A : *e*E [q] : : AB : EF [r] : : AD : EH [r]. Therefore

[p] Def. 5. 7.
[q] 14. 7.
[r] Def. 14. 4.

two

two upright prisms constituted on the bases ABCD, EFGH, of the same altitudes (aP, eQ) with the two solids Ac, Eg, will be similar, the one to the other [p]; and, therefore, in the triplicate ratio of the altitudes [s]. But the solids Ac, Eg, when taken as prisms, are respectively equal to the said upright ones; and, when taken as pyramids, are like parts of them [t]. Therefore the solids Ac, Eg are also in the triplicate ratio of the altitudes aP and eQ [u].

[p] Def. 5. 7.
[s] 24. 7.
[t] Cor. 3. to 8. 8.
[u] Cor. 1. 4.

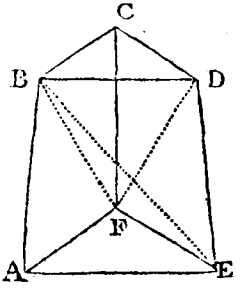
COROLLARY.

Because aP : eQ : : Aa : Ee : : AB : EF &c. it follows, that all similar prisms, and pyramids, are to one another, in the triplicate ratio of the homologous sides of the like planes by which they are bounded [y].

[y] Cor. 1. to 5. 4.

THEOREM X.

The frustum (ABCDEFA) *of any pyramid having a triangular base, is equal to a whole pyramid, of the same base and altitude, together with two other pyramids that are, in proportion thereto; the one, as any side* (BD) *of the upper base* (BCD) *is to its correspondent* (AE) *of the lower base* (AFE); *and the other, as the square of the former side is to the square of the latter.*



In the planes of the three sides, let the diagonals BE, BF, FD be drawn. Then will the part FBCD of the frustum, cut off by a plane extended by FB and FD, be a pyramid, on the base BCD, having the same altitude with the frustum itself [z], both solids being contained between the same [a] parallel planes AFE, BCD.

[z] Def. 16. 7.
[a] Def. 15. 7.

More-

Moreover, the remaining part FABDE of the frustum, if a plane be extended by FB and FE, will be divided into the two pyramids FBDE, FABE, having the same ratio, one to the other, as their bases BED, ABE [b], or as BD to AE [c]. But the latter of these pyramids (BAFE), taking B as the vertex thereof, has the same base and altitude with the frustum given: And the pyramid FBCD (first mentioned) is therefore, in proportion thereto, as the base BCD to the base AFE [d], that is (because the bases are similar [e]), as BD^2 to [f] AE^2: *whence the proposition is manifest.*

[b] 21. 7. and Cor. 4. to 8. 8.
[c] 7. 4.
[d] 21. 7. and Cor. 4. to 8. 8.
[e] Cor. to Lem. 5.
[f] 26. 4.

COROLLARY.

Since, of the three solids (FABE, FBDE, FBCD) into which the proposed frustum is divided, the ratio of the first and third, is the duplicate of that of AE to BD, or of the ratio of the first to the second [g]; it is evident, that these three solids are proportionals [h]. From whence it appears, that the frustum of any triangular pyramid is equal to two (whole) pyramids of the same altitude, on bases equal to the two opposite bases of the frustum, and to a third pyramid, which is a mean proportional between the two former. And it is also evident, that whatever is above demonstrated, in relation to triangular pyramids, must hold equally in all pyramids and cones, whatever: Because every such solid is equal to a triangular pyramid, of equal base and altitude [i]; and every frustum of the one, also equal to the corresponding frustum of the other [k].

[g] Cor. to 27. 4.
[h] Def. 7. 4.
[i] Cor. 3. to 8. 8.
[k] Cor. 5. to 8. 8.

LEMMA 6.

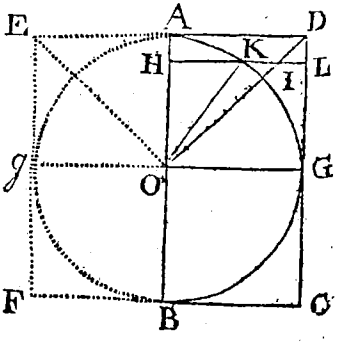
If with radii, respectively equal to the three sides of any right-angled triangle, three circles be described; that whose radius is equal to the hypothenuse, will be equal to both the other two, taken together.

It

It has been proved (*in Theor.* 3.) that circles are in proportion, as the squares of their radii; therefore the demonstration here, is the very same, as in similar right-lined figures (*Theor.* 29. *Book* 4.): Which (if necessary) you may reconsult.

THEOREM XI.

Every sphere is two-thirds of its circumscribing cylinder (Or, of a cylinder of equal diameter and altitude.)



Let AB be the axis about which the sphere and cylinder are generated, by the revolution of the semi-circle AGB and the rectangle ADCB[l]; let HL be any right-line perpendicular to AB, meeting DC in L, and the periphery of the semi-circle in K; and from the center O, let OK and OD, intersecting HL in I, be drawn.

[l] Def. 12. and 14.

Since AO is = AD[m], and HI parallel to AD[n], therefore is HI = OH[o]: But OHI being right-angled at H, the circle whose radius is OH (or HI) will (*by the preceding Lem. and Ax.* 5.) be equal to the difference of the two circles whose radii are OK (HL) and HK: Or, in other words, the circle[p] described by HI, or the section of the cone generated by the triangle AOD, in its revolution about the axis AB, will be equal to the difference of the two circles generated by HL and HK; that is, equal to the *annulas* described by KL[q], or the section of the solid which remains, when the sphere is taken out of the cylinder. Therefore, seeing these

[m] Hyp.
[n] 4. 1.
[o] 14. 4.
[p] Lem. 6.
[q] Ax. 2. 1.

these two sections are, every-where, equal to each other, the solids themselves will likewise be equal [r]; that is, the cone (EOD) will be equal to the excess of the cylinder (GDE*g*) above the inscribed hemisphere (GA*g*): whence, as the cone, or excess, is one third part of the cylinder [s], the hemisphere must necessarily be equal to the two remaining thirds. And what is here proved, with respect to the halves of the proposed solids, holds equally in the wholes. Therefore every sphere is two-thirds of its circumscribing cylinder.

[r] Lem. 4.

[s] 6. 8. and Cor. 3. to 8. 8.

COROLLARY I.

Hence, a cone, hemi-sphere and cylinder, of the same altitude, and standing upon equal bases, are in proportion, as the numbers 1, 2 and 3, respectively.

COROLLARY II.

Hence it also appears, that all spheres are to each other in the triplicate ratio of their diameters [t]; being in the same proportion as the circumscribing cylinders, whereof they are like parts.

[t] Cor. to 5. 8. and 24. 7.

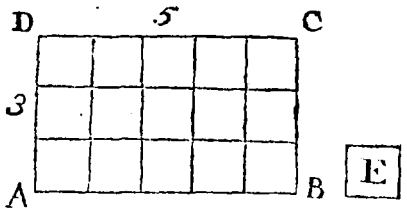
END *of the* ELEMENTS.

OF THE

MENSURATION

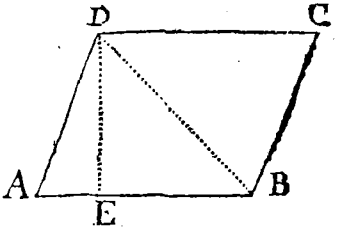
OF

Superficies and Solids.

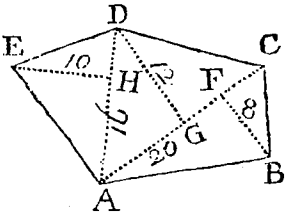


EVERY quantity is measured by some other quantity of the same kind; as a line by a line, a surface by a surface, and a solid by a solid: And the number which shews how often the lesser, called *the measuring unit*, is contained in the greater, or quantity measured, is called *the content of the quantity so measured.* Thus, if the quantity to be measured be the rectangle ABCD, and the little square E, whose side is one inch, be the measuring unit propounded; then, as often as the said little square is contained in the rectangle, so many square inches the rectangle is said to contain: So that, if the length DC be supposed 5 inches, and the breadth AD 3 inches; the content of the rectangle

rectangle will be 3 times 5, or 15 ſquare inches: Becauſe, if lines be drawn parallel to the ſides, at an inch diſtance one from another, they will divide the whole rectangle ABCD into 3 times 5, or 15, equal parts, of one inch each. And, generally, whatever the meaſures of the two ſides may be, it is evident (*from El.* 7. *of* 4.) that the rectangle will contain the ſquare E, as many times as the baſe AB contains the baſe of the ſquare, repeated as often as the altitude AD contains the altitude of the ſquare. Therefore, *to find the content of any rectangle, multiply the baſe by the altitude, and the product will be the anſwer.* Thus, let the length be 18 inches, and the breadth 15; then the content will be 15 times 18, or 270 ſquare inches.



The method of finding the content of a rectangle being thus known, the content of any parallelogram ABCD, or triangle ABD, will alſo be known; the former of theſe figures being equal to a rectangle of the ſame baſe and altitude; and the latter equal to the half of ſuch a rectangle (by Cor. 2. to 2. 2.) Therefore, *multiply the baſe by the perpendicular, for the content of any parallelogram; and the baſe by half the perpendicular, for that of any triangle.* Thus, for example, let the baſe AB be 18 feet, and the perpendicular DE 12 feet; then the content of the parallelogram will be 216, and that of the triangle 108, ſquare feet.



From the manner of finding the area of a triangle, the area of any right-lined plain figure, as ABCDE, may be determined, by dividing the whole into triangles and finding the content of each triangle. Thus, let the dividing lines AC and AD, be 20 and 16 inches, and the perpendiculars BF, DG, EH, falling thereon, 8, 12, and 10, reſpectively; then, the content of the

Triangle $\left\{\begin{matrix}\text{ABC}\\ \text{ACD}\\ \text{ADE}\end{matrix}\right\}$ being $\left\{\begin{matrix}80\\ 120\\ 80\end{matrix}\right\}$, it is evi-

dent, that the content of the whole figure will be the ſum of all theſe, or 280 ſquare inches. But, when the given lines are expreſſed by fractions, or very large numbers, the work will be ſomewhat ſhortened, by finding the content of every two triangles, having the ſame baſe, at one operation; that is, by firſt adding the two perpendiculars together, and then multiplying half their ſum by the common baſe of the two triangles. Thus, in the laſt example, the half-ſum of the two perpendiculars BF and DG being 10, if this number be, therefore, multiplied by 20 the meaſure of the common baſe AC, the product, which is 200, will be the content of the trapezium ABCDA; to which 80, the content of the triangle ADE, being added; the ſum will be 280, the ſame as before. But, if the polygon propoſed be a regular one, that is, one whoſe ſides, and angles are all equal, the ſhorteſt way

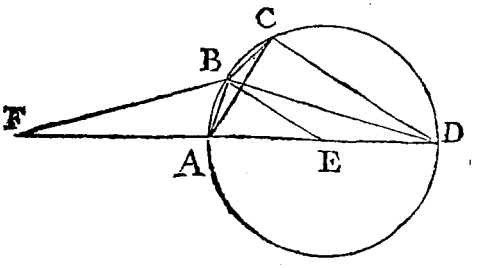
way of all, is, to *multiply half the ſum of all the ſides by the length of the line drawn from the middle of any ſide to the center of the polygon.* The reaſon of which is obvious, from the demonſtration to *Theor.* II. *B.* VIII.

Having ſhewn how the area of any right-lined figure may be computed, it will be proper here, to ſay ſomething with regard to the area, and periphery, of the circle.

It is well known, that to determine the *true* area of a circle, and to find a right-line *exactly* equal to the circumference thereof, are looked upon, by mathematicians, as abſolutely impoſſible: But, though neither the one nor the other can be *accurately* known, yet ſeveral Ways have been invented by which they may be approximated, to any aſſigned degree of exactneſs. That, which I am now going to lay down, though leſs expeditious than ſome others, ſeems, nevertheleſs, to be the moſt proper for this place, as depending on the moſt ſimple and evident principles: I ſhall therefore begin with premiſing the following

LEMMA.

If AD *be a diameter, and* AB, BC *two equal arcs of the ſame circle, and if the chords* DB, DC *be drawn; then, I ſay, that* $DB^2 = \frac{1}{2}AD \times DC + \frac{1}{2}AD^2$.



For, if in DA produced, there be taken AF = DC, and BF, BA, BC and the radius BE be drawn; then, the external angle FAB, of the trapezium ABCD, being equal to the internal oppoſite angle DCB (*by* 17. 3.) alſo AF = DC, and AB = CB (*by Hyp.*); it is evident, that FB is alſo = DB, and conſequently the angle F = FDB = DBE: And ſo the iſoſceles triangles DEB, DBF being equiangular, it will be as DE ($\frac{1}{2}$AD) : DB : : DB : DF (DC + AD); and conſequently $DB^2 = \frac{1}{2}AD \times DC + \frac{1}{2}AD^2$. *Q. E. D.*

COROLLARY.

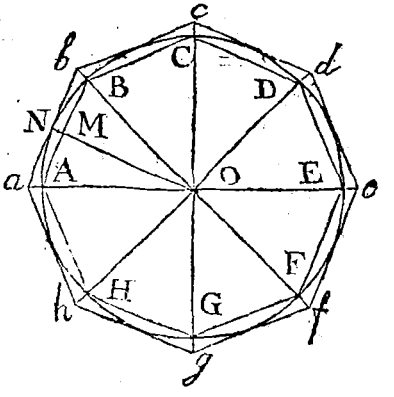
Hence, if the diameter AD be denoted by the number 2, the chord DB will be denoted by $\sqrt{DC + 2}$: whence, it appears, that, *if the meaſure of the ſupplemental-chord of any arch be increaſed by the number* 2, *the ſquare-root of the ſum will be the ſupplemental-chord of half that arch.*

Now, to apply this to the matter propoſed, that is, to the finding of the area and circumference of the circle; let the arch ABC be taken equal to $\frac{1}{3}$ of the ſemi-periphery ACD; then will the chord AC be equal to the radius AE (*by* 29. 5.); and

and therefore, ſince ACD is a right-angle (*by* 13. 3.) DC^2 ($= AD^2 - AC^2$, *by* 8. 2.) will be $= 4 - 1 = 3$; and conſequently $DC = \sqrt{3} =$ 1,7320508075, *&c.* Wherefore, ſeeing the ſupplemental-chord of $\frac{1}{3}$ of the ſemi-periphery is 1,7320508075, we ſhall, by the preceding Corollary,

$$\text{have}\left\{\begin{array}{l}\sqrt{2+1,7320508075} = 1,9318516525\\ \sqrt{2+1,9318516525} = 1,9828897227\\ \sqrt{2+1,9828897227} = 1,9957178465\\ \sqrt{2+1,9957178465} = 1,9989291743\\ \sqrt{2+1,9989291743} = 1,9997322757\\ \sqrt{2+1,9997322757} = 1,9999330678\\ \sqrt{2+1,9999330678} = \sqrt{3,9999330678}\end{array}\right\}\text{for the ſupplemental chord of}\left\{\begin{array}{l}\frac{1}{6}\\ \frac{1}{12}\\ \frac{1}{24}\\ \frac{1}{48}\\ \frac{1}{96}\\ \frac{1}{192}\\ \frac{1}{384}\end{array}\right\}\text{of the ſemi-periphery.}$$

Now, therefore, ſince it is found that 3,9999330678 is the ſquare of the ſupplemental-chord of $\frac{1}{384}$ of the ſemi-periphery, let this number be ſubſtracted from 4 the ſquare of the diameter, and the remainder 0,0000669322 will be the ſquare of the chord of the ſame arch; therefore the chord itſelf being $= \sqrt{0,0000669322} = 0,00818121$, let this number be multiplied by 768, or twice 384, and the product 6,28317 will be the perimeter of a regular polygon of 768 ſides, inſcribed in the circle; which, as the ſides of the polygon very nearly coincide with the circumference of the circle, muſt alſo expreſs the length of the circumference itſelf, very nearly.



But, in order to ſhew how near this is to the truth, let AB repreſent one ſide of a regular polygon of 768 ſides, inſcribed in the circle (whoſe length, we have found above, to be 0,00818121) and let *ab* be a ſide of another ſimilar polygon, deſcribed about the circle; and from the center O let ON be drawn, biſecting AB and *ab* in M and N: Then, ſince AM is $= \frac{1}{2}$AB $=$ 0,0040906, and AO $=$ 1, it is plain that OM^2 ($AO^2 - AM^2$) will be $=$ 0,99998327, and conſequently OM $=$ 0,99999163; whence, becauſe of the ſimilar triangles AOB, *aOb*, &c. we have 0,99999163 (OM) : 1 (ON) : : AB : *ab* : : 6,28317 (the perimeter of the inſcribed polygon) : 6,28322 the perimeter of the circumſcribed polygon. But the circumference of the circle being greater than the perimeter of the inſcribed polygon, and leſs than that of the circumſcribed one, it muſt, conſequently, be greater than 6,28317, and leſs than 6,28322; and muſt, therefore, be equal to 6,2832, very near; ſince this number exceeds the perimeter of the inſcribed polygon by no more than 0,00003, and is leſs than the perimeter of the circumſcribed one by 0 00002, only.

From the periphery thus found, the area of the circle will alſo be known; being equal to the product of half the periphery into the radius (*by* 2. 8.) that is, $= 3,1416 \times 1 = 3,1416$.

There-

Therefore, ſince it is proved (*in Theor.* 3. *and* 4. *of* 8.) that the peripheries of circles are in proportion as their diameters, and the circles themſelves as the ſquares of thoſe diameters; it follows, that, as 2 is to 6,2832, or as 1 to 3,1416 : : the diameter of any circle to its periphery; and as 4 to 3,1416, or as 1 to 0,7854 : : the ſquare of the diameter to the area.

But, if you had rather have the proportions in whole numbers, and the caſe propoſed does not require any great degree of accuracy; then, inſtead of the foregoing, thoſe of *Archimedes* may be uſed, *viz.* 7 : 22 : : diam. : circumf. and 14 : 11 : : ſquare diam. : area. Which proportions differ but little from thoſe above, as will appear from the following example: wherein the diameter of a circle being given 28, its circumference and area are required. Here, according to the firſt proportions, I multiply 28 by 3,1416 for the circumference, and the ſquare of 28 (or 784) by 0,7854 for the area; and there reſults 87,964 and 615,75, reſpectively. But, according to the proportions of *Archimedes*, the circumference will be found equal to 88, and the area 616; which differ very little from the former.

By knowing the proportion between the diameter of a circle and the circumference, and between the ſquare of the diameter and the area, the convex ſuperficies of ſolid bodies may be determined. Thus,

The convex ſuperficies of a cylinder is found, *by firſt finding the circumference of the baſe, and then multiplying by the altitude of the ſolid.* Therefore, if to that product, the area of the two circular ends be added, the ſum will be the whole ſuperficies of the cylinder.

To find the convex ſuperficies of a cone, *multiply half the length of the ſlant ſide thereof by the circumference of the baſe.*

The convex ſuperficies of any fruſtum of this ſolid is found, by *multiplying the ſum of the peripheries of the two ends into half the length of the ſlant ſide of the fruſtum.*

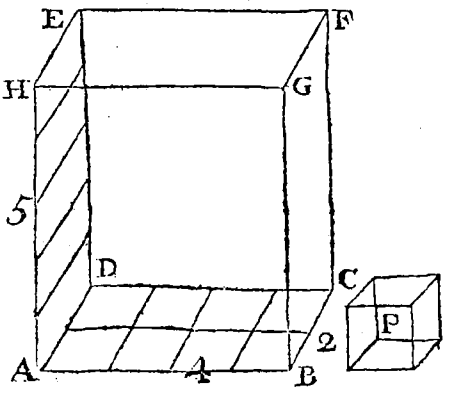
To find the ſuperficies of a ſphere, *multiply the periphery of the greateſt, or generating, circle by its diameter:* Or, *multiply the ſquare of the diameter by* 3,1416.

The convex ſuperficies of any ſegment of a ſphere is found, *by multiplying the periphery of the greateſt circle of the ſphere into the altitude of the ſegment.*

The demonſtration of theſe laſt rules, for finding the curve ſurfaces of ſolid bodies (which is not given in the *Elements*, for reaſons mentioned hereafter) is inſerted at the end of this ſection.

Of the Menſuration of Solids.

As every ſuperficies is meaſured by a ſquare, whoſe ſide is unity (as one inch, one foot, one yard, *&c.*) ſo every ſolid is meaſured by a cube whereof the ſide is alſo an unit. Thus,



let

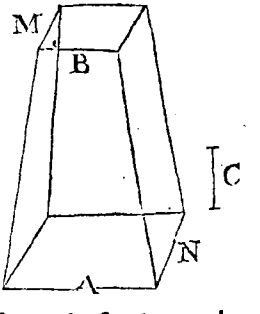
let the ſolid to be meaſured, be the rectangular parallelepipedon AF, and let the cube P, whoſe ſide is one inch, be the *meaſuring unit*; alſo let the length AB, of the baſe AC, be 4 inches, the breadth BC 2 inches, and the altitude AH of the ſolid 5 inches: Then, becauſe the area of the baſe ABCD is 2 times 4 (or 8) ſquare inches, it is eaſy to conceive, that, if the ſolid were to be only one inch high (inſtead of 5), the content thereof would be juſt the ſame number (8) of cubical inches; becauſe then, upon the eight equal ſquares into which the whole baſe ABCD is diviſible, a cube of one inch might be erected, ſo as to compoſe a parallelepipedon on that baſe, of one inch high. Therefore, ſeeing that the content of the ſolid, at one inch high, is 8 cubical inches, the whole content at 5 inches high, muſt conſequently be 5 times 8, or 40 cubical inches (ſince the whole ſolid AF may be conſidered, as compoſed of ſuch heights of cubes, one ranged above another). And, generally, whatever the dimenſions may be, it is manifeſt (*from* 21 and 22. of 7.) that the parallelepipedon will contain the cube P, as many times as the baſe ABCD contains the baſe of the cube, repeated as often as the altitude AH contains the altitude of the cube. Therefore *the content of any parallelepipedon will be found, by multiplying the area of the baſe by the altitude of the parallelepipedon.* Thus, for example, if the two dimenſions of the baſe be 16 and 12 inches, and the height of the ſolid 10 inches; then, the area of the baſe being 192, the content of the ſolid will be 1920 cubical inches.

From the content of a parallelepipedon, thus known, *that* of a priſm, or a cylinder, will likewiſe be known; every ſuch ſolid being (*by* 20. 7. *or* 5. 8.) equal to a parallelepipedon of equal baſe, and

and altitude. Therefore, *multiply the area of the baſe* (found by the rules for ſuperficies) *into the height of the priſm, or cylinder, and the product will be the content.*

Hence the content of any pyramid, or cone, is alſo obtained; being (*by Cor.* 3. *to* 8. 8.) equal to $\frac{1}{3}$ part of a priſm, or cylinder, of the ſame baſe and altitude. Therefore, *multiply the area of the baſe by* $\frac{1}{3}$ *of the altitude, and the product will be the anſwer.*

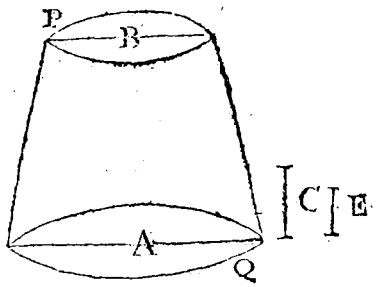
Every ſphere being (*by* 11. 8.) equal to $\frac{2}{3}$ parts of a cylinder of the ſame diameter and altitude; *the content of any ſphere will, therefore, be found, by multiplying the area of its greateſt, or generating, circle into* $\frac{2}{3}$ *of its diameter:* Or (becauſe the area of ſuch circle is to the ſquare of the diameter, in proportion as 0,7854 to 1), *let the cube of the diameter be multiplied by the fraction* ,5236 (= $\frac{2}{3}$ of 0,7854), *and the product will be the content.* Thus, if the meaſure of the diameter be 20, the cube thereof will be 8000; which, multiplied by ,5236, will give 4188,8 for the meaſure of the ſphere's ſolidity.



The manner of finding the content of any fruſtums of the ſolids above determined, is collected from *Theor.* 10. *and* 11. *B.* VIII. Let the fruſtum (MN), firſt propoſed, be that of a pyramid; then, having found the content of a whole pyramid, of the ſame given baſe and altitude; ſay, as any ſide A of the lower end or baſe, is to its correſpondent B of the upper, ſo is the ſaid content

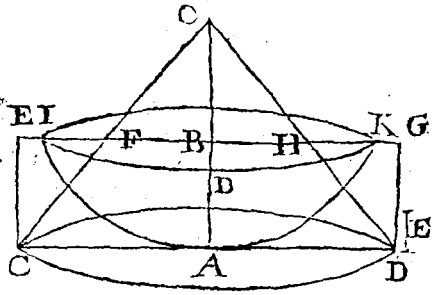
content to a fourth-proportional; and, as A is (again) to B, so is the quantity last found to another proportional: which two proportionals, added to the content first determined, will give the true content of the frustum. But when the opposite bases of the frustum are squares, the rule will be more simple, and put on a better form: For then the area of the base being A^2, the content of a whole pyramid thereon, of the same altitude with the frustum, will be equal to the parallelepipedon $C \times A^2$, C being $\frac{1}{3}$ of the given altitude of the frustum. But $A : B :: C \times A^2 : C \times A \times B$ (*by* 22. 7.) and $A : B :: C \times A \times B : C \times B^2$. Therefore $C \times A^2 + C \times A \times B + C \times B^2$ ($= C \times \overline{A^2 + A \times B + B^2}$, *by Schol. to* 20. 7.) is the true content, in this case.

Hence, *to find the content of the frustum of any square-pyramid, add the product of the two sides of the lower and upper ends to the sum of their squares, and then multiply the aggregate by* $\frac{1}{3}$ *of the pyramid's height.*



From the content here found, *that* of any conical frustum (PQ) is readily obtained; being in proportion to the content ($C \times \overline{A^2 + A \times B + B^2}$) of the frustum of a square pyramid circumscribing it, as the base of the former is to the base of the latter (*by Cor.* 5. *to* 8. 8.), or as the fraction ,7854 is to *unity*: And so, will be equal to the ,7854 part of $C \times \overline{A^2 + A \times B + B^2} = E \times \overline{A^2 + AB + B^2}$; by taking $E = ,7854 \times C =$ the ,6618 part of the whole given altitude. Therefore

fore, *to find the content of any fruſtum of a cone, add the product of the diameters of the two ends to the ſum of their ſquares; then multiply the aggregate by the fruſtum's height, and the product, again, by the fraction* ,2618.



Hence, and from *Theor.* 11. *B.* VIII. a Rule for finding the content of any ſegment IAK of a ſphere, may alſo be deduced: For, it appears, *from thence*, that the ſegment propoſed, IAK, is equal to the difference between a conical fruſtum FCDH and a cylinder ECDG of the ſame altitude, ſtanding upon a baſe, whoſe radius CA is equal to that (*AO*) of the ſphere itſelf. But the content of the fruſtum FCDH, if the two diameters CD, FH be repreſented *(as above)* by A and B, and the ,2618 part of the altitude (D) by E, will be $= E \times \overline{A^2 + A \times B + B^2}$ (that is, equal to a parallelepipedon whoſe altitude is E, and baſe $= A^2 + A \times B + B^2$): And the content of the cylinder ECDG will be $= 3E \times A^2$, or $E \times 3A^2$. Therefore the difference (or the content of the ſegment IAK) will be $= E \times \overline{2A^2 - A \times B - B^2}$ (*Schol. to* 20. 7. *and Ax.* 5. 1.) But $2A^2 - A \times B - B^2$ is compoſed of $A^2 - A \times B$ and $A^2 - B^2$; whereof the former part $A^2 - A \times B$ is $= \overline{A - B} \times A$ (*by* 5. 2.) $= 2D \times A$ (becauſe $A - B$ (or $CD - FH$) $= FF + HG = 2AB$ or $2D$); and the latter $A^2 - B^2 = \overline{A - B} \times \overline{A + B}$ (*by* 7. 2.) $2D \times \overline{2A - 2D}$: Whence the ſum of both will conſequently be $= 2D \times \overline{3A - 2D}$; and the content of the ſegment itſelf

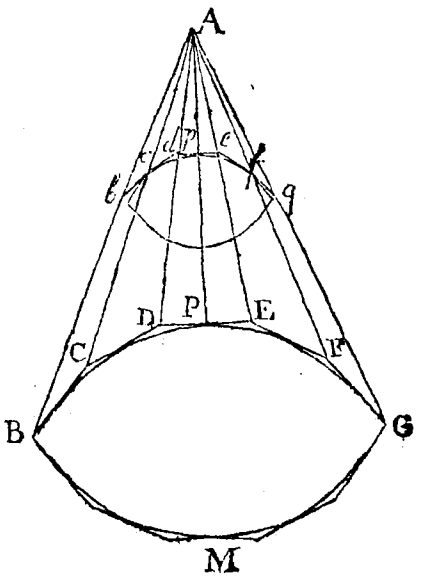
itself $= E \times 2D \times \overline{3A - 2D} = ,5236 \times D^2 \times \overline{3A - 2D}$ (because $2E = ,5236\,D$).

Therefore, *to find the content of any segment of a sphere, multiply the square of the segment's height by the excess of thrice the sphere's diameter above the double of that height; and then multiply by the fraction* ,5236.

The demonstration of the rules for determining the superficial content of the cylinder, cone and sphere, and of their several segments, or frustums, is collected from the two *Lemmas* here subjoined.

LEMMA 1.

The upper superficies, or the area of all the sides of a regular pyramid, in which a cone may be inscribed, is equal to a rectangle under the perimeter of the base and half the length of the cone's slant side.



For, let BCDE, &c. be the base of the pyramid, and BPGM that of the inscribed cone; and from the vertex A to the point P where any side DE of the polyg. touches the circle, let AP be drawn. Then, since the triangle ADE is $= \frac{1}{2}AP \times DE = \frac{1}{2}AB \times DE$; and as the like holds good with regard to every other side

of

of the pyramid, it is evident that the sum of all the sides, or the whole superficies of the pyramid (exclusive of the base) will be equal to $\frac{1}{2}AB \times \overline{DE + EF + \&c}$; that is, equal to a rectangle under $\frac{1}{2}AB$ and the whole perimeter of the base.

COROLLARY I.

Hence it will also appear, that all the sides of any frustum Bg of the pyramid, will be equal to a rectangle under half the length of each side and the sum of the perimeters of the two ends: For, the area of the side DE*ed* being $= \frac{1}{2}Pp \times \overline{DE + de}$, or $\frac{1}{2}Bb \times \overline{DE + de}$ (*by* 4. 2.), the area of all the sides will, therefore, be $= \frac{1}{2}Bb \times \overline{DE + de + EF + ef + \&c.}$

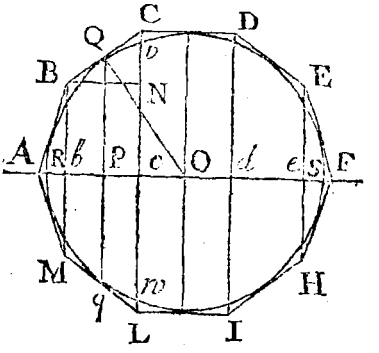
COROLLARY II.

Therefore, seeing that the foregoing conclusions hold universally, whatever the number of the sides may be; and as the pyramid, by increasing the number of its sides, approaches nearer and nearer, continually, to the inscribed cone, which is its limit; thence will the upper superficies of the cone (as well as that of the pyramid) be equal to a rectangle under half the length of its slant side and the perimeter of its base. And the convex superficies of any frustum of the cone will, *also*, be equal to a rectangle under half the length of its slant side and the sum of the peripheries of its two ends, or bases: Whence it likewise follows, that the convex surface of a cylinder will be equal to a rectangle under half its altitude and twice the periphery of its base (or under the whole altitude and once that periphery); because then the two ends are equal.—From this Corollary, the rules for finding the superficies of the cylinder and cone, are given.

LEM-

LEMMA 2.

If a regular polygon ABCDE, &c. *of an even number of sides, together with its inscribed circle* RQSq, *be supposed to revolve about the (produced) diameter* RS, *as an axis; the superficies of the solid generated by the polygon, will be equal to a rectangle under its axis* AF *and a right line equal to the circumference* RQSq *of the inscribed circle.*



From the center O, to the point of contact Q, of any side BC, let the radius OQ be drawn; also draw B*b*M, QP*q*, C*c*L, &c. perpendicular to AF, and BN perpendicular to CL.

Because the solid generated by the plane B*bc*C is the frustum of a cone, the convex superficies thereof, generated by BC, is equal to a rectangle under $\frac{1}{2}$BC and the sum of the peripheries of the two circles described by B*b* and C*c* *(by Cor. 2. to the precedent)*: But the sum of these two peripheries, as QP is an arithmetical mean between B*b* and C*c*, is equal to twice the periphery Q*q*; and therefore the convex superficies of the said frustum equal to $\frac{1}{2}$BC × 2 periph. Q*q* = BC × periph. Q*q*. But, because of the similar triangles OPQ, BNC, we have BC : BN (*bc*) : : OQ : PQ : : periph. RQS*q* : periph. Q*q* *(by* 4. 8.*)*; and consequently BC × periph. Q*q* = *bc* × periph. RQS*q* = the superficies

perficies generated by BC. By the very ſame argument, the ſuperficies generated by any other ſide CD is $= cd \times$ periph. RQSq: Whence it is manifeſt, that the ſuperficies of the whole ſolid is $= \overline{Ab + bc + cd + \&c.} \times$ periph. RQSq $=$ AF $\times$ periph. RQSq.

COROLLARY I.

Since the ſuperficies of the ſolid is, univerſally, equal to AF $\times$ periph. RQSq, let the number of ſides of the generating polygon be what it will; and as the ſaid ſuperficies, by increaſing the number of ſides, approaches nearer and nearer, continually, to the ſuperficies of the inſcribed ſphere, which is its limit; thence will the ſuperficies of the ſphere, itſelf, be alſo equal to a rectangle under its axis RS and periphery RQSq: And the convex ſuperficies of any ſegment thereof vRw, will likewiſe be equal to a rectangle under its axis (or height) Rc and the ſame periphery RQSq; ſince it is proved, that the correſponding ſuperficies of CBAML, is univerſally equal to Ac $\times$ periph. RQSq.

COROLLARY II.

Hence it alſo appears, that the ſuperficies of every ſphere is equal to four times its generating circle: Becauſe (*by* 2. 8.) the circle RQSq $= \frac{1}{2}$RS $\times \frac{1}{2}$ periph. RQSq $= \frac{1}{4}$RS $\times$ periph. RQSq.

In deriving theſe concluſions, as well as thoſe depending upon the preceding Lemma, the *Reader* muſt have obſerved, that ſomething is aſſumed, which is not demonſtrated in any part of theſe *Elements*. But this will not, I imagine, be conſidered as a fault, by Thoſe who know, that it is impoſſible to prove in a manner *perfectly* regular and geometrical, that a *curve ſurface*, of any kind, is equal to

to a *plane-one* of an assigned magnitude. Plane surfaces are compared with one another, in virtue of the 10*th Axiom*; in which, whatever relates to the equality of plane figures, has its original. But no principles have been yet admitted into the *common*, or *lower Geometry*, whereby a curve surface can be compared with a plane one; nor even by which the proportion of any one curve-line to a right-line can be known: Nor can it be demonstrated by all the Geometry in *Euclid's Elements*, that the periphery of a circle is less than the perimeter of its circumscribing square.—We can determine the proportion of solids bounded by curve-surfaces, by describing other solids in, and about them, so as to differ less from them, than by any assigned part however small. But in comparing of the surfaces, this method fails; because, let the number of sides of the inscribed, or circumscribed solid be ever so great, or let the solid itself approach ever so near to the proposed one; the two surfaces, after all, *will have no part in common* on which a demonstration can be formed, but will still be distinct things. Before such a comparison can possibly be made, in a regular and scientific manner, *new principles* must be laid down: But *these* belong to, and are best supplied in the *Modern Geometry*, or *Method of Fluxions*.

OF THE

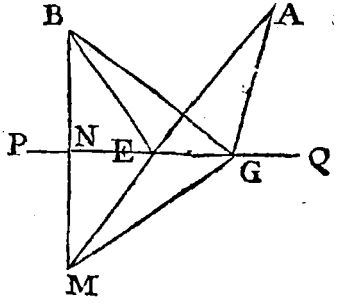
MAXIMA and MINIMA

OF

Geometrical Quantities.

THEOREM I.

If from two given points A, B, *on the same side of an indefinite line* PQ *(in the same plane with them) two lines* AE, BE *be drawn to meet on, and make equal angles* AEQ, BEP *with the said line* PQ; *the lines so drawn, taken together, shall be less than any other two* AG, BG, *drawn from the same points to meet on the same line* PQ.



For, let BNM be perpendicular to PNQ, and let AE be produced to meet it in M, also let MG be drawn.

Then the triangles MNE, BNE, having the angle MEN (= AEQ[a]) = BEN[b], MNE = BNE[c], and NE common to both; have also MN = BN, and ME = BE

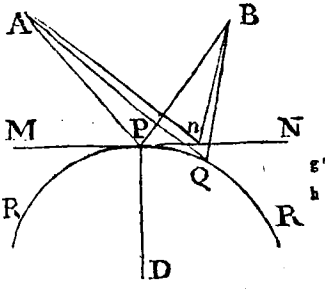
[a] 3. 1.
[b] Hyp.
[c] Constr.

BE [d]; whence alſo MG = BG [e]: But AM (AE + BE) is leſs than AG + MG [f], or, than its equal AG + BG. *Q. E. D.*

[d] 15. 1.
[e] Ax. 10. 1.
[f] 19. 1.

THEOREM II.

Of all right lines AP, BP; AQ, BQ, *that can be drawn from two given points* A, B, *to meet, two by two, on the convexity of a given circle* RPQR; *those two* AP, BP *taken together, ſhall be the leaſt, which make equal angles with the tangent* MPN *(or with the radius* DP*) at the point of concourſe* P.



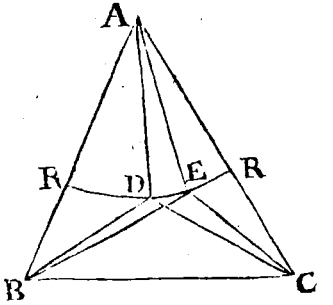
For, if to any point *n* in the part of the tangent intercepted by AQ and BQ, there be drawn A*n* and B*n*; then will AP + BP be leſs than A*n* + B*n* [g], and A*n* + B*n* leſs than AQ + BQ [h]: Conſequently AP + BP is leſs than AQ + BQ. *Q. E. D.*

[g] Theor. 1.
[h] 23. 1.

This demonſtration holds equally true, when the curve RPR is ſuppoſed of any other kind; provided all tangents to it, fall intirely without the curve.

THEOREM III.

If, in a given triangle ABC, *a point is to be determined, ſo that the ſum of all the three lines drawn from thence to the three angles, ſhall be the leaſt poſſible; I ſay, the poſition of that point muſt be ſuch, that all the angles formed about it by thoſe lines, ſhall be equal among themſelves.*

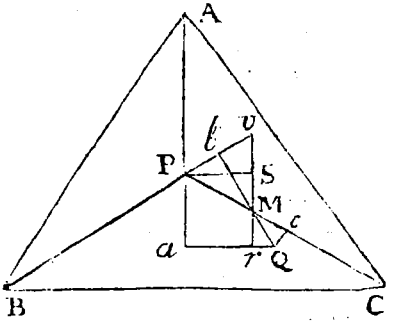


If you deny it, then let some point E, at which the angles BEA, CEA are unequal, be the required one.

Upon the center A, thro' E, let the circumference of a circle RER be described; and let D be that point in it, where the angles ADB and ADC are equal.

[1] Theor. 2. Because BD + CD is less than BE + CE [1], therefore is AD + BD + CD also less than AE + [k] Ax. 6. 1. BE + CE [k]; *which is repugnant.* Therefore no point at which the angles are unequal, can be the required one. *Q. E. D.*

The same otherwise.



Let the point P be that, at which all the angles APB, APC, BPC are equal *; and from any other point Q, upon the lines forming them, let fall the three perpendiculars Q*a*, Q*b*, Q*c*. I say, first, that the sum of the three distances A*a*, B*b*, C*c*, intercepted by those perpendiculars, and the three given points A, B, C, will be equal to the sum of the three

* *The determination of the position of a point, at which, line drawn from three given points, shall form any given angles, given among the Geometrical Constructions, in the next section.*

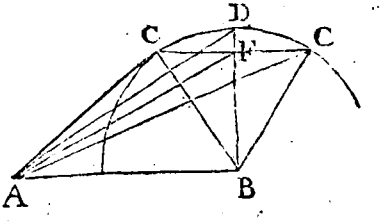
first

first distances AP + BP + CP. For, if thro' the intersection M of P*c* and *b*Q, *rv* be drawn parallel to A*a*, meeting B*b* (produced) in *v*, and PS, parallel to *a*Q, in S; it is evident (because the angle *v* = [l]BP*a* = *v*PM = $\frac{2}{3}$ of a right angle [m]) that the triangle P*v*M is equilateral; and that the right-angled triangles QM*c*, QM*r*, having Q[illegible]*c* (= PM*b* = *v*M*b*) = QM*r*, have also *c*M = *r*M; to which let MP = M*v* be added; so shall *c*P = *rv* = *a*P + *v*S = *a*P + P*b*. And, if to the first and last of these, AP + BP + C*c* be (again) added; then will AP + BP + CP = A*a* + B*b* + C*c*, *as was asserted:* Whence the Theorem itself is exceedingly obvious: For seeing that the sum AP + BP + CP is but equal to the sum of the three bases A*a*, B*b*, C*c*, it must necessarily be less than that of the three hypothenuses AQ, BQ and CQ. *Q. E. D.*

[l] Cor. to 8. 1.
[m] Hyp.

THEOREM IV.

The greatest triangle ABD *that can possibly be contained under two right-lines, given in length, and any other right line joining their extremes, will be when the two given lines* AB, BD *make right-angles with each other.*



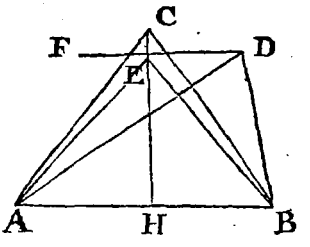
For, let BC be equal to BD, and the angle ABC either greater, or less than the right-angle ABD; let also CF be drawn parallel to AB, meeting BD (produced, if necessary) in F, and let A, F, and A, C be joined.

Then the angle BFC being a right-one [n], it is evident that BC (BD) is greater than BF [o]; and therefore the triangle ABD, being greater than ABF [p], is also greater than its equal [q] ABC. *Q. E. D.*

[n] 5. 1.
[o] 20 1.
[p] Ax. 2.
[q] Cor. to 2. 2.

THEOREM V.

Of all triangles ABC, ABD, *having the same base* AB, *and the sum of their other sides the same, the isosceles one* ACB, *is the greatest.*



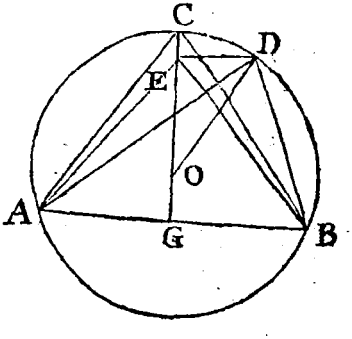
Let CH be perpendicular to AB, and DEF parallel to AB, intersecting HC (produced if need be) in E; likewise let AE and BE be drawn.

It is manifest that the angles AEF, BED are equal [r]; therefore AE + BE is less than AD + BD [s], or than its equal AC + BC [t]; and so the triangle AEB, falling within the triangle ACB [u], must be less than ACB [w]; and therefore ADB (= AEB [x]) must also be less than ACB. *Q. E. D.*

[r] 16. and 7. 1.
[s] Theor. 1.
[t] Hyp.
[u] 23. 1.
[w] Ax. 2. 1.
[x] Cor. 1. to 2. 2.

THEOREM VI.

Of all triangles ABC, ABD *standing upon the same base* AB, *and having equal vertical angles* ACB, ADB, *the isosceles one* ACB *is the greatest.*



Let ACDB be a segment of a circle, in which the equal angles ACB, ADB are contained [y]; make CEG perpendicular, and DE parallel, to AB; from the center O draw OD, and let A, E and B, E be joined. It is evident that CG, not only bisects AB [z], but also passes through the center O [a]. Therefore, OD (OC) being

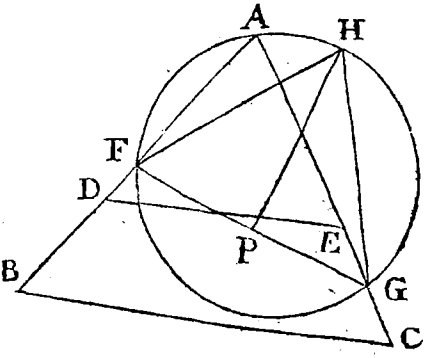
[y] 22. 5. and 11. 1
[z] 16. 1.
[a] Cor. to 2. 3.

ing greater than OE [a], the triangle ACB will also be greater than AEB, or than its equal ADB [b]. *Q. E. D.*

[a] 20. 1.
[b] Cor. 2. 2. and Ax. 2.

THEOREM VII.

Of all right-lines DE, FG *that can be drawn to cut off equal areas* ADE, AFG *from a given triangle* ABC, *that* DE *is the least, which makes the triangle* ADE *cut off an isosceles one.*

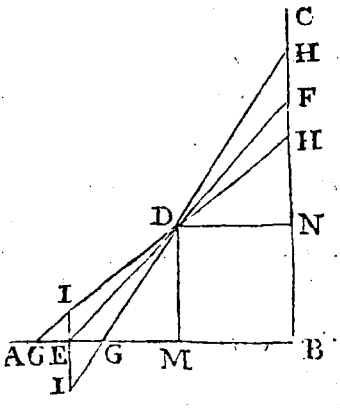


Let AFG be the circumference of a circle passing through the three points A, F, G; also let PH be perpendicular to FG, at the middle point P, meeting the circumference in H, and let FH and GH be drawn. The triangle FHG, being isosceles [c], is therefore greater than FAG [d], or than its equal [e] ADE: Whence, as the triangles FHG, ADE are equiangular [f], the base FG of the greater, must consequently exceed the base DE of the lesser. *Q. E. D.*

[c] Ax. 10.
[d] Theor. 6.
[e] Hyp.
[f] Hyp and Cor. to 10. 1.

THEOREM VIII.

Of all right-lines EF, GH, GH *that can be drawn thro' a given point* D, *between two right-lines* BA, BC *given in position; that* EF *which is bisected by the given point* D, *forms with them the least triangle* (EBF).



For, if EI, parallel to BC, be drawn, meeting GH in I; the[h] equiangular triangles DFH and DEI, having DF = DE[i], will be equal[k]; and DFH will therefore be less, or greater than DEG[l], according as BG is greater or less than BE. In the former case let DEBH common, be added to both; so shall FEB be less than HGB[k]. And if, in the latter case, DGBF be added, then will HGB be greater than FEB[m]; and consequently FEB (in this case also) less than HGB. *Q. E. D.*

[h] 3. and 7. 1.
[i] Hyp.
[k] 15. 1.
[l] Ax. 2.
[m] Ax. 6.

COROLLARY.

If DM and DN be drawn parallel to BC and BA; the two equal[o] triangles DEM, DFN, taken together (since EM = DN[o] = MB[p]) will be equal to the parallelogram DMBN[q]; and consequently the parallelogram DMBN = $\frac{1}{2}$BEF ⊐ $\frac{1}{2}$BGH. Whence it is manifest, that a parallelogram is always less than half a triangle in which it is inscribed, except when the base of the one is half the base of the other; in which case the parallelogram is just half the triangle.

[o] 15. 1.
[p] 24. 1.
[q] Cor. to 2. 2.

SCHO-

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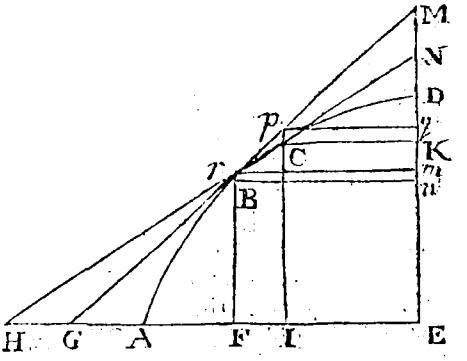
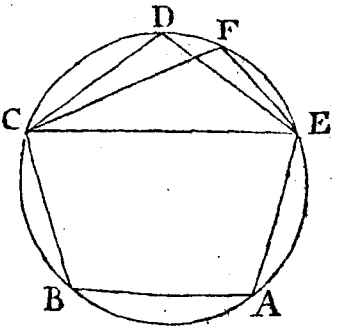


From the preceding Corollary alone, it may be very easily made to appear, that the least triangle EGM which can possibly be described about, and the greatest parallelogram EFB*n* that can be described in, any curve ABCD, concave to its axis AE, will be when the sub-tangent FG is equal to half the base EG of the triangle, or to the whole base EF of the parallelogram; and that the two figures will be in the ratio of *two* to *one*. For let HN be a side of any other circumscribing triangle (EHN) touching the curve in C, and meeting FB*r* in *r*: Then, the curve being concave to its axis, the point *r* will fall above B; whence, if *rm* be drawn parallel to B*n*, then will EGM = 2BE ⊐ 2*r*E ⊐ EHN. Again, if IC, parallel to EM, be produced to meet GM in *p*, and CK and *pq* be drawn parallel to AE; then, also, will BE = $\frac{1}{2}$EGM ⊏ *p*E ⊏ CE, *as was to be shewn.*

THEOREM IX.

Of all right-lined figures, contained under the same number of sides, and inscribed in the same circle, that is the greatest whose sides are all equal.

For,



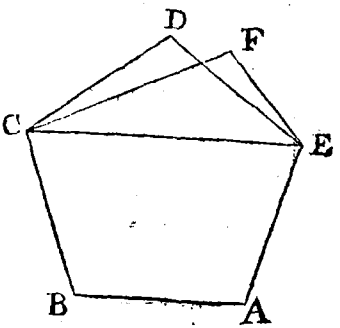
For, *if possible*, let some polygon ABCFE, whose sides CF, EF are unequal, be the greatest.

Let CDE be an isosceles triangle described in the same segment with CFE; which being greater than CFE [r], the whole polygon ABCDE will also be greater than the whole polygon ABCFE; *which is repugnant.* Therefore the polygon is the greatest when the sides are all equal.

[r] Theor. 6. and 11. 3.

THEOREM X.

Of all right-lined figures, contained under the same perimeter, and number of sides, the greatest is, when the sides are all equal.

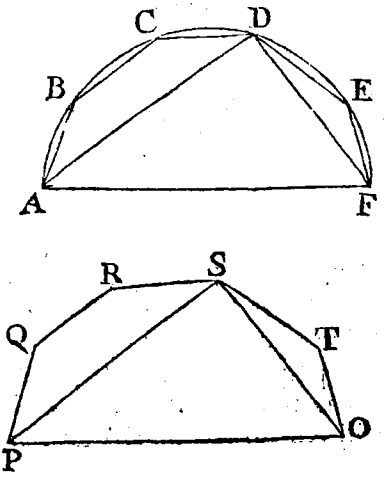


For, if ABCDE be the greatest possible, the triangle CDE must, manifestly, be greater than any other triangle CFE upon the same base, whereof the sum of the other sides is also the same. But, *by Theorem* V. the greatest triangle, when the base and the sum of the sides are given, is that whose sides are equal: Therefore DC and ED are equal. In the same manner it appears that BC = CD, *&c. Q. E. D.*

THEO-

THEOREM XI.

If all the sides of a polygon, except one, be given in length, and their position be required, so as to make the polygon itself the greatest possible; I say, their position must be such, that two lines drawn from the extremes of the unknown side to any angle of the polygon, shall form a right-angle.



For, if you would have the polygon ABCDEF to be the greatest possible, and yet ADF, subtended by the unknown side AF, not a right-angle: Then let PSO be a right-angle, contained under PS = AD, and OS = FD; and upon PS and OS, let the figures PSRQ and OST be described equilateral, and equal, to ADCB and FDE[s]. [s] 10. 6.

The triangle PSO is greater than ADF[t]; therefore, PSRQ being = ADCB, and OST = FDE[u], the whole polygon PQRSTO is also greater than the whole polygon ABCDEF[w], *which is repugnant.* [t] Theor. 4. [u] Constr. [w] Ax. 6, 1.

COROLLARY.

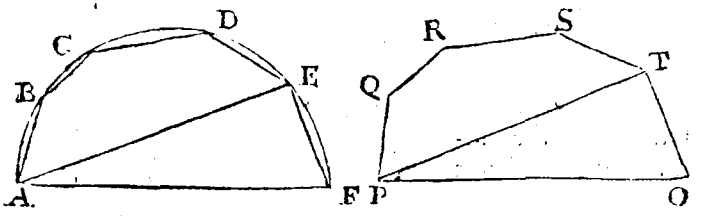
Hence, because the angle in a semi-circle is a right-angle[x]; it appears, that the greatest polygon that can be contained under any proposed number of given sides, and one other side any how taken, will be, when *it* may be inscribed in a semicircle, whereof the indetermined line will be the diameter. [x] 13. 3.

THEO-

THEOREM XII.

A polygon ABCDEA *in a circle, is greater than any other polygon* PQRSTP, *whatever, whose sides are the same both in length and number.*

Let AF be the diameter of the circle, and join E, F; also make the angle PTO = AEF, TO = EF, and let PO be drawn.

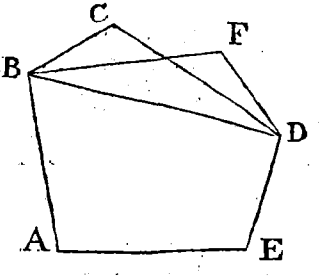


Because AB = PQ, BC = QR, CD = RS, DE = ST, and EF = TO[y], the polygon ABCDEF, being inscribed in a semi-circle, will be greater than the polygon PQRSTO[z]; and, if from these, the equal triangles AEF, PTO be taken away, there will remain AECDEA ⊏ PQRSTP. *Q. E. D.*

[y] Hyp.

[z] Theor. 11.

That the magnitude of the greatest polygon, which can be contained under any number of unequal sides, does not at all depend upon the order in which those lines are connected to each other, will appear, thus. Let ABCDE be the greatest, one way, or according to one order of the sides; and upon BD let a triangle BDF be constituted whose sides DF and BF are, respectively, equal to BC and DC; then, the triangles BCD, BFD being equal, the whole



whole polygons ABCDE and ABFDE will likewise be equal, notwithstanding their equal sides BC, DF, *&c.* are placed according to different orders.

THEOREM XIII.

Of all polygons, contained under the same perimeter, and number of sides; that whose sides, and angles, are equal, is the greatest.

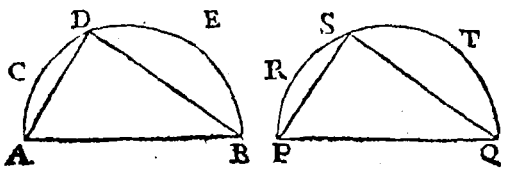
For, the greatest polygon that can be contained under a given perimeter, is one whose sides are all equal [a]. But, of all the polygons of this sort, that is the greatest which may be inscribed in a circle [b]: Therefore the greatest of all, is that whose sides are all equal, and which may be inscribed in a circle, or whereof the angles, as well as the sides, are all equal. *Q. E. D.*

[a] Theor. 10.
[b] Theor. 12.

THEOREM XIV.

The greatest area that can possibly be contained by one right-line, any how taken, and any other line or lines, whatever, whereof the sum is given; will be, when two right-lines drawn from the extremes of the unknown line first mentioned, to meet any where in the given boundary, make right-angles with each other.

For, if you would have the area ACDEBA, contained by some right-line AB, and ACDEB whereof the length is given, to be the greatest possible, and ADB, at the same time, not a right-angle: Then, let PSQ be a right-angle, contained under PS = AD, and QS = BD; and, having joined



PQ,

PQ, upon PS and QS conceive two figures PRS and QST to be formed, equal, and alike in all respects to ACD and DBE. Since the area PSQ is greater than ADB [c]; it is manifest, that the area PRSTQP, contained by the right-line (PQ) and PRSTQ (= ACDEB [d]) will also be greater than the area ACDEBA [e], *which is repugnant:* Therefore the area ACDEBA cannot be the greatest possible, unless the angle ADB be a right one. *Q. E. D.*

[c] Theor 4.
[d] Hyp.
[e] Ax. 6.

COROLLARY.

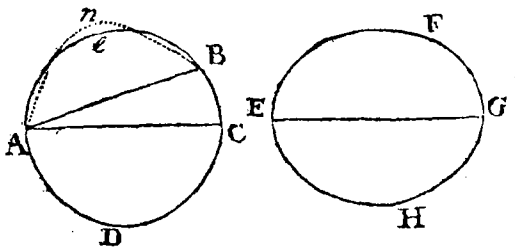
Hence, because the angle in a semi-circle is a right-angle [f], it is evident that the area will be the greatest possible, when the given length, or boundary, forms the arch of a semi-circle; whereof the indetermined right-line proposed is the diameter.

[f] 13. 3.

THEOREM XV.

Of all plane figures ABCD, EFGH, *contained under equal perimeters (or limits), the circle* (ABCD) *is the greatest.*

For, if the diameter AC be drawn, and EFG be taken equal to the arch ABC; then the area ABCA will *(by the precedent)* be greater than the



area EFGE, contained by EFG and the right-line EG; and ADCA will also be greater than EHGE: Therefore ABCD must, necessarily, be greater than EFGH. *Q. E. D.*

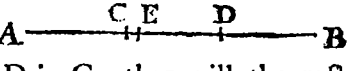
COROL-

COROLLARY.

Hence it appears, that the greateſt area that can poſſibly be contained by a right-line AB, and a curve-line A*e*B, both given in length; will be, when the latter is an arch of a circle. For, let A*n*B be any other curve-line, equal to A*e*B, and let the whole circle A*e*BCD be completed; which will (it is proved) be greater than the mixed figure A*n*BCD; and conſequently, by taking away the common ſegment ABCD, there will remain A*e*BA greater than A*n*BA.

THEOREM XVI.

The greateſt parallelepipedon that can be contained under the three parts of a given line AB, *any how taken, will be when all the parts are equal to each other.*



For, *if poſſible*, let two parts AE, ED be unequal. Biſect AD in C; then will the rectangle under AE (AC + CE) and ED (AC — CE) be leſs than AC^2 (or AC × CD) by the ſquare of CE^e. Therefore the ſolid AE × ED × DB will alſo be leſs than the ſolid AC × CD × DB [h]; *which is contrary to hypotheſis.* [e] 7. 2. [h] 21. 7.

COROLLARY.

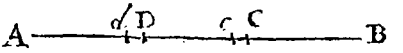
Hence, of all rectangular parallelepipedons, having the ſum of their three dimenſions the ſame, the cube is the greateſt.

THEO-

THEOREM XVII.

The greatest parallelepipedon $AC^2 \times CB$ *that can possibly be contained under the square of one part* AC *of a given line* AB *and the other part* AC, *any how taken; will be, when the former part is the double of the latter.*

For, let A*c* and B*c* be any other parts, into which the given line AB may be divided; and let AC and A*c* be bisected in D and *d*. So shall $AC^2 \times CB =$ $4AD \times DC \times CB$ [i] $= 4Ad \times dc \times cB$ [k] $(Ac^2 \times cB$ [l]$)$ *by the precedent. Q. E. D.*

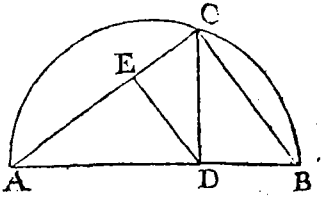


[i] Cor. to 6. 2. and 7.
[k] Cor. to 6. 2.

THEOREM XVIII.

The hypothenuse AB *of a right angled triangle* ABC *being given; the solid* $BC \times AC^2$ *contained under one leg* BC *and the square of the other* AC, *will be the greatest possible, when the square of the latter leg* AC *is double to that of the former* BC.

For, if CD be conceived perpendicular to AB, and DE to AC; it will be $AC^2 (AB \times AD$ [l]$)$: $AB^2 :: AD : AB$ [m] $:: DE : BC$ [n]; and consequently $AC^2 \times BC = AB^2 \times DE$ [o]; which (as AB^2 is given) will, evidently, be the greatest possible, when DE, or its square [p] (DE^2) is the greatest possible. But $DE^2 : AD^2 ::$ [q] $BC^2 (BD \times AB$ [l]$)$: $AB^2 :: BD : AB$ [m]; and therefore $DE^2 \times AB = AD^2 \times BD$ [o]; which (and consequently DE^2) will be the greatest possible, when AD is the double of BD [r]; that is, when AC^2 $(AD \times AB)$ is the double of BC^2 $(BD \times AB)$. *Q. E. D.*

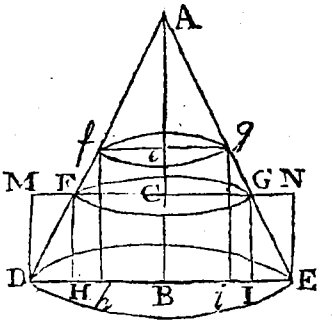


[l] Cor. to 19. 4.
[m] 21. 7.
[n] 14. 4.
[o] 23. 7.
[p] Cor. 2. to 6. 2.
[q] Cor. to 11. 4.
[r] Theor. 17.

THEO-

THEOREM XIX.

The altitude BC *of the greatest cylinder* HG *that can possibly be inscribed in any cone* ADE, *is one-third part of the altitude* AB *of the cone, and the cylinder itself* $\frac{4}{9}$ *parts of the cone.*



For, let *gh* be any other cylinder inscribed in the cone; and it will be, $AC^2 \times BC : CG^2 \times BC :: AC^2 :$ [s] $CG^2 :: Ac^2 : cg^2$ [t] $:: Ac^2 \times Bc :$ [s] $cg^2 \times Bc$; whence, *by alternation*, $AC^2 \times BC : Ac^2 \times Bc :: CG^2 \times BC : cg^2 \times Bc$: and so likewise is the cylinder HG to the cylinder [u] *hg*; but $AC^2 \times BC$ is greater [w] than $Ac^2 \times Bc$; therefore HG is also greater than *hg*. Again, since $AC = \frac{2}{3}AB$, and therefore $CG = \frac{2}{3}BE$; we also have, cylinder HG : cone ADE (or cylinder [y] DN) $:: CG^2 (\frac{4}{9}BE^2) :$ [z] $BE^2 :: \frac{4}{9} : 1$. *Q. E. D.*

[s] 22. 7.
[t] Cor. 11. 4.
[u] 3. and 5. of 8.
[w] Theor. 17.
[x] Ax. 4.
[y] Cor. 3. to 8. 8.
[z] 21. 7. and 3. 8.

SCHOLIUM.

From this proposition, by reasoning as in the Scholium to Theorem VIII. it will appear, that the least cone that can be described about, and the greatest cylinder that can possibly be described in, any solid generated by the rotation of a curve, concave to its axis, will be, when the sub-tangent is two-thirds of the altitude of the cone, or twice the altitude of the cylinder; and that the two figures will be in the ratio of nine to four. From whence the dimensions of the greatest and least cylinders and cones, that can be described in, and about solids generated by curves, to which the method of drawing tangents is known, may be readily determined.

THE

CONSTRUCTION

Of a great VARIETY of

Geometrical Problems.

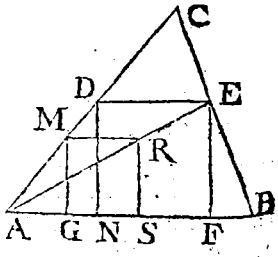
Being a farther

APPLICATION of what has been delivered in the Elementary Part of this Work.

PROBLEM I.

In a given triangle ABC, *to inſcribe a ſquare* DEFN.

CONSTRUCTION.



FROM any point M, in either ſide, upon the baſe AB, let fall the perpendicular MG; make MR perpendicular, and equal, thereto, and let ARE be drawn, meeting the other ſide of the triangle in E; then draw ED parallel, and EF and DN perpendicular, to AB; *and the thing is done.*

DE-

DEMONSTRATION.

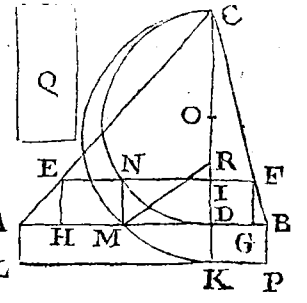
Let RS be drawn parallel to EF: Then *(by similar triangles)* RS (MG) : EF (: : AR : AE) : : MR : DE: Therefore, as MG and MR are equal, *by Construction*, EF and DE will likewise be equal.

By the same method a rectangle may be inscribed in a triangle, whose sides shall be in a given ratio; if MR and MG (instead of being equal) be taken in the given ratio; the rest of the Construction being exactly the same.

PROBLEM II.

In a given triangle ABC, *to inscribe a rectangle* EFGH *equal to any given right-lined figure* Q, *not exceeding half the triangle.*

CONSTRUCTION.



On the base AB *(by* 7. 6.) let a rectangle ABPL be constituted = Q; and let LP meet the perpendicular CD of the triangle (produced) in K. Then *(by* 17. 5.) let CD be divided in I, so that CI × DI = CD × DK (that is, let two semi-circles be described on CD and CK; drawing MN and NI parallel to CD and AB): So shall DI be the altitude of the required rectangle.

DEMONSTRATION.

Since *(by Constr.)* CI × DI (= NI^2 = MD^2) = CD × DK, thence will DI : DK : : CD : CI : : AB : EF *(by* 20. 5.); and consequently DI × EF *(by* 10. 4.) = AB × DK = Q. *Q. E. D.*

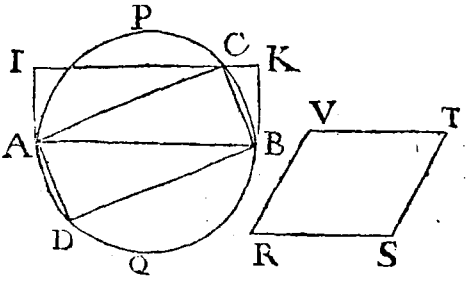
 That

That the Problem will be impossible, when Q is greater than half the triangle, is evident from the Construction, as well as from the Theorem on *p.* 200. It may also be observed, that there is another way, besides that used above, for dividing CD in the manner proposed; which (though not more obvious) is, in point of conciseness, rather preferable; and is thus. Having *(as before)* found a mean-proportional DM between CD and DK, and bisected CD in O; from M to CD apply MR = OD, and take OI = RD. So shall CI × DI = $OD^2 - OI^2$ *(by* 7. 2.) = $MR^2 - RD^2$ *(by Hyp.)* = DM^2 = CD × DK *(as before).*

PROBLEM III.

In a given circle APBQ, *to inscribe a rectangle equal to a given right-lined figure* RSTU, *not exceeding half the square of the diameter.*

CONSTRUCTION.



Upon the diameter AB describe the rectangle ABKI = RSTU *(by* 7. 6.); and from the point C, where the side KI intersects the periphery of the circle, draw CA and CB, parallel to which draw BD and AD; then will ACBD be the rectangle that was to be constructed.

DE-

DEMONSTRATION.

The lines AC, BD, and AD, BC being parallel *(by Constr.)* and the angle ACB a right-one *(by* 13. 3.) the figure ACBD is a rectangle *(by Cor. to* 24. 1.) and D is also in the circumference of the circle. But $ACBD = 2ACB = ABKI = RSTU$.

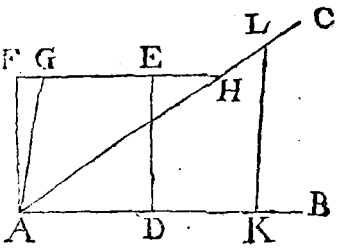
That the Problem will be impossible, when BK is greater than $\frac{1}{2}AB$, or when BI (RT) is greater than $\frac{1}{2}AB^2$, is manifest from hence, because KI will then fall intirely above the circle.

PROBLEM IV.

To draw a line KL *parallel to a given line* AG, *which shall terminate in two other lines* AB, AC, *given by position, so as to form with them a triangle* AKL, *equal to a given rectangle* ADEF.

CONSTRUCTION.

Let FE, produced, meet AG and AC, in G and H; and, in AB, take a mean-proportional AK between GH and 2EF; then draw KL parallel to AG, *and the thing is done.*



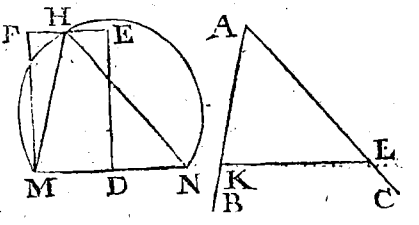
DEMONSTRATION.

The triangles AKL, HGA being equiangular, it will be $AKL : HGA :: AK^2$ ($= GH \times 2EF$, *by Constr.*) $: GH^2 :: EF : \frac{1}{2}GH$ *(by* 7. 4.) $:: EF \times AF : \frac{1}{2}GH \times AF$ ($= HGA$): Therefore, the consequents being equal, the antecedents AKL and $EF \times AF$ must also be equal. *Q. E. D.*

PROBLEM V.

Between two lines AB, AC, *given by position, to apply a line* KL, *equal to a given line* MN, *so that the triangle* AKL *formed from thence, shall be of a given magnitude.*

CONSTRUCTION.

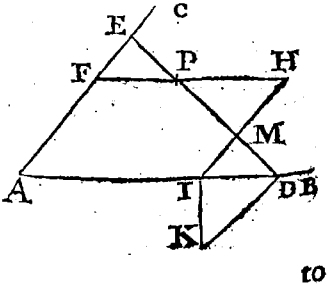


Having bisected MN in D, on MD describe a rectangle MDEF (*by* 7. 6.) = the magnitude given: Also on MN let a segment of a circle be described (*by* 22. 5.) to contain an angle = A; and from its intersection with EF, draw HM and HN; then make AK = HM, AL = HN, and join K, L. So shall the base KL be also = the base MN, and the triangle AKL equal, and like in all respects, to HMN; which last (*by Cor. to* 2. 2.) is, manifestly, equal to the magnitude given MDEF. *Q. E. D.*

PROBLEM VI.

Through a given point P, *to draw a line* EPD *to meet two lines* AB, AC, *given by position, so that the triangle* ADE *formed from thence, shall be of a given magnitude.*

CONSTRUCTION.



Draw FPH parallel to AB, intersecting AC in F; and on AF, let a parallelogram AFHI be formed, equal to the given area of the triangle: Make IK perpendicular to AI, and equal

6. 6

to

to FP; and from K, to AB, apply KD = PH; then draw DPE, *and the thing is done.*

DEMONSTRATION.

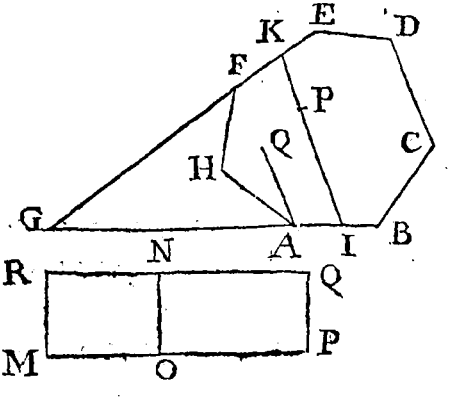
The triangles PHM, PFE and MDI, by reason of the parallel lines, are similar; and therefore, since the three homologous sides PH (KD), FP (IK) and DI are such, as to form a right-angled triangle (*by Constr.*), the triangle PHM on the first of them, is equal to both the other two FPE and MDI (*by* 29. 4.): and, if to these equal quantities, AFPMI be added; then will AFHI be also = ADE.

This Problem will be impossible, when KD (PH) is less than KI (PF); that is, when the area given is less than a parallelogram under AF and 2FP.

PROBLEM VII.

From a given polygon ABCDEFH, *to cut off a part* AIKFH, *equal to a given rectangle* MN, *by a line* (IK), *either parallel to a given line* AQ, *or passing through a given point* P.

CONSTRUCTION.



Let BA and EF be produced to meet in G; and upon ON let a rectangle OQ be constituted (*by* 7. 6.) equal to AGFH; then, by Prob. the 4th, or 6th, according to the case proposed,

 draw

draw IK, so as to make the triangle GKI = MQ, *and the thing is done.*

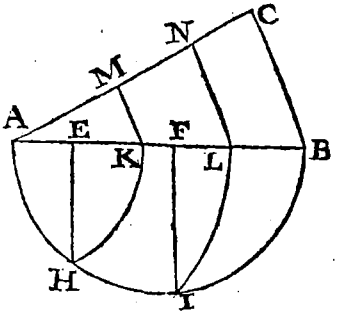
The Demonstration whereof is manifest from the Construction.

And, in the same manner, the polygon may be divided according to any given ratio; because, the whole being given, each part will be given.

PROBLEM VIII.

To divide a given triangle ABC *into any proposed number of parts* (AKM, KN, LC) *so as to have any given proportion to each other; by means of lines drawn parallel to one of the sides* BC *of the triangle.*

CONSTRUCTION.



Let AB be divided into parts, AE, EF, FB, having the same given proportion to each other, as the parts of the triangle are to have. Upon AB let a semi-circle AHIB be described; and perpendicular to AB, draw EH, FI, meeting the circumference in H and I: From the center A, through H and I, describe the arcs HK, IL, meeting AB in K and L; then draw KM and LN parallel to BC, *and the thing is done.*

DEMONSTRATION.

The triangles AKM, ALN and ABC, are in proportion, to one another, as AK^2 (AB × AE), AL^2 (AB × AF) and AB^2 (*by* 19. and 24. 4.); that is, as AE, AF and AB (*by* 7. 4.) Whence (*by division*) the proposition is manifest.

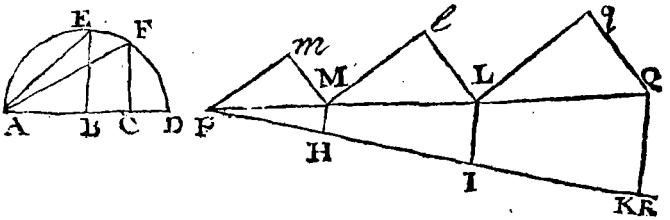
PRO-

PROBLEM IX.

To divide a given line PQ *into any proposed number of parts, so that similar right-lined figures* PMm, MLl, LQq *described upon them, shall have the same given ratio among themselves, as an equal number of right-lines* AB, AC, AD *assigned.*

CONSTRUCTION.

Upon the greatest AD of the given lines AD, AC, AB, describe a semi-circle AEFD; and perpendicular to AD, draw BE and CF, meeting the circumference in E and F; and, having drawn



PR, at pleasure, in it take PH = dist. AE, HI = dist. AF and IK = AD; draw KQ, and parallel thereto draw HM, IL; which will divide PQ, as required.

DEMONSTRATION.

PMm : LQq : : PM^2 : LQ^2 (*by* 26. 4.) : : PH^2 (= AE^2 = AB × AD, *by* 19. 4.) : IK^2 (AD^2) : : AB : AD (*by* 7. 6.) In the very same manner it appears, that MLl : LQq : : AC : AD. *Q. E. D.*

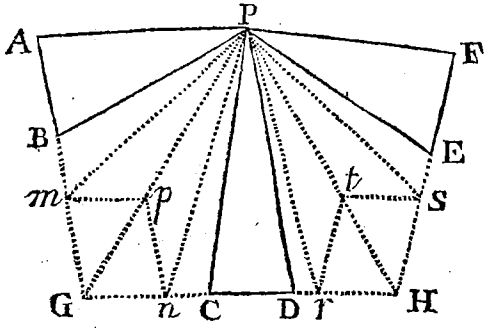
PRO-

PROBLEM X.

To determine the position of a point P, *so that lines drawn from thence to the extremes of three right-lines* AB, CD, EF, *given in length and position, shall form three triangles* ABP, CPD, EPF, *mutually equal to each other.*

CONSTRUCTION.

Let the given lines be produced to meet in G and H; in which take G*m* = AB, H*s* = EF, and



G*n*, H*r*, equal each to CD: Complete the parallelograms G*mpn*, H*rts*; and let the diagonals G*p* and H*t* be produced till they meet in P, *and the thing is done.*

DEMONSTRATION.

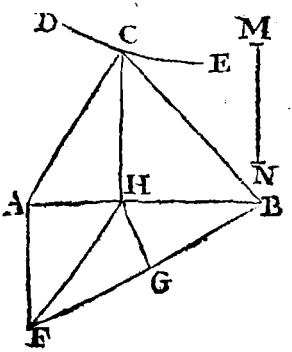
Let PA, PB, P*m*, &c. be drawn. The triangles GP*n*, GP*m*, having the same base GP, and equal altitudes (because GM*pn* is a parallelogram) are therefore equal to each other: But CPD = GP*n*, and APB = GP*m* (*by Cor. to* 2. 2.); whence CPD and APB are likewise equal. And, from the very same reasoning, it appears that CPD and FPE are equal. *Q. E. D.*

PRO-

PROBLEM XI.

From two given points A, B, *to draw two lines* AC, BC, *to meet in a line* DE *of any kind, given by position; so that the difference of their squares shall be equal to a square given* (MN^2).

CONSTRUCTION.



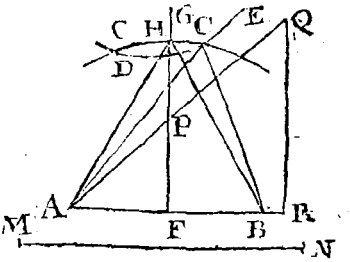
Make AF perpendicular to the line AB, and equal to MN; draw BF, which bisect with the perpendicular GH; and from its intersection with AB, draw HC perpendicular to AB, meeting DE in C; draw AC and BC, *and the thing is done.*

For $BC^2 - AC^2 = BH^2 - AH^2$ (*by* 9. 2.) $= FH^2 - AH^2 = AF^2 = MN^2$.

PROBLEM XII.

From two given points A, B, *to draw two lines* AC, BC, *to meet in a line* DE *of any kind, given by position; so that the sum of their squares shall be equal to a given square,* MN^2.

CONSTRUCTION.



Bisect AB with the perpendicular FG, in which take FP = FA, and draw APQ = MN; on AB (produced, if necessary) let fall the perpendicular QR; from A to FG apply AH = AR, and

about

about the center F, thro' H, let the circumference of a circle be described; and from its intersection C with the given line DE, draw CA and CB, *and the thing is done.*

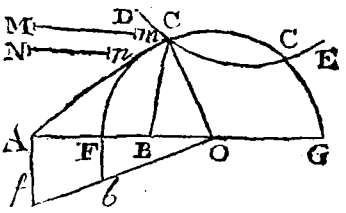
DEMONSTRATION.

Let BH be drawn; which being = AH = AR = RQ (because FP = AF); thence will $AC^2 + BC^2 = AH^2 + BH^2$ (*by* 20. 3.) $= AR^2 + RQ^2 = AQ^2 = MN^2$. *Q. E. D.*

PROBLEM XIII.

From two given points A, B, *to draw two lines* AC, BC, *meeting in a line* DE *of any kind, given by position; so as to obtain the ratio of two unequal right-lines* Mm, Nn *assigned.*

CONSTRUCTION.



Having joined the given points, divide AB in F (*by* 15. 5.) so that AF : BF : Mm : Nn; make A*f* and F*b* parallel to each other, taking the former = AF, and the latter = FB; and thro' their extremes draw *fb*O, meeting AB, produced, in O; from whence, with the radius OF, let the semi-circle FCG be described, cutting DE in C; then draw AC and BC, *and the thing is done.*

DEMONSTRATION.

Because OA : A*f* (AF) : : OF : F*b* (FB), we have (*by division*) OA : OF : : OF : OB; or OA : OC : : OC : OB. And so, the triangles OAC, OCB, having one angle FOC common, and the sides about it proportional, they must, therefore, be similar

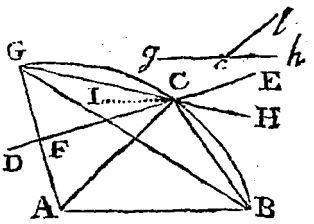
milar (*by* 15. 4.); whence the other ſides will alſo be proportional, or AC : BC : : OA : OC (OF) : : *Af* : *Fb* : : M*m* : N*n* (*by Conſtr.*) *Q. E. D.*

Note, When, in either of the two preceding Problems, the circle deſcribed, neither cuts nor touches the given line DE, the thing propoſed to be done, will be impoſſible; as no two lines drawn from A and B, to meet above the circumference, can poſſibly have their ratio, or the ſum of their ſquares, the ſame as two lines meeting in the circumference.

PROBLEM XIV.

From two given points A, B, *to draw two lines* AC, BC, *meeting in a right-line* DE, *given by poſition; ſo as to make therewith two angles* ACD, BCE, *whoſe difference ſhall be equal to an angle given,* bch.

CONSTRUCTION.



Make AFG perpendicular to DE, and FG = AG; and, having drawn GB, on it let a ſegment of a circle GCB be deſcribed (*by* 22. 5.) to contain an angle equal to the ſupplement (*bcg*) of the given one *bch*; and from its interſection (C) with DE, draw CA and CB; *and the thing is done.*

For, if BC and GCH be drawn, then will ACD = GCD = ECH = BCE — BCH (*bch*).

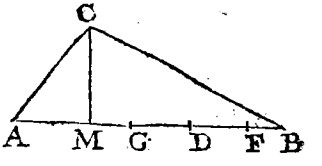
In the ſame manner, the Problem will be conſtructed, when, inſtead of the difference of ACD and BCE, *that* of ABC and BAC is given: Becauſe, when DE is parallel to AB, the latter difference

ference is equal to the former; and, in all other cases, differs from it by twice the given angle GCI, expressing the inclination of the said lines.—When the sum of the angles ACD and BCE is given, the angle ACB is also given: And here, nothing more is necessary, than barely to describe, upon AB, a segment of a circle to contain the said given angle ACB.

L E M M A.

If, of any three proportional lines AB, DB, FB, *the difference* AF *of the two extremes be bisected in* G; *and if on the greatest* AB, *as a base, a triangle* ABC *be so formed, that its lesser side* AC *shall be to the distance* MG *of the perpendicular from the bisecting point* G, *in the given ratio of* AB *to* DB; *then shall the greater side* BC *exceed the lesser* AC *by the given line* DB.

DEMONSTRATION.



Because FM exceeds AM by 2GM, BM will exceed *it* by 2GM + BF; and the rectangle under this excess and the whole base AB ($= 2MG \times AB + BF \times AB$) will therefore (*by* 9. 2.) be $= BC^2 - AC^2$. But (*by hypothesis and* 10. 4.) $MG \times AB = AC \times BD$, and $BF \times AB = BD^2$: Therefore $2AC \times BD + BD^2 = BC^2 - AC^2$; and, by adding AC^2, common, $AC^2 + 2AC \times BD + BD^2$ (or the square of $AC + BD$, *by* 6. 2.) will be $= BC^2$; and, consequently, $AC + BD = BC$. *Q. E. D.*

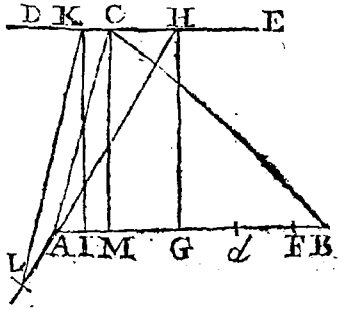
This *Lemma* is not only of use in the Problem next following, but will be found a ready instrument in the solution of many others; for which reason it is here put down.

PRO-

PROBLEM XV.

From two given points A, B, *to draw two lines* AC, BC *to meet in a right-line* DE, *given by position; so that their difference shall be equal to a given right-line* Bd.

CONSTRUCTION.



In AB, take a third-proportional BF to BA and B*d*; and, having bisected AF in G, take GI = B*d*; make GH and IK perpendicular to AB, meeting DE in H and K; and draw HAL, to which, from K, apply KL = AB; and parallel thereto draw AC, meeting DE in C; join B, C; *and the thing is done.*

DEMONSTRATION.

Let CM be perpendicular to AB.

Then, because of the parallel lines, it will be AC : KL (AB) : : HC : HK : : GM : GI (B*d*). Whence *(by alternation)* AC being to GM, as BA to B*d*; and AB, *d*B, FB being also proportionals; the whole Construction is manifest *from the Lemma* premised.

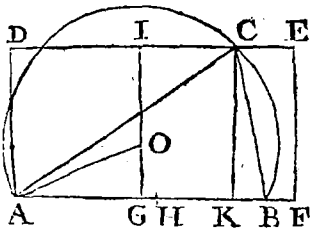
If the sum, instead of the difference, of the two lines (AC, BC) be given, the method of construction will be exactly the same, without the least alteration of any one step; provided that B*d* be first of all taken (in BA, produced) equal to the given sum, instead of the difference.

PRO-

PROBLEM XVI.

From two given points A, B, *to draw two lines* AC, BC, *so as to meet in a right-line* DE *parallel to that* (AB) *joining the said points, and that the rectangle* (AC × BC) *contained by them, shall be equal to a rectangle given,* ADEF.

CONSTRUCTION.



Bisect AB and AF, in G and H; and, having drawn GI perpendicular to AB, to it, from A, apply AO = AH; and from the center O, through A and B, let the circumference of a circle be described intersecting DE in C; from whence draw CA and CB, *and the thing is done.*

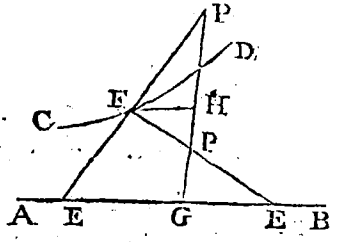
For, if CK be drawn perpendicular to AB, it is evident (*by* 25. 3.) that AC × BC = CK × 2AO = AD × AF. *Q. E. D.*

PROBLEM XVII.

Through a given point P, *so to draw a line* FPE, *that the parts thereof* PF, PE, *intercepted by that point and two lines* AB, CD, *given in position, shall obtain a given ratio.*

CON-

CONSTRUCTION.



Through P, to any point in AB, draw PG, in which (*by* 13. 5.) take PH to PG, in the given ratio of PF to PE; draw HF parallel to AB, meeting CD in F; then draw FPE, *and the thing is done.*

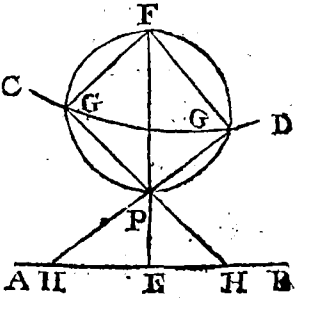
For, the triangles PGE, PHF being equiangular (*by* 3. *and* 7. 2.) thence is PF : PE : : PH : PG: And so PF and PE (as well as PH and PG) are in the ratio given.

In this Construction, it is necessary that one of the two given lines should be a *right-one:* The other may, it is manifest, be a line of *any kind* whatever.

PROBLEM XVIII.

Through a given point P, *to draw a line* GH, *terminating in a right-line* AB, *and in a line* CD *of any kind, given both by position; so that the rectangle under the two parts thereof* PG, PH, *shall be of a given magnitude.*

CONSTRUCTION.



Having drawn EPF perpendicular to AB, take therein PF (*by* 7. 6.) so that PE × PF = the magnitude given: Upon PF, as a diameter, let a circle be described, intersecting CD in G; then draw GPH, *and the thing is done.*

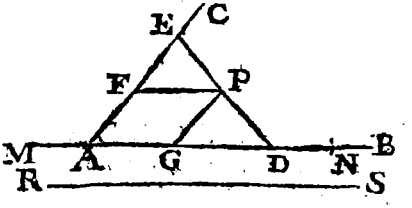
DEMONSTRATION.

If FG be drawn, the triangles PFG and PHE, having FPG = HPE, and FGP (= a right-angle, *by* 13. 3.) = PEH, will be equiangular: And, consequently (*by* 24. 3.) PG × PH = PF × PE = the magnitude given. *Q. E. D.*

PROBLEM XIX.

Through a given point P, *between two right-lines* AB, AC, *given by position, so to draw a line* ED, *that the sum of the segments* (AD + AE) *cut off by it from the two former, shall be equal to a given line* RS.

CONSTRUCTION.



Draw PF and PG parallel to AB and AC; in BA produced take AM = AF, and MN = RS; then divide GN in D (*by* 17. 5.) so that GD × ND = AM × AG; draw DPE, *and the thing is done.*

DEMONSTRATION.

By similar triangles, GD : GP (= AF = AM) :: FP (= AG) : FE; and consequently GD × FE = AM × AG = GD × ND (*by Constr.*): Whence FE = ND; and therefore FE + AF + AD (AE + AD) = ND + AM + AD = MN = RS. *Q. E. D.*

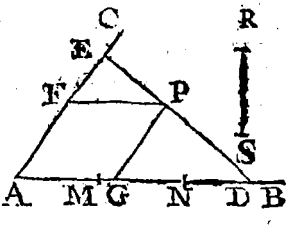
It appears from the Construction, that the Problem will be impossible, when (GN) the excess of RS above PF and PG, exceeds the double of a mean-proportional between these two quantities.

PRO-

PROBLEM XX.

Through a given point P, *between two right-lines* AB, AC, *given by position, so to draw a line* DE, *that the difference* (AD — AE) *of the segments cut off by it from the two former, shall be equal to a given line* RS.

CONSTRUCTION.



Draw PF and PG parallel to AB and AC; in AB take AM = AF, and MN = RS; then (*by* 18. 5.) let a line ND be added to GN, so that GD × ND, PG × PF; draw DPE, *and the thing is done.*

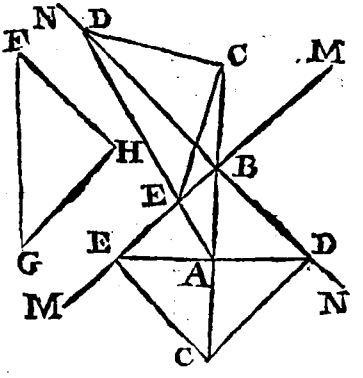
DEMONSTRATION.

Because of the similar triangles PGD, EFP, we have GD × FE = PG × PF = GD × ND (*by Constr.*) and consequently FE = ND; whence AD — AE = AN — AF (AM) = MN = RS. *Q.E.D.*

PROBLEM XXI.

Between two right-lines NBN, MBM, *given by position, to apply a line* DE *of a given length, and which (produced, if necessary) shall pass through a given point* A, *equally distant from the said given lines.*

CONSTRUCTION.



Let FG be the length given; and, having drawn AB, make the angles F, G equal each to ABD (or ABE); and in AB produced take AC (*by* 18. 5.) so that AC × BC = HG²: From C to NN apply CD = HG; then draw DAE (DEA), *and the thing is done.*

DEMONSTRATION.

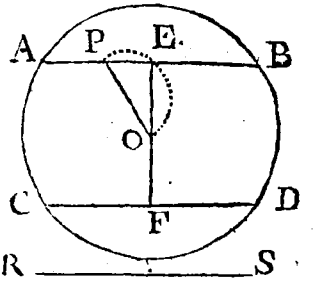
Because AC : CD : : CD : BC (*by Constr. and* 10. 4.) the triangles CAD and CDB (having one angle common, and the sides about it proportional) will be equiangular (*by* 15. 4.): And therefore, since CDA = DBC = ABE (*by hypothesis*), the circumference of a circle may be described through all the four points C, D, B, E (*by* 11. 3. *or* 19. 3.) And so the angle DEC, standing on the same subtense (DC) with DBC, will be equal to it; and, consequently, equal to EDC. Therefore the isosceles triangles EDC, FGH being equiangular, and having CD = HG, their bases ED and FG will also be equal. *Q. E. D.*

PROBLEM XXII.

In a given circle ABDC, *to apply a chord* AB *equal to a given line* RS (*less than the diameter*) *and which shall pass through a given point* P.

CON-

CONSTRUCTION.



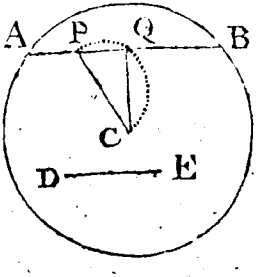
Inſcribe CD = RS, upon which, from the center O, let fall the perpendicular OF; alſo draw OP, upon which let a ſemi-circle be deſcribed, and in it apply OE = OF; laſtly, through P and E, let AB be drawn; which will, manifeſtly, be equal to CD (= RS, *by* 3. 3.) as being equally diſtant from the center, by conſtruction.

When the point given is placed without the circle, the conſtruction will be no-ways different.

PROBLEM XXIII.

Through a given point P, *ſo to draw a line* AB, *that the parts thereof* AP, BP, *intercepted by that point and the circumference of a given circle, ſhall have a given difference,* DE.

CONSTRUCTION.

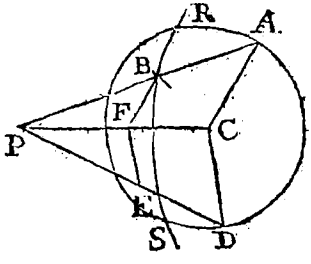


From the center C, draw CP, upon which let a ſemi-circle PQC be deſcribed, and in it apply PQ equal to half DE, producing the ſame, both ways, to meet the circumference in A and B: So ſhall AQ = BQ (*by* 2. 3); and therefore PB (= BQ + PQ = AQ + PQ = AP + 2PQ) = AP + DE, *which was to be done.*

PROBLEM XXIV.

From a given point P, *to the circumference of a given circle* C, *to draw a right-line* PBA, *so as to be divided in a given ratio by a line* RBS *of any kind, given in position.*

CONSTRUCTION.



To any point D in the circumference, draw PD, which divide at E in the ratio given; and, having drawn PC and the radius CD, parallel to the latter draw EF, meeting the former in F; from whence, to RS, apply FB = FE; then through B draw PA, *and the thing is done.*

DEMONSTRATION.

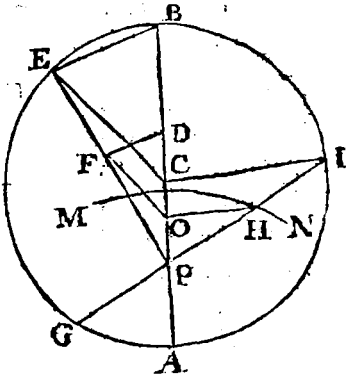
Let CA be drawn. Then, becauſe of the parallel lines CD, EF, it will be, CD (AC) : EF (FB) : : PC : PF; and ſo the triangles PAC, PBF will likewiſe be equiangular (*by* 16. 4.) And therefore PB : BA : : PF : FC : : PE : ED (*by* 13. 4.) *Q. E. D.*

PROBLEM XXV.

Through a given point P, *to draw a line* GH, *terminating in the circumference of a given circle* BGA, *and in a line* MN *of any kind, given by position; ſo that the rectangle under the two parts thereof* PG, PH, *ſhall be of a given magnitude.*

CON-

CONSTRUCTION.



Through P draw the diameter AB, in which let PD be so taken (*by* 7. 6.) that PA × PD = the magnitude given. From any point E in the circumference, draw EP and the radius EC; and, having joined BE, draw DF parallel to BE, and from its intersection with PE, draw FO parallel to EC, meeting PB in O; from whence, to MN, apply OH = OF; and through P draw HG for the line required.

DEMONSTRATION.

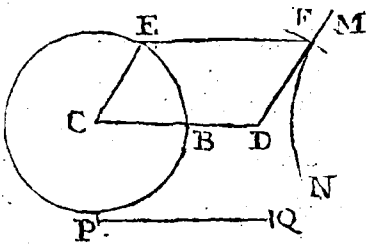
Let PH be produced (if necessary) to meet the circumference of the circle in I; and let C, I be joined.

The lines OF, CE being parallel, thence will PO : PC : : OF (OH) : CE (CI); and therefore OH and CI will likewise be parallels (*by* 16. 4.) Therefore PH : PI : : PO : PC : : PF : PE : : PD : PB; whence, alternately, PH : PD : : PI : PB : : PA : PG (*by* 21. 3. *and* 10. 4); and consequently PH × PG = PD × PA. *Q. E. D.*

PROBLEM XXVI.

From the circumference of a given circle C, *to a line* MN *of any kind, given by position, so to draw a right-line* EF, *as to be both equal and parallel to a given right-line* PQ.

CONSTRUCTION.

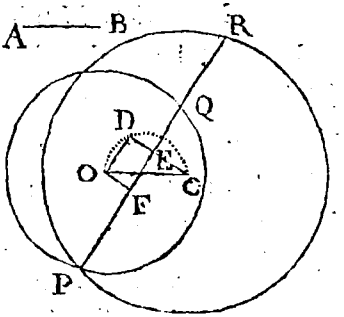


From the center C, draw CD equal and parallel to PQ; and, from D to MN, apply DF = the radius CB; draw CE equal and parallel to DF; then, EF being drawn, it will (*by* 26. 1.) be equal, and parallel to CD; *which was to be done.*

PROBLEM XXVII.

From the point of intersection P *of two given circles* O *and* C, *so to draw a line* PR, *that the part thereof* QR *intercepted by the two peripheries, shall be equal to a given line* AB.

CONSTRUCTION.



Upon the line OC joining the two centers, let a semi-circle ODC be described, in which apply OD = $\frac{1}{2}$AB; and parallel thereto, draw PR, *and the thing is done.*

DEMONSTRATION.

Let CD, and OF parallel to CD, be drawn, meeting PR in E and F. Then, the angle ODC being a right-one (*by* 13. 3.) and PR parallel to DO, E and F will also be right-angles, and EF = DO (*by* 24. 1): And so, PE — PF being (= DO) = $\frac{1}{2}$AB, it is manifest, that 2PE (PR) — 2PF (PQ) = AB, or that QR = AB. *Q. E. D.*

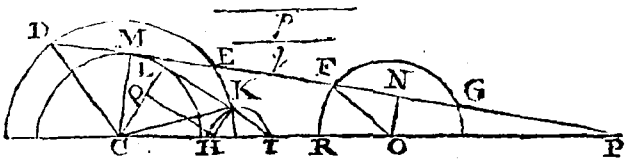
PRO-

PROBLEM XXVIII.

From a given point P, *in the line passing through the centers of two given circles* C *and* O; *so to draw a line* PD, *that the parts thereof* DE, FG, *intercepted by those circles, shall be in a given ratio* (viz. *as* p *is to* q).

CONSTRUCTION.

Take CH to OR, as *p* to *q*; and CI to OR, as PC to PO: Upon HI let a semi-circle be described, intersecting the circle CDE in K; through



which point draw IKL, and make CL perpendicular thereto; at which distance, from the center C, let a circle ML be described: Then, if from P a line PD be drawn to touch this circle, *the thing is done.*

DEMONSTRATION.

Let CD, CK, HK, FO be drawn, and also CM and ON, perpendicular to PD, and HQ to CL.

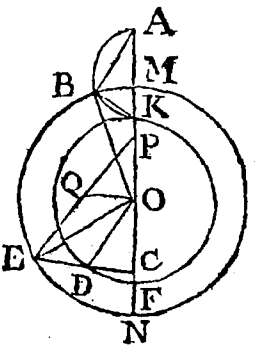
The right-angled triangles CDM, CKL, having CD = CK, and CM = CL, have also DM = LK = QH (since, *by* 13. 3. the angle HKI, as well as L, is a right-one). Moreover *(by Construction)* CI : OR (OF) : : PC : PO : : CM (CL) : ON; and so the triangles CIL, OFN (having their sides proportional) must be similar; and consequently OFN also similar to CHQ: whence, as QH (or DM) : FN : : CH : FO (OR) : : *p* : *q* *(by Constr.)* *Q. E. D.*

PRO-

PROBLEM XXIX.

To draw a line EC, *to make given angles with a line* MN *passing through the common center* O *of two given circles* MEN, KDF, *so that the parts thereof* CD, ED, *intercepted by that line and the two peripheries, shall obtain a given ratio.*

CONSTRUCTION.



Let QOM be the given angle, to which ECM shall be equal: In OM produced, let KA be so taken, as to be to the radius OK in the given ratio of ED to CD; upon which let a segment of a circle ABK be described to contain an angle equal to QOM; and, from its intersection with the circle MEN, draw BA; parallel to which, draw the radius OD; and then, through D, draw EC parallel to QO.

DEMONSTRATION.

Let BK, BO, EO be drawn, and also EP parallel to DO, meeting OQ and OM in Q and P.

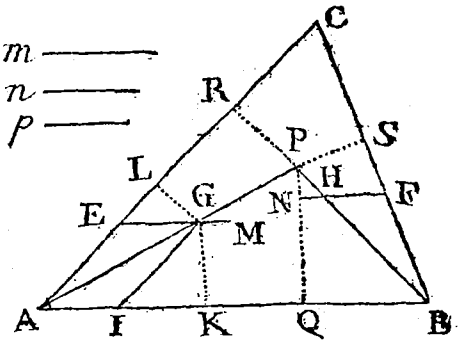
Then, the triangles AKB, POQ, having ABK = POQ *(by Construction)*, and KAB = OPQ *(by* 7. 1.) have their external angles OKB, EQO also equal *(by* 9. 1): Therefore, because EQ (= OD) = OK, and EO = BO, thence is OQ = KB *(by* 17. 1); and therefore, as the triangles POQ, ABK are equiangular, PQ is also = AK: But (because of the parallel lines) CD : ED :: DO (OK) : PQ (AK). *Q. E. D.*

PRO-

PROBLEM XXX.

To determine a point P, *so that three perpendiculars drawn from thence, to as many right-lines* AB, AC, BC, *given by position, shall obtain the ratio of three given lines* m, n *and* p, *respectively.*

CONSTRUCTION.



Take AE and BF each $= m$, and draw EM and FN parallel to AB; in which take EG $= n$, and FH $= p$; thro' G and H draw AP and BP, and the point of concourse P, *will be that which was to be determined.*

DEMONSTRATION.

Upon the sides of the triangle ABC let fall the perpendiculars GL, GK, PR, PQ, PS; and let GI, parallel to AC, be drawn.

The angle LEG being = LAK = GIK (*Cor.* 1. *to* 7. 1.) and L and K being both right-angles (*by Constr.*), the triangles EGL and GIK are similar; and therefore IG (m) : EG (n) : : GK : GL : : PQ : PR (*by* 21. 4). And, in the very same manner, $m : p$: : PQ : PS. *Q. E. D.*

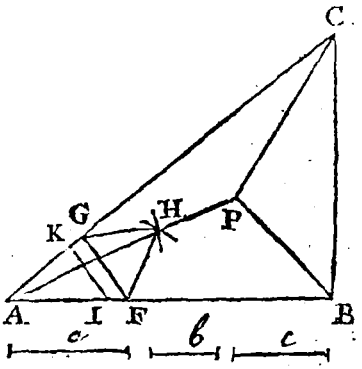
After the same way, the Problem may be constructed, when the lines drawn from P are required to make any given angles with the lines upon which they fall.

PRO-

PROBLEM XXXI.

To determine a point P, *so that three lines* PA, PB, PC, *drawn from thence to three given points* A, B, C, *shall obtain the ratio of three given lines* a, b *and* c, *respectively.*

CONSTRUCTION.



Having joined the given points, in AB take AF = *a*, and AI = *c*; make the angles AFG and AIK equal, each, to ACB; and from the centers F and G, with the radii *b* and AK, let two arcs be described, intersecting in H; from which point draw HF and HA; then draw BP to make the angle ABP = AHF, and it will meet AH (produced) in the point P, required.

DEMONSTRATION.

Let BP, CP, and GH be drawn.

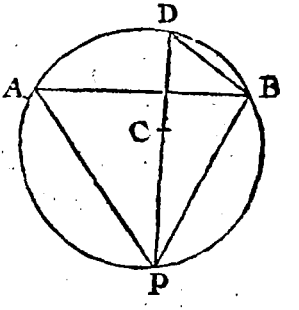
The triangles ABP, AHF being equiangular, it will be, AP : BP : : AF (*a*) : FH (*b*); also AB : AP : : AH : AF; and AB : AC : : AG : AF (because ABC and AGF are likewise equiangular, *(by Constr.)*. Now, seeing the extremes of the two last proportions are the same, the four means AP, AC, AG, AH *(by* 10. 4.) will therefore be proportionals; and so, the triangles ACP, AHG being equiangular *(by* 15. 4.) it will be AP : CP : : AG : GH (AK) : : FA (*a*) : AI (*c*). *Q. E. D.*

PRO-

PROBLEM XXXII.

To determine the position of a point P, *at which, lines drawn from three given points* A, B, C, *shall make given angles, one with another.*

CONSTRUCTION.



Draw AB, upon which (*by* 22. 5.) let a segment of a circle APB be described, capable of containing the given angle which the lines drawn from A and B are to include; and let the whole circle be completed; make the angle ABD equal to that which the lines drawn from A and C are to include; and from the point D, where BD meets the circumference, through C, let DP be drawn, meeting the circumference in P; *which is the point required.*

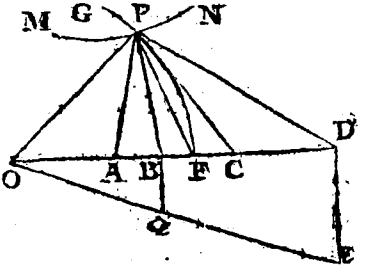
For, AP and BP being drawn, the angle APC will be = the given angle ABD (*by* 11. 3.) both standing on the same arch AD: And APB is also of the given magnitude by construction.

PROBLEM XXXIII.

Any two unequal segments AB, CD *of a right-line* AD *being given, as well in position as length; to determine a point* P *in a line* MN *of any kind, given by position; at which the two angles* APB, CPD, *subtended by those segments, shall (if possible) be equal to each other.*

CON-

CONSTRUCTION.



Make B*a* and D*c* parallel to each other, taking the former = BA, and the latter = DC; and thro' their extremes draw *ca*O, meeting DA produced, in O: Take OF a mean-proportional between OB and OC; and from the center O, with the interval OF, let a circle FG be described; which (when the Problem is possible) will cut (or touch) MN, and the point of intersection P, will be *that* required.

DEMONSTRATION.

Let PO, PA, PB, PF, PC, and PD be drawn.

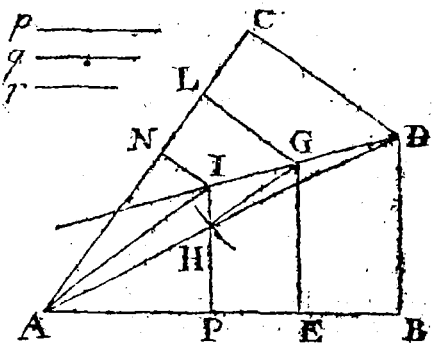
Because OD : D*c* (DC) : : OB : B*a* (BA), it will be *(by division)* OD : OC : : OB : OA; and consequently OD × OA = OC × OB = OP^2 *(by Constr.)*. Therefore, seeing that OA : OP : : OP : OD, and that the angle O is common to both the triangles OAP, OPD, these triangles must be equiangular *(by* 15. 4.) and consequently APF = OPF (OFP) — OPA (ODP) = DPF *(by* 9. 1). In the very same manner, because OB : OP : : OP : OC, the angle BPF will be = CPF; and, consequently, APB also = CPD. *Q. E. D.*

PRO-

PROBLEM XXXIV.

Two right-lines AB, AC *being given, both in length and position; from the point of their concourse* A, *so to draw another line* AI, *that two perpendiculars* IP, IN, *falling from the extreme thereof upon the two given lines, shall cut off alternate segments* BP, CN *in a given ratio each to the line* AI *so drawn.*

CONSTRUCTION.



Let the given ratio of AI, BP and CN be that of the lines p, q and r, respectively. Take BE $= q$, and CL $= r$; making BD, EG, CD and LG perpendicular to AB and AC, so as to meet in D and G: Draw DA and DG, and from G to AD, apply GH $= p$, and parallel thereto draw AI, meeting DG produced (if needful) in I, *and the thing is done.*

DEMONSTRATION.

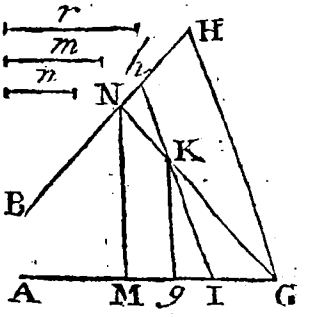
Because of the parallel lines, AI : GH (: : ID : GD) : : BP : BE; whence, alternately, AI : BP : : GH (p) : BE (q). In the very same manner AI : CN : : p : r. *Q. E. D.*

PRO-

PROBLEM XXXV.

Between two lines AG, BH, *given both in position and length, to draw a line* MN, *which shall be in a given ratio to each of the segments* MG, NH *cut off from the two given lines.*

CONSTRUCTION.



Let the given ratio of MN, MG and NH be that of r, m and n, respectively: In AG and BH take $Gg = m$, and $Hh = n$; and, having drawn GH, parallel to it draw hI; to which, from g, apply $gK = r$; draw GKN, meeting BH in N, and parallel to Kg draw MN, *and the thing is done.*

DEMONSTRATION.

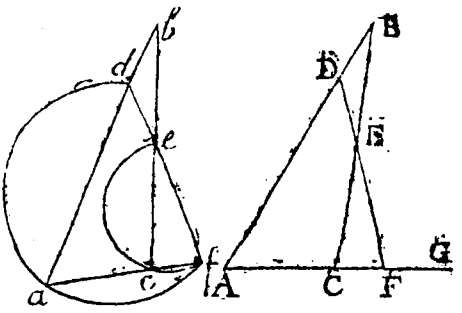
Because of the parallel lines, it will therefore be MN : GM : : gK (r) : Gg (m); and likewise MN : gK (r) : : GN : GK : : NH : Hh (n); which last *(by alternation)* is MN : NH : : r : n. *Q. E. D.*

PRO-

PROBLEM XXXVI.

To draw a line DF, *to cut three other lines* AB, AG, BC, *given by position; so that the two parts thereof* DE, EF, *intercepted by those lines, shall be respectively equal to two given lines,* de; ef.

CONSTRUCTION.



Upon the right-lines *df*, *ef* let two segments of circles, *daf*, *ecf*, be described, to contain angles, respectively equal to BAG and BCG: Then (*by Prob.* 27.) draw *fa*, so that the part *ac*, intercepted by the two peripheries, shall be equal to AC; join *a*, *d*, and take AF = *af*, AD = *ad*; then draw DF, *and the thing is done.*

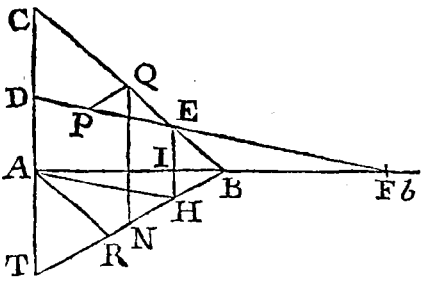
DEMONSTRATION.

The triangles FAD, *fad*, having AF = *af*, AD = *ad*, and A = *a* (*by Construction*), are equal in all respects; and therefore (if *fe* be drawn) the triangles FCE, *fce*, having F = *f*, FCE = *fce*, and CF = *ef*, will also be equal and alike: Therefore, seeing the wholes DF, *df*, and the parts FE, *fe* are equal, the remaining parts DE and *de* must likewise be equal. *Q.E.D.*

PROBLEM XXXVII.

Through a given point P, *to draw a line* DPF, *to cut three lines* AC, CB, ABb, *given by position; so that the parts thereof* DE, EF *intercepted by those lines, shall obtain a given ratio.*

CONSTRUCTION.



In CA produced, take AT to CA in the given ratio of DE to EF; and, having joined B, T, draw PQ parallel thereto; and from its intersection with BC, draw QN, parallel to CT; also draw AR parallel to BC: And in RB take RH (*by* 18. 5.) so that NH × RH = PQ × TR: Then draw AH, and DPEF parallel to it, *and the thing is done.*

DEMONSTRATION.

Because (*by Constr. and* 10. 4.) NH : TR : : PQ : RH : : QE : AR (*by* 14. 4.) it follows (*by alternation*) that NH : QE : : TR : AR : : TB : BC (*by* 12. 4.) : : BN : BQ: Therefore EH (when drawn) will be parallel to QN or AD (*by Cor. to* 12. 4); and so DEHA being a parallelogram, we have again (by similar triangles) as DE (AH) : EF : : IH : IE : : AT : AC. *Q. E. D.*

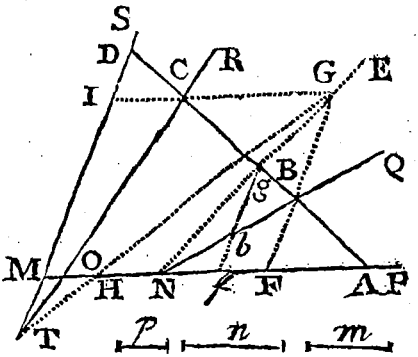
When the segments AD, BE, cut off from the given lines AC, BC, are required to be in a given ratio (instead of DE and EF), the construction will be

be the ſame; provided that CT be taken to BC in the ratio given. For, then AD (EH) : BE :: CT : BC.

PROBLEM XXXVIII.

To draw a line ABCD *to cut four other lines* MONP, NQ, ROT, SMT, *given by poſition; ſo that the three parts thereof* AB, BC, CD, *intercepted by thoſe lines, may obtain the ratio of three given lines* m, n, *and* p, *reſpectively.*

CONSTRUCTION.



From any point *f* in NP draw *fbg* parallel to TMS, interſecting NQ in *b*; and, having taken *bg* to *bf* in the given ratio of *n* to *m*, draw NgE: Moreover, having taken OH to MO in the given ratio of *n* to *p*, draw THG; and from (G) its interſection with NE, draw GI and GF parallel to MP and ST, reſpectively, cutting NQ and TR in B and C; thro' which two points draw ABCD, *and the thing is done.*

DEMONSTRATION.

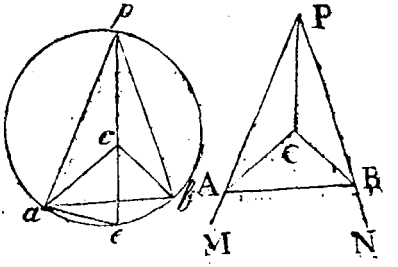
By ſimilar triangles $\begin{cases} AB : BC :: BF : BG :: bf : bg :: m : n; \\ BC : CD :: CG : CI :: OH : MO :: n : p. \end{cases}$

Q. E. D.

PROBLEM XXXIX.

To draw two lines CA, CB, *from a given point* C, *to terminate in two other lines* PM, PN, *given by position, and which, together with the line* AB, *joining their extremes, shall form a triangle,* ABC, *similar to a given one,* abc.

CONSTRUCTION.



Upon *ab* let a segment of a circle *apb* be described, to contain an angle equal to MPN, and let the whole circle be completed; draw PC, and also *ae*, making the angle *bae* = CPN, and intersecting the periphery in *e*; and through *c* draw *ep*, meeting the periphery in *p*; make the angles PCA and PCB respectively equal to *pca* and *pcb*; then join A, B, *and the thing is done.*

DEMONSTRATION.

If *pa* and *pb* be drawn; then will the angle *bae* (CPB) = *cpb*, both standing on the same arch *be*; therefore, APB being also = *apb* (*by Constr.*), the remaining angles APC and *apc* must consequently be equal; whence, as PCA = *pca*, and PCB = *pcb* (*by Constr.*), the triangles APC, *apc*, and BPC, *bpc* are equiangular; and therefore AC : *ac* (: : PC : *pc*) : : CB : *cb*. And so the triangles ABC, *abc*, having the sides about the equal angles ACB, *acb*, proportional, they are like to each other (*by* 15. 4). *Q. E. D.*

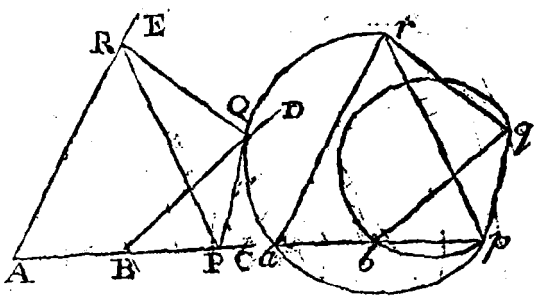
PRO-

PROBLEM XL.

To describe a triangle PQR, *equal and similar to a given triangle* pqr, *which shall have its angular points placed in three right-lines* ABC, BD, AE, *given by position.*

CONSTRUCTION.

Upon the two sides *pq*, *pr*, let two segments of circles *pbq*, *par* be described, to contain angles respectively equal to CBD and CAE: Then draw *pa* (*by Prob.* 27.) so that the part thereof *ba*, intercepted by the two peripheries, shall be equal to



BA; and, having joined *b*, *q*, and *a*, *r*, make BP = *bp*, BQ = *bq*, AR = *ar*, and let PQ, PR, and QR be drawn for the sides of the triangle required.

DEMONSTRATION.

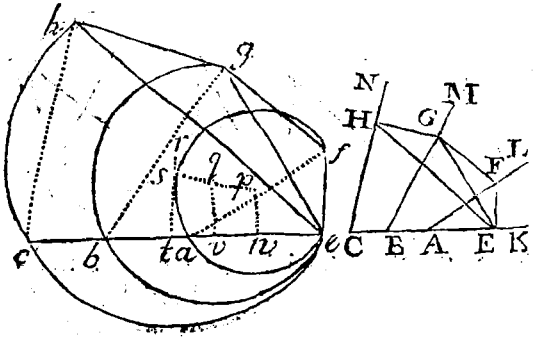
The triangles PAR, *par*; PBQ, *pbq*, being equal and alike in all respects (*by Constr.*), not only the sides PR, *pr*; PQ, *pq*, but the angles QPR, *qpr*, will be equal; and, consequently, the two triangles PQR, *pqr* also equal and like to each other. *Q. E. D.*

PROBLEM XLI.

To describe a trapezium similar to a given one efgh, *having its angular points placed in four right-lines* CN, BM, AL, CBAK *given by position.*

CONSTRUCTION.

Let three points *p*, *q*, *r* be found, from whence, as centers, segments of circles may be described, upon *ef*, *eg*, *eh*, to contain angles equal to the three given angles KAL, KBM, KCN, respectively; draw *pq*, in which, produced (if necessary) take *qs* to *qp* in the proportion of BC to AB;



and, having drawn *rst*, make *ec* perpendicular thereto, intersecting the three circles in *a*, *b*, *c*. Take AE : AB : : *ae* : *ab*; and make the angles CEH, CEG, and CEF, respectively equal to *ceh*, *ceg*, and *cef*; then let H, G and G, F be joined; and, I say, the trapezium EFGH will be similar to the given one *efgh*.

DE-

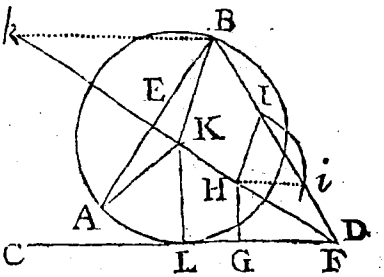
DEMONSTRATION.

Let *af*, *bg*, *ch*, be drawn, and also *pw*, *qv*, perpendicular to *ec*. So shall $ab = eb - ea = 2ev - 2ew = 2vw$; and $bc = ec - eb = 2et - 2ev = 2vt$: And therefore $ab : bc :: vw : vt :: pq : qs$ (*by* 13. 4.) :: AB : BC (*by Constr.*). Again, seeing the triangles EAF, *eaf*; EBG, *ebg*; ECH, *ech* are, respectively, equiangular (*by Constr.*) it will be EG : *eg* :: EB : *eb* :: AE : *ae* (*by Constr. and Composition*) :: EF : *ef* (*by* 14 4); and so the triangles GEF, *gef* (having the sides about the equal angles proportional) are similar. And, in the very same manner, the two remaining triangles EGH, *egh* (and consequently the whole trapeziums EFGH, *efgh*) will appear to be similar. *Q. E. D.*

PROBLEM XLII.

To describe the circumference of a circle through two given points A, B, *which shall touch a right line* CD *given by position.*

CONSTRUCTION.



Draw AB, which bisect in E by the perpendicular EF, meeting CD in F; from any point H in EF, draw HG perpendicular to CD; and, having drawn BF, to the same apply HI = HG, and parallel thereto draw BK, meeting EF in K; then from the center K, with the radius BK, let a circle be described, *and the thing is done.*

DEMONSTRATION.

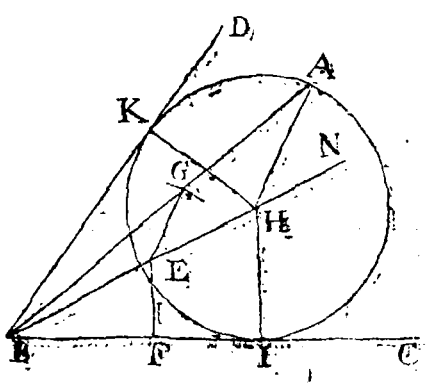
Join K, A, and draw KL perpendicular to CD: Then, because of the parallel lines, HG : HI :: KL : KB (*by* 21. 4.); whence, as HG and HI are equal, KL and KB are equal likewise: But it is evident, from the Construction, that KA is = KB; therefore KB = KL = KA. *Q. E. D.*

Because two equal lines HI, H*i* may be applied from H to BF (except, when DIH is a right-angle) the Problem will therefore admit of two solutions. But when a perpendicular let fall from H upon BF is less than HG; the Problem will be impossible. And the like is to be understood in the construction of the subsequent Problems.

PROBLEM XLIII.

To describe the circumference of a circle through a given point A, *which shall touch two right-lines* BC, BD *given by position.*

CONSTRUCTION.



From the point of concourse B of the two given lines, draw BA; and also BN, to bisect the angle CBD; from any point E in BN, upon BC, let fall the perpendicular, EF, and to BA apply EG = EF,

EF, parallel to which draw AH, meeting BN in H; then from the center H, with the interval HA, let a circle be described, *and the thing is done.*

DEMONSTRATION.

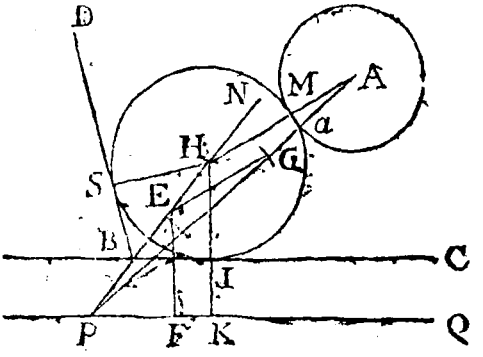
Upon BC and BD let fall the perpendiculars HI and HK; which are manifestly equal, because *(by Constr.)* the angle HBI = HBK: Moreover, as EF and EG are equal, HI and HA are also equal *(by* 21. 4). *Q. E. D.*

PROBLEM XLIV.

To describe a circle, which shall touch a given circle AaM, *and two right-lines* BC, BD, *given by position.*

CONSTRUCTION.

Draw PQ parallel to BC, at the distance of the radius Aa; and through the point of concourse B of the two given lines, draw NBP, bisecting the angle CBD, and meeting PQ in P; moreover, from any point E in PN, upon PQ, let fall the



perpen-

perpendicular EF, and from the same point, to PA, apply EG = EF; draw AH parallel to EG, meeting the periphery of the given circle in M, and the right-line PN in H, from which last point, as a center, through M, describe the circumference of a circle; *and the thing is done.*

DEMONSTRATION.

Draw HS perpendicular to BD, and HK to PQ, intersecting BC in I.

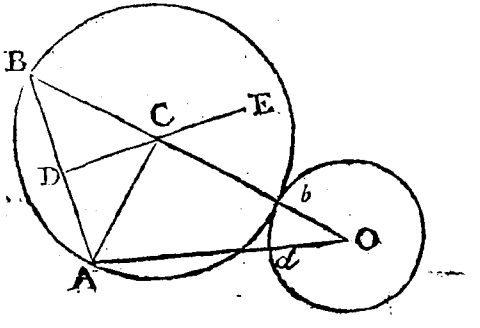
Then, because EG and EF are equal (*by Constr.*), HA and HK (*by* 21. 4.) are likewise equal; from which take away KI = AM (A*a*), and the remainders HI, HM will be equal: But, it is evident, that HI is = HS, because BH bisects the angle IBS; therefore HI = HM = HS. *Q. E. D.*

PROBLEM XLV.

To describe the circumference of a circle through two given points A, B, *and which shall also touch another circle* Odb, *given in position and magnitude.*

CONSTRUCTION.

Bisect the given distance AB with the perpendicular DE, in which (*by Prob.* 15.) let the point C



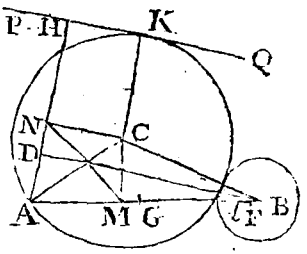
be

be so determined, that CO (when drawn) shall exceed CA by the radius O*d* of the given circle. Then that point, it is manifest, will be the center of the circle to be described.

PROBLEM XLVI.

Through a given point A, *to describe the circumference of a circle, which shall touch a given circle* B, *and also a right-line* PQ *given by position.*

CONSTRUCTION.



Make AH perpendicular to PQ, and BD to AH; and having drawn AB, in it take BF a third-proportional to AB and the radius B*d*, and let AF be bisected in G: Then draw MN (*by Prob.* 35.) so that MN, NH, and MG may be in the same given ratio, among themselves, as BD, BA, and B*d*: And at M and N let two perpendiculars be erected on AB and AG; which will intersect each other in the center C of the required circle.

DEMONSTRATION.

Let CA and CB be drawn, and also CK perpendicular to PQ. Because MN : NH : : BD : BA (*by Constr.*) : : MN : AC (*by* 22. 4.), thence is NH (CK) = AC. And since (*by Constr.*) NH (AC) : MG : : BA : B*d*, it is also manifest, from the Lemma on p. 222. that BC = AC + B*d*. *Q. E. D.*

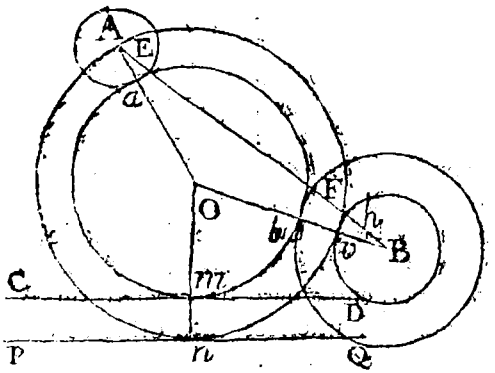
PRO-

PROBLEM XLVII.

To describe a circle O*amb*, *which shall touch two given circles* AEa, BFb, *and also a right-line* CD, *given by position.*

CONSTRUCTION.

From the radius BF of the greater circle, take away F*b* equal to the radius AE of the lesser; and from the center B, with the interval B*b*, describe the circle B*bv*; also draw PQ parallel to CD, at the distance of F*b* or AE: Then, by the



last Problem, let the center O of a circle be found, whose circumference shall pass thro' A, and touch PQ and B*bv*; and the same point O will, likewise, be the center of the required circle O*amb*.

DEMONSTRATION.

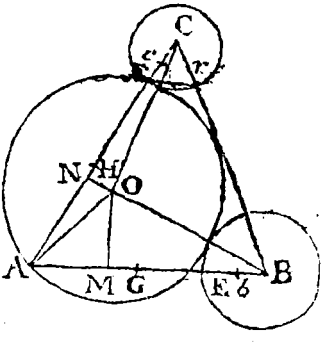
Draw O*n*, perpendicular to PQ, cutting CD in *m*; also let OA and OB be drawn intersecting the circles

circles A and B in *a* and *v*: Then, since *(by Constr.)* AO = *v*O = *n*O, and A*a* (AE) = *bv* (F*b*) = *nm*, let these last be respectively taken from the former, and there will remain O*a* = O*b* = O*m*. *Q. E. D.*

PROBLEM XLVIII.

To describe the circumference of a circle through a given point A, *which shall touch two other circles* B *and* C, *given in position and magnitude.*

CONSTRUCTION.



To the centers of the given circles, draw AB and AC; in which take B*b* = a third-proportional to AB and the radius BE; and C*c* = a third-proportional to AC and the radius CF: bisect A*b* and A*c* in G and H, and let F*r* be drawn parallel to AB, meeting BC in *r*. Then *(by Prob.* 34.) draw AO, so that OM and ON being drawn perpendicular to AB and AC, the three lines AO, GM and HN shall have the same given ratio among themselves, as AB, BE and F*r*. Then shall the point O be the center of the required circle: For, since *(by Constr.)* AO : GM : : AB : BE; and AO : HN (: : AB : F*r*) : : AC : CF; it is manifest, from the Lemma on p. 222. that OB = AO + BE; and OC = AO + CF. *Q. E. D.*

In

In the very same manner, a circle may be described to touch three given circles; the Problem amounting to no more than, *To find a point from whence lines, drawn to three given points, shall have given differences:* Since a point so found will always be the center of the required circle, as well when the three given circles are to touch that circle *inwardly*, as when they are all required to touch it *outwardly*.

The End of the GEOMETRICAL CONSTRUCTIONS.

NOTES

NOTES

GEOMETRICAL and CRITICAL

ON THE

Elementary Part of this WORK.

AXIOM 10. *Book* I. What is here laid down, as an Axiom, would, more properly, have been made a propoſition, had it admitted of ſuch a demonſtration, as is perfectly conſiſtent with geometrical ſtrictneſs and purity. But the laying of one figure upon another, whatever evidence it may afford, is a mechanical conſideration, and depends on no *Poſtulate*.

Theor. 4. *and* 5. *Book* I. There is ſcarce any thing more obvious to ſenſe, and at the ſame time more difficult to demonſtrate, than the firſt, and moſt ſimple properties of parallel lines. Even when we have (in Theor. 4.) proved the poſſibility of the exiſtence of ſuch lines, we cannot from thence infer, that their diſtance from one another is every-where the ſame; without having recourſe to an Axiom, which, though very evident to ſenſe, cannot be demonſtrated. Theſe difficulties wholly ariſe from our not having any properties, previouſly demonſtrated, whereby the progreſs of a right-line, produced out, can be traced, with reſpect to its diſtance from ſome other right-line

assigned;

assigned; nothing less being required here, than the proportionality of the sides of equiangular triangles; which is not proved before the middle of the Fourth Book, and which depends upon these very principles.

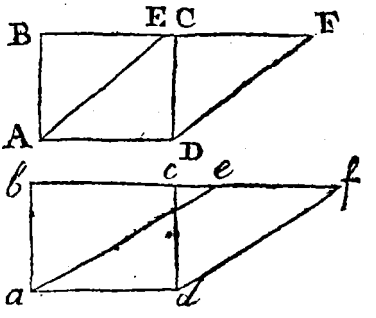
Schol. to Theor. 5. *Book* I. As there are several conditions requisite to make up the definitions of a rectangle and square, it was necessary to shew here, that the several properties ascribed to those figures, are not incompatible one with another. *Euclid* is very strict in this particular, and never undertakes to demonstrate any thing relating to a figure, till he has proved the possibility of the existence of such figure by an actual construction.

Theor. 22. *Book* I. This proposition, which is not in *Euclid*, is of considerable use, being often wanted in determining the *Maxima* and *Minima*, in mathematical enquiries.

Theor. 27. *Book* I. This Theorem, though not in *Euclid*, is also very useful, at least, to our design: by it we are not only enabled to divide a right-line into any number of equal parts, without the help of proportions, but also to demonstrate that most important proposition, *That the homologous sides of equiangular triangles are proportional.* It is true, the method pursued here, is not exactly conformable to the idea of proportions delivered in the 6th Def. of *Euclid's* 5th Book. But, even in that light, the demonstration will be equally easy, without inferring it from the proportionality of triangular spaces.

Theor.

Theor. 1. *Book* II. This propoſition, which is not mentioned by *Euclid*, may be thought unneceſſary; but it muſt either be demonſtrated, or aſſumed, as it is wanted in almoſt every propoſition of the ſecond book: By means of it, we alſo arrive at a general, and very eaſy demonſtration of that important Theorem, *That all parallelograms* (AEFD, aefd) *which ſtand upon equal baſes, and have equal altitudes, are themſelves equal:* For, when it is known that theſe parallelograms AEFD, *aefd* are equal, reſpectively, to rectangles ABCD, *abcd*, of the ſame baſe and altitude (which is proved in Prop. 2.) it is alſo manifeſt that they muſt be equal to one another, as their equal rectangles ABCD, *abcd* are ſhewn (*by Theor.* 1.) to be equal, the one to the other. This determination is more general than that given by *Euclid*, in the 36th Prop. of his firſt book; where he demonſtrates the equality of parallelograms, whoſe equal baſes are in the ſame ſtraight line: Which may, perhaps, be thought ſufficient for the whole; becauſe, if the baſes are not in the ſame right-line, one of the two figures may be conceived to be removed, and ſo placed, that its baſe ſhall be in the ſame right-line with the baſe of the other.—But, that theſe were not *Euclid*'s ſentiments, is evident from this; He hints at no ſuch thing: And had he approved of this ſort of demonſtration, his 36th Prop. would have been intirely uſeleſs; as nothing more (after the 35th) would be neceſſary, in order to a general demonſtration, than barely to place one baſe upon the



other. But it is certain, that this is a kind of demonſtration, which *Euclid* never has recourſe to, when the thing in hand can be done without it. For which reaſon I cannot help wondering a little, that that very accurate Geometer, *Profeſſor Simſon* of *Glaſgow*, ſhould make uſe of *it*, where (I apprehend) *Euclid* would not. The place, I have now particularly in my eye, is the addition (for it cannot properly be called a corollary) made by him to the firſt propoſition of the ſixth book: Which addition would have been quite unneceſſary, had what is above remarked, reſpecting the equality of parallelograms, been fully eſtabliſhed in the ſecond book: For then the demonſtration, that parallelograms, having equal altitudes, are as their baſes, would nothing differ from that whereby parallelograms, ſtanding between the ſame parallels, are proved to be in proportion as their baſes.

Theor. 9, 11, 12, 13. *Book* II. Theſe four Theorems, tho' not in *Euclid*, are of very conſiderable uſe, particularly the two firſt of them.

Theor. 1. *Book* III. This eaſy propoſition is added in order to give the learner a proper, and more preciſe idea of the quantity of an angle, and of its diviſion in practical uſes.

Theor. 2, 3, 4, 5, 6, 7, 8. *Book* III. Theſe ſeven propoſitions comprehend all that is moſt material in the firſt ſeventeen Theorems of *Euclid*'s 3d book.—As there is no where, in this author, ſo long a run of propoſitions together that are leſs entertaining to learners, or of leſs real importance, than the greater part of the ſaid Theorems; I thought it would be of uſe to reduce the ſubſtance of them into a leſs compaſs: And I flatter myſelf, that I have not ſucceeded ill, in this particular.

Theor.

Theor. 24, 25, 26, 27, 28. *Book* III. These Theorems, which are all of very considerable use, will not, I flatter myself, appear less plain by being proved independent of proportions, as the demonstrations here given are, not only more concise, but depend also upon fewer principles.

Def. 12. *and* 13. *Book* IV. The explication here given, is not strictly conformable to the idea of ancient *Commentators*, but is delivered in a sense somewhat more general. With them, *the composition and division of ratios*, extends to those cases only, where the sum, or difference of each antecedent and its consequent, is compared with the consequent. When the antecedent is compared with its excess above the consequent, this they call the *conversion of ratios*. But in such cases where the antecedent is less than the consequent, and where the sum, or difference of the antecedent and consequent is required to be compared with the antecedent; it does not appear that any *terms* have been given to signify such a comparison. *Professor Simson* thinks, that the definitions we have, are not *Euclid*'s, but an addition by *Theon*; which to me appears highly probable: This at least seems clear, that these definitions ought, either, to have been extended to a greater number, or else to have been rendered more general. For this reason I have, after the example of modern Geometers, given the signification of those terms, so as to include all the several cases: And this, I thought, might be done with the greater propriety, as the truth of whatever is here understood, depends upon the same demonstration.

Theor. 20, 21, 22, 23. *Book* IV. All these Theorems, tho' not in *Euclid*, are of considerable use;

by the two former, the demonſtration of ſeveral others is rendered more eaſy; and the latter have been applied, to good purpoſe, in the analytical determination of ſome difficult geometrical problems.

Prob. 3, 4, 5, 6. *Book* V. In the demonſtrations here given, it may be thought, that I have affected an unneceſſary exactneſs, by making them depend on *Axioms* alone. But I was willing that theſe fundamental propoſitions ſhould have the ſame foundation and evidence, as they appear to ſtand upon in *Euclid*; without referring to any thing derived in virtue of the 4th Poſtulate. But, whether I judged well, or ill, in this particular, is of little conſequence, as the demonſtrations here given, are not leſs plain, and but very little longer, than they otherwiſe would have been.—The Problems themſelves might, indeed, have been given along with the Theorems, as they became neceſſary, according to the method purſued by *Euclid*; whereby any objection, of this ſort, might have been avoided. But, beſides ſome ſmall convenience to the learner, there is a real advantage in having the Problems all together, after the Theorems; ſince, from a great choice of properties, ready demonſtrated to our hands, we are often able to arrive at a ſhorter and better conſtruction, than could poſſibly be given from ſuch properties alone as are antecedent to *Euclid*'s ſolution of the ſame Problem; his method of writing having obliged him to introduce the leading Problems as ſoon as poſſible, in order to evince and eſtabliſh the conſiſtence of his definitions, and to open his way in a regular manner to the many uſeful Theorems thereon depending. And it is for this reaſon alone, that many of his Conſtructions are not ſo well adapted to practice, as thoſe in common uſe.

Upon

Upon which account, ſome have been precipitate enough to blame him; not ſeeing, or conſidering, that ſuch Conſtructions, tho' not ſuited to anſwer every purpoſe, were the moſt proper for his plan, and the beſt that could be given in the places where they ſtand.

Prob. 8. *Book* V. The reaſoning in this propoſition, to prove that the two circles will cut each other, may, to ſome, appear needleſs. *Profeſſor Simſon* (at p. 359. of his *Euclid)* has been a little ſevere upon me, on this head, for attempting to ſupply, what I thought, a ſmall defect in *Euclid.* "Who is ſo dull (ſays he) tho' only beginning to "learn the Elements, as not to perceive that the "circle deſcribed from the center F, *&c.*" It is not without a real concern that I here ſee this able Geometer drop his own character ſo far, as to expreſs himſelf in a manner ſo very *ungeometrical.* If the thing is, indeed, eaſy to be perceived, it muſt be ſo, either, as an immediate object of the ſenſes, that is, in plain terms, by inſpection; or elſe it muſt be in conſequence of geometrical reaſonings antecedent to the thing itſelf. Now I am clear that he would not be thought to mean the former; and, as to the latter, nothing had been given from whence the evidence of the inference could be ſo clearly ſeen: For, tho' it is proved, that any two ſides of every triangle are greater than the third ſide, it would be abſurd to urge that concluſion in the caſe before us; becauſe the queſtion here, is, whether a triangle, under certain ſpecified conditions, can, or cannot be formed? and, therefore, to conclude any thing from the properties of triangles, would be ridiculous, and nothing leſs than begging the queſtion. That the determination propoſed limits the Problem, no body will diſpute: But then, is it not neceſſary that

this ſhould be proved, in Elements of Geometry, where a reaſon for every thing is, or ought to be given? From this conſideration, I cannot intirely approve of the emendation propoſed by this Editor to *Euclid*'s 24th Prop. Book I. For, tho' the addition there made, does indeed reſtrain the propoſition to one caſe; yet this ought to have been demonſtrated, by ſhewing that the point F (vid. p. 29.) muſt in conſequence of ſuch reſtriction, neceſſarily fall below the line EG; but this is not done. Many other inſtances might be produced to ſhew, that this gentleman, who often appears a little too haſty and ſevere in his cenſures, is not, himſelf, every-where equally guarded. In Prop. I. Book III. he bids you to draw a ſtraight-line within a circle, without ſpecifying that it muſt terminate in the circumference; and, what is a great deal worſe, he here very improperly uſes the word *within*; when the propoſition itſelf is laid down in order to prove, in the ſubſequent one, that ſuch line muſt neceſſarily fall *within* the circle. Theſe are, it is true, but little matters; but leſs than theſe have fallen under this gentleman's notice. At p. 358. M. *Clairaut* is glanced at, for an inadvertency of this ſort. And, in the note at the head of p. 415. it is ſaid, " The words, *for a ſtraight-line cannot* " *meet a ſtraight-line in more than one point*, are left " out, as an addition by ſome unſkilful hand; for " this is to be demonſtrated, not aſſumed." Now can it poſſibly ſhew any want of ſkill in an editor, to refer to an Axiom which *Euclid* himſelf had laid down (Book I. N° 14.) and not to have demonſtrated, what no man can demonſtrate?

Prob. 16, 17, 18. *Book* V. Theſe three Problems, tho' not ſo frequently wanted as ſome of the preceding ones, are nevertheleſs of very conſiderable uſe. The two laſt of them are the ſame, in

in effect, with the 28th and 29th of *Euclid*'s 6th book; but are here put down in a manner rather more commodious.

Axiom, p. 131. This Axiom is substituted instead of the common definition of equal solids, which, I really think, is too bad to be the work of *Euclid*. It is not a definition, but a proposition, "whose truth or falshood ought (as a very judi-"cious writer observes) to be demonstrated, not "assumed." Neither is it at all conformable to *Euclid*'s manner of writing, where he establishes the foundation whereon the equality of plane figures is grounded; which he does, not by means of a definition, but from the application of two of the most simple figures to each other; so that, from the coincidence of their bounds, their equality may appear manifest. And this method we have pursued in treating of solids; without which a clear and distinct idea of their equality can be no more obtained, than of the equality of plane figures independent of the 4th Prop. of the first book, which is our 10th Axiom. I should have said a good deal more on this head, but I find that *Professor Simson* has already placed this matter in so strong and clear a light, as to render any farther apology, or comment unnecessary here. Tho' I must confess, that, had this gentleman's work come into my hands * before the elementary part of my own had

* *This did not happen till the middle of last November; when being in town, in company with my bookseller, and being pressed by him to finish; I acquainted him that every thing was actually ready, except the Preface, which would cost me some pains, since it would be necessary to obviate some objections, particularly with regard to the reasons for my rejecting* Euclid's *definition of equal solids, and building upon a different foundation. On which, he immediately let me know, that Mr.* Rob. Simson *had already cleared*

had been intirely printed off, the definition of similar solids, which I have given, from *Euclid*, would have been delivered under a form somewhat different. For, tho' it involves no absurdity, as it now stands, yet there are certain cases (but such indeed as do not occur in any Elements of Geometry) where it will not afford the precise idea it ought to convey.

Postulatum, p. 131. The unsatisfactory and inconclusive demonstration given to *Euclid*'s 2d Prop. Book XI. (by *Euclid* himself, or some less skilful *Editor*) seemed to render something of this sort necessary. In that demonstration it is taken for granted, that one part of the triangle, at least, must be in the same plane. But it has been very justly observed, that a curve-surface may be bounded by three right-lines: Nor does it seem easy to form a clear idea, that even a part of any one of the three lines will be in the same plane with one of the others, unless by conceiving a plane to be turned about upon the one, till it meets with, or falls upon some point in the other. And I have the satisfaction in this particular, to see my sentiments exactly agree with those of a very good judge, whose name, I have, more than once, had occasion to mention in these notes. It is true, he makes that a Theorem, which I lay down as a Postulate. But, since a plane can no more be turned about upon a line, than a line can be drawn from one point to another, it seemed to me, that the one

up that point; and expressed his surprize that a work of so much repute, wherein (he told me) my own name was more than once mentioned, had not come into my hands. A copy of which I received from him the next morning. In consequence whereof I changed my first design of writing a long Preface; thinking it would be better to give what I had to offer, in notes, after the example of this Editor.

was

was as properly a Postulate, as the other. However, whether this be, or be not allowed, is immaterial, as the degree of evidence is precisely the same. There is, indeed, one reason, why this Theorem, or Postulate, ought to have preceded that gentleman's demonstration of Prop. 1. Book XI; It is there wanted: For, in the Corollary on which that demonstration is made to depend, the lines AB, BD, BC are supposed to be all in the same plane; *which ought by no means to be assumed in* the 1st of the 11th. *Euclid*'s 10th Axiom, which that Corollary is intended to supply, and by which the proposition is usually demonstrated, is not limited by any such restriction.

Theor. 12. *Book* VII. This proposition is added on account of its use, being the foundation on which the whole *art of perspective* in a manner depends.

Theor. 25. *Book* VII. From this Theorem, which is very extensive in its application, several others of considerable note may be deduced: one, or two of which, for the sake of the learner, I shall here derive, and put down by way of example.

Let A, B, C, D denote four lines in continual proportion.

Then, since $\left\{\begin{array}{l} A : B :: C : D, \\ A : B :: B : C, \\ A : B :: A : B, \end{array}\right\}$ it follows (*from Theor.* 25.) that

$A^3 : B^3 :: CBA : CBD :: A : D$ (*by* 22. 7.) or that, of four lines in continual proportion, the cube of the first is to the cube of the second, as the first line is to the fourth.

Again, let, $A : a :: B : b :: C : c$ (where A, a; B, b; C, c, may be supposed to represent the homologous sides of two similar parallelepipedons).

Then,

Then, since $\left\{\begin{array}{l} A : a : B : b \\ A : a :: C : c \\ A : a :: A : a \end{array}\right\}$ it alſo follows,

that, $A^3 : a^3 :: BCA : bca$; or, that ſimilar parallelepipedons are to one another, as the cubes of their homologous ſides.—The proportionality of ſimilar parallelepipedons, deſcribed upon proportional lines, is alſo included in the ſame Theorem; being no other than that caſe of it, where the propoſed ratios are all equal.

Theor. 5. *Book* VIII. The demonſtration of this Theorem might have been delivered under a form ſomewhat different, by aſſuming two other ſolids (without regard to figure) the one leſs, and the other greater than the propoſed parallelepipedon IP, and proving that cylinder muſt, *alſo*, be greater than the one, and leſs than the other: which is done by means of *Lem.* 1. that is, by taking P*m*, or P*t*, ſo ſmall a part of IP, as to be leſs than the difference between the given parallelepipedon and either of the ſaid ſolids: from whence the demonſtration will proceed on, in the ſame manner in which we have given it. But, as theſe additional conſiderations would have increaſed the number of ſchemes, and lengthened the proceſs, without adding one jot to the degree of evidence, it was thought proper, for the ſake of the learner, to omit them.

Theor. 8. *Book* VIII. This Theorem is not ſo uſeful as the Corollaries that follow from it, which are all of very great importance: In the 3d and 4th of them, the proportion of all kinds of priſms and pyramids is aſſigned, without the aſſiſtance of the uſual demonſtrations given for this purpoſe; which, tho' ſufficiently evident in themſelves, are often found a little perplexing to learners, on account

count of the ſchemes, wherein ſo great a number of lines is neceſſary.

Nothing, in the courſe of theſe notes, has been ſaid relative to the Theorems on proportions, tho' ſo nice and critical a ſubject, and tho' the method I have therein purſued may ſtand in need of ſome apology. But, indeed, the whole of what I have to offer on this head, was too much to be comprized in the compaſs of one ſingle note, and could not ſo properly be delivered in ſeveral detached ones: For which reaſon, I ſhall here throw together all that I have to advance on that ſubject.

There are two objections that may be brought againſt the method in which proportions are treated of in this work; the one, grounded on the impoſſibility of dividing every magnitude into equal parts; and the other, on the incommenſurability of two or more magnitudes of the ſame kind, when compared with each other. The firſt of theſe objections appears, to me, to have very little weight. For, tho' a magnitude may be ſo conſtituted, that the diviſion of it into an aſſigned number of equal parts cannot be, *actually*, effected by any geometrical conſtruction; yet it is no leſs evident, for that reaſon, that every ſuch magnitude has not really its third, fourth, or other aſſigned part, tho' we are at a loſs how to take it; or, in other words, it ſeems very clear to conceive, that in every propoſed magnitude, whatever its figure may be, a leſs magnitude is contained, which, repeated an aſſigned number of times, ſhall be equal to the magnitude given. If, as the moſt rigid judges allow, every plane figure is equal to *ſome* ſquare, and every ſolid equal to *ſome* parallelepipedon; then the parts of the ſquare, or parallelepipedon, which are actually determinable by a geometrical

metrical conſtruction, will alſo be like parts of the figure firſt propoſed, and ſuch as we conceive to be taken.

The other objection, depending on the incommenſurability of magnitudes, is a matter of real difficulty; which we have taken ſome pains to obviate, in the *Scholia* to our 3d and 7th Theorems. *Euclid*, himſelf, ſeems to have been not a little embarraſſed with *it*, if we may be allowed to judge from the different methods he has left us in his 5th and 7th books; the former whereof, which is ſuited to include the buſineſs of incommenſurables, being nothing near ſo eaſy and natural as the latter: It has, it is true, the advantage of being general; but, that the principles whereon it is grounded, are neither ſo ſimple, nor ſo evident as might be wiſhed for, the many diſputes about them, ſince *Euclid*'s time, by Geometers of the firſt rank, will in a great meaſure evince. And farther, it ſeems ſufficiently plain, from *Euclid*'s own authority, that he himſelf was not intirely pleaſed with his own performance on this head; or that he was convinced (at leaſt) that it had not *every advantage:* For, otherwiſe, it will be very difficult to account for his having demonſtrated many things in his 7th book, by another method, whoſe demonſtrations had been actually given before, in the 5th, under a different form. For theſe reaſons, when I ſee the extravagant commendations that have been laviſhed on this 5th book of *Euclid*, I am no farther convinced by them, than that great men may ſometimes launch out too far in behalf of opinions which they have adopted. And I believe that, whoever has read the notes on the 5th book, by that great reſtorer of *Euclid*, *Profeſſor Simſon*, will be apt to conclude, that thoſe high encomiums are a little miſapplied. Indeed, if all that is advanced in thoſe notes be allowed of, I think the author of them has proved

too

too much; and this ſuperb fabric of proportions, reared with ſo much art, ſtands upon a tottering foundation. It is not by choice that I go out of my way to play the *critic*; but as the writers againſt the vulgar and indiſtinct notion of proportions (as they term it) are very ſevere in their cenſures, and aſſume a great ſuperiority, from the boaſted *accuracy of their reaſonings*, it may be neceſſary to ſhew my reader, that, tho' what he is *here* taught on proportions, is liable to ſome objections, the method which ſome ſo greatly prefer, has *alſo* its difficulties; and that there are *other objections* to it beſides its *obſcurity*. And this I ſhall make appear from this learned Commentator's own authority and conceſſions; and in order thereto, ſhall firſt refer to his note on Prop. 10. which proceeds thus. "It was "neceſſary to give another demonſtration of this "propoſition, becauſe that which is in the *Greek*, "and *Latin*, or in other editions, is not legitimate. "For the words *greater*, *the ſame*, or *equal*, *leſſer*, "have a quite different meaning when applied to "magnitudes and ratios, as is plain from the 5th "and 7th definitions of Book 5. by help of theſe let "us examine the demonſtration of the 10th Prop. "which proceeds thus, *&c.*" He then goes on, in a long note, to ſhew the inſufficiency of a demonſtration, which had been received, by all, as perfectly genuine and ſatisfactory; and at laſt comes to this concluſion. "Wherefore the 10th Propo-"ſition is not ſufficiently demonſtrated. And it "ſeems, that he who has given the demonſtration "of the 10th Propoſition, as we now have it, in-"ſtead of that which *Euclid* or *Eudoxus* had given, "has been deceived in applying what is manifeſt, "when underſtood of magnitudes, unto ratios, "*viz.* that a magnitude cannot be both greater "and leſs than another. That thoſe things which "are equal to the ſame are equal to one another,

"is

" is a moſt evident Axiom when underſtood of " magnitudes, yet *Euclid* does not make uſe of it " to infer that theſe ratios which are the ſame to " the ſame ratio, are the ſame to one another; " but explicitly demonſtrates this in Prop. 11. of " Book 5. The demonſtration we have given of the " 10th Prop. is no doubt the ſame with that of " *Eudoxus* or *Euclid*, as it is immediately and di- " rectly derived from the definition of a greater " ratio, *viz.* 7th of 5."

Here the weight of the objection reſts on its not having been proved, that, of three given magnitudes A, B, C, the ratio of A to C could not, at the ſame time, be both greater and leſs than that of B to C. But, if in the demonſtration, here rejected as inſufficient, there is any real flaw, it is chargeable on the definition of a greater and leſs ratio, as the reaſoning from it, is clear, ſtrong, and perfectly ſcientific. And I would ſeriouſly aſk the Contemners of *the vulgar and confuſed notion* of proportions, if a definition, by which it cannot be known, whether the ratio of the firſt to the ſecond of four given magnitudes, may not, at the ſame time, be both greater and leſs, than that of the third to the fourth, is really calculated to afford thoſe very accurate ideas, they pretended to? This Commentator has too much penetration not to be aware of the force of this objection, which he has attempted to obviate in one particular caſe. But the new propoſition given by him, for that purpoſe, ought to have preceded the 10th, and to have been demonſtrated independent of it. This he alſo ſeems apprized of, when he ſays, that " it cannot " be eaſily demonſtrated without the 10th, as he that " tries to do it will find." But, be this as it will, I am not at all clear that his " demonſtration of " the 10th, is the ſame with that of *Eudoxus* or " *Euclid*." *Euclid* or (if you pleaſe) *Eudoxus*, does never

never (that I know of) refer to any definition, till it has been proved, either by an actual construction, or by some demonstration previous to that in hand, that such definition involves no absurdity, or conditions that are incompatible one with another. If, therefore, it was conceived possible, that the definition of a greater and less ratio, could involve so great an absurdity, as that, *by it*, the ratio of A to C might at the same time be both greater and less than that of B to C; this point, according to the method prescribed by *Euclid*, ought to have been cleared up, not by means of propositions derived in virtue of that very definition, but by others antecedent thereto, and independent thereupon. And, to me, the 8th Prop. seems the proper place for the doing of this, where it might be easily introduced, either in the Prop. itself, or by way of Corollary. It is there proved, that if, of three magnitudes A, B, C, the first A is greater than the second B, then certain equimultiples of A and B may be taken such, that being compared with some multiple of C, the multiple of A shall be greater, and that of B less than the said multiple of C. Whence, by the definition of a greater ratio, the ratio of A to C is greater than that of B to C. To which might be added—And, because A is greater than B, no equimultiples of A and B can possibly be so taken, that the multiple of A shall be equal to, or less than some multiple of C, and that of B greater than the same; because A being greater than B, the multiple of A is ever greater than that of B, and therefore also greater than that of C, which is equal to, or less than the multiple of B. Wherefore the ratio of A to C cannot (by the definition) be less (as well as greater) than the ratio of B to C.

But, notwithstanding all that has been proved on this head, either here, or by that gentleman himself, the same objection occurs again in Prop. 13. where

where it remains in its full force. For, tho' it be allowed, that "there are ſome equimultiples of C "and E, and ſome of D and F ſuch, that the "multiple of C is greater than the multiple of D, "but the multiple of E not greater than the mul-"tiple of F;" yet it is not demonſtrated, nor in any ſort ſhewn, that other equimultiples of thoſe quantities cannot be taken ſuch, that the very contrary ſhall happen.—If the demonſtration of the 10th Prop. has been juſtly rejected by this gentleman himſelf, as inſufficient, becauſe the impoſſibility of a contrary concluſion had not been ſhewn; can it be thought that this 13th Propoſition is, at this day, ſufficiently demonſtrated, where the ſame objection occurs, and that in a much greater latitude? I have a much better opinion of this Editor's diſcernment, than to imagine, that his paſſing this matter over in ſilence, proceeded from his not being aware of the difficulty; but it ſeems to me, that his great diſlike to the *vulgar* idea of proportion (ſo often teſtified in the courſe of his notes) would not permit him to borrow any thing from thence, however evident, and though this objection, that ſtrikes deep at the very root of proportions, might by means thereof be very eaſily removed. I ſay, the very root of proportions is deeply ſtruck at in this objection; becauſe both the alternation and equality of ratios (*ex æquali ſc. diſt.*) are grounded on the ſaid 13th Prop. and which, therefore, till the objection is removed, muſt be allowed to ſtand upon an uncertain foundation.

The principle hinted at above, whereby the difficulty might be obviated, is, that if a magnitude of any kind be given, or propounded, there may (or can) be another magnitude of the ſame kind which ſhall have to it any ratio aſſigned. This aſſumption Mr. *Profeſſor* will by no means admit of (tho'

(tho' *Euclid* himself, in Prop. 2. of his 12th book, has used it); and, in a long note on Prop. 18. is angry with *Clavius* for having recourse to it; affirming "That the demonstration (given by means thereof) "is of no force;" and that "the thing itself can-"not (as far as he can discern) be demonstrated by "the preceding propositions, so far is it from de-"serving to be reckoned an Axiom, as *Clavius*, "after other Commentators, would have it." That the assumption cannot be generally demonstrated by the preceding propositions (nor even by all the propositions in the Elements) I readily assent to: but then, because a thing, exceedingly obvious in its own nature, cannot be demonstrated, is it therefore less proper for an Axiom? I should rather take the other side of the question, and maintain that nothing ought to be made an Axiom, which can be demonstrated. But we are not, it seems, allowed to have any idea of proportion but what is contained in the 6th and 8th (or, as this Author makes them, the 5th and 7th) definitions of *Euclid*'s 5th book. And, in his note on the new Prop. marked A, He is again displeased with *Clavius*, for thinking it sufficiently evident, *from the nature of proportionals*, that if, of four proportional magnitudes, the first antecedent is greater than its consequent, the second antecedent will also be greater than its consequent. "As if there was "(says he) any nature of proportionals antecedent "to that which is to be derived and understood "from the definition of them." Now I cannot help thinking, with *Clavius*, that there was a nature, or idea of proportion antecedent to that given in the 6th and 8th definitions of *Euclid*'s 5th book: For, that mankind, long before the time of *Euclid*, had some way to shew, or express, in what degree one magnitude was greater or less than another, cannot be doubted: And this was

the firſt, and natural idea of proportion: And I look upon thoſe definitions, as refinements, only, on the ſimple and natural idea, in order to take in the buſineſs of incommenſurables; whereby the original notion is ſo much obſcured, that it requires ſome ſkill, even to ſee that it is at all contained in theſe definitions. I intirely agree with this gentleman, that every demonſtration ought to be ſtrictly derived from principles before eſtabliſhed: But then, whether is it more eligible, to have recourſe to an Axiom founded (as all other Axioms are) on the evidence of ſenſe and reaſon, or to an obſcure and perplexed definition, which may, for any thing that has been proved to the contrary, involve an abſurdity?

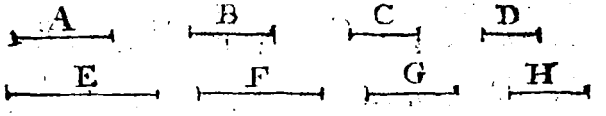
That there is ſomething very ingenious and ſubtle in the doctrine of proportions, as delivered in *Euclid*'s 5th book, cannot be denied. All that I contend for, is, that the principles on which it is built are *obſcure*, and not ſo firmly eſtabliſhed, as to authorize its partiſans to aſſume that great ſuperiority they lay claim to, in point of geometrical ſtrictneſs.

I have intimated above, that the principle is rejected, by which the conſiſtence of the definition of a greater and leſs ratio might be eſtabliſhed, without much difficulty: But I would not be thought to mean, that the ſame thing cannot poſſibly be effected any other way, becauſe I am ſatisfied that it may be done from the conſideration of multiples alone: But a demonſtration of this ſort is not eaſy.—Were I to treat of proportions from the plan laid down in the 5th book of *Euclid*, I would intirely reject the 10th and 13th propoſitions, and every thing elſe founded on the definition of a greater and leſs ratio, as being of no other uſe in the Elements, than to open the way to thoſe important Theorems on the alternation and equality of ratios; which may be better demonſtrated without them, from the definition of equal ratios alone; which,

which, from the conditions of it, can admit of no absurdity; and whose consistence is evinced in Prop. 15. and still more clearly in the first of the sixth.

In the 14th of the 5th, whereon the alternation of ratios is grounded, it is necessary to demonstrate, "That if, of four proportional magnitudes, of the "same kind, A, B, C, D, the first be greater than "the third, the second shall be greater than the "fourth; and if equal, equal; and if less, lesser." Which may be very easily done, independent both of the 10th and 13th, in the manner following.

First, let A be greater than C.



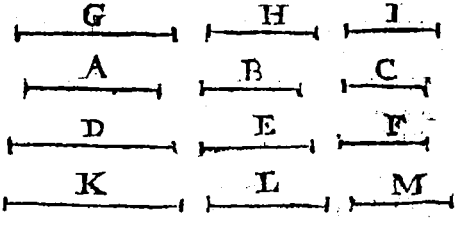
Of A and C (*by Prop.* 8.) let such equimultiples be taken, that the multiple of A shall be greater, and that of C less, than some multiple of B; let E and G be any two such equimultiples of A and C, and F the multiple of B; so that E shall be greater than F, and G less than F; and let H be the same multiple of D, as F is of B. Therefore, because E and G are equimultiples of the first and third, and F and H also equimultiples of the second and fourth; and seeing that (*by Hyp*) E is greater than F; it is evident, from the definition of equal ratios, that G must likewise be greater than H: Therefore much more shall F (which exceeds G) be greater than H; whence also B shall be greater than D (*by Ax.* 4.) B and D being like parts of F and H.

When A is less than C, it will be demonstrated in the same manner, that B is also less than D. But when A is equal to C, no new demonstration is necessary; since neither the 10th nor the 13th have any thing to do in this case.

Again, in the 20th Prop. (in which the 10th and 13th also enter) we are to prove, "That if there be "three

"three magnitudes (A, B, C) and other three (D, "E, F) which taken two and two have the same ra- "tio (A : B : : D : E, B : C : : E : F) if the first (A) "be greater than the third (C), the fourth (D) shall "be greater than the sixth (F); and if equal, equal; "and if less, lesser." Which may likewise be done without the assistance of either the 10th or the 13th, in the same manner, above specified.

For, if A be greater than C; then of A and C (*by Prop.* 8.) such equimultiples may be taken, that the multiple of A shall be greater, and that of C less, than some multiple of B; let G and I be two such equimultiples of A and C, and let H be the multiple of B, so that G shall be greater than H, and I less than H; moreover take L the same multiple of E, as H is of B; and K and M the same equimultiples of D and F, as G and I are of A and C. Therefore, since of the four proportionals A, B, D, E, equimultiples G, K of the first and third, and equimultiples H, L of the second and fourth, are here taken, it is manifest, from the definition of equal ratios, seeing G is greater than H (*by Hyp.*) that K must also be greater than L. And in the very same manner, because B, C, E, F are proportionals, and H is greater than I, L will likewise be greater than M: Therefore much more shall K, which exceeds L, be greater than M. And consequently (*by Ax.* 4.) D shall also be greater than F. —When A is less than C, the demonstration is the same: The other case, when A is equal to C, does not require, nor indeed admit of any improvement.



F I N I S.