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Empirical Studies of the Motion of Long Waves in the Westerlies

by

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EMPIRICAL STUDIES OF THE MOTION OF LONG WAVES IN THE WESTERLIES

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I. Introduction

Among the basic tools used by the Extended Forecast Section of the United States Weather Bureau are five-day mean charts of pressure at 10,000 feet, representing the average of 10 twice-daily synoptic charts. These charts are at present prepared three times a week so that some overlap occurs. An extensive discussion of the importance and practical use of mean 10,000 foot charts in long-range forecasting is contained in previous reports of the Extended Forecast Section[1]. Here it is sufficient to point out that these charts generally reveal fairly simple, quasi-sinusoidal pressure waves, as distinct from the more complex patterns generally found on daily synoptic charts. These waves move, on the average, along latitude circles from west to east. Since it is one of the objectives in the forecasting procedure to construct prognostic mean pressure patterns for periods a half week (3 or 4 days) or a week in advance of the latest availa ble mean charts, it is obvious that one of the basic problems is to forecast the west-to-east motion of the mean troughs and ridges making up the pressure waves. This report deals with certain statistical studies of methods of forecasting trough displacement originally suggested by the theoretical work of C.-G. Rossby[2,3].

II. Statistical Studies of the Rossby Formula Assuming Constant Velocity

A. Formula for Trough Displacement.

Rossby's well-known formula for the motion of long waves in the westerlies is:

$$(1) \qquad C = \bar{U} - \frac{\beta L^2}{4\pi^2}$$

where C is the instantaneous west-to-east speed of the wave, $\bar{U}$ is the average speed of the westerlies, β is the rate of change of the Coriolis parameter with latitude, and L is the wavelength.

This formula has been used in practice for a number of years, but it has been only recently that enough data has been available to check the formula by statistical means. However, a preliminary check by J. Namias (unpublished) revealed essentially the same results as the more extensive work to be described below. The present study is based on approximately 3 years of 5-day mean 10,000 foot charts.

Before we can use the above formula to obtain an estimate of trough displacement, it is necessary to make certain assumptions regarding the terms, C, $\bar{U}$, and L. In the over-simplified theoretical model used by Rossby in deriving the formula, the pressure waves are assumed to be sinusoidal, and L and $\bar{U}$ are assumed everywhere constant. In practice, it is not only difficult to decide what to use for L, due to the non-sinusoidal nature of observed pressure waves, but having decided, it is then found that both L and $\bar{U}$ vary widely from place-to-place and from time-to-time. It is therefore necessary to adopt a more or less arbitrary definition for L and $\bar{U}$ to obtain best results. The definitions used in this study are modifications of those suggested by H. Wexler[4].

First it is necessary to select a certain latitude circle along which the displacements of the mean pressure troughs are to be measured. In the first part of this study, the fixed latitude chosen is 45°N. The displacements may then be conveniently expressed in degrees of longitude at the chosen latitude circle.

A trough or ridge line is defined as a line drawn through the minimum or maximum latitude points of the isobars in the troughs or ridges. The advantage of this definition over one involving maximum curvature points is mainly one of practical simplicity, and has been discussed more fully elsewhere(5). The position of a trough or ridge on the chosen latitude circle is simply the longitude of the point where the trough or ridge line intersects the latitude circle. The wavelength is twice the difference in longitude between a given trough and the ridge to its west.

The definition of $\bar{U}$, the average zonal velocity, was originally taken as the mean west-to-east geostrophic wind between trough and ridge and in the latitude zone extending 5 degrees to the north and south of the chosen latitude circle. Later, as will be explained below, it was found more satisfactory to use a value of $\bar{U}$ covering the entire range of longitudes included in the available data, or from about 130°W to 30°W.

On substituting the so-obtained values of $\bar{U}$ and L in formula (1) an estimate of the instantaneous speed of the trough is obtained. To obtain the displacement over a given time interval, it is necessary to make some assumption regarding the variation of speed with time. The most obvious assumption is that the speed of the trough will remain constant. The estimated displacement is then given by the simple formula:

(2) $$S = CT$$

Where S is the displacement, expressed in degrees of longitude at the chosen latitude circle, C the speed in degrees of longitude per week obtained from formula (1), and T is the time interval, usually a half-week, (3 or 4 days) or a week.

B. Statistical Treatment and Division of Data

Before discussing the results of applying this method, it is desirable to make a few general remarks concerning the nature of the available data and the statistical methods employed.

The method of comparing estimated displacement with observed displacement is by means of the well-known simple correlation coefficient, reference to which may be found in any standard textbook on statistics. Other equally elementary statistical techniques, such as determination of linear regression equations, are used throughout the report. The principal objective of the study is to find the general order of magnitude of such coefficients, but from time-to-time comparison is made between one coefficient and another. In making such comparisons, no attempt is made to use statistical tests of significance.

The data consist of 2-1/2 to 3 years of 5-day mean 10,000 foot charts, covering the period from January 1941 to April, 1943. These were first divided according to season, the standard seasonal divisions being used (December, January, and February, for example). The troughs selected for the tests were taken within the range of available data, from about 130° to 30° west longitude, and, for the present study, at 45° north latitude. Cases of uncertain continuity were first eliminated. As explained in a previous publication[5] these constitute up to 10% of the available cases. The remaining data for each season were then divided in accordance with certain criteria suggested in the earlier study of J. Namias, mentioned above.

It was found as a result of that study that one of the most important causes for the failure of formula(2) is the development of new troughs or the disappearance of old ones during the time interval covered by the forecast. This is evident when we realize that such occurrences in the vicinity of the trough we wish to extrapolate constitute a major violation of many of the assumptions upon which formula(1) is derived. Furthermore, they cause sudden changes in wavelength which in turn result in changing the speed of the troughs. Thus it is desireable to eliminate such cases when testing the Rossby formula. However, because of the spacial limitation of the data and because of the fact that only two or three mean maps are constructed a week, it is necessary to adopt an arbitrary definition of trough development or disappearance. A case is defined as one in which a trough develops or disappears during the week if on the chart a half week later than the current one there is evidence that such an event caused a sudden modification of the wavelength as measured between the trough concerned and the ridge to the west. Cases of the appearance or disappearence of troughs on the chart a full week in advance, but not on the half week chart, are not considered, as it is assumed that the occurrence took place too late to greatly effect the motion of the trough.

Another suggestion arising from the preliminary study is that those computations which work best are made well within the main belt of the westerlies, as implied in the derivation of formula(1). The definition of westerly flow used in this study is not entirely satisfactory, as it could not be made rigid. However, it is concerned with the amplitude of the pressure waves and the presence or absence of closed centers at 45°N, as well as other factors. As will be shown later, cases classed as "not in the westerlies" actually give better results than the others, and so were no longer considered as a separate group. Therefore, the definition used will not be discussed further.

The division of the data in accordance with the above criteria is shown in the following table:

TABLE 1.

Division of Data (Figures are number of cases)

	Troughs develop or disappear	Troughs not in Westerlies	Other	Total
Winter	12	15	37	64
Spring	6	22	36	64
Summer	7	12	29	48
Fall	6	6	35	47
Totals	31	55	137	223

C. Results of Statistical Tests.

The first test of formula (2) was made using a value of the strength of the westerlies between the trough and the ridge to the west. The extrapolations were made for a half week and a full week in advance. The correlations are shown in Table 2. The column headings show the type of data used, the same division of data being used as in Table 1. Thus, Column 1 of Table 2 refers to those troughs which are within the main belt of westerlies and in the vicinity of which no troughs developed or disappeared. The number of cases used in each correlation is also indicated.

TABLE 2.

Correlations for Motion at 45°N, Rossby Trough Formula, Using Trough-to-Ridge Wavelength and Westerlies

	1	2	3	4	5	6	7	8
	Full Week		Half-Week		Full Week		Full Week	
	In westerlies no new development	Number Pairs	In westerlies no new development	Number Pairs	Not in westerlies	Number Pairs	Development or disappearance	Number Pairs
Winter	.59	37	.62	38	.26	15	-.15	11
Spring	.38	35	.46	38	.71	20	-.67	6
Summer	.27	29	.29	31	.42	12	.07	7
Fall	.39	35	.52	37	.63	6	.12	5

The correlations in Column 1 of Table 2 suggest that there is a definite relationship, especially in winter, between the observed displacements for a full week in advance and those computeded by the Rossby formula. The correlations are not remarkably high, but this may be in part due to the inadequate nature of the data. Thus, while the location of troughs and ridges at the beginning of the weekly interval is fairly definite, this is not true at the end of the week, as they are then often well out in the Atlantic, where, until the latter half of 1942, data were very scarce. This unavoidable difficulty results in poor definition of the final trough positions and, therefore, a poor estimate of observed trough displacement.

Another factor which may account for the low correlations is the length of time covered by the extrapolation. Thus, since formula (1) only gives us at best the instantaneous velocity of the troughs, it is not a fair test of its validity to extrapolate over too great an interval of time, since there may be significant changes in velocity during that time. Therefore, if the extrapolation covers a shorter time interval, we should expect to find higher correlations. To test this conclusion extrapolations were made for a half week (3 or 4 days) in advance. The resulting correlations are shown in column 3 of Table 2. These correlations were made using essentially the same cases as for the full week interval. However, there are a few additional cases due to greater ease in determining continuity. It will be noted that for all seasons the correlations are higher for the half week extrapolations, as we should expect.

Column 5 shows the correlations for those cases which were not in the main belt of the westerlies, but for which there was no new trough development. Except for the winter season, these coefficients are considerably higher than those in column 1. The reason for this rather surprising result might possibly be found in the inadequate definition of the westerlies, but the most probable explanation is that troughs and ridges of large amplitudes(which are classed as not in the main belt of the westerlies) are better defined than the others. From the standpoint of continuity, the advantage of this better definition is probably enough to counteract the more limited applicability of formula (1).

Column 7 shows the results of those cases where new troughs formed or old ones disappeared during the interval of a week. As we should expect, these correlations are low or even negative. However, it should be pointed out that this result is somewhat inconclusive because of the extremely small number of cases.

In an attempt to find a more representative value of the zonal westerlies ($\overline{U}$) than that given by the mean geostrophic wind between trough and ridge, a few other definitions were tried. One of these was a value of the geostrophic wind taken between 40° and 50°N, and covering the entire range of longitudes included in the data rather than the more limited range. Such a value of the mean zonal westerlies is known as the 10,000 foot index (I_3). The correlations obtained using this definition of the westerlies are shown in Table 3. Only the cases for which there was no new trough development are shown.

TABLE 3.

Correlations for full-week and half-week movement at 45°N, Rossby trough formula, using total I_3 and T to R wavelength, and eliminating cases of new trough development.

	Full-Week		Half-Week	
Winter	.66	37	.67	36
Spring	.44	35	.51	35
Summer	.35	29	.42	29
Fall	.42	35	.63	35

TABLE 4.

Correlations for full-week and half-week movement at 45°N, using current wavelength alone, and eliminating cases of new trough development.

	Full-Week		Half-Week	
Winter	-.64	52	-.71	36
Spring	-.54	54	-.46	35
Summer	-.38	39	-.33	29
Fall	-.51	40	-.55	35

It will be noted by comparing these correlations with those in columns 1 and 3 of Table 2 that they are larger in every case, suggesting that the total zonal westerlies are more representative than the rather limited values obtained between trough and ridge.

The linear regression equations between observed and computed full week displacement corresponding to the cases just discussed are shown in Table 5. In this table E' is the estimate of the observed motion and E is that computed by means of the Rossby formula. Both are expressed in degrees of longitude at 45°N.

TABLE 5.

Regression Equations of Observed on Computed Full-week Displacement at 45°N, Rossby Trough Formula, Using Total I_3 and Trough to Ridge Wavelength, and Eliminating Cases of New Trough Development

Winter	$S' = 14.78 + 0.156\ S$
Spring	$S' = 17.14 + 0.114\ S$
Summer	$S' = 9.23 + 0.139\ S$
Fall	$S' = 12.23 + 0.179\ S$

These equations are currently used in the Extended Forecast Section in obtaining estimates of displacement by means of the Rossby formula.

A point of considerable interest in regard to these equations is that they by no means indicate a one-to-one relationship between observed and computed displacement, in spite of the fact that the correlations indicate that the theory has a fair degree of validity. Thus, in the case of the equation for winter, it will be noted that on the average the Rossby formula indicates a retrogression (westward motion) of almost 100 degrees when the observed motion is zero. Turning to equation (1) we see that this must mean that $\overline{U}$ is on the average too small in comparison to $\frac{\beta L^2}{4\pi^2}$. However, due to the normal increase of the westerlies with height, one can presumably find a level such that the value of $\overline{U}$ will be large enough. This level at which the Rossby formula is satisfied is known as the level of non-divergence, for the formula was derived on the assumption of no divergence in the westerly flow. Two years ago Namias attempted to find this level of non-divergence by computing the $\overline{U}$ necessary to make the Rossby formula fit the observed results, and locating the level of this zonal velocity through the observed distribution of $\overline{U}$ with elevation His results (unpublished) showed that most of the time the level of non-divergence lies between 1 and 6 Km, although the scatter is great and numerous cases of levels above and below this interval are found. It is not the purpose of this report to go into detail concerning the theory involved in the concept of the level of non-divergence. Those interested may consult a recent publication dealing with that subject(6). Nevertheless, it is suggested here that the reason for the lack of a one-to-one relationship between observed and computed displacements is because of the fact that the 10,000 foot level lies on the average below the level of non-divergence.

D. General Conclusions

It may be helpful to summarize the principle conclusions of Part I. They are as follows:

1. A fairly good relationship is indicated between observed displacements, as far as a week in advance, and those computed from the Rossby formula (assuming constant velocity), provided that cases of trough formation and disappearance are eliminated.

2. Computations for shorter intervals of time show even better correlations, thus further establishing the validity of the formula.

3. The total 10,000 foot index is a more representative value to use for the speed of the westerlies than a local index taken between trough and ridge.

4. The regression equations indicate that the 10,000 foot level may be, on the average, below the level of non-divergence.

II. Empirical Study of Trough Displacements

Having completed the statistical study described above, it was thought desirable to attempt to find a purely empirical formula for trough displacement based on the variables which were indicated by the theoretical studies as being most important. Besides the variables, $\bar{U}$ and L, another variable was added, the amplitude (A) of the pressure waves.

Thus, it is assumed that the displacement (S) can be expressed as a function of these three variables, or:

$$S = F\ (\bar{U},\ L,\ A) \tag{3}$$

The problem, then, is to determine this functional relationship by means of the data alone. Unfortunately, in meteorological studies unlike experimental aerodynamics, it is not possible to hold all but one of the variables constant and study the effect of the remaining one. Nevertheless, there are two statistical methods which can be used to obtain the desired functional relationship, if any, provided a sufficient amount of data is available. These are the method of graphical correlation, and the more objective, but closely related, method of curve fitting by least squares. It is not the purpose of this report to do more than mention the use of such methods. Details may be found in most standard statistical textbooks.[7] However, it must be emphasized again that for conclusive results by either of these two methods, a large amount of data must be available, and this was hardly true in the case of the present study, as can be seen from Table 1.

Nevertheless, both methods indicate the same results, namely, that the only important variable related to the displacement is the wavelength, and in the case of the least squares method the first power of the wavelength is the important factor. No other combination of the three variables seems to produce significantly better results.

Surprisingly enough, this appears to suggest that the zonal velocity $\bar{U}$, which enters Rossby's formula, is not an important variable. However, it should be stated that in studying the problem by means of graphical correlation methods, it was found that on eliminating certain questionable data, $\bar{U}$ became important[8]. Furthermore, the trough to ridge value of $\bar{U}$ was used in this particular study, and was probably somewhat unrepresentative, as indicated previously. Therefore, these results cannot be interpreted to mean that $\bar{U}$ is not important in determining trough displacement.

The correlations of observed displacement with wavelength are shown in Table 4(page 9). The correlations show a negative relationship, as is to be expected from the Rossby formula.

These correlations were made using essentially the same data as in the case of the Rossby formula (Table 3, page 9). The greater number of cases for a full-week (Table 4) is due to the fact that cases classed as not in the westerlies were included.

In comparing tables 3 and 4, it is interesting to note that, while correlations for a full week using wavelength alone average slightly higher than the corresponding ones for the Rossby formula, the half week correlations average slightly lower. Furthermore, unlike the case of the Rossby formula there does not seem to be a systematic decrease in the correlations with time.

III. Statistical Study of the Rossby Formula Assuming Changing Velocity

A. Formula for Trough Displacement.

It is now of interest to consider a slightly different approach to the problem of trough displacement which may throw light on the reasons for the differences in the correlations of Tables 3 and 4, and may also help to explain other observed peculiarities of trough behavior.

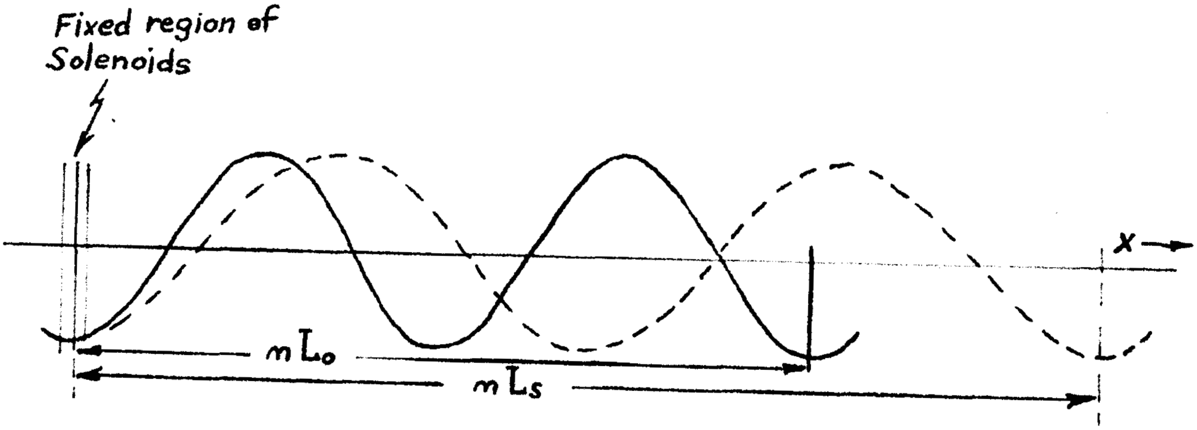
It will be remembered that one of the assumptions made in regard to the use of the Rossby formula in Part I of this report was that the speed of the troughs remains constant during the period covered by the extrapolation.

However, the behavior of individual troughs at 10,000 feet as shown on observed mean five-day 10,000 feet charts, reveals that troughs seldom move with constant velocity, but rather may move eastward for three or four days or perhaps even a week, and then come to a stop or perhaps retrograde. It is particularly true in the summer season that troughs move in an apparently irregular fashion, with rapid changes in both speed and direction of motion. Even in the winter season, when the simple assumption of constant velocity seems more justified, troughs do not continue to move in the same direction for very long periods of time. It seems desirable, then, to develop some method to account for the changing velocity of the troughs.

According to formula (1) the velocity of a trough may change either because of a change in speed of the westerlies ($\bar{U}$) or because of a change in wavelength (L). It is obvious from a study of observed charts that such changes actually occur, but in order to make use of them in obtaining a better estimate of displacement, it is necessary to show that they can be expressed as a systematic function of time. At present, there is no known theory to account for a systematic change in the speed of the westerlies, but there is a very simple scheme to account for a systematic change in wavelength. In fact, this scheme follows directly from the theory of trough formation. According to ideas expressed by Rossby and others, troughs in the westerlies form when an approximately west-to-east current of air strikes a region where there is a pronounced meridional concentration of solenoids. Such a region is to be found off the east coast of Asia and tends to remain fixed for long periods of time. The air current acquires vorticity on passing through this region and a wave-train is set up to leeward which, in a completely balanced state, will have a stationary wavelength (L_s)

given by formula (1), where C is set equal to zero. Let us suppose, however, that at some initial moment the wave-train is not completely balanced, but has a wavelength shorter than the stationary wavelength (see Fig. 1). All the troughs except the one fixed by the solenoid field will than start to move eastward with a velocity given by formula (1).

Fig. 1



Let us now consider the particular trough which, at the initial moment, is separated from the solenoid field by n whole wavelengths. In other words, its distance from the solenoid field is:

(4) $$x_0 = n\ L_0$$

where x_0 is the coordinate of the trough line on the east-west axis, n is the number of whole wavelengths between trough and solenoid field, and L_0 is the wavelength of the wave-train at the initial moment. We will assume that as this trough moves eastward there will be no change in the speed of the westerlies, and there will be no formation of new troughs or disappearance of old ones.

In this case, it is obvious that since the field of solenoids remains fixed, the wavelength must increase as the trough moves eastward, and therefore the speed of the trough must decrease in accordance with formula (1). The trough will continue to move eastward, however, until the wavelength is equal to the stationary wavelength (L_s) at which time the distance of the trough from the solenoid field will be:

(5) $$x_s = n L_s$$

The total displacement of the trough will then be:

(6) $$S_t = n L_s - n L_o = n (L_s - L_o)$$

This very simple and obvious formula states that the trough cannot go on moving indefinitely, but can only move a maximum distance given by formula (6). Furthermore, it is clear that this formula does not depend on any particular theory of trough motion but depends merely on the assumption that there is some stationary wavelength at which the wave-train will be completely balanced. It is interesting to point out, in connection with formula (6) that if n and L_s do not vary greatly from one case to the next, then the displacement will be a linear, negative function of the wavelength, as suggested by the results of Part II.

However, we are interested in knowing what the displacement will be for any given time interval t. In order to obtain a formula for displacement as a function of time, it is necessary to adopt some theory of wave motion. Such a theory is expressed in formula (1). Before we use formula (1) in this connection, it is necessary to point out that, in so doing, we are violating one of the assumptions involved in its derivation. This is the assumption that there will be no change in shape of the wave perturbations, and the waves we are considering definitely change their shape as the wavelength changes. However, we will assume that changes in wavelength are so slow that they will not constitute a serious violation of this assumption.

Considering a given trough at any time after the initial moment, we have, then:

$$x = nL \tag{7}$$

$$C = \frac{dx}{dt} = \bar{U} - \frac{\beta L^2}{4\pi^2} \tag{8}$$

Substituting (7) in (8) we get

$$\frac{dx}{dt} = \bar{U} - \frac{\beta x^2}{4\pi^2 n^2} \tag{9}$$

If we assume that $\bar{U}$ and n are constant, we can integrate the above equation, and after consideration of the initial conditions, we obtain the following equation for the displacement as a function of time:

$$E = \frac{n(L_s - L_0)\left(1 - e^{-Bt}\right)}{\left(\frac{L_s - L_0}{L_s + L_0}\right) e^{-Bt} + 1} \qquad B = \frac{\beta L_s}{2\pi^2 n} \tag{10}$$

where all the terms have been defined previously.

It will be realized, of course, that n and L_s will vary from one individual case to the next. In fact, as will be shown later, n does not even have to be a whole number, because ridges as well as troughs may form the fixed point of the wave-train.

B. Qualitative Verification of Displacement Formula

For the purpose of the following statistical discussion, it may be pointed out that under certain special conditions given below, the displacement obtained from formula (10) will be a linear, negative function of wavelength for any given time interval.

First, it is necessary that the term $\frac{(L_s - L_o)}{(L_s + L_o)} e^{-\beta t}$ be small in comparison to unity. This will be true when the observed wavelength is close to the stationary wavelength and for fairly large intervals of time. Within the range of wavelengths actually observed on mean maps, and for time intervals of a half-week or more, the above term is generally quite small in comparison to unity, but its omission may, in special cases, introduce errors up to 40%. Nevertheless, it will be assumed in the discussion which follows that in the first approximation it may be safely neglected.

Next, we may make the same approximations as in the case of the simple formula (6), and assume that n and L_s do not vary greatly from one individual case to the next. Under these circumstances, the displacement will be a linear function of wavelength for any given time interval.

It may be pointed out that, for the purpose of statistical analysis, it is not necessary that n and L_s remain constant. As long as the frequency distributions of n and L_s for a large number of cases correspond to the ideal "normal" distribution, and as long as these two variables are not related functionally to the variable L_o, then the regression of displacement on wavelength will still be linear, and as will be discussed later, we may determine the average values of n and L_s from these regression equations.

It is now desirable to emphasize a few of the properties of a family of lines of the form:

$$S = n\,(L_s - L_o)\,(1 - e^{-\beta t}) \tag{11}$$

where average values are used for n and L_s. In Figure 2, the lines have been drawn assuming n and β equal to unity, and L_s equal to 100. The figures inserted in the lines indicate the corresponding value of time (t).

Some of the obvious properties of these lines are that they all have the same intercept on the x axis, where the wavelength is equal to the stationary wavelength. Furthermore, due to the exponential character of the function, the displacement corresponding to a given wavelength will not double when the time interval is doubled.

It is now interesting to compare these theoretical lines qualitatively with the observed linear regressions of displacement on wavelength for a half week and a full week (Fig. 3). These regression lines were computed using the same data as for the correlations in line 1 of Table 4. It will be noted that qualitatively the observed behavior of troughs seems to agree with the theoretical behavior illustrated by Fig. 2. The two regression lines have nearly the same intercept on the x axis (within 2 degrees longitude), and the displacement for a full week for a given wavelength is not twice the displacement for a half week. The corresponding regression curves for the other three seasons show approximately the same results. However, corresponding to the low correlation of the summer season, there is a rather large discrepancy of nine degrees in the intercepts on the x axis. The same statistical treatment was applied to two other latitude circles, showing essentially the same results.

C. <u>Determination of Constants in Displacement Formula</u>

It might now be supposed that this theory of trough displacement can be tested quantatively by simply substituting the proper values of B and L_s from the Rossby formula into formula (11) and assigning some arbitrary value to n. More will be said about the problem of the quantatitive verification of the theory, but it must be pointed out here that it cannot be done in the simple manner outlined above.

Not only is it not possible to determine the average value of n in this way, but it will be remembered from Part I of this report that, while the Rossby formula seems to be of the correct form, it does not give results of the correct magnitude. Therefore, it is necessary to determine the three constants n, L_m, and B from the regression curves themselves. Furthermore, since there are three unknown constants, while only two can be determined from each regression curve, it is necessary to make use of both curves to determine them. It is also obvious that, in order to get a unique solution for the constants, it is necessary that both curves pass through the same point on the x axis. Therefore, the full-week regression, as being assumed the most unreliable of the two, has been arbitrarily adjusted by shifting it parallel to itself along the y axis until it passes through the point representing the x intercept of the half-week curve. This is illustrated by the dotted line in Figure 3, and it will be noted how small an adjustment is necessary. The final arithmetical process of determining the constants, given formula (11) and the two regression equations for a half-week and a full-week, is perfectly straightforward although somewhat tedious, and will not be discussed here. The constants for all seasons of the year are shown in Table 6.

TABLE 6.

Constants in Displacement Formula (Formula 11) Obtained From Two Regression Curves

Lat. 45°N

	n	B	$T_e = 1/B$	L_e
Winter	.62	1.5	.69	98
Spring	.49	2.1	.48	100
Summer	1.1	.55	1.8	75
Fall	1.1	.95	1.1	81

The values of n are shown in Column 1. They average close to 1/2 for winter and spring. This result can probably be explained by the effect of the semi-permanent pressure ridge found over the Rocky Mountain Area. Thus in deriving formulae (6) and (10) it was assumed that a trough is the fixed point of the wave train, and that therefore in an individual case n is always a whole number. However, there are several regions in the atmosphere where ridges, rather than troughs, tend to remain fixed for long periods of time. Such a region is the Rocky Mountain Area of the North American continent. Here, orographic influences, as well as solenoidal fields created by differential heating, tend to the establishment of a semi-permanent ridge. Therefore it is clear that for the bulk of the troughs considered in this study, initially located in the region east of the mountain area, n will be close to 1/2 under circumstances leading to a semi-permanent ridge over the plateau. Thus it is probably the Rocky Mountain ridge which is responsible for the average value of n being close to 1/2.

It is not so easy to explain the larger values of n for the summer and fall seasons. Actually, there is every reason to think that these values should be smaller, if anything. It was pointed out before that motions of troughs seem to be more irregular in the warmer season than in the colder. When we realise that the motion will more nearly approximate motion at constant velocity the larger the values of n, then we should expect n to be larger during the winter season. We must then conclude that the large values in summer and fall may be fictitious. The possible reason for fictitiously large values may be due to the poorer determination of continuity for the ill-defined troughs and ridges of summer and early fall.

In column 2 of table 6 are listed the values of B. This constant is not of such immediate interest as it's reciprocal, listed in column 3. The reciprocal of B is a measure of the "effective" length of time necessary to complete a displacement. Obviously, since the displacement reaches its final value exponentially, the time necessary to complete a displacement is theoretically infinite. In such cases, however, it is customary to choose as a measure of the effective time, the time necessary to complete 2/3 of the final displacement. This is known as the time constant. In this case the time constant (T_e) is given by the reciprocal of B. It is seen to be close to half a week for winter and spring and somewhat larger for the warmer months. These latter values may again be fictitious due to the poor continuity.

In the last column of table 6 is listed the stationary wavelengths obtained from the x intercepts of the half week regression lines. It will be noted that the spring value is slightly larger than that for winter, which may be due to the shortness of the record. The summer value is the smallest, as is to be expected.

It is interesting to compare these values of the stationary wavelength with some other wavelength values obtained from mean charts. These are indicated in Table 7.

TABLE 7

Selected Values of Wavelength Obtained From Mean 10,000 Foot Charts.

	L_s	L_n	L_o
Winter	98	110	83
Spring	100	100	75
Summer	75	82	65
Fall	81	94	76

In the first column are found the stationary wavelengths obtained from the regression curves as listed in Table 6. The second column contains the normal wavelengths (L_n) obtained from normal monthly pressure charts at 10,000 feet (9). These values are obtained by taking twice the difference in longitude between the trough and ridge to its west, on the normal charts. There is no reason to think that these normal wavelengths, which represent stationary conditions, should differ materially from the stationary wavelengths obtained from the regression curves. The slight differences indicated in Table 7 can probably be accounted for by the small amount of data from which L_s was computed. More specifically, if we assume that the stationary wavelength is a direct function of the zonal velocity, as is indicated by the Rossby formula, then the somewhat smaller values in column 1 might be accounted for by lower than normal values of the zonal velocities. The data indicates that the strength of the westerlies were actually below normal for our sample of data, particularly during the summer season.

The last column of table 7 shows the average wavelength (L_o) obtained by simply averaging the individual wavelengths for the 2-1/2 years of data. These values are all considerably smaller than either L_s or L_m, as we should expect, for this accounts for the observed average eastward displacement of the mean troughs.

D. Empirical Displacement Formulas

Let us now return to the constants in Table 6. If we assume that these represent good first approximations to the constants in the complete formula for displacement (formula 10), then we may substitute them in that formula and obtain a final complete expression for displacement. The final displacement formula for the winter season is indicated graphically in Figure 4. Numbers inserted in the curves of Figure 4 represent the time interval in days. However in the formula itself, also shown in the figure, the factor t is the time interval in weeks. Thus the displacement for two days is obtained by substituting 2/7 for t. This set of curves as well as similar ones for the other seasons and for latitudes 35°N and 55°N, are currently used in the Extended Forecast Section as an aid in forecasting mean trough motions. Because of lack of data the curves for 55°N are based on only two seasons, the warm season (April to September) and cold season (October to March).

In view of the possible usefulness of such formulae to others, the constants in formula (10) for latitudes 35°N and 55°N are listed in Table 8. Also included in the same table are the correlations and number of observations upon which the corresponding constants are based.

TABLE 8

Constants in the Formula for Trough Displacement and Other Pertinent Data, for Latitudes 35°N and 55°N

	Full week		Half week		Constants		
35° N	r	#obs.	r	#obs.	n	B	L_e
Winter	-.32	47	-.39	75	.28	2.41	82
Spring	-.51	62	-.50	80	.57	1.36	74
Summer	-.25	59	-.19	71	.81	0.29	54
Fall	-.54	29	-.61	41	.56	4.80	67
	Full week		Half week		Constants		
55°N	r	#obs.	r	#obs.	n	B	Le
Cold Season	-.43	18	-.61	26	.32	5.12	101
Warm Season	-.61	28	-.53	38	.47	.80	113

In order to obtain the complete formula for displacement, the constants in Table 8 are simply substituted for the corresponding values in formula (10). It should be remembered that Le, as well as trough displacement (δ) and current wavelength (L_c) are always expressed in degrees longitude at the appropriate latitude. The factor t is the time interval in weeks. It is of course obvious that the practical usefulness of the formulae is limited by the magnitude of the correlations upon which they are based. This is especially true at 35°N and in the summer season at 35°N, 45°N, and 55°N. It should be also noted that the correlations at 55°N are based on a small amount of data. More will be said in the following paragraphs about the use of the displacement formulae in practice.

E. Example of Application of Formulae

An individual example of the application of the fall displacement graph is illustrated in Figure 5. In this figure the ordinate is the date corresponding to the last day of the 5-day period, and the abscissa is the longitudinal position of the trough. The observed trough positions are indicated by dots, while those obtained from the fall displacement graph, using the observed wavelength for the period ending November 7, are indicated by crosses. This case was not included in the data from which the graph was computed. It was chosen because it was a fairly simple case, with n approximately equal to 1/2, as shown by the fixed position of the plateau ridge throughout the period. The largest discrepancy between computed and observed position is 4 degrees, on November 10. The other three points are within two degrees of the observed position.

F. Refinement of Extrapolation Formulae

As explained previously, the derivation of the empirical displacement formulae was based on the assumption that one can obtain the correct average values of n and L_s and the correct value for B even though n and L_s vary in individual cases. If this assumption is justified, then it is clear that one can improve the estimate of displacement if it is possible to find the values of n and L_s in individual cases. It was mentioned previously that if the stationary wavelength (L_s) were a direct function of the zonal velocity, then values of the zonal velocity greater than normal should be accompanied by greater than normal values of L_s. Therefore, this will result in values of the displacement greater than those obtained from the graphs. In order to test this idea the stationary wavelength as given by the Rossby formula was used. However, it will be remembered that observed values of the 10,000 foot index are too small to satisfy the formula, and will therefore result in values of the stationary wavelength which are also too small. In order to correct this scale error, the Rossby stationary wavelength may be multiplied by the ratio between the observed average stationary wavelength (Table 6) and that obtained by substituting the normal value of $\bar{U}$ in the Rossby formula. The resulting equation for the stationary wavelength for winter is:

(12) $$L_s = 1.53\,L_{sr} \qquad L_{sr} = 2\pi\sqrt{\bar{U}/\beta}$$

This formula was substituted for L_s in the empirical formula for winter and the resulting equation used to compute the half-week displacement. Unfortunately the results were not as good as those obtained using wavelength alone. This may have been due to a poor sample in this particular case, for more recent tests suggest that the zonal velocity is important as a supplement to the wavelength graphs[10]. As a working rule it is assumed that greater than normal values of the zonal westerlies will result in somewhat greater displacements than those obtained from the graphs using wavelength alone.

An attempt has also been made to determine values of n in individual cases. General synoptic experience, obtained by inspection of the character and motion of mean waves, can often be used to tell which trough or ridge is likely to remain fixed. For example, it is well known that pronounced cold troughs or warm ridges, extending to high levels in the atmosphere, can be depended on to remain fixed for long periods of time. An attempt has also been made to determine n by the relationship between the troughs and ridges and the mean solenoid fields between sea level and 10,000 ft., but preliminary results do not seem very promising.

C. Quantitative Verification of Displacement Formula

It is now desirable to discuss further the quantitative verification of formula (10). In deriving the empirical formula for trough displacement it was assumed that to the first approximation the relationship between displacement and wavelength was linear. This was done in order to simplify the statistical problem. However, it is possible, although considerably more difficult, to fit the complete formula (10) to the data for a half week by the method of least squares. Since in doing so the term in the denominator is included in the process of curve fitting, all three constants will be obtained directly. One may then use the resulting empirical formula to obtain an estimate of the full week displacements and compare the results with those observed. However, a curve of the form of equation (10) can be treated in this way only with difficulty, and furthermore, due to the smallness of the term in the denominator, a large amount of data is necessary in order to obtain conclusive results. Therefore this method has not been used.

Another method is to use the empirical formulae already derived to obtain the displacement for a different time interval. These hypothetical displacements can then be compared with those observed. This was done, using a time interval of two days. Because the construction of mean maps two days apart was only begun about the first of April, 1943, the amount of data is necessarily limited. The correlation between observed two-day displacement and that obtained from the empirical formula for winter, together with the number of observations is:

$$r_{\psi} = .67 \qquad n = 16$$

Thus the correlation between computed and observed displacement is about what we should expect, and therefore seems to constitute a satisfactory verification of the empirical formula. However, when we examine a graph of observed displacement plotted against computed displacement, we see that the regression line runs parallel to the line representing perfect agreement, but somewhat to one side, indicating that in this case the observed displacements averaged somewhat greater than the computed ones. This may indicate an error in the theory but it may also suggest that during the short period covered by the test the 3 km index was much higher than normal. This was actually found to be the case.

In discussing the question of the validity of formula (10) as applied to trough displacement, it is appropriate to mention another hypothesis which is of quite a different nature but which could possibly lead to essentially the same results. This is a hypothesis based on statistical reasoning alone, and may be called the "null" hypothesis. It is based on the known fact that if one selects a series of random values of a normally distributed variate, then the probability that a given value will be followed by another closer to the average is greater than one-half. In fact it can be shown that if the observed change from one value to the next is correlated with the departure of the first value from the average, a coefficient of 0.71 will be obtained. This reasoning may be applied to trough displacement if it is assumed that there is some preferred, or normal, region of trough concentration. Troughs to the west of this position, for example, would have a probability greater than 1/2 of moving eastward, and if there were no relation between the positions of a trough on two successive maps, the correlation between departure of trough position from normal and displacement from one map to the next would be 0.71. Due to the observed fact that there is serial correlation between successive positions of troughs the above coefficient would be somewhat less than .71. It can also easily be shown that if this "null" hypothesis is correct the regression lines of observed displacement on departure from normal, both for a full week and a half week, will have the same intercept on the x axis, and the full week regression line will have the greater slope. Furthermore, since the ridge over western North America also tends to concentrate about a normal position, the wavelength will be negatively correlated with the departure of the trough position from normal. This means that the above statistical hypothesis is not independent of the more physical one expressed in formula (10), which may be called the wavelength hypothesis. That is, if one hypothesis is correct, then positive results will be obtained upon adopting the other. Thus it might be postulated that this accounts for the success obtained on correlating displacement with wavelength. The only way to find out which hypothesis is closer to the truth is to see which one produces the best results.

A comparison of correlations obtained by these two hypotheses for approximately the same set of data is shown in Table 9.

TABLE 9

Correlations For Summer And Winter Season Based On Two Different Hypotheses Of Trough Displacement

	Full Week				Half Week			
	"Null" method		"Wave-length" method		"Null" method		"Wave-length" method	
	r	n	r	n	r	n	r	n
Winter	.51	36	-.64	52	.64	36	-.71	36
Summer	.17	29	-.38	39	.18	29	-.33	29

It will be seen that the correlations for the "wavelength" method (obtained from Table 4) are higher in all cases.

H. Application to Day to Day Trough Displacements

It should be emphasized that the formulas for trough displacement presented above apply only to five-day mean 10,000 foot charts. No attempt has been made to find out how they should be modified for use on daily synoptic charts.

However, it is of interest to point out that the Rossby trough formula may have more success when applied to mean charts than when applied to daily charts. Thus, if we may assume that the formula is theoretically as valid for mean charts as for daily charts, then it should produce better results because of the much better approximation to ideal sinusoidal waves.

Statistical evidence apparently favoring this idea is contained in research recently completed at the Massachusetts Institute of Technology. Using different data, both Brooks(11) and Yehle(12) found extremely low correlations when applying the Rossby formula to the 24-hour motion of daily troughs.

A somewhat different approach to daily trough displacement was made at M.I.T. by Lieutenant Craig (in an unfinished study). Instead of using for a value of wavelength the distance between the trough concerned and the next trough to the west he used the distance between the trough and the next trough having approximately the same depth. On correlating this value of the wavelength with the subsequent 24-hour motion he found considerably higher coefficients than those obtained by Brooks and Yehle. It is possible that this result is due to the fact that the selection of the most prominent troughs is equivalent to smoothing the pressure patterns by constructing five-day means.

I. Summary of Results

The principle results of this section of the report may be summarized as follows:

1. Statistics indicate a definite correlation between displacement and wavelength, as indicated by the Rossby formula, but they also indicate a systematic variation of velocity with time.

2. These statistical results agree qualitatively with a theoretical formula for displacement based on the Rossby formula (Formula 10).

3. Preliminary quantitative tests of the formula further verify the theory.

4. Although the results of an application of formula (10) based on wavelength alone are not materially better than those assuming constant velocity, they may be improved qualitatively by consideration of the strength of the zonal westerlies and other factors.

IV. Conclusions

The importance of the results outlined in this report does not lie in the correlation coefficients of Table 4, upon which the graphs of trough displacement depend, for even the largest of these coefficients accounts for only 50% of the variability in trough displacement. The significance of the results lies in the apparent verification of the theory, and in the promise that if better methods can be developed for determining stationary wavelengths and areas of semi-permanent troughs or ridges in individual cases, then correlations such as those in Tables 3 and 4 will be materially improved. Recently a theoretical study has been completed in the Extended Forecast Section(13), which suggests that amplitude, as well as zonal velocity, is important in determining the stationary wavelength. Preliminary tests of this modified theory suggest that it may have some advantages over the Rossby method, but these are so doubtful and inconclusive that they have not been presented here.

It must also be admitted that any theory leading to improved methods of estimating stationary wavelength, will without doubt also lead to different formulae for trough velocity, and therefore trough displacement. But it is not likely that the formulae will differ materially from formula (10).

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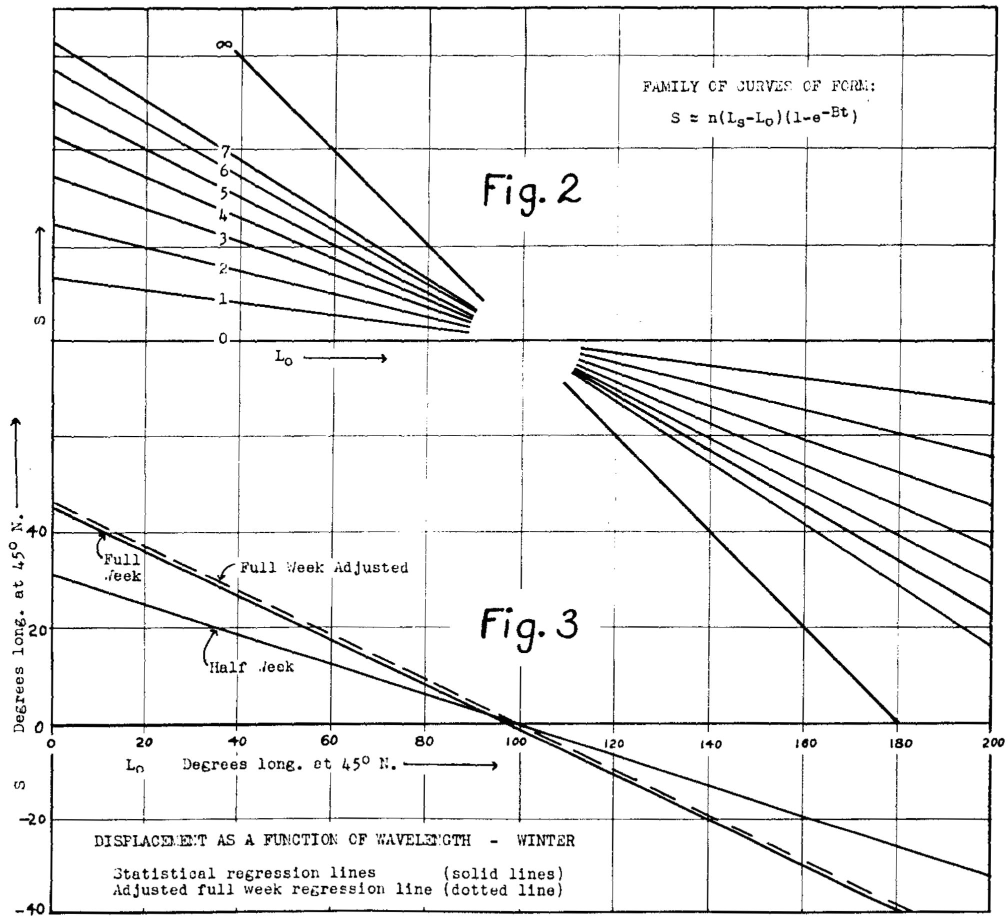
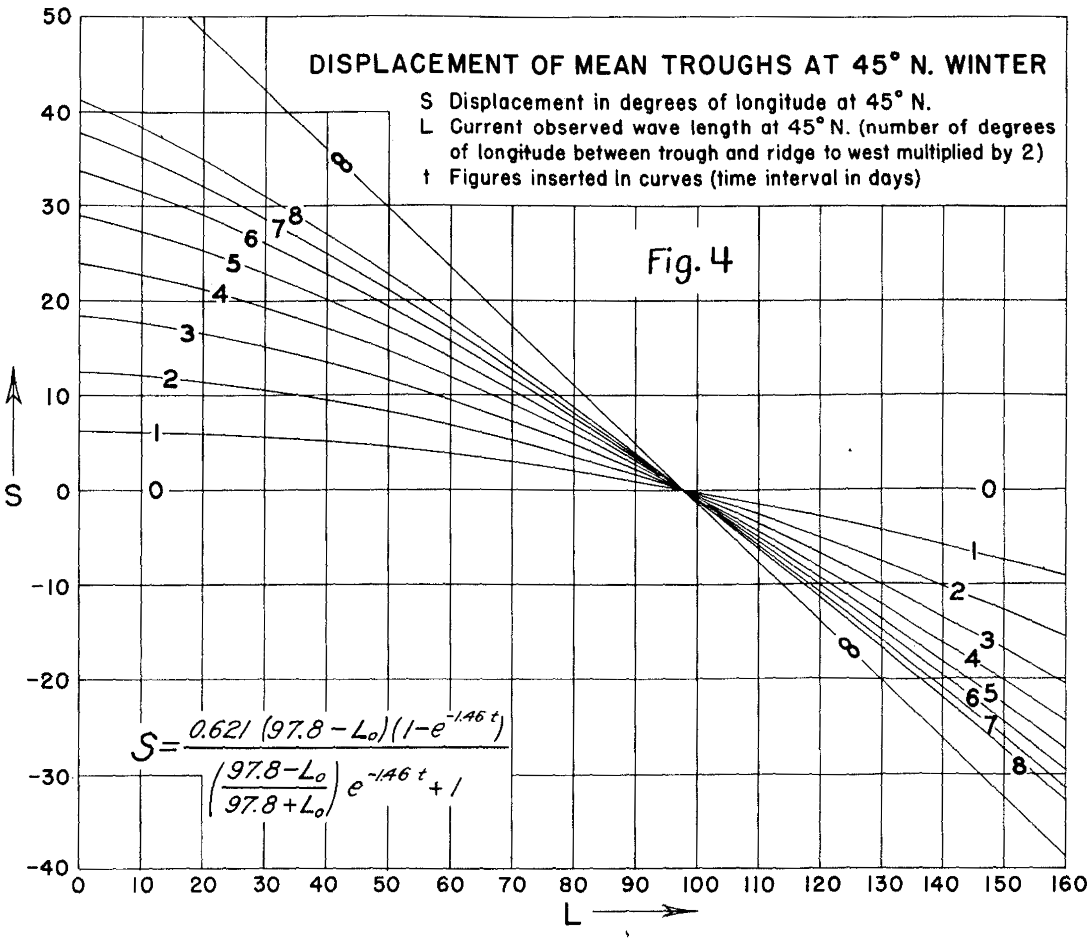


FAMILY OF CURVES OF FORM:
$S = n(L_s - L_o)(1 - e^{-Bt})$
Fig. 2
Fig. 3
L_o
Full Week
Full Week Adjusted
Half Week
Degrees long. at 45° N.
L_o Degrees long. at 45° N.
S
DISPLACEMENT AS A FUNCTION OF WAVELENGTH - WINTER
Statistical regression lines (solid lines)
Adjusted full week regression line (dotted line)



DISPLACEMENT OF MEAN TROUGHS AT 45° N. WINTER
S Displacement in degrees of longitude at 45° N.
L Current observed wave length at 45° N. (number of degrees of longitude between trough and ridge to west multiplied by 2)
t Figures inserted in curves (time interval in days)
Fig. 4
$S = \dfrac{0.621\,(97.8 - L_o)(1 - e^{-1.46\,t})}{\left(\dfrac{97.8 - L_o}{97.8 + L_o}\right) e^{-1.46\,t} + 1}$
S
L

DISPLACEMENT OF INDIVIDUAL MEAN TROUGH

November 7 to 17, 1943