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RESEARCH PAPER NO. 35

The Application of the Hydraulic Analogy to Certain Atmospheric Flow Problems

by
MORRIS TEPPER

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PREFACE

In studying a meteorological paper, the reader is prone to classify it in his mind either as "theoretical" or "practical"; and, depending on his own background, interest, and specific professional assignment, may peruse it in the minutest detail or leaf through it with hardly more than the mildest of interest. This is quite understandable. Meteorology, today, is split between two groups. One is characterized by a researcher who is primarily interested in the physical aspects of cause and effect and in the understanding of the mechanics of weather processes. He attacks his problem by the mathematical manipulation of basic physical principles which he has applied to some model considered to contain the essence of the meteorological situation. The second group, on the other hand, is characterized by the practicing meteorologist whose primary concern is in the interpretation of the weather today in terms of a prognosis for the weather tomorrow. His tools may not have mathematical rigor nor adhere to strict physical principles, but they are time-tested and for the most part, they work. It is only natural then that the theoretical meteorologist would tend to have little patience with the empiricism of a "practical" paper and the practicing meteorologist would consider of little value the mental exercise of a "theoretical" paper.

This situation is unfortunate. Although the immediate efforts of these two groups are directed along different lines, the final goal is basically the same, namely, to unravel the intricate pattern of the cause and effect of the weather. Thus, the findings of either group must be a step forward toward this final goal. The importance of an active interchange of ideas and information between these groups was emphasized at a symposium on this problem in a recent American Meteorological Society meeting.[1] The author has intended this paper to fall within the scope of both of these groups. In leafing through the paper, the reader will find both mathematical formulae and synoptic maps. It would be unfortunate, however, if the sections on the mathematical development were to be read only by the "theoretical" meteorologist and the sections containing the applications only by the "practical" meteorologist. Such a dissection would undoubtedly defeat the purpose of this paper.

The author is particularly anxious to interest the forecaster, the meteorologist who concerns himself with the analysis of weather information on a current basis, in the ideas suggested here. In the last analysis, the only measure of the usefulness of a theory is in the extent of its applicability in explaining weather situations and in anticipating their occurrence. The efforts presented here are admittedly limited. Nevertheless, it is believed that a start has been made. It is hoped that those meteorologists who analyze today's weather and forecast tomorrow's will find some additional analytical and forecast tools here; by understanding the basis for the proposals made here will enlarge the range of the applications; and as a result of their own rich background in synoptic meteorology will arrive at a more complete set of useful forecasting criteria.

[1] Symposium on Coordinating Meteorological Research and Weather Forecasting, 115th National Meeting, American Meteorological Society, Washington, D. C., May 1952.

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THE APPLICATION OF THE HYDRAULIC ANALOGY TO CERTAIN ATMOSPHERIC FLOW PROBLEMS

MORRIS TEPPER

U. S. Weather Bureau, Washington, D. C.
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ABSTRACT

The nature of the hydraulic analogy is discussed, following which it is applied in the study of two stratified liquids. A "three-layer" atmosphere is then introduced and shown to be equivalent to the two-liquid model. A discussion of the basic assumptions contained in the atmospheric model reveals some of its deficiencies. Suggestions are made for coping with these deficiencies.

In the application of this model to atmospheric flows, a hypothesis is presented to account for the erratic behavior of prefrontal squall lines. By adopting the hydraulic analogy, these meteorological phenomena can be logically explained as manifestations of pressure jump lines, which are the atmospheric analogue of hydraulic jumps in liquid flows and of shock waves in gas flows. Illustrative cases of pressure jump lines reveal consistency with the theory.

In the Summary, suggestions are made relative to the inclusion of the theory discussed in this paper in the day-to-day analysis of the weather. Indications are also given for the extension of the hydraulic analogy beyond the point covered in this paper.

Some mathematical remarks concerning the single-wave problem are given in Appendix A and the conclusions are shown to correspond to those found in the text. Finally, in Appendix B, an analysis of the general steady-state two-liquid problem is given and attention is focused on the results as they pertain to the unsteady-state solution.

INTRODUCTION—THE HYDRAULIC ANALOGY

The term "hydraulic analogy" as used in this paper has a very restricted meaning. The reader is cautioned against interpreting it to mean the analogy to liquid flow in general. The exact meaning of the term will be brought out in the succeeding paragraphs where it will be stressed that "hydraulic analogy" refers only to the comparisons made in the table of analogies (table 1).

With the development and growth of aerodynamics as a science in its own right, and with the particular interest in the behavior of air flight in the supersonic region, it became essential to develop laboratory models through which air flow characteristics could be studied. Since wind tunnels were exceedingly expensive and relatively difficult to operate, a study of the desired information was attempted through some sort of analogy to liquid flow.

Several investigators [35], [43], [44], [45], [64],[1] etc., showed that the equations of motion and continuity for an unsteady state, isentropic, one dimensional perfect gas are completely analogous to the unsteady state, incompressible, one dimensional open channel flow in a gravity field, and furthermore, the momentum and continuity equations are also analogous for steady state conditions across a shock wave in the former and an hydraulic jump in the latter. Thus, the analogy may be applied in two ways. For unsteady state flows, it applies to the region where there are no energy losses (i. e., entropy changes in a gas or turbulence in a liquid). In the steady state it may be generalized to include energy losses provided these losses are not excessive. The details of this analogy are given in the first two columns of table 1 and shall not be elaborated on further at this time. For an appraisal of this analogy the reader is referred to the references cited above as well as to [11], [20],

[1] The numbers in brackets correspond to the list of references to be found at the end of the treatise.

TABLE 1—*Table of analogies—the hydraulic analogy*

	Gas	One liquid	Two liquids	Three-layer atmosphere
Analogous Domains	Hypothetical compressible gas $p=\text{const}\times\rho^{\gamma}$ $\gamma=\frac{c_p}{c_v}=2$	Incompressible fluid with a free surface in a gravity field	Interface separates two incompressible homogeneous liquids (ρ, ρ'); one dimensional motion in lower liquid; level free surface. (See Chapter I)	Three layers with an inversion separating two fluids, each autobarotropic (θ, θ'); one dimensional motion in lower fluid, Coriolis force neglected; $\frac{\partial H}{\partial x}\ll\frac{\partial h}{\partial x}$ (See Chapter II)
Analogous Magnitudes	velocity (u)	velocity (u)	velocity (u)	velocity (u)
	density (ρ)	height of free surface (h)	height of interface (h)	height of inversion (h)
Analogous Concepts	compression wave	wave of elevation	wave of elevation	elevation-type wave
	expansion wave	wave of depression	wave of depression	depression-type wave
	sound velocity $a=\left(\frac{dp}{d\rho}\right)^{\frac{1}{2}}$	wave velocity $c=(gh)^{\frac{1}{2}}$	wave velocity $c=\left[\left(1-\frac{\rho'}{\rho}\right)gh\right]^{\frac{1}{2}}$	wave velocity $c=\left[\left(1-\frac{\theta}{\theta'}\right)gh\right]^{\frac{1}{2}}$
	sonic speed (a)	critical speed (c)	critical speed (c)	critical speed (c)
	Mach No. $M=\frac{u}{\left(\frac{dp}{d\rho}\right)^{\frac{1}{2}}}$	Froude No. $F=\frac{u}{(gh)^{\frac{1}{2}}}$	modified Froude No. $\mathsf{F}=\frac{u}{[(1-\rho'/\rho)gh]^{\frac{1}{2}}}$	modified Froude No. $\mathsf{F}=\frac{u}{[(1-\theta/\theta')gh]^{\frac{1}{2}}}$
	supersonic flow $M>1$	shooting water $F>1$	supercritical flow $\mathsf{F}>1$	supercritical flow $\mathsf{F}>1$
	subsonic flow $M<1$	streaming water $F<1$	subcritical flow $\mathsf{F}<1$	subcritical flow $\mathsf{F}<1$
	shock wave	hydraulic jump	hydraulic jump	pressure jump

and [51]. The accuracy and usefulness of this analogy has been investigated numerically and by laboratory experiments. These have been reported on in [6], [23], [30], [31], [34], [40], [41] and [42], to mention but a few.

The importance of this analogy is reflected through the increased interest in the study of high-velocity flow in a channel and particularly in those features of water flow which are analogous to the desired characteristics of gas flow (see [2], [4], [18], [28], [29], [47], and [57]). For the most part what has been done has been to take the theory of the flow of a compressible gas as given in any current standard textbook on aerodynamics (e. g. [9], [13], and [38]) and to apply these solutions to the flow of water in an open channel. A review of any sample of the papers cited above will reveal that this analogy has been utilized with a high degree of success.

Extending this analogy, we shall show that if we are dealing with two stratified incompressible fluids, under certain conditions, the behavior of the flow and of the interface are analogous to the behavior of the flow and of the free surface in one liquid. (See [14] or ch. I.) Consequently, the table of analogies, table 1, may be extended to the third column.

For the sake of simplicity of solution, meteorologists have adopted the convenience of approximating an autobarotropic atmosphere by an equivalent homogeneous incompressible atmosphere. This approximation is discussed in [5], [14], [15], [22], and [53]. This same approximation is made here and a discussion of the validity of this approximation is given in chapter II. Having made this approximation we may consider columns 3 and 4 of table 1 as analogous.

While the relationships between adjacent columns of table 1 had already been known and utilized by many investigators, it was Freeman [14] who generalized on these relationships by showing that the flow properties of a particular meteorological situation (the subtropical easterlies) are analogous to the flow properties of a supersonic gas. Table 1, the table of analogies, represents the "dictionary" which translates the properties of the flow in one domain into those of another.

In chapter I, we pursue in finer detail the solution and properties of the flow for the domain represented by column 3. The equations of motion and continuity are derived much as Freeman did [14] and [17]; however, in the solution of the simultaneous set of nonlinear partial differential equations and its graphical representation we shall follow Clauser [8]. For a more extensive discussion of the solution of nonlinear partial differential equations by the method of characteristics the reader is referred to [3], [32], and [48], or any standard book on aerodynamics of supersonic flow. The emphasis of this chapter lies not in the actual "solution" of the basic equations, but rather in the extraordinary properties of the flow which are expressed by the characteristic lines.

In chapter II, we investigate the equivalence of columns 3 and 4 of table 1 and make some remarks on the general type of atmospheric problems which can be handled by the methodology outlined in chapter I. The remainder of the treatise deals with specific applications of this theory and is illustrated by case histories which demonstrate certain features of the atmosphere which may be explained very simply and very directly by the theory as outlined in the table of analogies.

There is an additional remark to be made on the general nature of the problem. Basically, the phenomenon discussed here, "the pressure jump line," is an internal gravity wave. Earlier discussions of gravity waves, whether in a compressible or incompressible medium relied extensively on perturbation theory ([25], [26], [36], [37], [50], [55]) or on a symmetrical form for the solution of a solitary finite wave ([10], [37], [56]). It will be exceedingly obvious that the gravity type wave discussed here approximates most nearly an hydraulic jump in appearance—and as such can be described neither as a perturbation nor as a symmetrical wave. This lends considerable support to the explanation of the phenomenon by means of the hydraulic analogy.

It has already been mentioned that for experimental purposes it has proven useful to investigate the properties of the flow of a single liquid in a channel (column 2) in attacking problems in aerodynamics. Similarly, it is recommended to investigate the properties of two stratified liquids in a channel (column 3) in studying the properties of certain atmospheric motions. Such a beginning has been begun under the direction of Dr. George Benton in the Hydraulics Laboratory of the Civil Engineering Department, Johns Hopkins University, and part of the work reported on here is the result of this program in which the author had the pleasure of participating.

CHAPTER I

THE UNSTEADY TWO-LIQUID FLOW PROBLEM

In this chapter we shall investigate with considerable detail the properties of the flow indicated in column 3 of the table of analogies, table 1 of the Introduction. In this flow we assume:

1. Two incompressible fluids, $\rho > \rho'$, with h equal to the height of the interface, describe the nature of the flow domain.
2. The horizontal velocity (u) of the lower liquid is independent of the lateral coordinate (y) and of the vertical coordinate (z).
3. Both the surface ($z=0$) and the top of the upper layer (the free surface, $z=H$) are horizontal surfaces.
4. The pressure is hydrostatic (i. e., vertical accelerations are neglected).

In these assumptions and in the mathematical development to follow, the motion of the upper liquid is not considered. However, as we shall see later, the assumptions listed above together with some of the conclusions which we shall reach imply a definite restriction on the nature of the flow of the upper liquid.

The equation of motion for the lower fluid is given by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

and the equation of continuity by

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu)=0. \tag{2}$$

Since the pressure is hydrostatic, we have

$$p=(H-h)\rho' g+(h-z)\rho g \tag{3}$$

and therefore,

$$\frac{\partial p}{\partial x}=(\rho-\rho')g\frac{\partial h}{\partial x}. \tag{4}$$

Substituting into (1),

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\left(1-\frac{\rho'}{\rho}\right)\frac{\partial h}{\partial x}. \tag{5}$$

Equations (2) and (5) represent our system of simultaneous differential equations, in the dependent variables u and h. For convenience, we introduce a new variable c, given by

$$c^2=\left(1-\frac{\rho'}{\rho}\right)gh. \tag{6}$$

Substituting into (5) and (2) we get

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+2c\frac{\partial c}{\partial x}=0 \tag{7}$$

$$2\frac{\partial c}{\partial t}+2u\frac{\partial c}{\partial x}+c\frac{\partial u}{\partial x}=0. \tag{8}$$

We have thus transformed our original set of partial differential equations in u and h into a set of partial differential equations in u and c.

First adding and then subtracting equations (7) and (8), we arrive at

$$\left[\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}\right](u+2c)=0 \tag{9}$$

$$\left[\frac{\partial}{\partial t}+(u-c)\frac{\partial}{\partial x}\right](u-2c)=0. \tag{10}$$

(Compare equations (9) and (10) with the equation for a compressible isentropic gas given by Lamb [37, p. 482]).

From elementary calculus we may write that if $f=f\{x(t),t\}$ then the *total derivative*

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\frac{dx}{dt}.$$

Thus if we consider any line Γ in the x,t-plane, the total change with time of the function f along this line in the x,t-plane is given by the expression above. In this expression $\frac{dx}{dt}$ refers to the slope[1] of the line Γ at the point where the total derivative is being computed. *Only in the special case where*

[1] The reader is to keep in mind that in this paper a slope of a line in the x,t-plane is always given by $\frac{dx}{dt}$, never by $\frac{dt}{dx}$.

Γ *prescribes the motion of a particle may we interpret* $\frac{dx}{dt}$ *as a particle velocity.* In the more general case $\frac{dx}{dt}$ merely refers to the slope of a line along which the total derivative of a function is desired.

Thus, equations (9) and (10) specify that along those lines in the x,t plane whose slopes are given by $\frac{dx}{dt}=u+c$ for (9) or $\frac{dx}{dt}=u-c$ for (10), the functions $f(x,t)=u+2c$ or $f(x,t)=(u-2c)$ respectively have a zero total derivative with time.[2]

These lines are called *characteristic lines.*

We repeat then, that along the characteristic lines given by

(11) $$\frac{dx}{dt}\ (\text{characteristic line})=u\pm c,$$

we have the relation that

(12) $u\pm 2c=\text{constant}=\alpha,\ \beta$, respectively.

If we suppose for convenience that $c>u$, then figure 1 will represent schematically the statements given in (11) and (12). It should be noted that since u and c may both vary in the field, not only will α and β vary, but the characteristic lines represented by these parameters need not be straight lines. We shall now introduce three definitions which relate to the properties of these characteristic lines.

Definition: A wave is a family of characteristic lines. By (11) and (12) we have the alternate definition that a wave expresses a functional relationship involving the dependent variables in the x,t-plane.

Definition: An identifiable point on the wave is given by a specific value of α (or β).

Definition: The speed of a point on the wave at a given point in the x,t-plane is given by the slope of the characteristic line passing through that point.

Thus, in figure 1, the α family, expressing the functional relationship $u+2c=\text{constant}$, represents a wave travelling in the positive x-direction and the β family, expressing the functional relationship $u-2c=\text{constant}$, represents a wave travelling in the negative x-direction. Furthermore, the progress of any point of the α wave, say α_i, is given by the characteristic line α_i in the

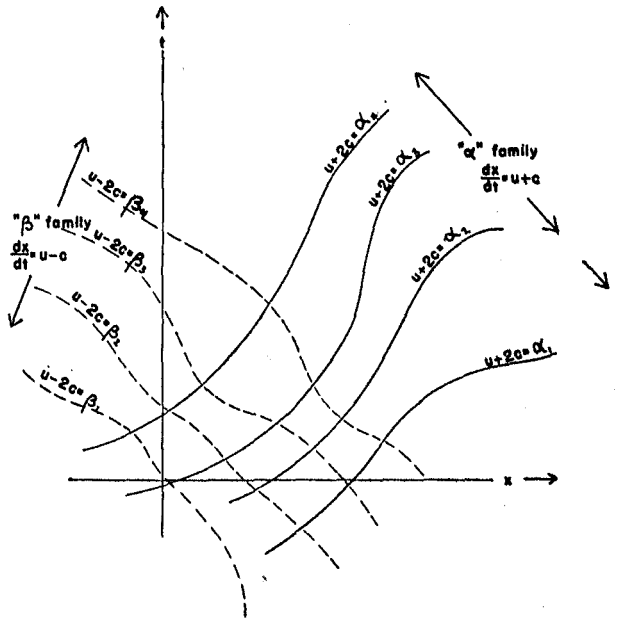

FIGURE 1.—Two-family characteristics diagram.

x,t-plane, and the speed of the point is given by the local slope of the characteristic line α_i.

We shall see later that these definitions are in perfect accord with our physically intuitive concepts of what are a wave, a point on a wave, and the speed of a point on a wave.

SINGLE WAVES

Referring to figure 1, it is apparent that situations may arise (as when $c\gg u$) in which the two families of characteristic lines diverge sufficiently such that after some time the two families no longer intersect. Each family will then represent a single wave travelling in the direction indicated by the slope of the characteristic lines. In the meteorological problems to which this theory has been applied c is usually much larger than u and indeed the evidence seems to be that we are dealing with a single wave only.

Consequently, we shall now restrict our attention to those solutions of our basic problem which shall yield single waves only. For convenience we take the positive x-direction in the direction of the wave propagation. Thus, we can say that we are dealing with the α family only, while the β family is missing. Another way of looking at this is that the "points" on the wave of the β family are indistinguishable one from the other. This implies

[2] A point in the x,t-plane refers to a location and a time for the flow. At this point, the flow has a velocity (u) and a value for c, given by (6) and the interface height. Thus, at every point there is a value of $u+c$ (or $u-c$). The required slopes ($u\pm c$) thus exist.

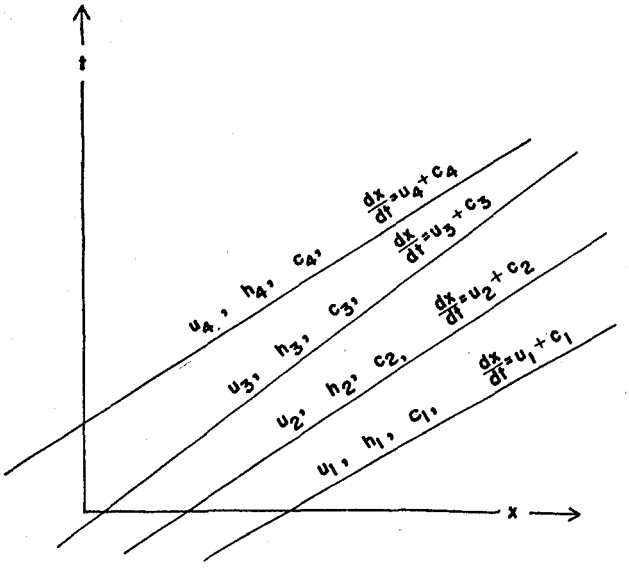

FIGURE 2.—Characteristics diagram of a single wave.

$$\beta_1=\beta_2=\beta_3= \ . \ . \ . \ =\beta.$$

By (12)

$$u-2c=\beta=\text{constant} \tag{13}$$

over the entire x,t-plane or from (6).

$$u-2\left[\left(1-\frac{\rho'}{\rho}\right)gh\right]^{\frac{1}{2}}=\text{constant} \tag{14}$$

over the entire x,t-plane.

If we have given the undisturbed values of the flow velocity u and the interface height h, we may solve for the constant in (14). This equation prescribes that any subsequent value of u defines the corresponding h, and vice versa.

We shall also find useful the following relation derived from (13)

$$\Delta u=2\Delta c \tag{15}$$

over the entire x,t-plane. Through the definition of c, (6), equation (15) prescribes the relation between the variation of u between two points and the corresponding variation in the interface height.

We have then for a single wave

$$\begin{cases} u+2c=\alpha_t \\ u-2c=\beta. \end{cases} \tag{16}$$

Along a given characteristic line, e. g., α_t, these two equations yield a solution for u and c in terms of α_t and β. Since α_t is constant along a characteristic line, we have the extremely important result that both u and c are constant along any characteristic line. Again through (6), this means that not only the flow velocity (u) but the interface height (h) is constant along any characteristic line. Furthermore, from (11) the slope of a characteristic line must be constant at every point of the characteristic line.

Thus, for a single wave, the characteristic lines are straight lines along which u and c (and from (6) h also) are constant. (See fig. 2.)

We shall now demonstrate that the definition and properties of a wave as discussed above are in accord with the customary physical and intuitive ideas about waves. Consider a symmetrical steady state single wave (fig. 3) and its position at various times. The physical picture of the event is given in the diagram to the left. On the right the event has been transposed on to an x,t-diagram, where instead of drawing the wave profile we have merely indicated the height values at the appropriate time and place. Since the wave is steady-state, we may draw parallel lines through the same values of the wave height. Each line represents a specific wave height (c. f., definition of point on a wave) and furthermore the slope of the line is the speed of that wave height (c. f., definition of speed of point on wave). The entire family of lines passes through the wave at any time (c. f., definition of wave) and expresses the functional relationship that along each line the wave height is constant.

METHODS OF SOLUTION

In this section we shall be concerned with the following problems:

1) What distribution of values of flow speed (u) and interface height (h) must be given so that we may find other values of u and h in space and time (x,t-plane)?
2) Given the required distribution, how may we solve for these other values?

We have already seen that by (14) if we have the values of both u and h at one point, then any other value of one variable prescribes the one corresponding value of the other. To find the solutions at other points, we shall consider three possibilities:

Case I. Given (a) an initial value of flow velocity and interface height (u_0, h_0) plus (b) the spacial wave profile (or velocity distribution) at a time ($t=0$) (fig. 4).

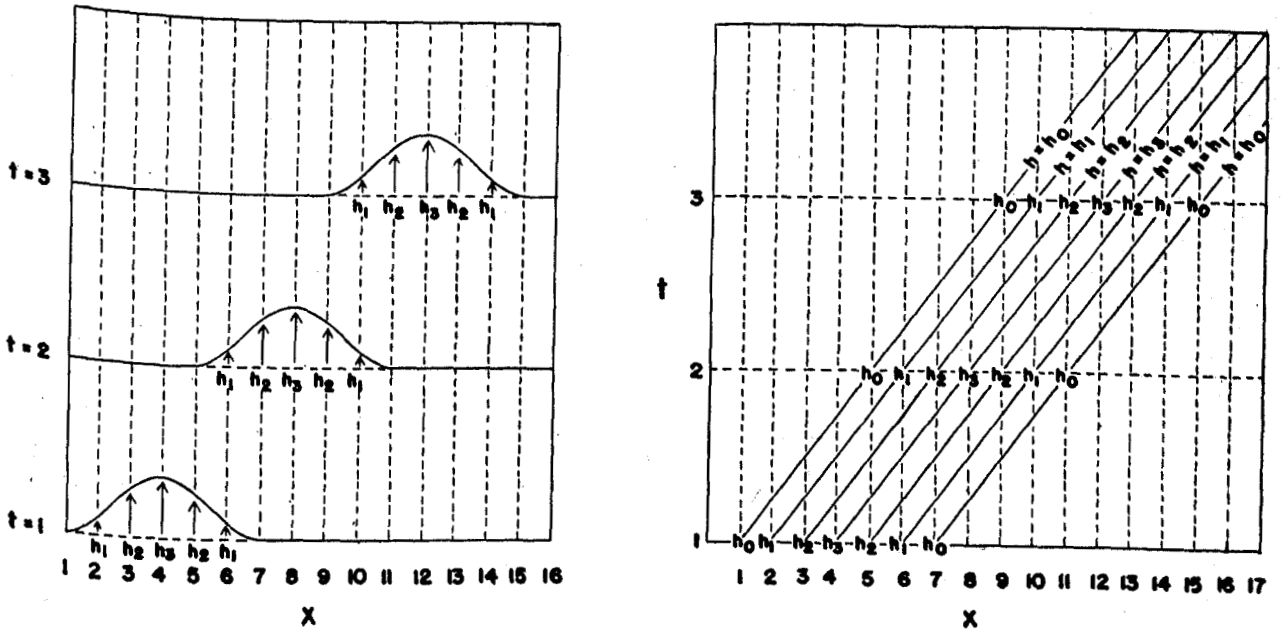

FIGURE 3.—Symmetrical steady-state single wave.

Step 1: Using (6), replace each value of the interface height (h) by its corresponding value of c.

Step 2: Every point on the line $t=0$ now has a value of u and c. At each point construct a line with a slope equal to $(u+c)$. These are characteristic lines along which the flow velocity (u) and the interface height (h) will be constant.

Case II. Given (a) an initial value of the flow velocity and interface height (u_0, h_0) plus (b) the temporal wave profile (or velocity distribution) at a single point ($x=0$) (fig. 5).

Step 1: Using (6), replace each value of the interface height (h) by its corresponding value of c.

Step 2: Every point on the line $x=0$ now has a value of u and c. At each point construct a line with a slope equal to $(u+c)$. These are the characteristic lines along which the flow velocity (u) and the interface height (h) will be constant.

Case III. Given (a) an initial value of the flow velocity and interface height (u_0, h_0) plus (b) a distribution of interface

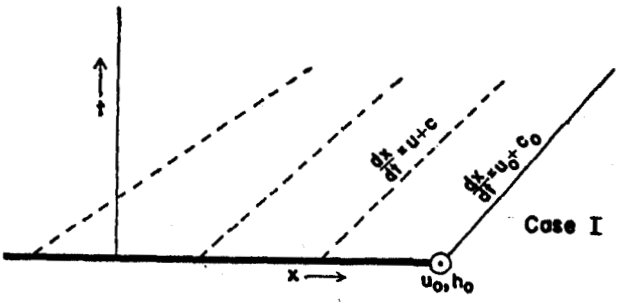

FIGURE 4.—Method of solution, Case I.

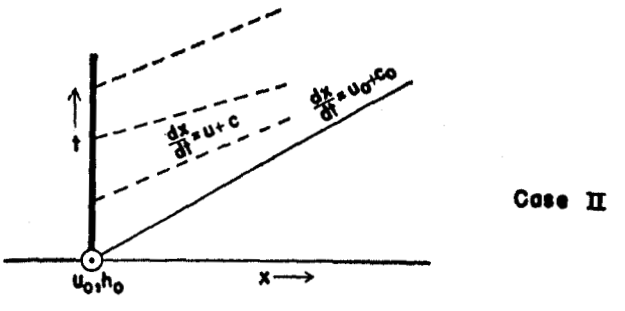

FIGURE 5.—Method of solution, Case II.

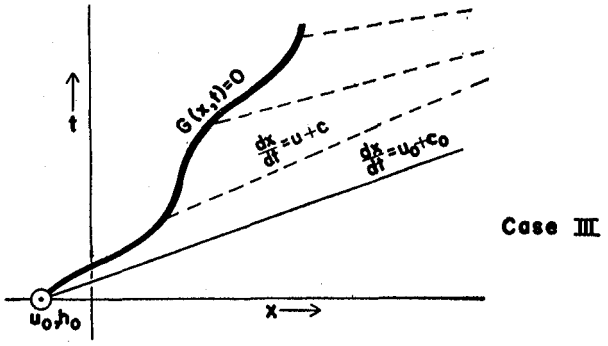

FIGURE 6.—Method of solution, Case III.

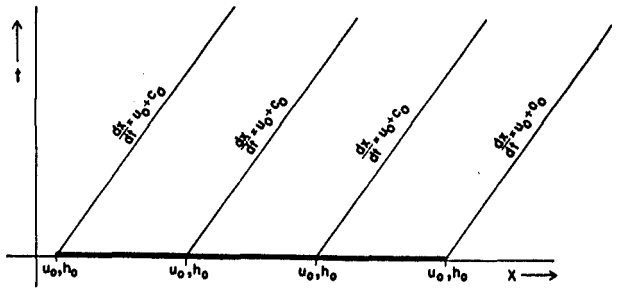

FIGURE 7.—Characteristics diagram of uniform flow.

heights (or velocity) along any arbitrary line $G(x,t)=0$ in the x,t-plane (fig. 6).

Step 1: Using (6) replace each value of the interface height (h) by its corresponding value of c.

Step 2: Every point on the line $G(x,t)=0$ now has a value of u and c. At each point construct a line with a slope equal to $(u+c)$. These are the characteristic lines along which the flow velocity (u) and the interface height (h) will be constant.

(*Note:* The simplest procedure to follow is to convert h into c whenever it appears and to solve for u whenever necessary through the simple relation (15) rather than the more cumbersome (14).)

CLASSES OF WAVES

In the preceding section we noted the procedure to be followed in constructing the characteristic lines of a particular flow when certain initial conditions are given. The general pattern formed by characteristic lines can fall in any one of three classes. In the following discussion we shall consider the initial conditions as for Case I above. The argument for the other cases follows in an identical manner.[3]

UNIFORM FLOW (NO WAVE)

Given: At time $t=0$ a uniform interface height (h_0) or a constant flow velocity (u_0). (Fig. 7.)

By (6) and/or (15) every point on the line $t=0$ has the same value of u and c (i. e., u_0 and c_0). Consequently the characteristic lines are all parallel and all have the identical values of the variables along them. Thus *uniform flow (or no wave) is characterized by parallel characteristic lines.*

DEPRESSION-TYPE WAVE

Given: Initial values of u and h (u_0, h_0) plus at time $t=0$ a distribution of interface height (h) such that h decreases in the negative x-direction in some interval ($x_1 \leq x \leq x_3$) (fig. 8a). Outside of this interval the interface is undisturbed, such that

$$h=h_1 \text{ for } x \leq x_1$$

$$h=h_3 \text{ for } x \geq x_3$$

(Recall that the positive x-direction is always taken in the direction of propagation of the wave.)

Consequently $h_i < h_j$ for $x_1 \leq x_i < x_j \leq x_3$. By (6) and/or (15) it follows that for $x_i < x_j$, $(u+c)_i < (u+c)_j$. Since the slopes[4] of the characteristic lines must decrease as we go toward the negative x-direction, the characteristic lines will fan out in the direction of propagation of the wave.

To see what changes this wave undergoes with increasing time, we take three cuts at time $t=0,1,2$ and using the values given by the characteristic lines, construct the wave profiles (fig. 8b). We note that with increasing time, the slope of a depression-type wave becomes shallower. Thus, *a depression-type wave is characterized by diverging characteristic lines. A depression-type wave becomes shallower with increasing time.*

ELEVATION-TYPE WAVE

Given: Initial values of u and h (u_0, h_0) plus at time $t=0$ a distribution of interface heights (h) such that h increases in the negative x-direction in some finite interval ($x_1 \leq x \leq x_3$) (fig. 9a).

[3] The following is essentially a graphical discussion of the classes of waves and of their properties. A mathematical discussion yielding the same results is given in Appendix A.

[4] See footnote 1, p. 4.

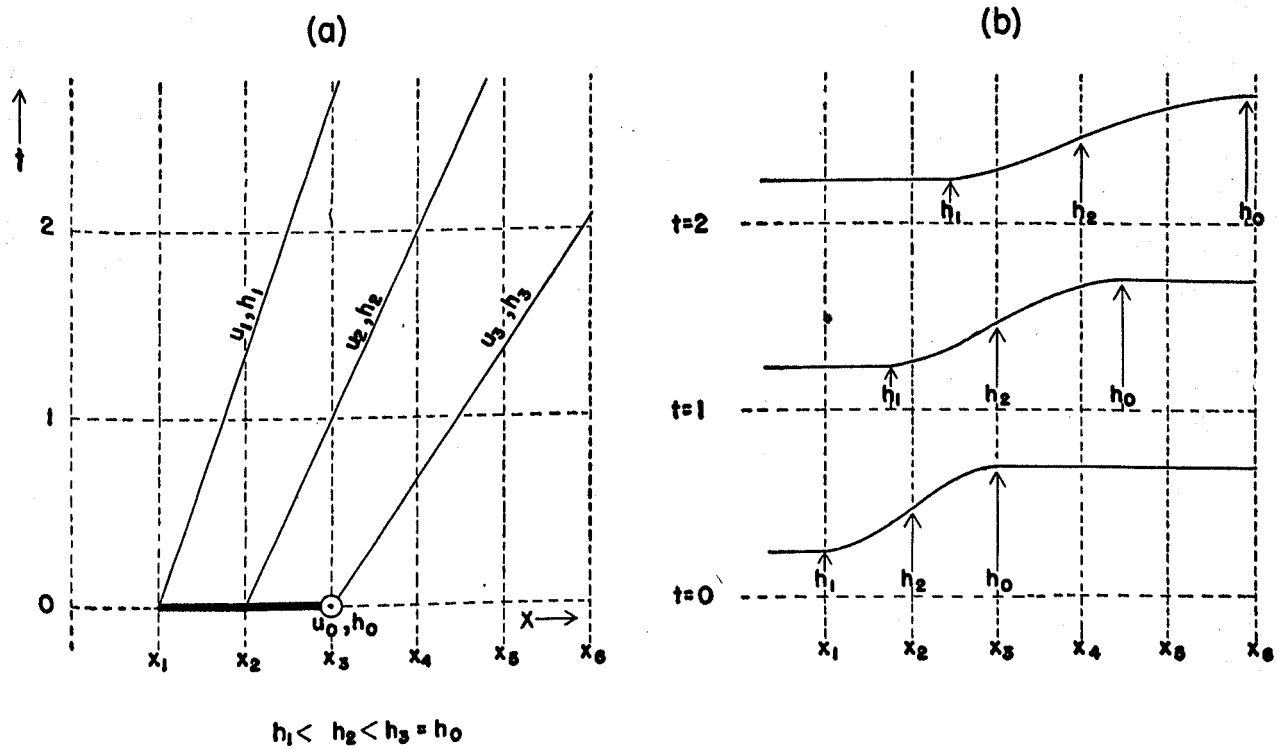

FIGURE 8.—Depression-type wave.

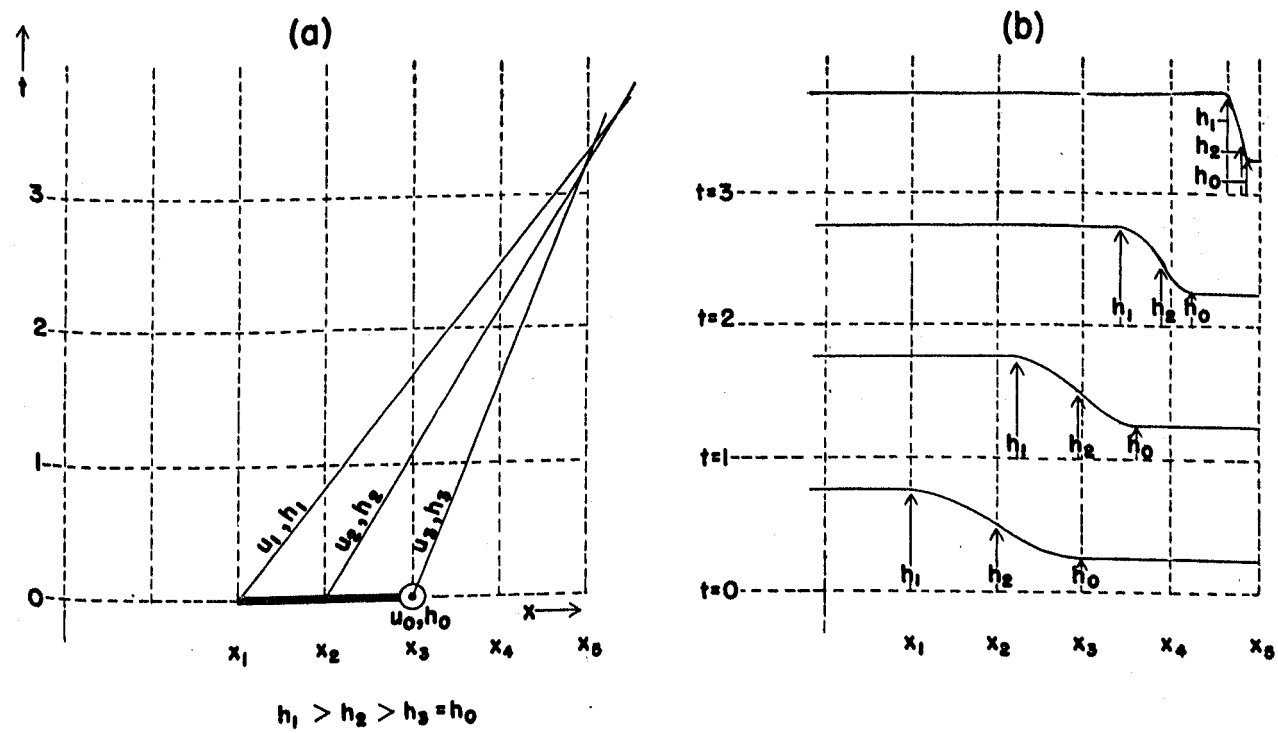

FIGURE 9.—Elevation-type wave.

Again, we suppose that the interface is undisturbed outside of this interval; such that

$$h=h_1 \text{ for } x\leq x_1$$

$$h=h_3 \text{ for } x\geq x_3$$

Consequently $h_i>h_j$ for $x_1\leq x_i<x_j\leq x_3$.

By (6) and/or (15) it follows that for $x_i<x_j$, $(u+c)_i>(u+c)_j$. Since the slopes of the characteristic lines must increase as we go towards the negative x-direction the characteristic lines will converge in the direction of propagation of the wave.

To see what changes this wave undergoes with increasing time, we take four cuts at time $t=0,1,2,3$ and using the values given by the characteristic lines, construct the wave profiles (fig. 9b). We note that with increasing time, the slope of an elevation-type wave becomes steeper and steeper. Thus, *an elevation-type wave is characterized by converging characteristic lines. An elevation-type wave becomes steeper with increasing time.*

THE HYDRAULIC JUMP

Obviously the conditions portrayed by figure 9b cannot proceed very far beyond the point where the wave front has a vertical slope since thereafter we would have denser fluid (the lower fluid) overriding the lighter fluid (the upper fluid). What happens (fig. 10) is that the wave "breaks" similar to the breaking of a wave on a beach. As the wave overturns with considerable turbulence, our assumption 2 relating to the one dimensionality of the flow is violated and the theory ceases to hold.

This situation is handled on the x,t-diagram by drawing the envelope of the converging characteristic lines and attributing the effect of all the characteristics along this line—the hydraulic jump.

If an envelope is not readily defined such as in the intersection of just a few characteristic lines, a good approximation customarily used is the angle bisector of the intersecting characteristic lines.

The slope of the jump line is, of course, the speed of propagation of the jump. From the properties of these three classes we can make an important observation about the nature of waves admissible under our theory. The above discussion implies that if a wave exists, the characteristic lines cannot

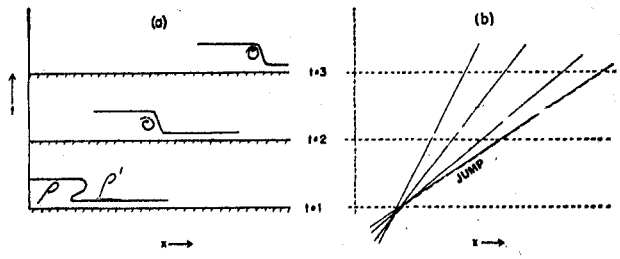

FIGURE 10.—Formation of an hydraulic jump.

be parallel. On the other hand, the discussion on page 8 produced the result that the characteristic lines of a steady state wave are indeed parallel. Consequently, we must exclude the possibility of steady state wave solutions in our theory.

THE NATURE OF THE UPPER FLOW IN A SINGLE WAVE

It is our purpose now to investigate the restrictions which we have implicitly placed on the upper liquid. We shall first demonstrate that the flow of the upper liquid must vary with height (z) in order to satisfy the previous solution:

Assume that the flow of the upper liquid is independent of the vertical coordinate (z). Thus $u'=f(x,t)$, where as before the primes refer to the upper liquid. Therefore

$$\frac{\partial u'}{\partial t}+u'\frac{\partial u'}{\partial x}=-\frac{1}{\rho'}\frac{\partial p'}{\partial x}=g\frac{\partial H}{\partial x}=0,$$

or

$$\frac{du'}{dt}=0. \tag{17}$$

The continuity equation for the upper liquid is

$$\frac{\partial}{\partial t}(H-h)+\frac{\partial}{\partial x}[(H-h)u']=0.$$

Combining this equation with the continuity equation for the lower liquid (2), we have

$$\frac{\partial}{\partial x}[(H-h)u'+hu]=0,$$

which when integrated yields

$$(H-h)u'+hu=k(t).$$

From (14), we conclude that

$$u'=u'(h,t). \tag{18}$$

Case 1.—Assume $u'=u+c$, or the paths of the particles of the upper liquid coincide with the characteristic lines. By (17) this assumption implies that the individual particles in any vertical column of the upper liquid always remain over the same point on the wave. Thus, along the characteristic lines,

$$\frac{\partial h}{\partial t}+u'\frac{\partial h}{\partial x}=0.$$

But from the equation of continuity for the upper liquid

$$\frac{\partial h}{\partial t}+u'\frac{\partial h}{\partial x}+(h-H)\frac{\partial u'}{\partial x}=0.$$

Consequently $\frac{\partial u'}{\partial x}=0$ or $u'=$constant depending on time only.

By our assumption we have the conclusion that $\frac{dx}{dt}=u+c=$constant, at any given time, for all points on the wave. This implies parallel characteristic lines, which means that we have no wave at all.

Case 2.—Assume that the upper liquid particle path lines on the x,t-diagram do not coincide with the characteristic lines ($u'\neq u+c$). At time $t=t_1$ choose two points not on the wave but on the same side of the wave. These two points will have the same velocity U, and by (17) the particle path lines through these two points will be straight parallel lines. By our assumption it follows that there must exist a time $t=t_2$, such that one of the two particle path lines intersects the wave while the other does not.

Let u'_1 be the velocity on the particle path line which intersects the characteristic line

u'_2 be the velocity on the other particle path line

α^* be the characteristic line intersected by the particle path line having the velocity u'_1. All starred quantities refer to the values on the characteristic line α^*.

At time $t=t_2$, we have

$$h^*\neq h_2,$$

where h_2 is the interface height given at time $t=t_2$ on the particle path line with velocity u'_2. If $u'_1(h^*, t_2)\neq u'_2(h_2, t_2)$, we contradict our assumption that $u'_1=u'_2=U$.

If no two points on one side of the wave yield the result that $u'_1(h^*, t_2)\neq u'_2(h_2, t_2)$ then the motion of the upper fluid on that side of the wave must be given by $U=u'_1(h^*)$ for all t. Since h^* is constant in α^*, U must be constant in α^* and all the particle path lines are parallel to the characteristic line α^*. The argument can now be repeated for any other characteristic line α^{**}. But the particle path lines cannot be parallel to two characteristic lines at the same time, and we reach another contradiction. Thus, $u'\neq u+c$ is also not valid.

Since neither $u'=u+c$ or $u'\neq u+c$ is possible, we conclude that our original premise is not valid and the upper flow must depend on the z coordinate. However, we shall show that even here we must impose restrictions. If the upper liquid is not independent of the vertical coordinate, we may write

$$\frac{\partial u'}{\partial t}+u'\frac{\partial u'}{\partial x}+w'\frac{\partial u'}{\partial z}=-\frac{1}{\rho'}\frac{\partial p'}{\partial x}=g\frac{\partial H}{\partial x}=0$$

$$\frac{\partial u'}{\partial t}+\frac{\partial}{\partial x}(u')^2+\frac{\partial}{\partial z}(u'w')-u'\left(\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}\right)=0.$$

Integrating from h to H after dropping $\left(\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}\right)$ due to continuity considerations, we have

$$\int_h^H\frac{\partial u'}{\partial t}dz+\int_h^H\frac{\partial u'^2}{\partial x}dz-u'_h w'_h=0.$$

Reversing the process of differentiation and integration, we have

$$\frac{\partial}{\partial t}\int_h^H u'dz+u'_h\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\int_h^H u'^2dz+u'^2_h\frac{\partial h}{\partial x}-u'_h w'_h=0;$$

but

$$w'_h=\frac{\partial h}{\partial t}+u'_h\frac{\partial h}{\partial x}.$$

Now define

$$\overline{u'}=\frac{1}{H-h}\int_h^H u'dz,$$

$$\overline{u'^2}=\frac{1}{H-h}\int_h^H u'^2dz,$$

where the expressions with a bar denote averages. After substituting, we have

$$\frac{\partial}{\partial t}(H-h)\overline{u'}+\frac{\partial}{\partial x}(H-h)\overline{u'^2}=0.$$

Introducing the variance $\sigma^2=\overline{u'^2}-\overline{u'}^2$,

$$(H-h)\frac{\partial \overline{u'}}{\partial t}+(H-h)\overline{u'}\frac{\partial \overline{u'}}{\partial x}+\overline{u'}\frac{\partial}{\partial x}(H-h)\overline{u'}+$$

$$\overline{u'}\frac{\partial}{\partial t}(H-h)+\frac{\partial}{\partial x}(H-h)\sigma^2=0.$$

Finally, by applying the equation of continuity, we arrive at the following for our equation of motion

$$\frac{\partial \overline{u'}}{\partial t}+\overline{u'}\frac{\partial \overline{u'}}{\partial x}=-\frac{1}{H-h}\frac{\partial}{\partial x}(H-h)\sigma^2. \tag{19}$$

Combining the continuity equations for the upper and lower liquids, we have in addition

$$(H-h)\overline{u'}+hu=k(t). \tag{20}$$

If σ^2 and its horizontal derivative are negligible relative to the terms on the left-hand side of equation (19), equations (19) and (20) reduce, as expected to equations (17) and (18) which represent the conditions where the flow is independent of the vertical coordinate. This we have shown to be unsatisfactory since we have no wave. Thus, σ^2 or its horizontal derivative must be significant and then (19) represents the acceleration furnished to the mean flow by the passage of the wave.

In order to have some estimate of the required variance, let us consider a steady state flow varying linearly with height, or

$$u'=u'_h+\mu(z-h)$$

where u'_h is the upper flow speed at the interface and μ is a constant. Substituting this equation into the definition for the averages, it follows that

$$\frac{-\frac{1}{H-h}\frac{\partial}{\partial x}(H-h)\sigma^2}{\overline{u'}\frac{\partial \overline{u'}}{\partial x}}=\left[\frac{u'_H-u'_h}{u'_H+u'_h}\right]^2$$

where u'_H is the wind speed at H. For flows where $u'_H>3u'_h$ the value of the fraction exceeds 25 percent. Thus, we see that variances which are not at all excessive will yield sufficiently large relative values to the right-hand side of (19), and our equations will not degenerate to those implying no wave.

We have shown that our solution requires that the upper fluid vary with height, and, furthermore, that this restriction is not at all prohibitive to our model. On the other hand, nowhere in our solution have we included the effect on the solution of the flow of the upper liquid. As a matter of fact, we have implied that there is no such effect. This, however, violates one's physical reasoning on the nature of the flow. It seems reasonable to expect that some effect does exist. A discussion of this problem is given in Appendix B, where the steady-state two-liquid problem is discussed and the effect of the upper flow analyzed in greater detail.

CHAPTER II

THE "THREE-LAYER" ATMOSPHERE

ASSUMPTIONS

In the previous chapter, we have discussed the properties of a two-liquid flow (column 3, table 1 of the Introduction). In this chapter, we shall specify a "three-layer" atmosphere and indicate to what extent it may be described by the two-liquid domain. Having established this relationship, we may deduce the properties of our atmospheric model simply by reference to the properties of the two-liquid domain (i. e., by comparing columns 3 and 4, table 1).

We characterize a "three-layer" atmosphere by the following:

1. The atmosphere consists of three layers. The bottom two are autobarotropic, $\theta'>\theta$, and the top layer is arbitrary. The height of the inversion between the two bottom layers is denoted by h and the boundary between the middle and top layer by H.
2. The horizontal velocity (u) of the lowest layer is independent of the lateral and vertical coordinates (y and z). We do not consider the motion of the middle and top layers.
3. The surface $z=0$ is horizontal. The boundary between the second and third layers (H) is disturbed only slightly by the passage of a disturbance on h, such that $\frac{\partial H}{\partial x}\ll\frac{\partial h}{\partial x}$.
4. The pressure is hydrostatic.
5. The effects of the Coriolis force are neglected.

We shall now proceed to show that the equation of motion for this atmospheric model may be transformed exactly into an equation of motion for a flow in a two-liquid domain. Furthermore, we shall show that the equation of continuity for this flow may be approximated to a reasonable degree to the equation of continuity for the flow in the same two-liquid domain. Thus, we shall be able to apply all of the results of chapter I to our three-layer atmospheric model. (For a slightly different development, see [14] and [17].)

THE EQUATION OF MOTION

We shall start with the equation of motion for the lower fluid,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1}$$

the condition of hydrostatic pressure in the lower and middle fluid,

$$0=-g-\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{2}$$

$$0=-g-\frac{1}{\rho'}\frac{\partial p'}{\partial z}, \tag{3}$$

and the condition of autobarotropy for the two layers,

$$p=A\rho^{\gamma},\quad A=\text{constant},\quad \gamma=\frac{c_p}{c_v}=1.4 \tag{4}$$

$$p'=A'\rho'^{\gamma},\quad A'=\text{constant} \tag{5}$$

As before all primed quantities refer to the upper of the two bottom layers.

Consider the quantity $\frac{\delta p}{\rho}$, where δp refers to any partial variation in p. From (4)

$$\frac{\delta p}{\rho}=A^{\frac{1}{\gamma}}\frac{\delta p}{p^{\frac{1}{\gamma}}}=A^{\frac{1}{\gamma}}\left(\frac{\gamma}{\gamma-1}\right)\delta\left(p^{\frac{\gamma-1}{\gamma}}\right).$$

Define $\pi=\frac{\gamma}{\gamma-1}p^{\frac{\gamma-1}{\gamma}}$. Therefore

$$\frac{\delta p}{\rho}=A^{\frac{1}{\gamma}}\delta\pi. \tag{6}$$

similarly from (5),

$$\frac{\delta p'}{\rho'}=A'^{\frac{1}{\gamma}}\delta\pi',\ \text{where}\ \pi'=\frac{\gamma}{\gamma-1}p'^{\frac{\gamma-1}{\gamma}}. \tag{7}$$

From (2),

$$A^{\frac{1}{\gamma}}\frac{\partial\pi}{\partial z}=-g.$$

Integrating through a layer from z arbitrary in the lower fluid to $z=h$, we have

$$A^{\frac{1}{\gamma}}[\pi-\pi_h]=g(h-z). \tag{8}$$

Similarly from (3) and (7),

$$A'^{\frac{1}{\gamma}}[\pi'_h-\pi'_H]=g(H-h). \tag{9}$$

Since $\pi'_h=\pi_h$, we may substitute (9) into (8) and obtain

$$A^{\frac{1}{\gamma}}\left[\pi-\left(\frac{1}{A'}\right)^{\frac{1}{\gamma}}g(H-h)-\pi'_H\right]=g(h-z). \tag{10}$$

Differentiating partially with respect to x, we have

(11)

$$A^{\frac{1}{\gamma}}\frac{\partial\pi}{\partial x}-g\frac{\partial h}{\partial x}\left[1-\left(\frac{A}{A'}\right)^{\frac{1}{\gamma}}\right]=g\left(\frac{A}{A'}\right)^{\frac{1}{\gamma}}\frac{\partial H}{\partial x}+A^{\frac{1}{\gamma}}\frac{\partial\pi'_H}{\partial x};$$

but

$$\frac{\partial\pi'_H}{\partial x}=p'^{-\frac{1}{\gamma}}_H\frac{\partial p'_H}{\partial x}=-\frac{\rho''_H g}{p'^{\frac{1}{\gamma}}_H}\frac{\partial H}{\partial x}$$

(where the double prime refers to the top fluid); therefore the right-hand side of equation (11) becomes

$$g\left(\frac{A}{A'}\right)^{\frac{1}{\gamma}}\frac{\partial H}{\partial x}\left[1-\frac{\rho''_H}{\rho'_H}\right].$$

Since A and A' are constants, we choose to evaluate these constants at h where $p'_h=p_h$. Therefore

$$\left(\frac{A}{A'}\right)^{\frac{1}{\gamma}}=\left(\frac{\rho'}{\rho}\right)_h=\text{constant} \tag{12}$$

Therefore equation (1) may now be written as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\left[1-\left(\frac{\rho'}{\rho}\right)_h\right]\frac{\partial h}{\partial x}-\left(\frac{\rho'}{\rho}\right)_h g\left[1-\left(\frac{\rho''}{\rho'}\right)_H\right]\frac{\partial H}{\partial x}.$$

Since

$$\left(\frac{\rho'}{\rho}\right)_h\sim\left(\frac{\rho''}{\rho'}\right)_H\sim 1,\ \frac{\partial H}{\partial x}\ll\frac{\partial h}{\partial x}$$

we have finally

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\left[1-\left(\frac{\rho'}{\rho}\right)_h\right]\frac{\partial h}{\partial x}. \tag{13}$$

Since $\left(\frac{\rho'}{\rho}\right)_h$ is a constant by (12), we have transformed the equation of motion for a "three layer" autobarotropic atmosphere to the equation of motion for a two liquid flow, where the ratio of the densities is given by $\left(\frac{\rho'}{\rho}\right)_h$.

In actual cases it is useful to use the ratio of the potential temperatures θ/θ' rather than the ratio of the densities at the inversion surface $\left(\frac{\rho'}{\rho}\right)_h$. The relationship is found readily by applying the gas law $p=R\rho T$ and the potential temperature relation $\theta=T\left(\frac{1000}{p}\right)^{\kappa}$ ($\kappa=.288$) to each of the two layers and equating the pressures at the inversion level. The relationship is found to be

$$\left(\frac{\rho'}{\rho}\right)_h=\frac{\theta}{\theta'}. \tag{14}$$

THE EQUATION OF CONTINUITY

The equation of continuity for the lower fluid of the atmospheric motion considered is given by

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial z}(\rho w)=0. \tag{15}$$

Expanding and rearranging, we have

$$\frac{1}{\rho}\frac{d\rho}{dt}+\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0.$$

Integrating from $z=0$ to $z=h$ and recalling that u is independent of z, we obtain

$$\int_0^h\frac{1}{\rho}\frac{d\rho}{dt}dz+h\frac{\partial u}{\partial x}+w_h=0. \tag{16}$$

But

$$w_h=\frac{dh}{dt}=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}; \tag{17}$$

therefore (16) may be written as

$$\int_0^h\frac{1}{\rho}\frac{d\rho}{dt}dz+\left(\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x}\right)=0, \tag{18}$$

where the quantity in parenthesis represents the equation of continuity of an incompressible layer

of depth h. We shall endeavor to evaluate the relative importance of the first term, i. e.,

$$\int_0^h \frac{1}{\rho}\frac{d\rho}{dt}\,dz.$$

Expanding this term, we get

$$\frac{1}{\rho}\frac{d\rho}{dt}=\frac{1}{\rho}\frac{\partial\rho}{\partial t}+\frac{u}{\rho}\frac{\partial\rho}{\partial x}+\frac{w}{\rho}\frac{\partial\rho}{\partial z}, \tag{19}$$

but $\frac{1}{\rho}\frac{\partial\rho}{\partial t}=\frac{1}{\rho a^2}\frac{\partial p}{\partial t}=\frac{g}{a^2}\left[1-\left(\frac{\rho'}{\rho}\right)_h\right]\frac{\partial h}{\partial t}$, where a is the velocity of sound. Similarly

$$\frac{1}{\rho}\frac{\partial\rho}{\partial x}=\frac{g}{a^2}\left[1-\left(\frac{\rho'}{\rho}\right)_h\right]\frac{\partial h}{\partial x}.$$

The last term of (19) may be written as

$$\frac{w}{\rho}\frac{\partial\rho}{\partial z}=\frac{w}{\rho}\frac{d\rho}{dp}\frac{\partial p}{\partial z}=-g\frac{w}{a^2}.$$

Integrating from 0 to h, we get

$$\int_0^h \frac{w}{\rho}\frac{\partial\rho}{\partial z}\,dz=-\frac{g}{a^2}\int_0^h w\,dz=-\frac{gh}{a^2}w^*\sim-g\frac{hw_h}{2a^2},$$

where $w^*\equiv\frac{1}{h}\int_0^h w\,dz\sim\frac{w_h}{2}$.

From (17) and (19) we may now write (18) as

$$\left(1+\frac{M^2}{\mathsf{F}^2}-\frac{M^2}{2F^2}\right)\frac{\partial h}{\partial t}+\left(1+\frac{M^2}{\mathsf{F}^2}-\frac{M^2}{2F^2}\right)u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x}=0, \tag{20}$$

where $M=\frac{u}{a}$=Mach number,

$F=\frac{u}{(gh)^{\frac{1}{2}}}$=Froude number,

$\mathsf{F}=F/\left[1-\left(\frac{\rho'}{\rho}\right)_h\right]^{\frac{1}{2}}$=modified Froude number.

Note the appearance of the ratios $\frac{M}{\mathsf{F}}$ and $\frac{M}{F}$ in equation (20). These ratios express the relative importance of the Froude numbers denoting the gravitational properties of the flow relative to the Mach number denoting compressibility properties. For small values of these ratios relative to 1, we may drop them from equation (20) and have remaining the exact continuity equation for an incompressible fluid.

Since $\mathsf{F}^2\gg F^2$, we need focus our attention only on the relative importance of the term $\frac{M^2}{2F^2}=\frac{gh}{2a^2}$. Since in the atmospheric problems to which this theory has been applied h usually ranges from 2,000 feet to 10,000 feet and has a modal value at about 5,000 feet, $\frac{M^2}{2F^2}$ will range from .03 to .15 and have its most frequent value at about .075.

Thus in order to reduce (20) to an equation of an incompressible fluid, we drop terms which are in the range of 10 percent of the retained terms,[1] and arrive at

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x}=0. \tag{21}$$

We have shown then, that within the limits of certain approximations a "three layer" atmosphere containing two autobarotropic layers, separated by an inversion surface ($z=h$) and obeying the assumptions specified on page 13 may be considered as equivalent to a two-layer liquid domain whose interface is at $z=h$ and whose density ratio is given by (14). To these types of situations the results of Chapter I follow directly.

DISCUSSION OF ASSUMPTIONS

In many fields of science and specifically in the science of meteorology there are two broad approaches open to the investigator. On the one hand he may amass a considerable amount of observational data, seek some sort of statistical relationships among the various elements and present his interpretation of the phenomenon under discussion, either descriptively or by means of empirical numerical relationships. While contributing a great deal to describing the behavior of the phenomenon and even suggesting a pattern for its future behavior, this approach contributes little to the basic understanding of why the behavior is as it is. Furthermore, it is difficult by means of this approach to predict any behavior of the phenomenon inconsistent with its behavior during the period covered by the observational data.

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[1] In order to avoid making this restrictive assumption, one should define a modified height ζ and a modified velocity v such that

$$\frac{\partial h}{\partial t}\left(1+\frac{M^2}{\mathsf{F}^2}-\frac{M^2}{2F^2}\right)+u\frac{\partial h}{\partial x}\left(1+\frac{M^2}{\mathsf{F}^2}-\frac{M^2}{2F^2}\right)+h\frac{\partial u}{\partial x}=\frac{\partial\zeta}{\partial t}+v\frac{\partial\zeta}{\partial x}+\zeta\frac{\partial v}{\partial x}$$

The modified height and velocity will have to satisfy the equation of motion as well. The author has been unsuccessful in finding solutions for this equation satisfying the required criteria.

On the other hand, the investigator may apply the rules of the science already at his disposal and in this manner *deduce* the behavior of the phenomenon. In meteorology this usually means the application of certain physical "laws" such as the equations of motion and continuity and the First and Second Laws of Thermodynamics to the atmosphere. Unfortunately, the atmosphere involves so many complex features, that the equations completely describing the atmospheric flow have to this date defied mathematical solution. However, the benefits to be derived from this type of approach are so numerous that the investigator compromises his situation and resorts to mathematical models. These models are admittedly fictitious since they involve several assumptions about the atmosphere which do not in fact describe the atmosphere at all. The investigator, however, is able to solve his model mathematically and arrive at specific results. In the last analysis, the realistic criterion as to whether or not the choice of a model has been a good one, depends on whether repeated applications of the results of the model describe the behavior of the true phenomenon and predict its future action.

We shall now proceed to analyze the nature of the assumptions of our model and show which assumptions appear to contain weakness. In the succeeding chapters we shall show that despite these weaknesses, we have been able to explain a great deal about certain atmospheric phenomena. From that point of view we may consider the model as a successful one.

The assumptions which we have used follow:

1. *The atmosphere consists of three layers, the lower two of which are autobarotropic and separated by an inversion of zero thickness.* In all the cases to which this theory has been applied an inversion of temperature is invariably found. The *maximum* thickness of this inversion is under 50 mb. (about 500 m.). Considering the fact that the recording of a temperature lapse by means of radiosondes is not a continuous process and that in the evaluation of the radiosonde recorder trace considerable smoothing is practiced, the actual width of the inversions may indeed be even shallower. On the other hand, estimates of the amplitude of the jump indicate that a conservative value for its normal amplitude is about 1,000 meters. Thus, we might reasonably expect that the fact that the inversion is not a sharp discontinuity, should not present too serious a difficulty. Thus, if we choose a height (H) where the effect of the jump is essentially negligible, we may consider the atmosphere as consisting of three layers: (1) The bottom layer below the inversion, (2) the middle layer extending from the inversion to the height H, and (3) the rest of the atmosphere above H.

We have postulated that the lower two layers are autobarotropic. In synoptic studies it has been observed that the vertical lapse rate of temperature is basically constant and that the stratification is usually conditionally unstable, i. e., the vertical lapse rate of temperature lay between the dry and moist adiabatic lapse rates $\left(0.6\times10^{-4}\leq-\frac{\partial T}{\partial z}\leq1.0\times10^{-4}\,c.\,g.\,s.\right)$. It can be easily demonstrated that a horizontally homogenous layer, in which the pressure is hydrostatic and which has a constant vertical lapse rate of temperature, is barotropic and may be expressed in the form $p=K\rho^a$, where K and a are constants only in the undisturbed state.[2] Thus in developing the equations for atmospheric layers having a constant lapse rate of temperature, equation (11) would contain additional terms containing partial derivatives of the form $\frac{\partial}{\partial x}K^{\frac{1}{a}}$. These terms may be expected to be significant for large values of $\frac{\partial K}{\partial x}$ (i. e., large horizontal variations in pressure and density) or large values of $\frac{\partial}{\partial x}\left(\frac{1}{a}\right)$ (i. e., significant departures of the vertical lapse rate of temperature from the adiabatic lapse rate). Conversely, for relatively weak jumps and lapse rates of temperature close to the adiabatic, the model should be quite satisfactory.

We might give the following physical interpretation to this assumption of replacing a nonadiabatic layer by one which is adiabatic. Since air parcels in an adiabatic stratification are in thermodynamic equilibrium, all energy changes which they undergo are due to variations at the inversion level only. On the other hand in a nonadiabatic stratification, energy changes are due to variations in the entire

[2] The proof of this theorem follows:

Let $-\frac{\partial T}{\partial z}=\alpha$.

By the gas law and hydrostatic relation

$$\frac{\partial p}{p}=\frac{\partial\rho}{\rho}+\frac{\alpha}{\rho g T}\partial p.$$

Separating the variables and integrating, we have

$$p=K\rho^a \text{ where } \frac{1}{a}=1-\frac{\alpha R}{g}.$$

column. Thus, in replacing a nonadiabatic stratification by one which is adiabatic, we are in effect concentrating the effects of variations throughout the column, at one level, the inversion height.

2. *The horizontal velocity* (u) *of the lowest layer is independent of the lateral and vertical coordinates* (y *and* z). *We do not consider the motion of the middle and top layers.* One may interpret the first sentence as follows: We know that the wind is rarely constant with elevation. Nevertheless, if we take the average of the wind through the layer, we may consider that our final results describe this mean flow rather than the actual one. However, this interpretation intrinsically places a serious restriction on the variation of the wind with height, if we are to apply the equations of motion and continuity to the mean flow. We may demonstrate this by proceeding as for the discussion in Chapter I. We thus show that in defining an average through the layer as

$$\bar{\lambda}=\frac{1}{h}\int_0^h \lambda dz$$

the equation of motion reduces to

$$\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\frac{1}{h}\frac{\partial}{\partial x}(h\sigma^2)=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

where $\sigma^2=\overline{u^2}-\bar{u}^2$ is the variance. This equation will be identical with the equation of motion for a flow which is independent of height if we may neglect the term $\frac{1}{h}\frac{\partial}{\partial x}(h\sigma^2)$ relative to the other terms. This immediately places a restriction on the variance and its space variation.

In Chapter I we took as an illustration a steady state flow varying linearly with height, thus

$$u=u_0+\mu z,$$

where u_0 is the surface wind speed and μ is a constant, and found a relationship which here takes the form:

$$-\frac{\frac{1}{h}\frac{\partial}{\partial x}(h\sigma^2)}{\bar{u}\frac{\partial \bar{u}}{\partial x}}=\left[\frac{u_h-u_0}{u_h+u_0}\right]^2$$

where u_h is the wind speed at h. In order that this fraction be less than $\frac{1}{10}$ it is necessary that $\frac{u_h}{u_0}<1.9$. We see then that if we are to use the equation of motion for the mean flow without the term involving the variance, the wind at the top of the layer must not be too different from the wind at the surface. In the last analysis then, in applying the theory, each case must be carefully studied for the distribution of wind with height in order to determine whether the variance is sufficiently large so as to affect the use of the basic equation of motion.

Omitting the y component of the horizontal velocity is not at all serious if we keep in mind throughout that our results apply only to the variations of that component of the wind which is in the direction of the wave propagation.

In investigating the effect of the flow of the upper liquid in the general steady state two-liquid problem (see Appendix B) we find that its effect tends to be small for weak jumps and for speeds of the upper flow near the jump velocity. For strong jumps or for upper flow speeds significantly different from jump speeds, the motion of the upper liquid appears to have a marked effect on the solution. Thus one might reasonably expect that in the unsteady state problem, the upper flows would contribute to the nature of the solution. Unfortunately, the effect of this upper flow does not appear in the unsteady state model. Of course, some estimate (even if only qualitative) of this effect may be made by reference to the steady state solution given in the Appendix B.

3. *The surface* $z=0$ *is horizontal and* $\frac{\partial H}{\partial x}\ll\frac{\partial h}{\partial x}$. The first part of this assumption holds fairly well in the flat Plains area. However, the atmospheric waves discussed here are also found in areas where topography changes from sea level to 3,000–4,000 feet. Since the speed of the wave is partly a function of the inversion height above a horizontal surface, this variation in topography must have a marked effect. In actual practice it is recommended that a mean elevation be used for the surface $z=0$ and that the problem be solved as required by the model. It should be expected, however, that the results would indicate excessive jump speeds where the topography exceeds the mean elevation and reduced speeds where the topography is lower than the mean.

A similar problem exists for the height of the inversion. The theory implicitly assumes that the inversion height is level in the undisturbed state. This too may not always be the case. The pro-

cedure that is recommended is identical with the one for topography. A mean inversion height may be selected for the computations. Again, discrepancies in the values of the jump speed as predicted by the model must be expected whenever the inversion height is substantially different from the average.

The latter part of the assumption, that a level H exists such that $\frac{\partial H}{\partial x} \ll \frac{\partial h}{\partial x}$ is usually quite satisfactory. Physical intuition and observational evidence indicate that the amplitude of the jump decreases with height such that eventually some level H will be reached satisfying the criterion that $\frac{\partial H}{\partial x} \ll \frac{\partial h}{\partial x}$. However, it should be noted that in practice if the level H is very high the assumption that the layer between h and H is autobarotropic may not be satisfied.

4. *The pressure is hydrostatic.* This assumption is a reasonably good one, provided we stay away from the jump zone. In this zone there is no doubt that vertical accelerations are appreciable. Furthermore, we shall see later (ch. III) that under proper thermodynamic conditions, copious precipitation may be associated with a jump. This too may have an effect on the surface pressure. Thus, we are in the unfortunate position of not knowing how much of the pressure rise is due to the vertical acceleration or falling rain and how much is due to the jump mechanism. Our only alternative is to consider the pressure as hydrostatic, but we should keep it well in mind that this assumption is a particularly poor one in the jump zone.

Lamb [37, p. 425] in discussing Russell's solitary wave type argues that a steady state solution is possible for this finite wave specifically because of the vertical accelerations. It may very well be that the omission of a steady state jump in our solution arises directly from the assumption of hydrostatic pressure.

There is another aspect of this assumption which is important. The condition of hydrostatic pressure together with assumption 3 implies that in the undisturbed state the surface pressure is constant in the x-direction. This of course is rarely the case. However, pressure variations not due to the jump are usually balanced by geostrophic winds in the y-direction. In practice it is wise to investigate only the pressure changes caused by the jump rather than to consider the absolute values of the pressure. In this manner any non-uniformity in the initial distribution of pressure does not enter.

5. *The effects of the Coriolis force are neglected.* In the cases to which this theory has been applied, the speed of propagation of the disturbance is much greater than the speed of the wind in the lowest layer. Consequently, any particle is under the influence of the disturbance only a very short time. As such in the neighborhood of the disturbance the particles move in response to the pressure gradient force only and are affected by the Coriolis force only slightly. Fultz [19] has introduced the Rossby number as a measure of relative importance of the inertia forces to the Coriolis force. We are interested in the effect of the Coriolis force on the wave propagation, thus the Rossby number in our model is given by

$$R_0 = \frac{v_j}{Lf}$$

where R_0 = Rossby number
v_j = speed of jump
L = characteristic length
f = Coriolis parameter

Representative values of these parameters are (in c. g. s. units): $v_j = 2 \times 10^3$, $L = 8 \times 10^5$ while $f \sim 10^{-4}$. Thus, $R_0 = 25$, and we may neglect the effect of the Coriolis force as being only a few percent of the inertia forces. However, in the early stages of the elevation-type wave, when it covers a relatively large distance and/or time, individual particles remain under the influence of the wave for an appreciable period. Under these conditions the Coriolis force would tend to deviate the particles to the right.

6. *No mixing.* While we have not stated it explicitly, our model implies that there is no mixing of air between the layers, i. e., that the inversion h is preserved as a simply connected boundary. In the undisturbed state, this assumption is quite valid. However, with the passage of the jump, appreciable convection takes place and not infrequently the lower layer pierces the inversion, and may even destroy it completely. Although the jump may have progressed onward by the time this destruction of the inversion has taken place, it is not unreasonable to suppose that the effects of this inversion destruction may be propagated too, and in some manner influence the future history of the jump.

In the discussion of the various assumptions many more problems were raised than could be answered. The model has been demonstrated to be deficient in many ways and only in a small part of these, indications are given in what manner these deficiencies may be handled. Consequently, with our current model we cannot possibly hope to achieve more than a general description of atmospheric events.

We have spoken of a jump in the three layer autobarotropic atmosphere. We shall see later that this disturbance is identified primarily by the jump in pressure on a microbarogram. Consequently, the name "pressure jump" and "pressure jump line" have been assigned to an individual disturbance of this kind and to a distribution of such disturbances along a synoptic line ([14] and [58]).

CHAPTER III

METEOROLOGICAL APPLICATIONS OF THE HYDRAULIC ANALOGY

EARLIER ATTEMPTS

There is no record of when man first contemplated weather processes by analogy to the flow of water. It was probably among the first and most obvious approaches that suggested themselves to the serious investigator of meteorological phenomena. As pertaining to the three layer model discussed here, it is possible that Helmholtz [27] was among the first (if not the first) to think of atmospheric waves on inversion surfaces by analogy to internal water waves on density discontinuities. Shaw [52] in 1913 really approached the essence of the hydraulic analogy (table 1) in describing a line squall. (He did not distinguish at the time between a cold front and a nonfrontal squall line.) He said, "these phenomena recall those of the advance of a bore up a tidal river which could give in a like manner a sudden increase of pressure upon a recording gage if we imagine it to be placed at the bottom of the river." The first known attempt at a more direct application of the hydraulic analogy as described in table 1 was by McGurrin [39] in 1942. McGurrin tried to apply this analogy to develop a mechanism for tornado formation. While it may be debatable whether his conclusions are indeed realistic, the fact that he arrived at these conclusions through the use of the hydraulic analogy is noteworthy. In 1945, Rossby [46] suggested that internal waves in the atmosphere are capable of developing sharp forward boundaries by analogy to the properties of dispersive waves in the ocean. While he did not use the hydraulic analogy in the strict sense in which we have described it here, his discussion of atmospheric waves with sharp forward boundaries (pressure jumps) by analogy to the flow of water is indeed interesting. The first attempt at the application to meteorological situations of the unsteady state features of the hydraulic analogy was by Freeman [14] in 1948. Freeman argued that since Coriolis force effects are negligible in the tropics and moreover, since the stratification of the atmosphere in the tropics approximates a two layer autobarotropic atmosphere, we may apply the analogy. Using the mean wind in the lower layer as reported on the 6-hourly pilot balloon observations, and the variation of the inversion as reported by the 12-hourly radiosonde observations at one station (Hollandia), he was able to construct the characteristic lines and show that the jump with the proper intensity arrived at a station 350 miles away (Biak) at the appropriate time. In 1949 Abdullah [1] demonstrated that if we assume that the atmosphere behaves as described in column 3, table 1, we arrive at many of the features of the flow described in Chapter I. Abdullah's method of solution follows the mathematical method of Hugoniot and for details the reader is referred to [1] or [21]. In essence, Abdullah discusses an elevation type wave which progresses on a cold frontal surface and steepens as it moves forward. Abdullah's main interest was to show that when this wave breaks on the surface (after descending the cold front) more than sufficient energy is liberated for cyclogenesis to occur. This hypothesis he demonstrated with a case study.

The first generalization of the problem with regard to the application of the analogy to varied atmospheric situations was in 1949 by Tepper and Freeman [63] who stated the general atmospheric conditions under which the hydraulic analogy may be utilized. Essentially, these conditions are the assumptions listed in Chapter II.

APPLICATION TO PREFRONTAL SQUALL LINES

A review of the meteorological literature will disclose that the term "squall line" is among the oldest in meteorology, but is perhaps the least clearly defined. Prior to the general adoption by meteorologists of the frontal theory of cyclones, it was customary to designate as a squall line, any line of storms projecting in a general southerly and easterly direction from a depression. Such storms were characterized as being of strong intensity, accompanied by strong wind gusts, wind shifts, pressure rises, temperature falls, and heavy rain of the showery type. With the adop-

tion of the frontal theory, some of these lines of storms were redesignated more descriptively as cold fronts. There remained the lines of storms which appear in general in the warm sector of cyclones, roughly parallel to the cold front and along which there is intense convective activity. This type of squall line has been referred to as a pre-cold-frontal squall line and as an instability line. All subsequent discussion of squall lines will pertain to this latter type of line of storms only.

Unfortunately, very little is known about the dynamics and kinematics of squall lines so that their identity must be established on a surface synoptic map almost exclusively by the precipitation that accompanies them. Very frequently the analyst may have considerable difficulty in following a squall line through a series of maps because the associated showery precipitation belt appears to move very erratically, to fluctuate considerably in intensity, and to die out suddenly and reappear just as suddenly. Furthermore, due to their similar synoptic features, it is not uncommon to confuse a squall line with a cold front when the two are geographically proximate.

In 1950, Tepper [58] proposed that this showery precipitation belt may at times be a sufficient, though not a necessary, criterion for establishing the presence of a squall line. Instead, he proposed a new classification—the pressure jump line—of which the prefrontal squall line is only a particular example. The proposed pressure jump line is, of course, the phenomenon described in column 4 of the table of analogies.

Motivated by a detailed descriptive account by Williams [65] of seven squall lines which traversed 55 automatic recording stations in the Wilmington, Ohio, area, Tepper [58] reexamined the original data for the purpose of finding some explanation for the series of events accompanying the squall line passage. The following relationships were found, some of which had already been known to meteorologists ([24], [54], [65], and others).

1. The squall line roughly parallels the cold front; it lies to the east of the cold front and outruns the cold front.
2. Horizontal pressure gradients associated with squall lines are extremely intense. Gradients of the order of 1.15 mb./km. are not rare. Moreover, the equilibrium geostrophic wind for such a gradient would have to be of the order of 2,500 m. p. h. This indicates that geostrophic balance is never even approximately reached, meaning that individual air particles remain under the influence of the intense pressure gradient only a very short time.
3. The leading edge of the pressure gradient—the pressure jump—is clearly defined as a continuous (though undulating) line on micro-maps. (See fig. 11.)
4. In the region of the intense pressure gradient, the winds blow practically perpendicular to the isobars in response to the pressure gradient force only; and, moreover, on the back side of the ridge of maximum pressure, the winds appear to maintain their original direction (i. e., into the ridge) but with diminished speed since here they are being decelerated by a reverse pressure gradient force. Again, at no time is geostrophic balance reached.
5. The variations in the pressure patterns show decided unsteady state conditions.
6. In ordering chronologically the meteorological parameters associated with a squall line, it is clear that the onset of the rapid rise in pressure precedes all of the other parameters. This sequence of events seems to indicate that the onset of the rapid rise in pressure is independent of such elements as precipitation, wind shift, temperature fall, and wind gustiness.
7. The speed of propagation of the squall line far exceeds the wind speeds recorded below 10,000 feet.
8. The air mass into which the squall line propagates is usually characterized by two quasi-adiabatic layers separated by an inversion of temperature.

The conclusions drawn from these data were that we are indeed dealing with a gravitational wave on an inversion between two adiabatic layers. Tepper showed that by adopting the hydraulic analogy we might consider a prefrontal squall line (i. e., the pressure jump line), as an analogue to the hydraulic jump in hydraulics or to the shock wave in gas flows. He suggested that the initiating mechanism for the pressure jump line is the acceleration of the cold front, whose behavior would be much like the acceleration of a piston in a tube of fluid in producing a

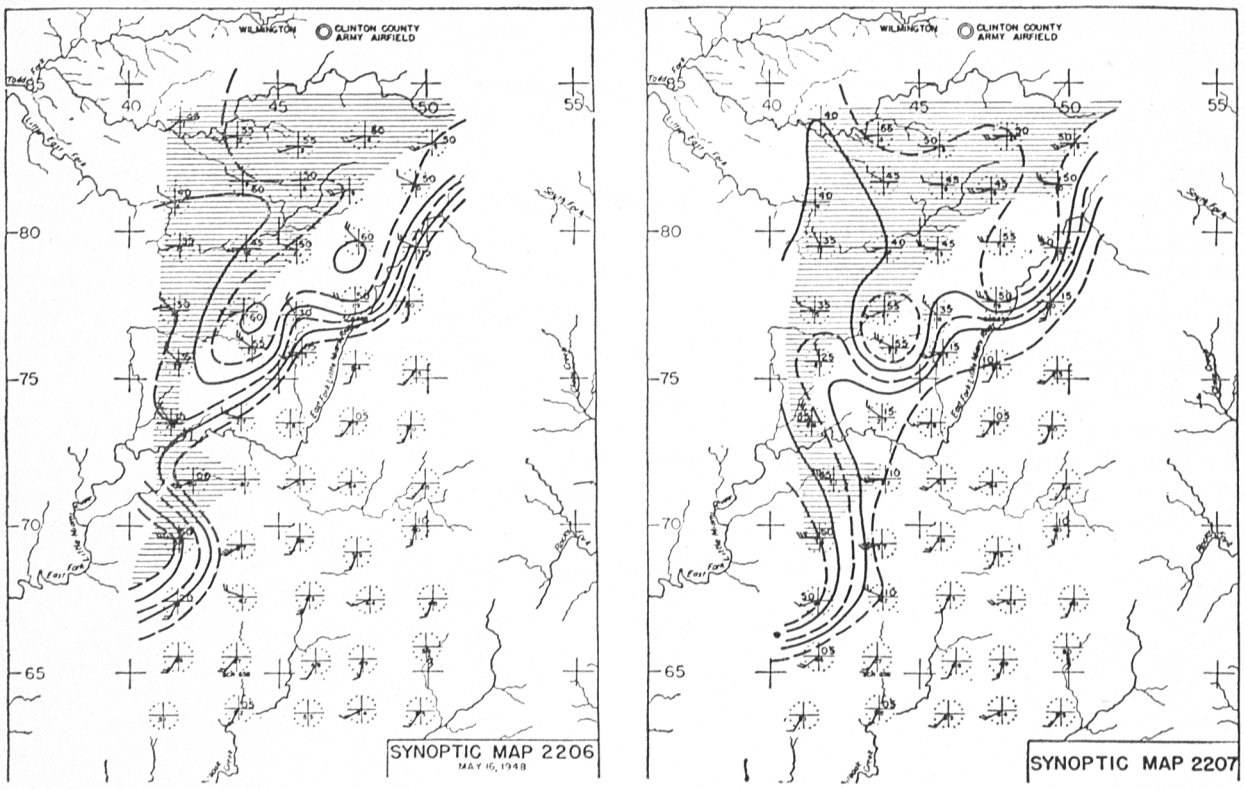

FIGURE 11.—One-minute synoptic maps indicating the progress of a pressure jump line over a micro-network near Wilmington, Ohio. The average distance between stations is 2 miles. The lines are isobars in .001" Hg. and represent the rise in pressure above its undisturbed value; the winds are given in terms of ½ barb=1–5 m. p. h., 1 barb=6–10 m. p. h., etc.; precipitation is shaded.

shock wave. He showed that the properties of the pressure jump line could account for all of the idiosyncrasies of the prefrontal squall line.

One particular generalization resulting from this hypothesis is that pressure jump lines may exist independent of any associated precipitation. Consequently, a much more accurate identification of this phenomenon is to be made through the pressure jump behavior rather than the customary precipitation pattern. Thus, one may speak about "wet" pressure jump lines and "dry" pressure jump lines. Obviously, the determining factor as to which of the two will actually occur depends on the thermodynamic properties of the atmosphere and the amount of lift produced by the pressure jump line. It was foreseen that a single pressure jump line might at times be wet and at times be dry.

Meterologically speaking, the pressure jump phenomenon, as experienced at any one station, is of very short time duration, the entire rise in pressure occurring in a matter of a few minutes. Consequently, for positive identification of a pressure jump line passage, ordinary barographs currently in use by the U. S. Weather Bureau, are considered inadequate. These barographs complete one revolution of the drum in more than four days with the result that the time resolution of the pressure traces must suffer. On an experimental basis, and for the purpose of studying pressure jump lines, accelerated barographs have been utilized in two studies by the United States Weather Bureau. The first was a small pilot project network in the Washington, D. C., area during the spring of 1950, and the second, a much larger network in the Midwest during the spring and summer of 1951. The latter network is to continue its operation during the 1952 season.

WASHINGTON PRESSURE JUMP PILOT PROJECT

Eighteen automatic recording stations within a radius of about 50 miles of Washington, D. C., participated in this program (table 2 and fig. 12). Of these stations, nine had each a microbarograph, hygrothermograph, triple register (to record wind speed and direction), and a recording

TABLE 2.—*Stations participating in the Washington Pressure Jump Pilot Project, spring 1950*

Station	Code letters	Station	Code letters
Andrews Air Force Base, Md.	ADW	Mt. Weather, Va	MW
Annapolis, Md	ANP	Patuxent River, Va	NHK
Baltimore, Md	BAL	Quantico, Va	NYG
Beltsville, Md	BL	Rectortown, Va	RC
Chantilly, Va	CY	Rockville, Md	RK
Charlotte Hall, Md	CL	Warrenton, Va	WR
Elkins, W. Va	EKN	Waterford, Va	WF
Frederick, Md	FDK	Washington, D. C	CO
Front Royal, Va	FRR	Washington National Airport.	WNA

rain gage; one other had a microbarograph and recording rain gage; and the remaining eight stations had only a microbarograph each. The microbarographs and hygrothermographs were modified with special gears to accelerate the motion of the drum. For the remaining instruments, it was not deemed feasible to make a similar modification. Figure 13 shows an installation having all of the equipment. Upper air data were available from the observations at the Washington National Airport Weather Bureau Station.

In addition to verifying the general features of the pressure jump line hypothesis discussed in the first part of this chapter and more fully in [58], this study of 12 pressure jumps traversing the area during the period of observation, revealed several additional features.

1. The intensity of pressure jump lines in the Washington area is relatively weak. The average maximum pressure rise was under 1.5 mb.
2. The origin of pressure jump lines in the Washington area appears to be related to the mountains to the west. Of the 12 cases studied, only one pressure jump line could be extended westward across the mountains. (This was done through the examination of the pressure data from regular reporting stations.) In one case, data were not easily available for study. In the remaining 10 cases, the first indication of a pressure jump line was over the mountains or east of the mountains.
3. Some pressure jump lines pass across the Washington area and are in no way reflected in current standard meteorological analyses.[1]

[1] Throughout this treatise the analyses of the WBAN Analysis Center (a joint Weather Bureau, Air Forces and Navy weather analysis center located in Washington, D. C.) are taken as representative of the best possible analysis of the weather situation made by current standard meteorological practices. Whenever the WBAN Analysis Center analysis is mentioned, therefore, it is done so in this regard only.

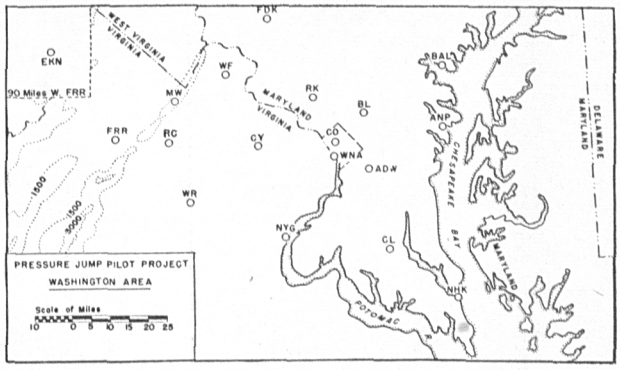

FIGURE 12.—Geographic distribution of stations participating in the Washington Pressure Jump Pilot Project, Spring 1950. The contours are in feet above m. s. l.

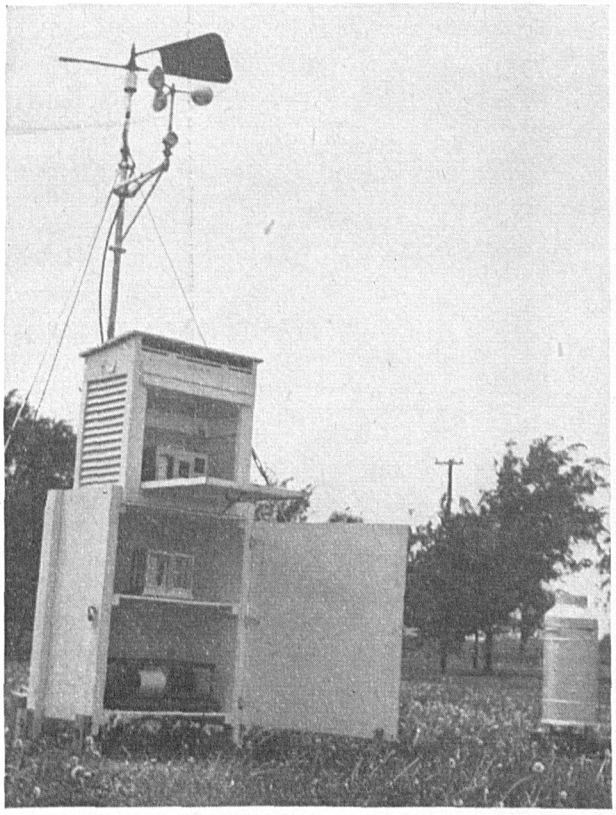

FIGURE 13.—An automatic recording station typical of those in the Washington Pressure Jump Pilot Project and in the Tornado and Severe Storm Network in the Midwest. The shelter contains a hygrothermograph on the top shelf, a microbarograph on the center shelf, and a triple register on the bottom shelf. The anemometer and wind vane are behind the shelter and a recording rain gage is to the right.

4. Evidence for a completely "dry" pressure jump line, predicted by theory, was found to have occurred on March 15, 1950. Figure 14 shows the sections of the microbarograms which went into the analysis. Figure 15 is the isochronic analysis itself

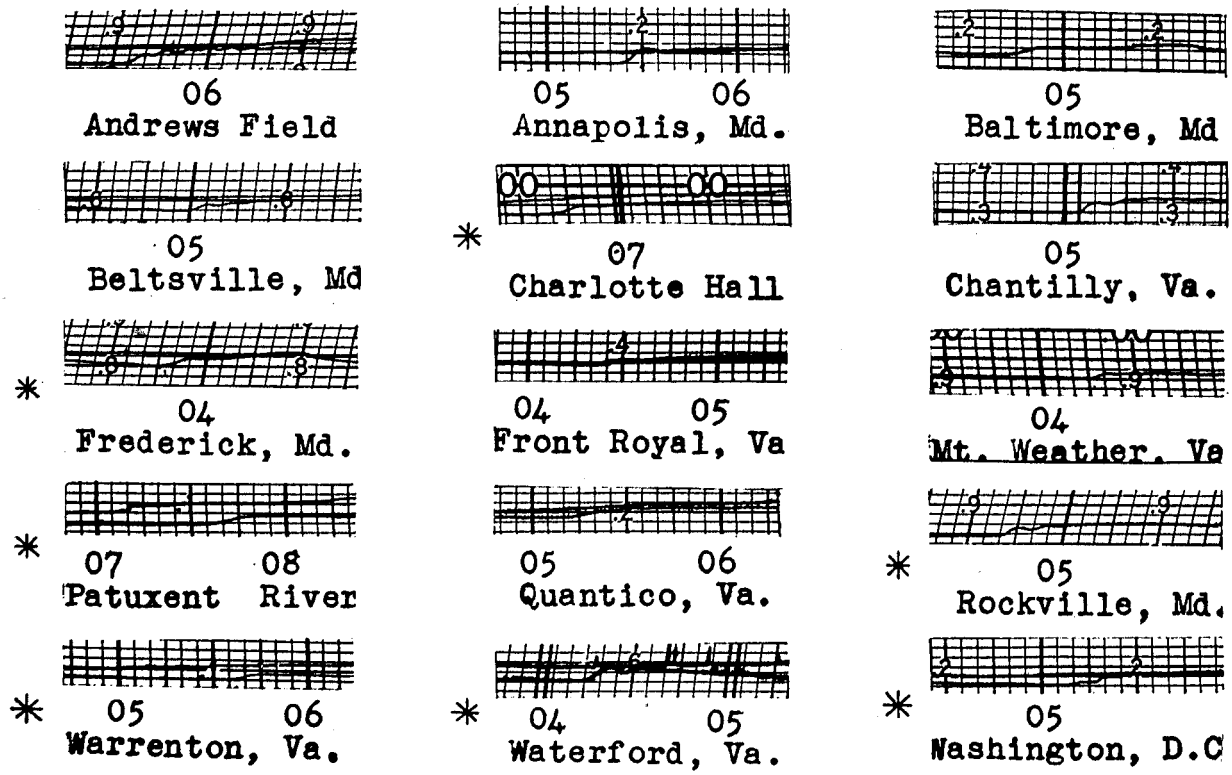

FIGURE 14.—Microbarograms, March 15, 1950. The numerals below each microbarogram show Eastern Standard Time. Stars indicate stations for which precipitation records were available. No precipitation was recorded at these stations at the time of pressure jump passage.

and indicates the progression of the pressure jump line. In this figure, as in subsequent isochrone charts, the number to the right of the station circle refers to the time of passage of the pressure jump while the ratio to the left refers to the rise in pressure divided by the duration of the rise in minutes. Figure 16 is the upper air sounding. It shows not only that the stratification approximates that of the three-layer atmosphere model, but also that the air was very dry. As a consequence we would expect that even considerable lift by the pressure jump might be inadequate to produce condensation and precipitation. The star symbols in figure 14 refer to those stations which had recording rain gages but which did not record any precipitation associated with the pressure jump line passage. It can be noted that all of the corresponding traces show marked pressure rises nevertheless. This must rule out the hypothesis that the rise in pressure in squall lines is due only to falling precipitation or the associated cold air downdrafts (see [7], [49], etc.). While this particular case was the only one of the 12 cases studied in which no station recorded precipitation, it is of further interest that in only *one* other case did *all* of the stations report precipitation. The remaining cases contained numerous individual examples of pressure jumps unaccompanied by precipitation.

5. Sporadic pressure jumps which could not be systematically aligned were also found. The properties of such jumps need further investigation. Since our objective here is to study cases of pressure jumps which are aligned, we shall pursue this point no further at this time.

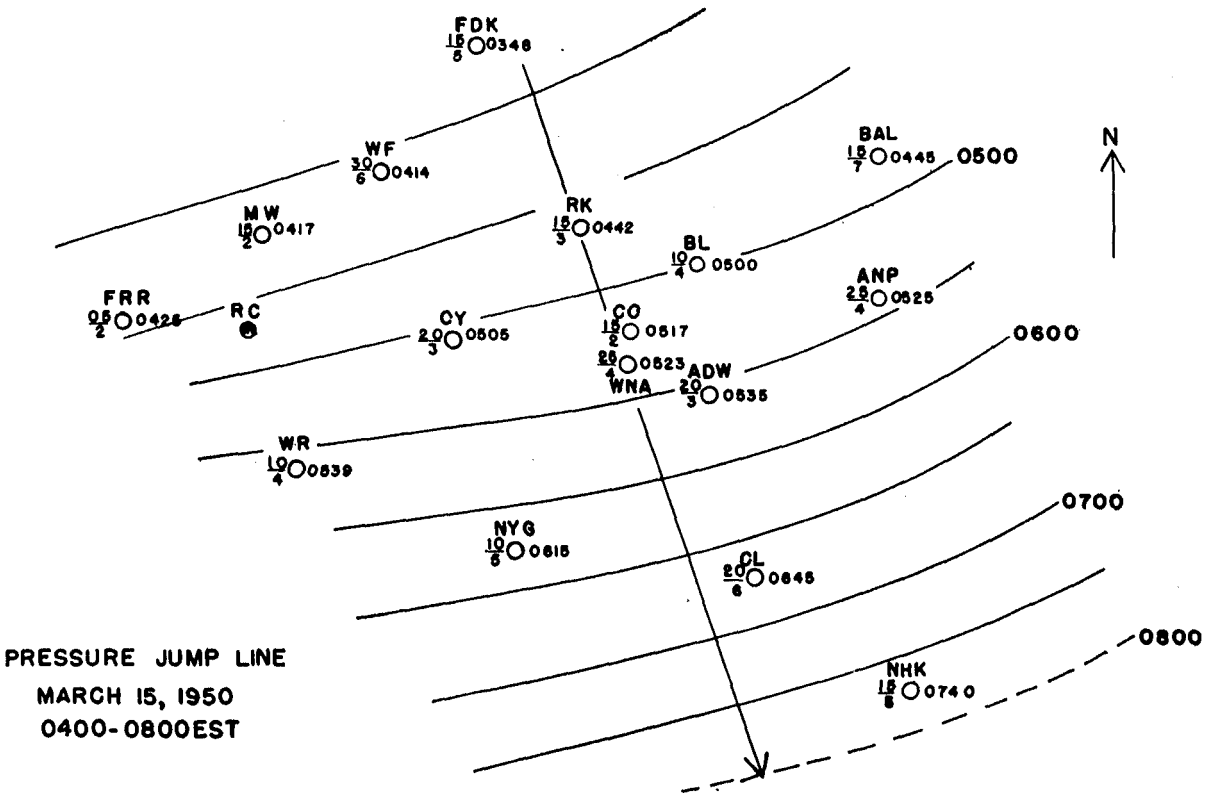

FIGURE 15.—The isochrone analysis of the pressure jump line of March 15, 1950 over the Washington, D. C. area. The numbers to the right of each station show Eastern Standard Time. The numerator of the ratio to the left of the station is the pressure rise in .001" Hg. and the denominator is the duration of this rise in minutes.

TORNADO AND SEVERE STORM NETWORK IN THE MIDWEST

Subsequent to the operation of the pilot project in the vicinity of Washington, D. C., it was considered advisable to enlarge on the observational network and to relocate it in an area more conducive to the study of storms. A special network of surface stations was established in the Kansas-Nebraska-Oklahoma area and included approximately 135 stations (see fig. 17). Of these only about 50 were at existing weather stations, the remainder were serviced by cooperative observers. The basic instrument was again the accelerated microbarograph to permit detailed observations of short period pressure fluctuations. Within this large network, an arc-shaped area was selected as the area of maximum tornado incidence over the past 17 years. At the 39 stations located in the "arc," there were installed high speed recording rain gages, hygrothermographs, and wind equipment, in addition to the microbarograph. Some additional 300 radiosonde and rawinsonde soundings were made during critical periods to afford a better picture of the weather elements above the surface. (For a more detailed description of this project, see [62].)

The analysis of the data for the midwestern project is still in progress and final results will not be available for perhaps another year. Two examples from these data will be presented here. These will illustrate clearly some of the salient features of the pressure jump line as a gravitational wave on an inversion.

AN OSCILLATORY PRESSURE JUMP

Case 1: The pressure jump line of March 10, 1951, in western Kansas.—Unfortunately this pressure jump line traversed that part of the network which was only sparsely covered by accelerated microbarographs. Nevertheless, some evidence for the existence of the line could be found at 6 stations. The microbarograms from Goodland, Garden City,

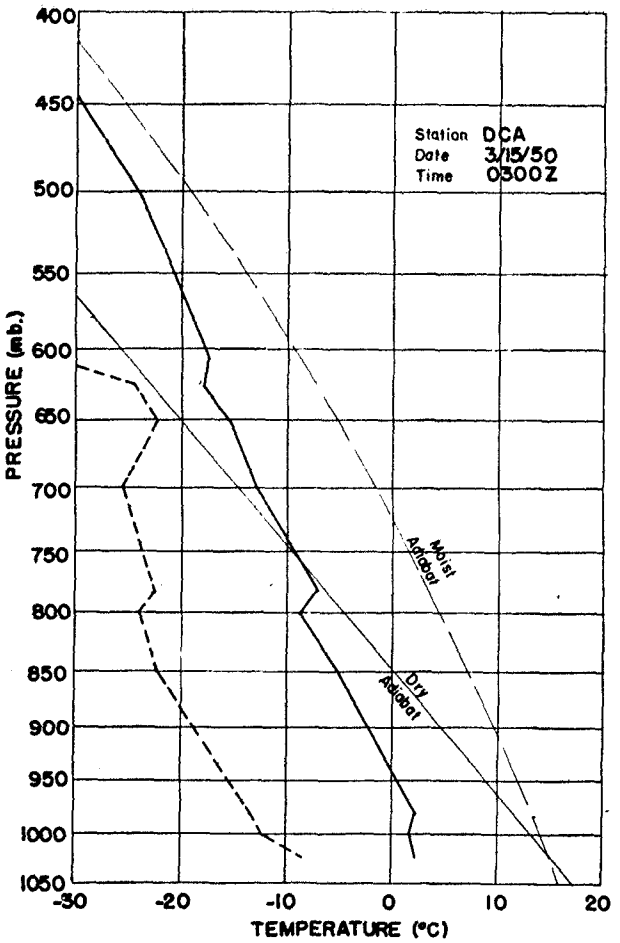

FIGURE 16.—Upper air sounding for 0300 GMT, March 15, 1950 taken at the Washington National Airport Weather Bureau Station. The solid line is a plot of temperature and the dashed line is a plot of the dew point temperature. For comparison there are included lines indicating both the dry and the moist adiabatic lapse rates of temperature.

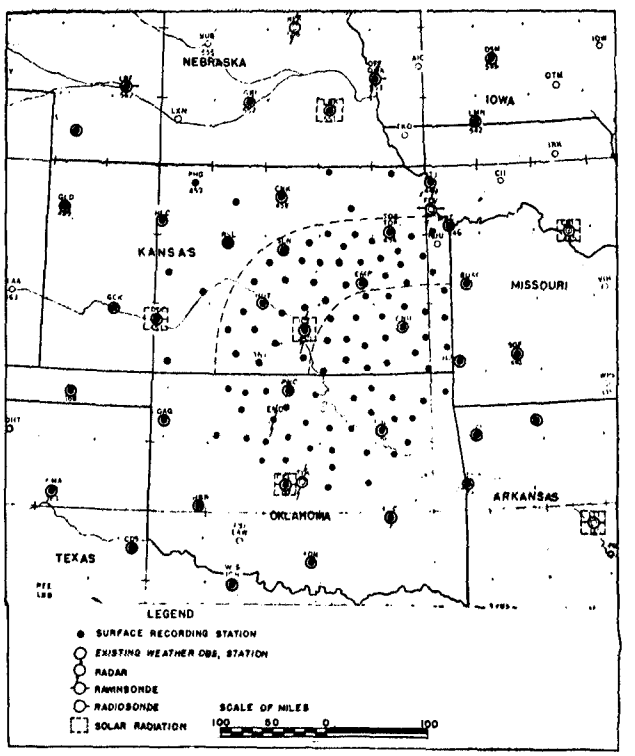

FIGURE 17.—Geographic distribution of stations participating in the Tornado and Severe Storm Network in the Midwest during 1951.

and Dodge City (fig. 18) were all so very interesting that a closer examination of the event was made. It will be noted that the Goodland trace represents a sudden rise in pressure—a pressure jump—followed by an oscillation which has all the appearances of a damped wave. A closer examination will reveal that the period of the first two waves is about 8 minutes and that of the succeeding wavelets is about 5 minutes. The traces for Garden City and Dodge City do not produce the identical wave structure, but, in general, it is clear that the over-all pattern is more or less the same.

The answers to the following two questions were sought:

(1) What produced the initial and pronounced rise in pressure at these three stations?

(2) What is the explanation for the damped wave which followed the initial pressure rise?

Three alternatives may be considered as possible causes for the initial pressure rise: (1) Falling precipitation, (2) advection of colder and denser air, and (3) some sort of wave activity.

The first alternative may be disposed of in rapid order. The weather (fig. 19) preceding and following the rise in pressure at the three critical stations [2] indicated that aside from fog at Goodland there was no precipitation in any form that could be associated with the pressure rise. Furthermore, drizzle was the only form of precipitation reported in all of western Kansas and southwestern Nebraska during the entire period in question. This immediately rules out the alternative that the rise in pressure was produced by falling precipitation. We may note in passing that we rule out as well downdrafts from thunderstorms as a possible cause for the marked rise in pressure. No thunderstorm activity nor even convective-type clouds were reported anywhere in the vicinity.

[2] The time of the rapid rise in pressure is given to the right of figure 19 and denoted by an arrow.

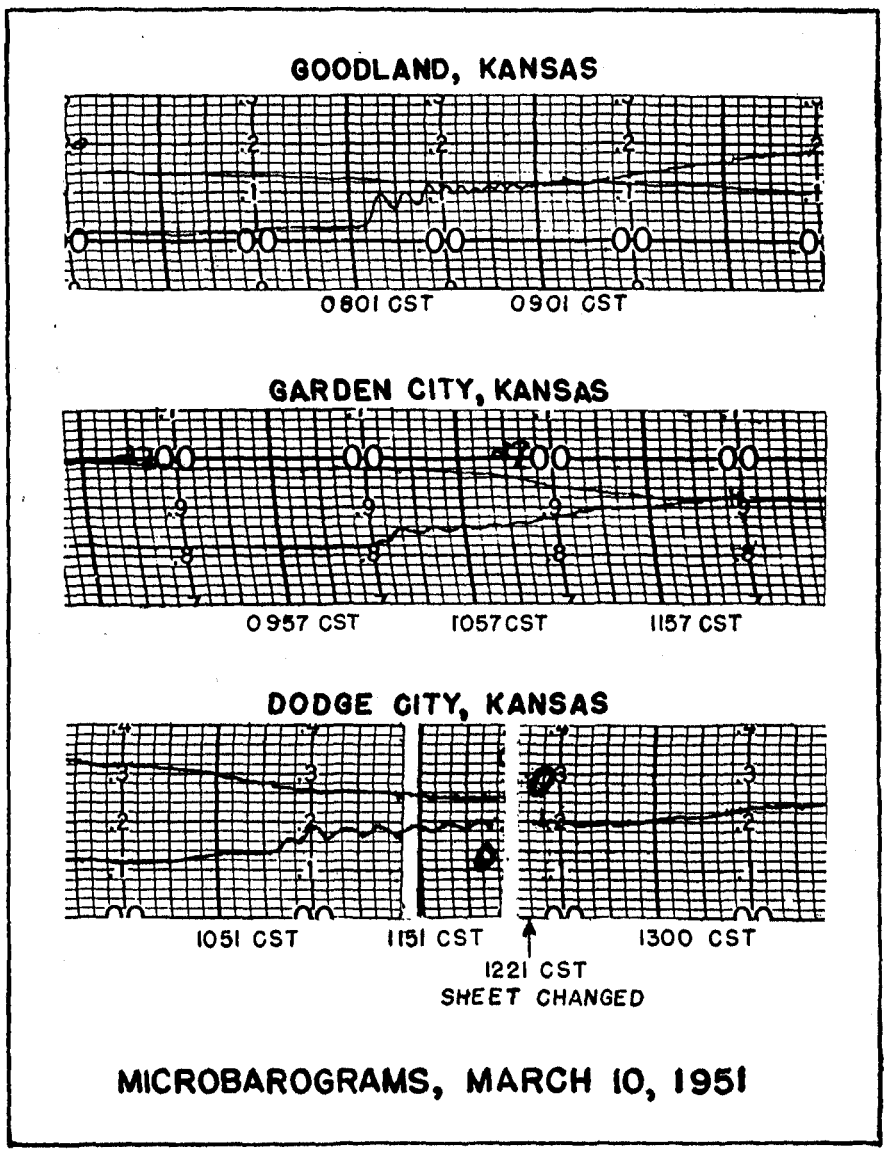

FIGURE 18.—An "oscillatory" pressure jump.

By advection of cold air as a cause for the pressure rise, we refer to possible cold frontal activity. Indeed on the day in question, March 10, 1951, a cold front did traverse the area. On figure 20 we have plotted the synoptic features of the day on an x, t-diagram. The x-axis was taken normal to the orientation of both the pressure jump line and cold front. From the regular WBAN Analysis Center 3-hourly surface synoptic analyses, it was possible to locate the surface position of the cold front on the diagram. These successive 3-hourly positions are indicated with solid triangles. The successive positions of the pressure jump line as taken directly from the microbarograms are plotted as dots. It is apparent that the analyst who interpreted the weather of the day considered the points G and H as cold frontal positions. In other words his interpretation was that the pressure jumps were caused by the rapid advection of cold air behind a cold front. Referring to figure 20 we note that this picture implies unreasonable conclusions about the cold front. If B and G are two successive 3-hourly positions of the cold front, then the front speed must have *averaged* more than 70 m. p. h. during the 3-hour period 0630–0930 CST. Connecting

GOODLAND, KANSAS

0625	M7	⊕	15		102	25	23	↑↖	6		976	
0703	E7	⊕	15					↑↖	6		977	
0728	E7	⊕	15		108	26	24	↑↖	2		977	
0802	E7	⊕	4	F				↖	5		978	← 0806
0814	E7	⊕	4	F				↑	15	↑0810	984	
0825	E7	⊕	1	F	135	27	25	↙	7		984	
0848	E4	⊕	1/4	F				↑↙	17		987	
0907	W1	x	1/8	ZL-F				↙	12		988-⁎-	
0930	W1	x	1/8	ZL-SG-F	156	26	25	↙	17		990	
1003	W1	x	1/8	ZL-SG-F				↙	15		993	
1027	W1	x	1/4	ZL-SG-F	176	26	25	↙	12		995	

GARDEN CITY, KANSAS

0828	W4	⊕	3	F	108	34	32	↑↖	18		979	
0910	E250	-⊕5-⦶	8					↑↖	16		979	
0928	E250	-⊕5-⦶	8		102	39	35	↑↖	14		979	
1010	E200	⦶5-⦶	10					↑↖	14		979	
1028	E180	⦶	10		095	52	36	↑	15		979	← 1032
1106	E120	⊕	10					↓	16	↑1056		
1128	E200	-⊕120⦶	10		125	41	34	↑↙	23+		985	
1140	W5	⊕	8					↑↙	20			
1210	W5	⊕	8					↑↙	22		990	
1228	W4	⊕	3	L-F	142	35	32	↑↙	25+		990-⁎-	
1310	W4	⊕	3	L-F				↑↙	24+		991	
1328	W4	⊕	3	L-F	156	33	30	↑↙	26+		993	

DODGE CITY, KANSAS

0928	W2	x	3	F	115	35	35	↑	17		982	
1005	W5	⊕	5	F				↑	17			
1028	E200	⊕	10		105	40	36	↑	28+		980	
1100	E200	-⊕140⦶	10					↑	20+		979	← 1110
1128	E200	-⊕140⦶	15		132	46	38	↖	4	↑1125	988	
1200	E17	-⊕⦶	15					↑↖	1		986	
1228	E17	⊕	15		129	49	38	↑↖	10		988	
1306	M10	⊕	7					↑↙	18		990-⁎-	
1328	W5	⊕	5	F	146	38	34	↑↙	22+		991	
1408	W2	x	3/4	L-F				↑↙	23+		992	
1428	E3	⊕	8		146	37	33	↑↙	27+		991	

FIGURE 19.—Station weather observations, March 10, 1951.

B and G with any smooth curve in order to allow the front continuous velocities will indicate that sometime during this 3-hourly period the front speed must have exceeded 100 m. p. h. On that day, however, at no elevation did the wind exceed 45 m. p. h. behind the front. Furthermore, figure 19 indicates that in each case, we can find a later time (represented by a star) which represents the cold frontal passage much more clearly. We must thus eliminate the advection of cold air as a cause for the pressure jumps.

The cause of the rapid rise in pressure lies in the third alternative—i. e., in wave motion. As indicated in the previous paragraph the weather observations at each of the three key stations (fig. 19) show that following the time of the rapid rise in pressure there is, at some later time, a definite indication of a cold frontal passage.

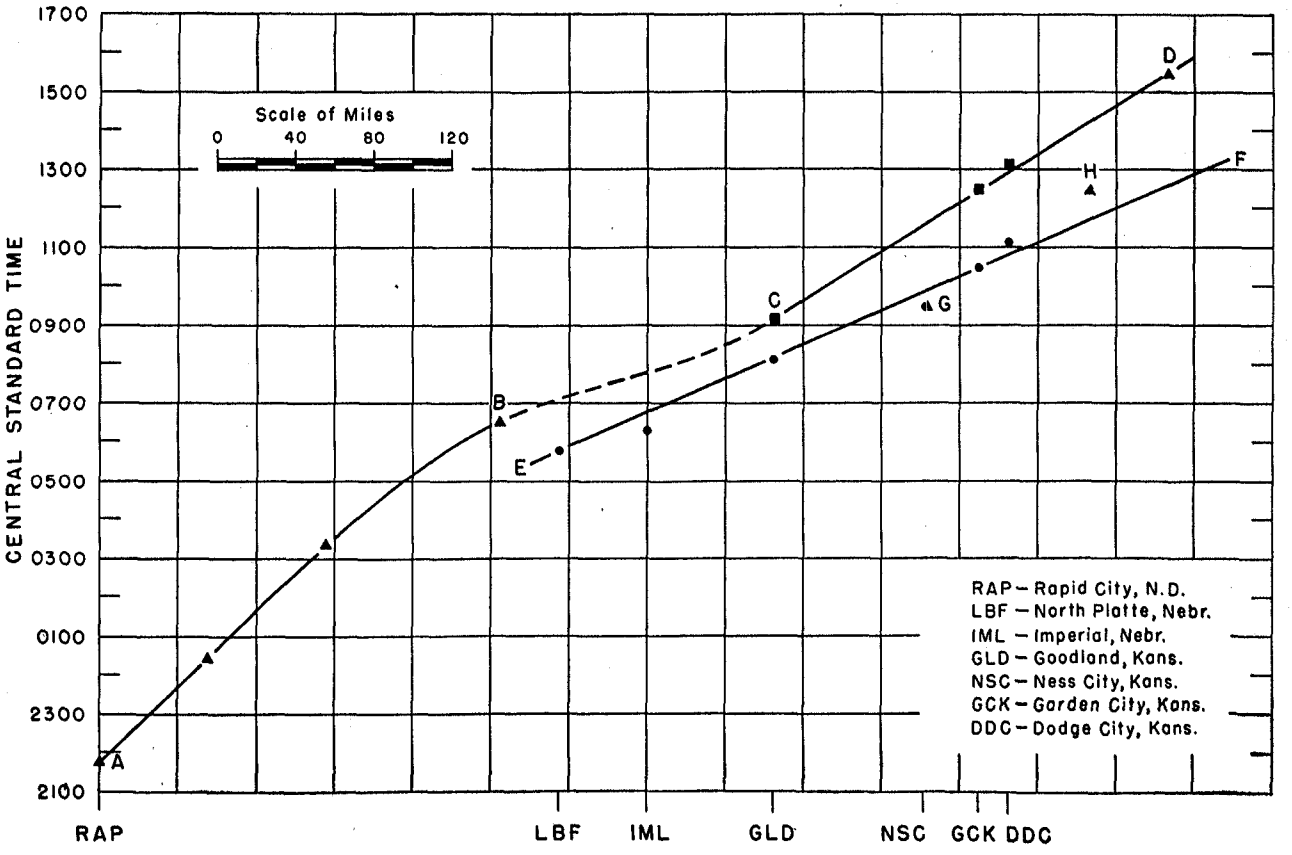

FIGURE 20.—x, t-diagram, March 9-10, 1951.

These times have been plotted on figure 20 as solid squares. If we draw a line through these points, the line intersects D which is a subsequent Analysis Center position for the cold front. The proposed sequence is as follows: The cold front progressed from A and one or two hours before B it began to accelerate. During this period of acceleration it set off a prefrontal pressure jump line (E–F). The cold front's subsequent motion is given by CD. Points G and H now more closely represent pressure jump line positions than cold front positions. Finally, we might add that one hour prior to the arrival of the pressure jump line the upper air sounding at Dodge City showed an 18° C. temperature inversion between 840–900 mb.

We repeat our conclusion. The synoptic evidence is compatible with the conclusion that a cold front accelerating into a three-layer atmosphere set off a pressure jump line which travelled out ahead of it. This is another instance of a dry pressure jump line. The reason that this jump line was dry was not because the air was exceedingly dry (as for the jump line of March 15, 1950—see p. 23) but because the lift due to the jump was unusually small. We may estimate this lift by using the formula developed in chapter II,

$$\frac{\Delta p}{\rho}=g\left[1-\frac{\theta}{\theta'}\right]\Delta h,$$

where Δ refers to a local variation. Using the maximum rise at Goodland (2.5 mb.) and the upper air data for Dodge City, we find

$$\Delta h=\frac{5}{2}\times 10^3\times\frac{300}{18}\ cm=420\ m.$$

This is an exceedingly small lift and in this case insufficient to produce precipitation.

We have now to answer our second question, i. e., what is the nature of the oscillations?

Bakhmeteff and Matzke [2] report having found experimentally that hydraulic jumps with $\frac{h_2}{h_1}<2$

are oscillatory. To date no explanation for this phenomenon has been put forth. For $\frac{h_2}{h_1}>2$ the oscillations cease (fig. 21). If this relation holds for the three-layer atmosphere as well, then in the case in question where $\frac{h_2}{h_1}\sim 1.8$ there occurred the atmospheric analogue to the oscillatory hydraulic jump. It is quite possible that all the oscillations indicated on the microbarograms (fig. 18) are due to the oscillatory pressure jump.

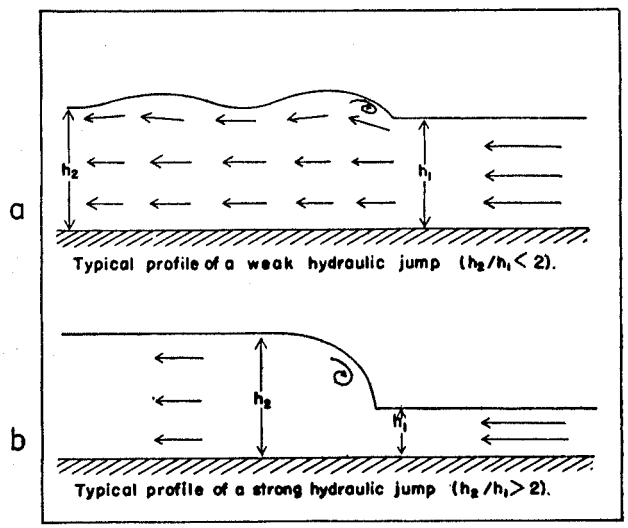

FIGURE 21. Typical profiles of stationary hydraulic jumps.

On the other hand, it may be that the period set up by the oscillatory pressure jump was resonant with a very short natural period of oscillation of the atmosphere (see [12]). It is difficult to answer the question at this time as to which of these alternatives is the more likely.

There is one further bit of evidence to support the contention that the oscillations of the microbarogram reflect the oscillations of the inversion surface. We would expect that every time the surface rose, cooling and condensation would take place, and every time the inversion surface fell heating and evaporation would occur (c. f. [33]). Fortunately, the solar radiation records from Dodge City were available for this comparison. On figure 22 there are superimposed the solar radiation trace and the microbarogram for Dodge City. *This relationship is striking:* A maximum in pressure (due to the maximum rise of the inversion surface) corresponds in time with a minimum value of solar radiation on the ground (due to the interruption of the solar beam by condensed water vapor). A minimum in pressure (due to a maximum fall of the inversion surface) corresponds in time with a maximum value of solar radiation on the ground (due to the decrease in cloud particles, accompanying the evaporation of water droplets). The fact that the solar radiation oscillation persists beyond the pressure oscillation may mean that the former (in this case)

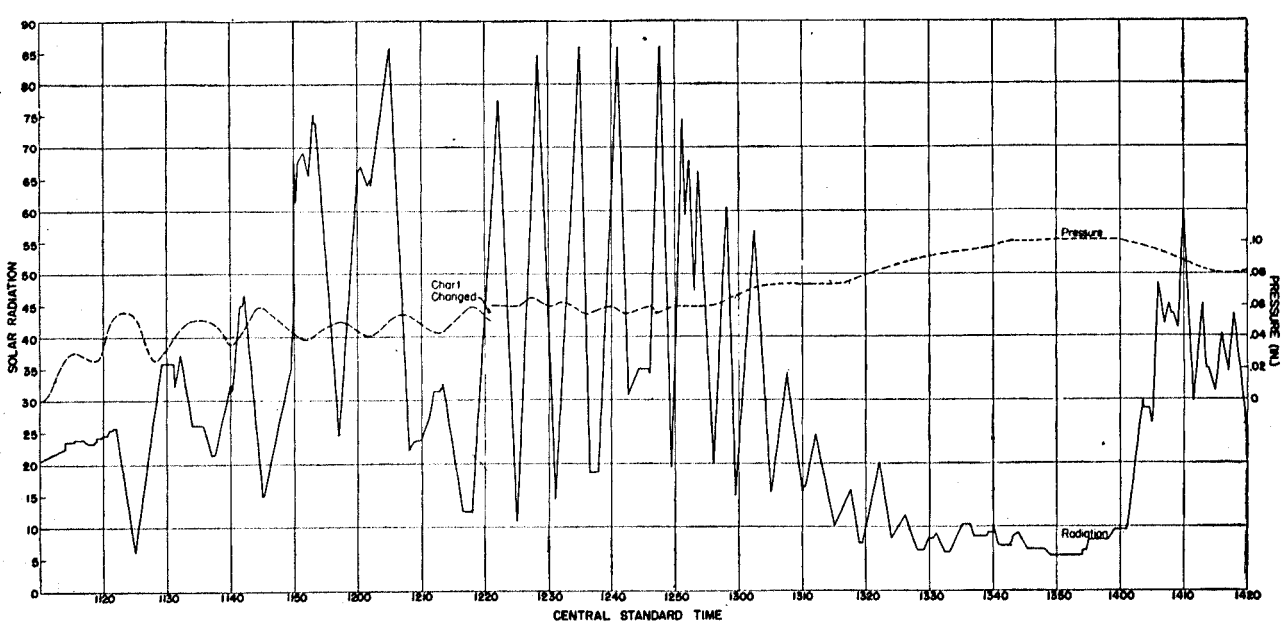

FIGURE 22.—Radiation trace and barogram for Dodge City, Kans., March 10, 1951.

is more sensitive to small amplitude oscillations than the latter.

AN EXAMPLE OF THE USE OF THE x, t-DIAGRAM

Case II: The pressure jump line through central Kansas, April 24–25, 1951.—One of the very first steps taken in the analysis of the microbarogram data from the United States Weather Bureau project in the Midwest, is to inspect each trace for pressure jumps. The time and intensity of a jump as it passes each station is plotted on a map and then isochrones are drawn to indicate the progress of the pressure jump line. Figure 23 is the isochrone map for the pressure jump line through Kansas on April 24–25, 1951. It reflects the analysis of the data as taken directly from the microbarograms. Note that along the line indicated by the arrow the pressure jump line is not clearly defined until Lyons (LYO). An explanation for this feature will be apparent from the subsequent discussion. Quite coincidentally, the lateral extent of the pressure jump line did not extend much beyond the geographic border of the State of Kansas. (The letter N on figure 23 implies that in the inspection of the pressure traces no evidence of a sharp pressure jump could be found.)

This case is presented in some detail as an illustration of how the acceleration of a cold front and the subsequent development of a pressure jump line might be handled by the method of characteristics described in chapter I. For the x-direction, we have taken a line running E–W through central Kansas. This line is indicated by an arrow on figure 23.

Synoptic maps: The surface synoptic maps and the 850-mb. maps (as analyzed by the WBAN Analysis Center) indicate that a cold front lay more or less stagnant through central Kansas. A "cold front aloft" approached from the west. As the cold front aloft reached the position of the surface cold front, the former disappeared and the latter began to accelerate toward the east. Suddenly a "squall line" appeared ahead of the cold front. Furthermore, an open wave developed on the surface cold front.

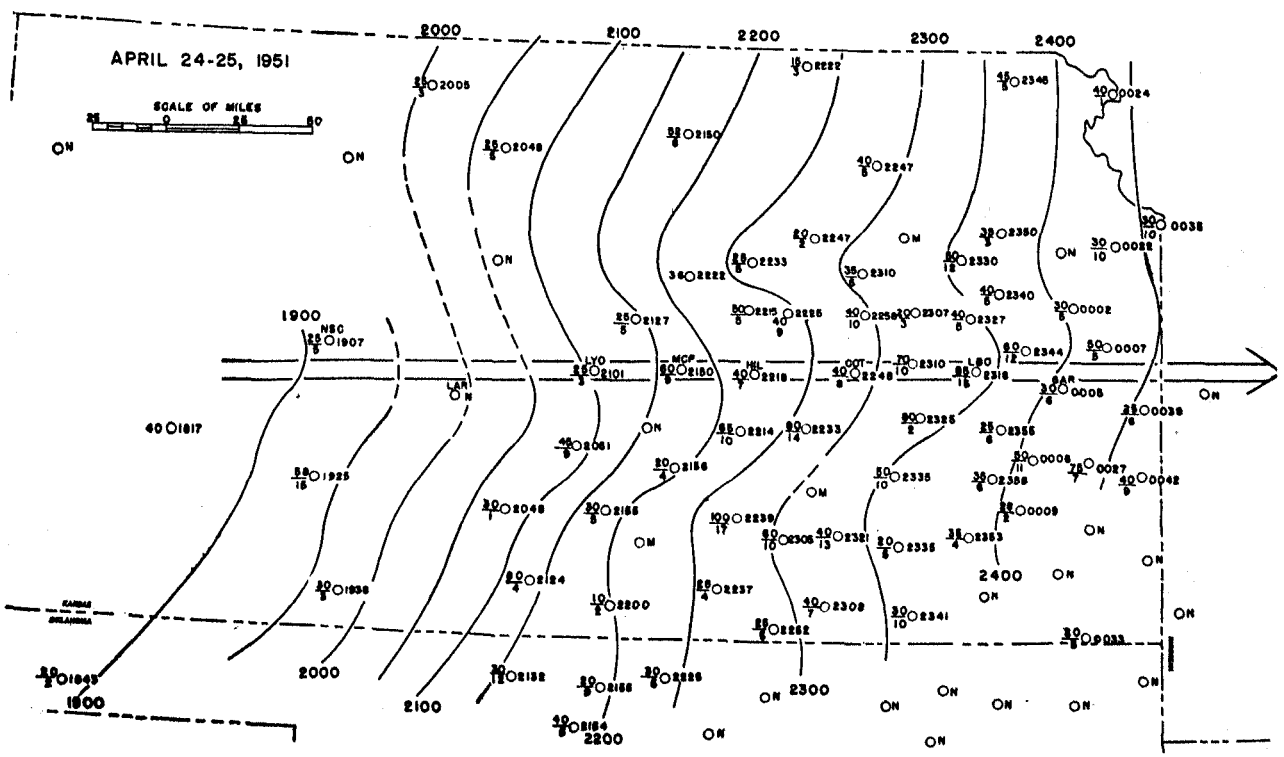

FIGURE 23.—Isochrone map of pressure jump line passing through Kansas. The time is Central Standard. The arrow shows the direction chosen for the x, t-diagram shown in figure 24.

Interpretation: The interpretation of these events according to the pressure jump hypothesis may be as follows. The cold front is quasi-stationary in western Kansas. A new push of cold air crosses the Rockies (analyzed by the WBAN Analysis Center as a "cold front aloft"). This additional push produces a wave on the cold frontal surface and this disturbance moves along as a gravitational wave. (See [1].) A marked temperature inversion ($\Delta\theta=8°$ C.) at about 5,000 feet m. s. l. exists ahead of the cold front. As the wave on the cold front pushes against the air beneath the inversion, it produces an elevation-type wave which moves out ahead of the cold front on the inversion. This elevation-type wave "breaks" into a pressure jump line in a manner described in Chapter I. Meanwhile the original wave on the cold front surface "breaks" on the surface causing the front to accelerate eastward. According to Abdullah [1], this action is conducive to cyclogenesis and indeed a wave does develop on the surface front.

Graphical Representation on the x,t-Diagram (fig. 24): The successive positions of the surface cold front are plotted as circles and a smooth curve is drawn through these points to represent the progress of the cold front. The cold front motion is far from constant but rather represents first a constant speed (up to about 1500 CST, April 24), then a deceleration (up to about 1830 CST), then a uniform speed (up to 2100 CST), then an acceleration (up to 2400 CST), then a uniform speed (up to 0330 CST April 25), and finally a deceleration. Undoubtedly, some of these variations may be due to the coarseness of the reporting station network by means of which any pinpointing of a frontal position is difficult. However, in lieu of an alternative, we shall accept as correct these positions as determined without any bias by the WBAN Analysis Center.

Since the height of the inversion in the air mass ahead of the front was at 5,000 feet m. s. l., we have included the position of the intersection of the frontal slope with the 850-mb. surface. Presumably, it is that part of the slope from the surface to the 850-mb. level which would be pushing against the air beneath the inversion. The progress of the 850-mb. position of the cold front was much more difficult to follow on the x, t-diagram since the upper air maps are drawn only every 12 hours. The April 25, 0300 GMT (April 24, 2100 CST) map yielded the one position at Lyons. The subsequent map yielded another point at 1500 GMT (0900 CST) off the diagram, and so a line drawn connecting these two points may be considered as representing the mean motion of the cold front at 850 mb. between these two times. By constructing a time cross section at Dodge City, we could interpolate for another position of the cold front at 850 mb. (1930 CST). We may draw a line between this point and the position of the front at 850 mb. at 1500 GMT, April 24 (0900 CST April 24, off the diagram) to represent the mean motion of the front at 850 mb. between 1500 GMT April 24 and 0130 GMT April 25. We now have specified completely the progress of the front at 850 mb. except between 1930 and 2100 CST. In order to join the two lines with a smooth curve, we are compelled to draw a curve with a point of inflection. This means that the front first accelerated and then decelerated.

Thus, we have specified completely on an x,t-diagram both the surface and 850-mb. positions of the cold front. We have represented these positions by continuous lines so that we have the continuous motion specified.

The 1830 CST position (at NSC) of the wave on the cold front surface (the front aloft) was available from the analysis of the surface synoptic maps. From previous maps we determine its speed and represent that speed by a line through its known position at 1830 CST. All significant synoptic lines have now been represented on the x,t-diagram.

We need make one modification, however. The theory as we have developed it implies that we are dealing with a horizontal lower surface and uniform velocities (through the vertical) below the inversion layer. Naturally, a cold front having a slope other than vertical, will not allow us to meet these conditions. We must replace then, the actual cold frontal slope below the inversion, with a vertical cold frontal slope. This approximation may be made if we take the midpoint of the positions of the cold front at the surface and at 850 mb. as the position of the "vertical" cold front. On the x,t-diagram the line starting from A represents the undisturbed motion of the "vertical" cold front. We stop this line at the point where it intersects the line representing the wave on the cold front. At this point the latter im-

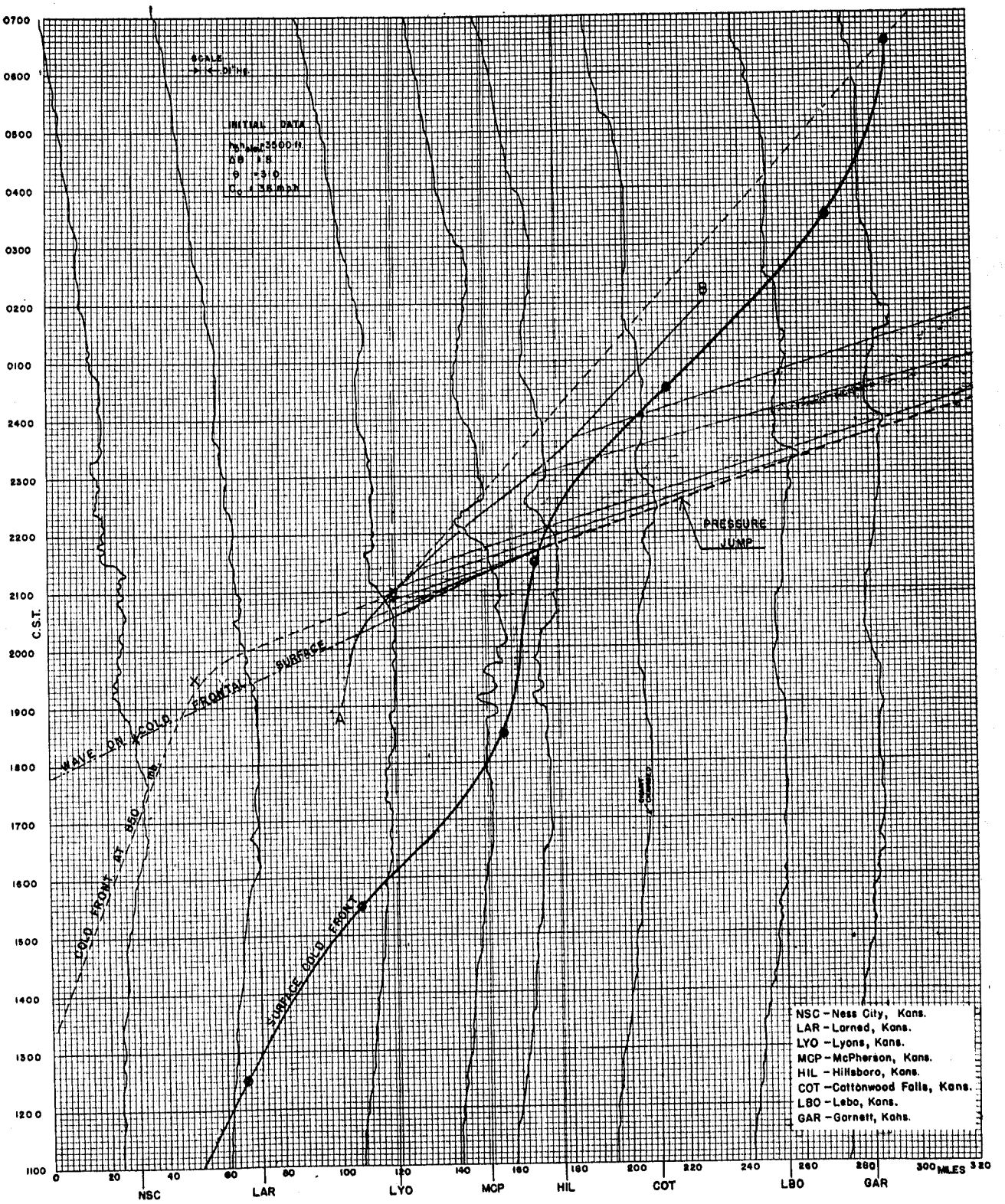

FIGURE 24.—x, t-diagram, April 24-25, 1951.

parts an acceleration to the vertical cold front. Similarly the line ending at B represents the final motion of the "vertical" cold front. We join these two lines with a smooth curve (without intersecting, of course, the 850-mb. and surface positions of the front). Line AB represents the motion of the accelerating "vertical" cold front.

The picture is now complete. The wave on the cold frontal slope hit the "vertical" cold front and accelerated it. This acceleration set off an elevation-type wave on the inversion surface ahead of the front. The wave then broke into a pressure jump. There is a suggestion of a deceleration following the acceleration. This deceleration produced a depression-type wave which followed the pressure jump in time. (The details for the construction of the characteristic lines are omitted here, but may be found in ch. I.)

The crosses and circles reflect *all* of the synoptic data used in the construction of the x,t-diagram and the characteristic lines. We shall now compare our diagram with the microbarograms for the stations located along the distance line. These traces have been drawn in their appropriate place (pressure increasing to the left). Each small division represents .01″ Hg. or about .3 mb. We shall consider the pressure traces in finer detail:

Ness City (*NSC*): As the wave on the cold front passed over the station (1830 CST) there was a marked pressure increase.

Larned (*LAR*): The wave on the cold front passed at about 1926 CST, approximately the time that the pressure began rising.

Lyons (*LYO*): The elevation-type wave had already been formed and it passed LYO at exactly the time that the marked pressure rise began (2032 CST). The total rise in pressure must be attributed not only to the pressure jump line but, in addition, to the wave on the cold frontal slope which passed shortly afterwards.

Reference to the x,t-diagram will reveal that the time of the rapid rise in pressure at the remaining stations fits very well with the calculated time of arrival of the jump as computed by the graphical method of characteristics.

Note the marked fall in pressure following the rise at LYO, MCP, and HIL. Our theory does not anticipate this. However, it is very interesting that this drop occurred only while the wave on the cold frontal surface travelled down the slope of the cold front. It may be that as the wave traveled down the slope its shape continued to change in such a way as to account for the pressure variation. The gentler fall in pressure at COT, LBO, and GAR (following the pressure jump) fits very well the depression-type wave characteristic lines (see ch. IV).

We also note that there was considerable agitation in pressure at MCP and HIL between 1900 and 2000 CST. We omit any explanation of this behavior except to point out that there may be some connection between this activity and the deceleration and stagnation of the cold front during that period.

We conclude then that our interpretation of the synoptic data of the day by means of the pressure jump hypothesis is consistent with the pressure data taken over a relatively dense network.

Finally, it is of interest to add the following precipitation data which cover the entire period being discussed:

LYO	None.	COT	Trace.
MCP	.01″	LBO	None.
HIL	.02″	GAR	None.

In other words, measureable precipitation fell only during the interval when the wave on the cold front and the pressure jump line were fairly close together. Their individual effects were inadequate to produce any measureable amount of rainfall.

CHAPTER IV

OTHER HYDRAULIC MODELS

In the previous chapter we have specified only one mechanism for the formation of a pressure jump line—the acceleration of a cold front into a region which could be described as a three-layer atmosphere. This mechanism is described schematically in figure 25. The cold front accelerates from its speed given by line AB to the speed given by line CD. The curved line BC indicates the acceleration (its slope is continually increasing, implying its velocity $\frac{dx}{dt}$ is also increasing). We prescribe that the air below the inversion and adjacent to the cold front moves with the speed of the front. We have then a distribution of velocity along a line in the x,t-plane. If in addition we know the initial inversion height, then by Case III, p. 7, we may solve for the flow. We have seen that this type of a distribution of conditions (p. 8) indicates an elevation-type wave. Figure 25 shows this elevation-type wave and the formation of a pressure jump line moving out ahead of the cold front.

This pattern is by no means the only way in which a pressure jump line may be produced. Following, we shall present other models for the formation of pressure jump lines as well as a propagating depression-type wave. This list is far from complete and the author is certain that other models can be and will be suggested by any investigator concerning himself with this theory.

THE STANDING PRESSURE JUMP LINE

Freeman, Bailey, and Byers [16] have found evidence of a pressure jump formed in a manner very similar to the formation of a standing hydraulic jump at the base of a spillway (fig. 26). This phenomenon, well known to the hydraulic engineer, is formed by the acceleration of water down the spillway. If this acceleration produces supercritical speeds in the liquid, then at the base of the spillway the water level changes abruptly from the high energy level (shooting water and low depth) to the low energy level (streaming water and high depth). Freeman, Bailey, and Byers proposed that in the atmosphere accelerating air down the cold frontal slope could by analogy

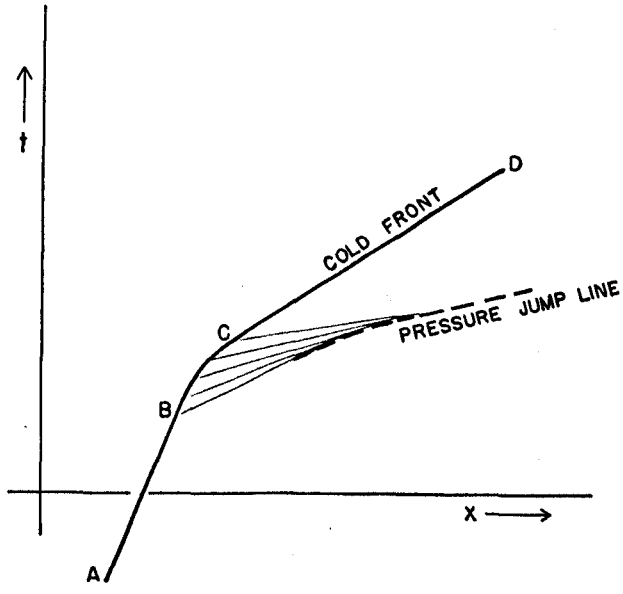

FIGURE 25. x, t-diagram showing genesis of pressure jump line as a result of the acceleration of a cold front. (Schematic).

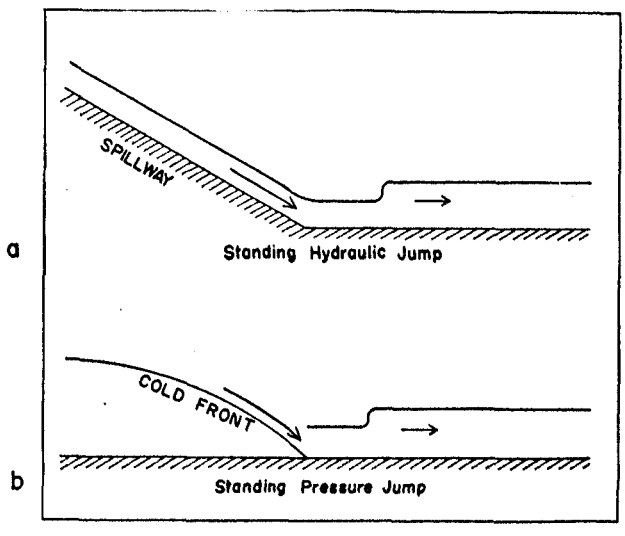

FIGURE 26. Models of standing hydraulic and pressure jumps.

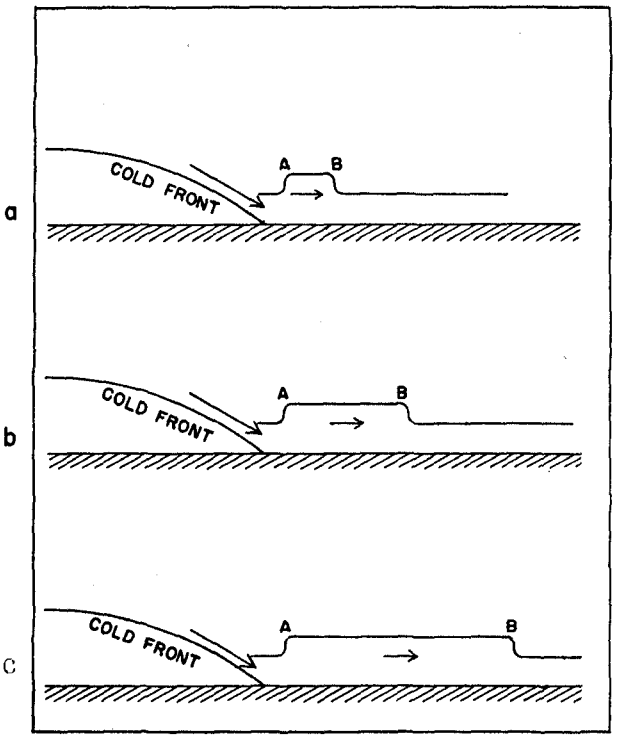

FIGURE 27.—The "double" pressure jump.

produce a standing pressure jump ahead of the front. They present synoptic data and show that this evidence is compatible with their hypothesis. This type of pressure jump line is more or less stationary relative to the cold front. It may, of course, move, but any such motion would be due to the fact that the cold front itself moves. Thus, if the cold front moved to the right (fig. 26b) a station would experience a *pressure fall* with the passage of the pressure jump line.

THE "DOUBLE" PRESSURE JUMP LINE

It is of particular interest that the synoptic case cited by Freeman, Bailey, and Byers relative to the standing pressure jump line contained an additional feature. A region of precipitation B (fig. 27) traveled out ahead of the front and, with time, the distance AB increased. The authors suggest that the precipitation was due to the thunderstorm activity associated with the pressure jump, A. However, this alone cannot explain why the distance AB increased with time. A more complete explanation may be given as follows:

We start with an undisturbed inversion height level. We then let air accelerate down the cold front slope. These accelerations produce a "bump" on the inversion surface. The forward half of this bump would be an elevation-type wave (see p. 8) which we have seen would move out to the right and develop into a jump. This jump could then produce precipitation in region B. With time, the length of the distance AB would naturally increase. On the other hand the backward half, A, would be the standing pressure jump described earlier. This experiment may easily be duplicated in a water channel. As a matter of fact, Freeman, Bailey, and Byers present a photograph of just such hydraulic evidence in their paper.

The double pressure jump line was inferred in Tepper's [60] model to explain the E–W oriented pressure jump lines which move northward into the Midwest. In this model, it was suggested that the fast-moving easterlies coming around the back side of the Bermuda High and meeting the land surface of the Gulf States would produce an E–W oriented pressure jump line which would move northward. Actually, a double pressure jump line would be formed. One would be a standing pressure jump where the supercritical flow would jump to a subcritical flow at a higher elevation and the second would be the propagation of this higher elevation northward and its development into a pressure jump line. The author, unfortunately, has no synoptic evidence to prove that this pattern actually occurs.

THE BACKWARD-MOVING PRESSURE JUMP LINE

Another mechanism for the formation of a pressure jump line is proposed in [59]. In the synoptic case cited, the pressure jump line moved in a direction opposite to the air flow in the lower layer, consequently the choice of the name "backward"-moving pressure jump. The mechanism for the genesis of the pressure jump line is as follows: The air flow beneath the inversion was turned from a general southerly flow to an easterly flow and directly normal to a stationary front which was oriented N–S. As in the case of water flowing against a barrier and producing a backward-moving hydraulic jump (fig. 28), it was suggested that by analogy a similar situation took place in the atmosphere. The air below the inversion flowed against the stationary front producing a pressure jump line which moved "backward" with respect to the air flow.

The explanation for the genesis of the pressure jump line follows from figure 9. We recall that the x-direction is the direction of propagation of the pressure jump line. In this case x is positive to the east. At some initial time $t=0$, we have a distribution of velocities such that the easterlies diminish toward the "barrier" (i. e., the stationary front). Since u is positive for a westerly wind, we might interpret the previous remark as indicating $u_i > u_j$ for $x_i < x_j$. This is exactly the condition for an elevation-type wave and its final form—a pressure jump line.

GENERAL PRESSURE JUMP LINE MODELS

For a moving pressure jump line, the criterion described on p. 8 must be satisfied. We must have a situation which produces either

(a) $h_i > h_j$ for $x_i < x_j$ (an elevation-type wave)

or (b) $u_i > u_j$ for $x_i < x_j$ (an acceleration of the air below the inversion).

From equation (14), chapter I, we have seen that (a) implies (b), and (b) implies (a). However, in any specific situation it might be more evident that we are dealing with one or the other.

For stationary pressure jump lines we must have conditions as specified in Appendix B. We must be dealing with supercritical flow below an inversion and some pattern that disturbs this flow.

The models discussed so far relate to pressure jump lines only. In fact, pressure jump lines are of particular interest because the vertical motion which they impart to the atmosphere into which they surge produces violent reactions (turbulence, thunderstorm activity, precipitation, etc.). However, the theory described in chapter I allows for depression-type waves as well. Just as elevation-type waves and pressure jumps would tend to produce upward motion, depression-type waves should tend to produce descending motion. As such we might expect to find clearing and general improvement of weather conditions with the passage of a depression-type wave.

DEPRESSION-TYPE WAVE

A very remarkable instance of the passage of the depression-type wave was observed over Washington, D. C., in December 1949 and reported on by Tepper [61]. On December 6, 1949, an accelerated microbarograph in Washington produced a unique trace at the time that a peculiar cloud bank passed overhead. The rear (western) edge of this cloud bank ended very abruptly and had the appearance of the edge of the Antarctic Ice Barrier. A study of the synoptic conditions of the day revealed that this pressure variation was produced by a depression-type wave which traveled eastward from the Midwest and was propagated on an inversion surface with a speed far in excess of the wind speed in that layer. The cloud bank also traveled at the level of the inversion surface, but was advected by the prevailing winds. The conclusion was reached that the depression-type wave which progressively overtook the cloud bank was associated with a rapid drop of the isentropic surfaces which in turn produced the adiabatic heating necessary to desiccate the cloud partially in the west and completely over Washington. Confirmation of this conclusion was found in the solar radiation records (fig. 29) of the day. The general feature of the solar radiation traces is that they contain two significant points—one is a marked decrease in radiation and the other is a subsequent sharp increase in radiation.

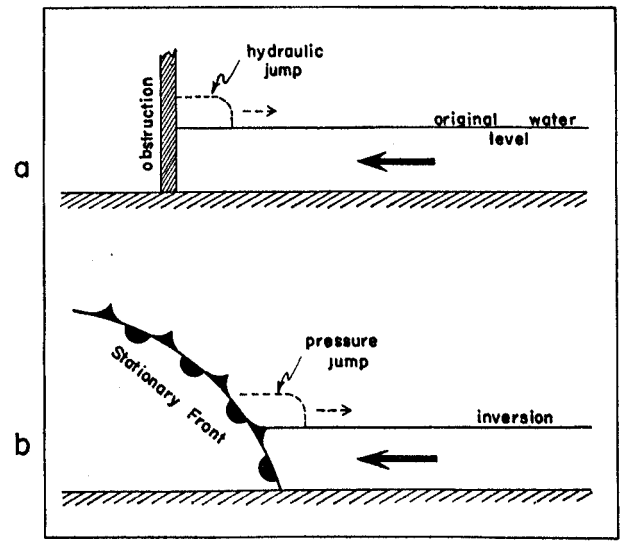

FIGURE 28.—The backward moving pressure jump.

In table 3, we compare (1) the time of decrease in radiation with the time of onset of the cloud bank and (2) the time of increase in radiation with the time of passage of the depression-type wave. Considering the errors that may arise in reading the traces, the difference in location of the instru-

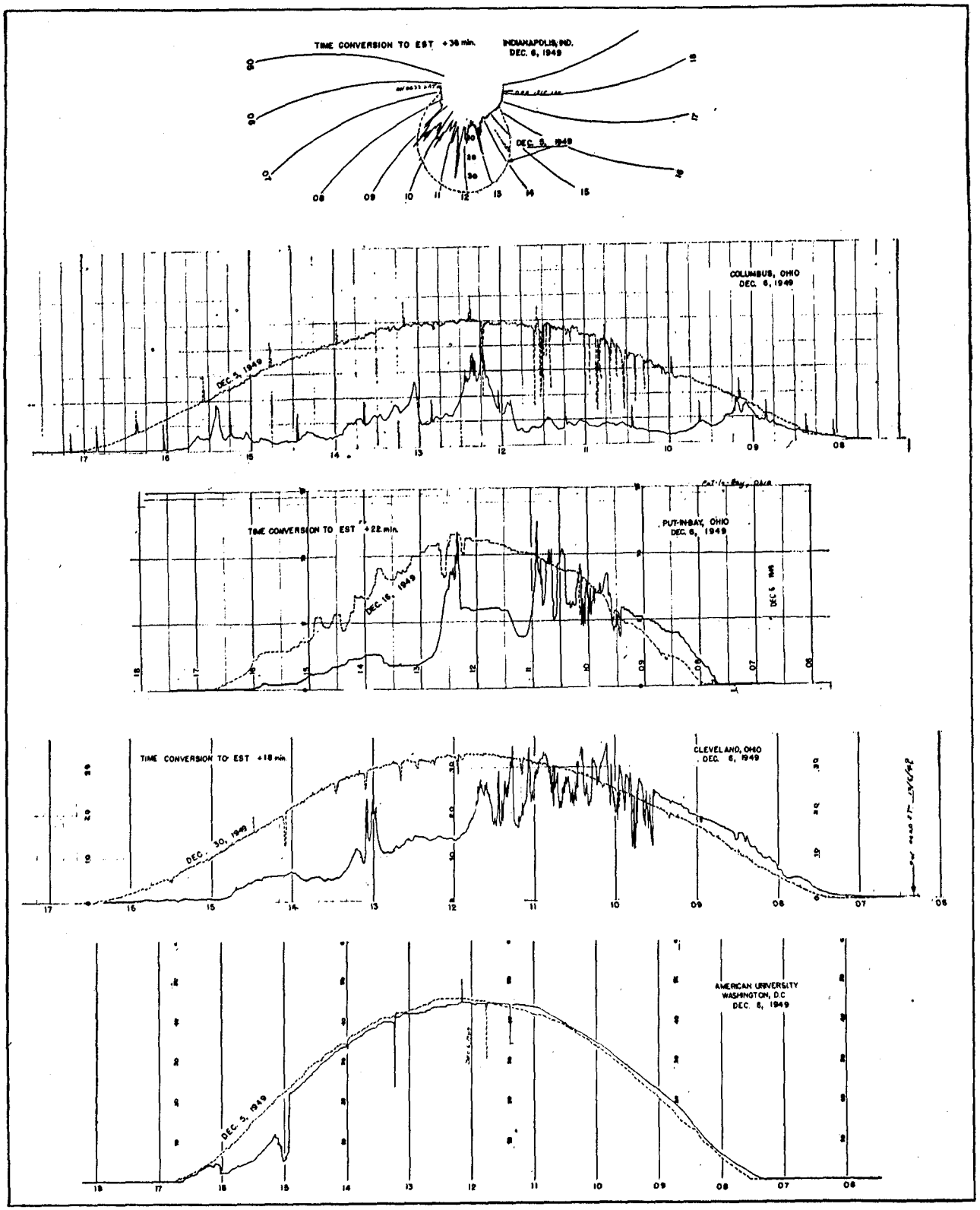

FIGURE 29.—Radiation traces for Washington, D. C., Cleveland, Put-In-Bay, and Columbus, Ohio, and Indianapolis, Ind., December 6, 1949. A relatively clear-day radiation trace is indicated by the dashed line.

TABLE 3.—*Times of sudden decrease and increase in solar radiation. Figures in parentheses refer to time of arrival of the cloud bank and figures in brackets to time of passage of pressure pulse*

	Time (EST)			
	Decrease		Increase	
Indianapolis, Ind	----	(0500)	0922	[0920]
Columbus, Ohio	0914	(0925)	1148	[1215]
Put-in-Bay, Ohio	0947	(1000)	1240	[1230]
Cleveland, Ohio	1200	(1203)	1315	[1320]
Washington, D. C	1455	(1455)	1602	[1610]

ments at any one city, and the possible lack of accurate time considerations in taking observations, the agreement in the time values is remarkable. We may safely associate the decrease in radiation with the onset of the cloud bank and the sudden increase in radiation with the passage of the pressure pulse.

For further details, the reader is referred to the original article [61].

CHAPTER V

SUMMARY

In this paper it has been demonstrated that by applying the hydraulic analogy to special classes of meteorological situations, a consistent synoptically significant phenomenon may be arrived at. This phenomenon is a pressure jump line.

The pressure jump line is a particular kind of gravitational wave traveling along an inversion surface. Its main characteristic is that it has a very steep slope so that it appears on microbarograms as a jump in pressure. This very steep slope (estimated at about 1:5 as compared with 1:80 for cold fronts and 1:200 for warm fronts) is responsible for marked forced lifting of the atmosphere. Under favorable moisture conditions this lift produces violent convection and thunderstorm activity.

In order to identify a pressure jump line, a meteorologist must use accelerated microbarographs. These instruments give a finer resolution of the time scale and as such can differentiate between pressure jumps and more gradual pressure rises. Channels could be developed by means of which information about pressure jumps, identified by accelerated microbarograms, could be communicated to a central analysis unit. By plotting the time of the jumps on a map, isochrones marking the progress of the line could be indicated. Then by reference to the thermodynamic character of the atmosphere, the meteorologist could anticipate the type of weather to expect with the progress of the pressure jump line.

This procedure involves the knowledge about the existence of a pressure jump line. Can anything be said about forecasting its development? Indeed there can, but unfortunately not too much. First of all, the meteorologist must become suspicious whenever a cold front is moving into a region described in this paper as a three-layer atmosphere. It is recommended that in such cases he choose a direction normal to the cold front as his x-direction and begin plotting an x,t-diagram, similar to those presented here. On this diagram he should plot

(1) The successive positions of the cold front.

(2) The weather at the various stations along his x-direction.

(3) A continuous plot of the pressure, if available, of the various stations along the x-direction. In the absence of this information, he may plot any pressure information available to him.

If the cold front begins to show rapid accelerations, the meteorologist could then construct the resulting characteristic lines. The weather elements along the direction of these lines should be carefully scrutinized for any indications of a pressure jump line.

We might go one step further and look for early indications that a cold front might be expected to accelerate in the near future. One such indication might be the appearance of a fresh surge of colder air within the cold air mass itself (somewhat like Abdullah's type of wave referred to in ch. III). We have seen (fig. 24) that such a surge behind the cold front could produce subsequent accelerations of the cold front. Another sequence producing cold front accelerations, and noted in several pressure jump cases, seems to be as follows: A quasi-stationary E–W cold front lies across the Central States to the Rockies. When a small low pressure system crosses over the mountains south of the cold front, the portion of the front lying west of the low center rushes southward. This acceleration produces a pressure jump line. These are but two mechanisms by means of which cold fronts seem to accelerate. Undoubtedly, there are many others. Once the practicing meteorologist begins to look for pressure jump lines, it is inevitable that a comprehensive listing of mechanisms which produce accelerations of cold fronts will follow.

What about pressure jump lines that are started by conditions not related to a cold front? We have seen that any meteorological situation producing either an acceleration of the air below the inversion, or an elevation of the inversion surface could yield a pressure jump line. It is left, then, to the ingenuity of the individual meteorologist to recognize such a situation and handle it in a man-

ner similar to the one suggested here for the accelerating cold front.

The bulk of this paper has concerned itself with the pressure jump line and a discussion of its meteorological properties resulting from a consideration of the analogy between a pressure jump line, a hydraulic jump, and a shock wave. However, the analogy need not stop here. The inherent properties of the flows, as described in table 1, suggest some interesting extensions of the analogy. We might, for example, look for highly specialized properties of the phenomena in their respective domains and by means of the "dictionary" contained in table 1, translate these specialized properties so as to apply in another domain. One such procedure was suggested by Tepper [60] in a speculation concerning the origin of tornadoes.

In gas flows, it has been demonstrated (see [13] and [9]) that when two shock waves intersect, they produce a vortex sheet in the flow behind them. Across this sheet, the flow is parallel but of different speed so that a concentrated zone of vorticity is formed. Translated into the language of the three layer atmosphere, we might state: *When two pressure jump lines intersect, they produce in the flow behind them a vortex sheet. Across this sheet, the flow is parallel but of different speed so that a concentrated zone of vorticity is formed.* The suggestion was then made that tornadoes could form in this zone of vorticity. Several instances consistent with this hypothesis were presented in [59] and [60], where it was shown that tornadoes were apparently associated in both time and space with the space-time locus of the intersection zone of two (or more) pressure jump lines.[1]

In conclusion, we may state that the analogy discussed in this paper between certain atmospheric flows and liquid and/or gas flows opens up a new avenue of study in meteorology, one which it is hoped will lead to a better understanding of certain meteorological processes.

[1] In a study of the 1950 tornado season, the author found that out of the 130 individual tornado occurrences studied, 71 (or 55 percent) could be located within 60 miles of their corresponding location on the locus of what might be interpreted as an intersection of two pressure jump lines. What was even more significant was the fact that many of the occurrences which did not verify were nonetheless associated with one pressure jump line. This seems to indicate that there undoubtedly are mechanisms by which tornadoes are formed other than that suggested by the intersection hypothesis, but that these other mechanisms are associated with some feature of a pressure jump line.

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APPENDIX A

SOME MATHEMATICAL REMARKS ON THE SINGLE WAVE PROBLEM

In chapter I we investigated graphically the three possible classes of single waves resulting from our basic equations. Herewith, we shall present the same results but from a more mathematical point of view.[1]

Let us investigate the local change in the shape of the wave profile as we progress with a point on the wave, i. e., along a single characteristic line. Thus, we are interested in $\frac{d}{dt}\left(\frac{\partial h}{\partial x}\right)$ where the differentiation is along a given characteristic line such that

$$\frac{d}{dt}=\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}. \tag{1}$$

Differentiating equation (6), Chapter I, yields

$$2c\frac{\partial c}{\partial x}=\left(1-\frac{\rho'}{\rho}\right)g\frac{\partial h}{\partial x}.$$

Since c is constant along the characteristic line, we have

$$\frac{\partial h}{\partial x}\propto\frac{\partial c}{\partial x} \tag{2}$$

along the same characteristic line.

Thus, to investigate the change in shape of the wave profile as we progress with the point on the wave, we prefer to study $\frac{d}{dt}\left(\frac{\partial c}{\partial x}\right)$. Applying (1), we may write

$$\frac{d}{dt}\left(\frac{\partial c}{\partial x}\right)=\frac{\partial}{\partial t}\left(\frac{\partial c}{\partial x}\right)+(u+c)\frac{\partial^2 c}{\partial x^2}. \tag{3}$$

But by equation (13), Chapter I,

$$u+c=3c+\beta. \tag{4}$$

Therefore equation (3) may be written as

$$\frac{d}{dt}\left(\frac{\partial c}{\partial x}\right)=\frac{\partial}{\partial x}\left[\frac{\partial c}{\partial t}+(u+c)\frac{\partial c}{\partial x}\right]-3\left(\frac{\partial c}{\partial x}\right)^2.$$

Since c is constant along the characteristic line, the expression in the brackets vanishes, and we have remaining

$$\frac{d}{dt}\left(\frac{\partial c}{\partial x}\right)=-3\left(\frac{\partial c}{\partial x}\right)^2. \tag{5}$$

The solution of (5) is given immediately by

$$\left(\frac{\partial c}{\partial x}\right)^{-1}=3(t-t_0)+\left(\frac{\partial c}{\partial x}\right)_0^{-1}, \tag{6}$$

where $\left(\frac{\partial c}{\partial x}\right)_0$ represents the local slope of the profile at an initial time $t=t_0$.

Uniform flow (no wave). If $\left(\frac{\partial c}{\partial x}\right)_0\to 0$, then for all t, $\left(\frac{\partial c}{\partial x}\right)\to 0$. By (4) this means $\frac{\partial}{\partial x}(u+c)\to 0$ or the characteristic lines are parallel.

Depression-type wave. In this case $\left(\frac{\partial c}{\partial x}\right)_0>0$. Thus, with increasing time $\left[\frac{\partial c}{\partial x}\right]^{-1}$ must increase. This means that the wave flattens out. By (4) we have $\frac{\partial}{\partial x}(u+c)$ decreases or the characteristic lines must fan out.

Elevation-type wave. In this case $\left(\frac{\partial c}{\partial x}\right)<0$. Thus, with increasing time and until the value of t given below, $\left|\frac{\partial c}{\partial x}\right|$ must increase. This means that the wave steepens. By (4), $\frac{\partial}{\partial x}|u+c|$ must increase or the characteristic lines converge.

Equation (6) is particularly useful, in that it yields the time at which the wave of elevation (in the vicinity of the point of the wave under consideration) steepens into a vertical slope (the jump). The time is given for

$$\frac{\partial c}{\partial x}\to\infty \text{ or } 3(t-t_0)=-\left(\frac{\partial c}{\partial x}\right)_0^{-1}.$$

[1] The author is indebted to Professor D. C. Lewis of the Mathematics Department, Johns Hopkins University, for suggesting the approach given here.

APPENDIX B

THE GENERAL STEADY STATE TWO-LIQUID PROBLEM

In chapter I, we succeeded in arriving at a solution for the unsteady state flow in the two-liquid problem. However, in so doing, we have had to ignore completely the nature of the upper flow and consequently have been unable to evaluate the effect of the upper flow on the over-all pattern.

We shall now restrict our attention to the steady state problem, but here we shall include the upper flow. In this manner we shall attempt to learn something specific about the importance of the upper flow. In particular, we shall seek the answers to the following questions: (1) Under what conditions does the upper flow produce negligible effects? In these circumstances we might expect that our unsteady state solution may be applicable. (2) When the effect of the upper flow is not negligible, in what manner does it affect the over-all flow pattern? In these circumstances, the unsteady state solution may be considered as a first approximation only.

The Problem: We establish our coordinate system with the jump, so that the liquids appear to enter the jump at one level and leave it at another. All velocities are relative to the jump speed. The variables used in the problem are given in figure 30. The subscripts 1 refer to the conditions on that side of the jump where the lower liquid appears to enter the jump, and the subscripts 2 refer to the conditions on the other side of the jump where the lower liquid appears to leave the jump. We retain assumptions 1, 2, and 4 given at the beginning of chapter I. In addition, we specify that while the surface $z=0$ is a horizontal surface, the height of the free surface, $z=H$, need not remain constant across the jump. This latter condition will later be amended in order that our model remain physically the same.

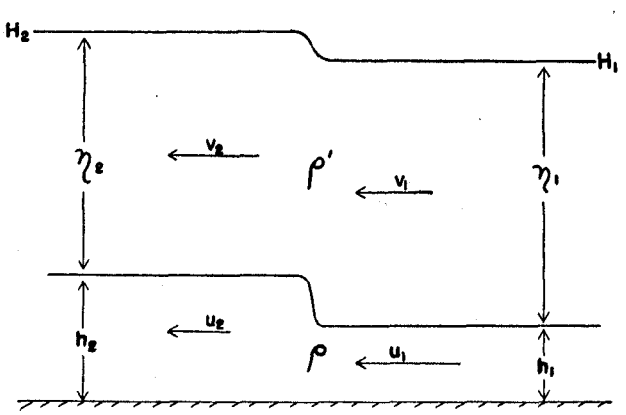

FIGURE 30.—The general steady-state two-liquid problem.

There are two continuity equations

$$(1) \qquad u_1h_1=u_2h_2$$
$$v_1\eta_1=v_2\eta_2$$

while by definition

$$(2) \qquad h_1+\eta_1=H_1$$
$$h_2+\eta_2=H_2$$

We have assumed the pressure hydrostatic, so that

$$(3) \quad p \text{ (at height } z \text{ in the lower liquid)}= \rho' g(H-h)+\rho g(h-z)$$
$$p' \text{ (at height } z' \text{ in the upper liquid)}=\rho' g(H-z')$$

In order to write the momentum condition, that the difference in the pressure force equals the difference in momentum flux, we note that the difference in the pressure force

$$\Delta P=\int_0^{H_1} p_1 dz-\int_0^{H_2} p_2 dz=\frac{\rho' g}{2}(H_1^2-H_2^2)-(\rho-\rho')\frac{g}{2}(h_2^2-h_1^2)$$

while the difference in momentum flux is given by

$$\Delta M=\rho h_2u_2^2+\rho'\eta_2v_2^2-\rho h_1u_1^2-\rho'\eta_1v_1^2$$

Equating $\Delta P=\Delta M$, using (1), and defining for convenience

$$(4) \qquad x=\frac{h_2}{h_1} \qquad F^2=\frac{u^2}{gh} \qquad s=\frac{\rho'}{\rho}$$
$$y=\frac{\eta_2}{\eta_1} \qquad \phi^2=\frac{v^2}{g\eta} \qquad r_1=\frac{h_1}{\eta_1},$$

the momentum condition reduces to

$$(5)\quad r_1^2(x-1)\left[\frac{x+1}{2}-\frac{F_1^2}{x}\right]+ s(y-1)\left[\frac{y+1}{2}-\frac{\phi_1^2}{y}\right]=sr_1(1-xy)$$

Since we have not restricted the free surface to be a constant across the jump, equation (5) represents the momentum equation for the general two-liquid steady-state problem. In any given situation, r_1, and s are considered known since they refer to undisturbed properties of the liquid. Thus, if we are given in addition, either the flow velocities (represented by F_1 and ϕ_1) or the wave profiles (represented by x and y) this equation reduces to a single equation in two unknowns. We need specify another condition in order that the problem be determined. In our model this other condition is that H, the height of the free surface, is a constant. Before we apply this restriction, we shall inspect equation (5) briefly and draw two very simple relations from this general equation.

1. A trivial solution is given by $x=y=1$, the case of no jump.
2. Consider the case $s=0$, or $\rho'\ll\rho$ (the condition for a single liquid)

Equation (5) reduces immediately to

$$(6)\qquad \frac{x+1}{2}-\frac{F_1^2}{x}=0$$

which is the standard equation relating the height ratio to the Froude number.

We shall now return to the conditions of our basic model and impose the additional condition that

$$(7)\qquad H_2=H_1$$

It follows readily that

$$(8)\qquad y=r_1+1-r_1x.$$

Substituting into (5), we have

$$(9)\qquad \frac{x+1}{2}-\frac{\mathsf{F}_1^2}{x}+\Phi_1^2\frac{s}{r_1(r_1+1-r_1x)}=0$$

where F_1 and Φ_1 are modified Froude numbers given by

$$(10)\qquad \mathsf{F}^2=\frac{F^2}{1-s}\qquad \Phi^2=\frac{\phi^2}{1-s}.$$

The case where $\Phi_1=0$ produces immediate results. In this case (9) reduces to

$$(11)\quad \bullet\bullet\bullet\qquad \frac{x+1}{2}-\frac{\mathsf{F}_1^2}{x}=0$$

which is analogous to (6), the equation for a single liquid where, however, we replace the Froude number by a modified Froude number. This equation can be solved for u_1 in terms of $x\left(=\frac{h_2}{h_1}\right)$ to give the speed of a jump moving as a steady state phenomenon into a region where the lower liquid is at rest and the upper liquid is moving with the speed of the jump. The solution yields

$$(12)\qquad v_j=-u_1=-\sqrt{\frac{1}{2}\frac{h_2}{h_1}g(h_1+h_2)\left(1-\frac{\rho'}{\rho}\right)}.$$

Returning to the basic equation (9) where Φ_1 need not be zero, we note that this equation is a cubic in x.[1]

A graphical solution for x is given by a successive application of figure 31 which is a nomogram solution for the equation

$$(14)\qquad \frac{x+1}{2}-\frac{\mathsf{F}_1^2}{x}+\Phi_1^2\frac{s}{r_1}=0.$$

The procedure for the use of this nomogram is as follows:

Step 1: Enter the left-hand side with the initial values of r_1 and s and find the corresponding value of $a_1=\frac{s}{r_1}$ on the ordinate.

Step 2: Enter the right-hand side with this value of a_1 and the proper value of Φ_1. Find the corresponding point on the b curves.

Step 3: Follow the b curve until it intersects the proper value of F_1. This will give the first approximation for the solution of (9) x_1.

Step 4: Compute $a_2=\frac{s}{r_1(1+r_1-r_1x_1)}$

---

[1] Dr. Benton has indicated to the author that for a sufficiently deep upper liquid $y=1+r_1-r_1x\sim 1$, and (9) may be put into the "standard form", viz.

$$(13)\qquad \frac{X+1}{2}-\frac{\mathfrak{F}_1^2}{X}=0$$

where $\quad X=\frac{x}{1+2\theta}\quad \mathfrak{F}=\frac{\mathsf{F}}{1+2\theta}\quad \theta=\Phi_1^2\frac{s}{r_1}$

Equation (13) would express the relation between a modified height ratio and a further modified Froude number; in each case, the modification is due to the flow velocity in the upper liquid.

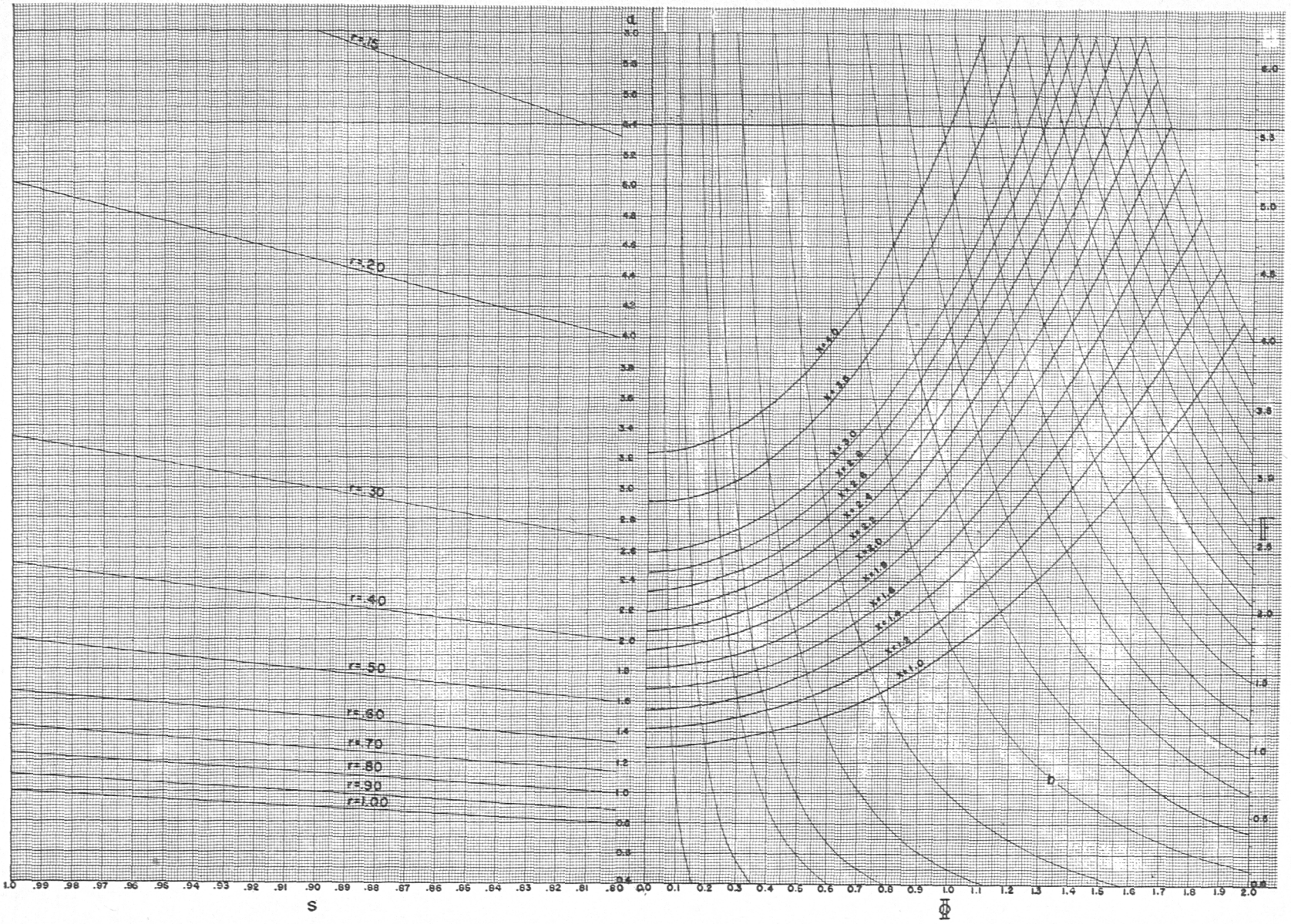

FIGURE 31.—Nomogram solution for the equation

$$\frac{x+1}{2}-\frac{F_1^2}{x}+\Phi_1^2\frac{\delta}{r_1}=0.$$

Step 5: Repeat Step 2 etc. using a_2 instead of a_1 and find the next approximation x_2.

Step 6: The series $x_1, x_2, x_3, \ldots$, if it converges, will converge to the solution x.

Several numerical examples in the range of values under interest have indicated that the series $x_1, x_2, x_3 \ldots$ converges very rapidly so that the solution for (9) can readily be found.

There are several conclusions that we may draw about the nature of the flow by reference to equation (14) or figure 31. The conclusions are drawn only for our case, namely that in which H remains constant.

1. In order that a jump occur on the interface (i. e., $x>1$) the modified Froude number, F_1, must also exceed 1. By equations (10) and (4) of this appendix and (6) of chapter I, this implies that $|u_1|>|c_1|$. If we interpret c_1 as a "critical wave speed" characterizing the conditions ahead of the jump, then this result implies that a (stationary) *jump exists only for supercritical flow.* Furthermore, since we have considered u_1 as the flow ahead of the jump relative to the jump, then for a jump propagating into a liquid at rest $|u_1|=|v_j|$ where v_j is the speed of the jump. Thus, *the speed of the jump exceeds the critical wave speed ahead of the jump.*

   In gas flow and in one-liquid flow (columns 1 and 2, table 1) analogous relations exist. In addition, it can be shown that for these flows, the flow behind the shock wave, or hydraulic jump, is subsonic or subcritical, respectively. In two-liquid flow, the flow that we are investigating, this relation is valid only for the case $\Phi_1=0$. It readily follows from (14), (10), and (1) that for $x>1$, $|u_2|<|c_2|$ provided $\Phi_1=0$. Thus, if we interpret c_2 as a "critical wave speed" behind the jump, we may state that *when the upper liquid is at rest relative to the jump, the flow behind the jump is subcritical.* This rule has not been proved for large departures of the flow speed of the upper liquid from the jump speed.

2. Figure 31 reveals another important effect of the flow of the upper liquid. On the right-hand side of the graph it can be noted that all the x-equals-constant curves have their minimum at $\Phi_1=0$. This means that for the same initial depths of the liquid (h_1, η_1) and the same final depths (h_2, η_2), the value of the jump speed (for a moving jump, the jump speed is $|v_j|=|u_1|=\mathsf{F}_1\ [(1-s)\ gh]^{1/2}$) is a minimum when the upper liquid is moving with the speed of the jump. For any other value of the speed of the upper liquid, the jump speed is greater.

3. It will be recalled that in the unsteady state development we ignored the effect of the upper liquid. Thus we shall be particularly interested in those ranges where variations in the upper flow produce little modification in the results. From figure 31, it will be noted that such a range is given in the neighborhood of $\Phi_1=0$. For weak jumps (i. e., x near 1), the range of values over which Φ may extend without causing a large percentual change in F_1 is wide. However, for large values of x, such a range for Φ_1 is limited more closely to the neighborhood of $\Phi_1=0$.

   For moving jumps we may interpret this as follows: For relatively weak jumps the speed of the jump is altered but little for a relatively wide range of departures of the speed of the upper liquid from the jump speed. However, for strong jumps, significant departures of the flow of the upper liquid from jump speed tend to increase the speed of the jump relative to what it would have been were the upper liquid flowing with jump speed.

Energy Considerations. So far we have restricted our attention to the momentum and continuity considerations only. While these yield important relations, they fail to shed any light on the required direction of flow relative to the jump. We have explicitly assumed that the lower liquid enters the jump at a lower level than the one at which it leaves it. This restriction that $x>1$ is, of course, not necessary for the mathematical solution of the momentum equation. Consistent solutions are possible involving $x<1$. These solutions on occasion will yield $\mathsf{F}_1<1$.

We shall now prove that in most cases $x<1$ is not a *physically* realizable situation and that in the remaining cases it applies to situations which are not of interest to us. This we shall do by applying the following energy requirement: Since the flow is exceedingly turbulent in the jump zone we postulate that there must be a decrease in

energy flux as the two liquids pass through the jump.

The energy flux (per unit width per unit time) of the lower and upper liquids are given respectively by

$$(15)\qquad e=\left(\frac{\rho u^2}{2}+\rho g h+\rho' g\eta\right)uh$$

$$e'=\left(\frac{\rho' v^2}{2}+\rho' g h+\rho' g\eta\right)v\eta.$$

We shall consider two cases.

CASE I: u and v have the same sign. In this case, we postulate that the total energy flux must decrease through the jump since an exchange of energy between the two liquids is possible. Thus, using (1) and (7) and (15)

$$\Delta E\equiv(e+e')_1-(e+e')_2=$$

$$\left[\frac{u_1^2-u_2^2}{2}+g(h_1-h_2)+sg(\eta_1-\eta_2)\right]\rho u_1h_1+$$

$$\left[\frac{v_1^2-v_2^2}{2}\right]\rho' v_1\eta_1$$

Eliminating u_2 and v_2 by the equations of continuity and after a few simple algebraic manipulations, we arrive at

$$\boldsymbol{E}=\frac{\Delta E}{(1-s)\rho g u h h_1}=(x-1)\left[\frac{\Phi_1^2}{2}\frac{x+1}{x^2}-1\right]+$$

$$\frac{s}{2}\Phi_1^2\frac{v_1}{u_1}\left(\frac{\eta_1}{h_1}\right)^2\left(1-\frac{1}{y^2}\right)$$

Substituting the momentum condition (9), this reduces to

$$\boldsymbol{E}=\Phi_1^2\frac{s}{2}\frac{(x^2-1)}{r_1xy}\left[1-\frac{v_1}{u_1}\frac{x(y+1)}{y(x+1)}\right]+\frac{(x-1)^3}{4x}$$

where $\boldsymbol{E}$ must be larger than zero.

(a) If $x>1$, then the energy condition will be satisfied provided

$$\frac{v_1}{u_1}<\frac{y(x+1)}{x(y+1)}\left[\frac{(x-1)^2r_1y}{2s\Phi_1^2(x+1)}+1\right]$$

(b) If $x<1$, then the energy condition will be satisfied provided

$$\frac{v_1}{u_1}>\frac{y(x+1)}{x(y+1)}\left[\frac{(x-1)^2r_1y}{2s\Phi_1^2(x+1)}+1\right]>1.$$

CASE II: u and v have opposite signs. In this case we postulate that the energy flux of each liquid separately must decrease through the jump since an exchange of energy in the direction of flow is not possible. Thus,

$$0<\Delta e=\left[\frac{u_1^2-u_2^2}{2}+g(h_1-h_2)+sg(\eta_1-\eta_2)\right]\rho uh$$

$$0>\Delta e'=\left[\frac{v_1^2-v_2^2}{2}\right]\rho' v\eta.$$

Eliminating u_2 and v_2 and applying the momentum condition (9) to Δe we have

$$0<\boldsymbol{E}=\frac{\Delta e}{(1-s)\rho g u h h_1}=(x-1)\left[\Phi_1^2\frac{s}{2}\frac{(x+1)}{r_1xy}+\frac{(x-1)^2}{4x}\right]$$

$$0>\boldsymbol{E}'=\frac{\Delta e'}{(1-s)\rho' g v\eta\eta_1}=(y-1)\left[\Phi_1^2\frac{y+1}{2y^2}\right].$$

The first of these is possible if and only if $x>1$. This implies $y<1$ which automatically satisfies the second condition.

If we transpose our coordinate system so as to be relative to the earth and assume that the steady state jump is moving into a liquid at rest, we may interpret these results as follows:

The necessary conditions for the existence of a jump under these conditions are:

$$\text{(a)}\quad x>1,\ \frac{v_1}{u_1}<\frac{y(x+1)}{x(y+1)}\left[\frac{(x-1)^2r_1y}{2s\Phi_1^2(x+1)}+1\right]$$

whenever the upper liquid is moving in the same direction as the jump but usually with a slower speed.

$$\text{(b)}\quad x<1\ \text{ and }\ \frac{v_1}{u_1}>\frac{y(x+1)}{x(y+1)}\left[\frac{(x-1)^2r_1y}{2s\Phi_1^2(x+1)}+1\right]$$

whenever the upper liquid is moving in an opposite direction than the jump.

(c) $x>1$

whenever the upper liquid is moving in the same direction but faster than the jump.

In the atmospheric situations to which this theory has been applied, the upper liquid is almost always moving in the same direction as the jump. Thus, since (b) is rarely if ever applicable, we need restrict our attention only to the other two possibilities, namely when $x>1$.

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